Scientific Computing and FEM
Exercise sheet 4

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Exercise (A)
We are given the PDE

$$
\begin{aligned}
-u''(x) + au(x) &= f(x), \quad \text{for} \quad x \in (0,1),\\
u(0) &= 0,\\
\frac{\partial u(1)}{\partial \vec{n}} &= \alpha(g_b - u(1))
\end{aligned}
$$

for $a, \alpha, g_b \in \mathbb{R}$.

Solution (A)
We use integration by parts and choose v with $v(0) = 0$ such that unwanted terms vanish. After plugging in the Neumann data and bringing it to the other side we get the variational formulation:

$$
\begin{cases}
\text{Find } u \in V_0, \text{ such that}\\
a(u,v) = \langle F, v\rangle, \quad \forall v \in V_0,
\end{cases}
$$

with

$$
\begin{aligned}
&a(u,v) = \int_0^1 u'(x)v'(x) + au(x)v(x)dx + \alpha u(1)v(1),\\
&\langle F, v\rangle = \int_0^1 f(x)v(x)dx + \alpha g_b v(1),\\
&V_0 = \{v \in H^1(0,1) \colon v(0) = 0\}.
\end{aligned}
$$

For the discretization, we choose $S_h^1(0,1) \subseteq H^1(0,1)$ as the space piecewise linear, continuous basis functions. For $N+1$ equidistant nodes $x_0 < x_1 < \ldots < x_N$, $x_i = ih$ with distance $h = \frac{1}{N}$, the basis functions have the form

$$
\varphi_k(x) = \begin{cases}
\frac{1}{h}(x - x_{k-1}) & \text{for } x \in (x_{k-1}, x_k),\\
\frac{1}{h}(x_{k+1} - x) & \text{for } x \in (x_k, x_{k+1}),\\
0 \quad \text{else},
\end{cases} \qquad \forall k = 0, \ldots, N.
$$

This gives us the discrete variational formulation

$$\begin{cases} \text{Find } u_h \in V_h \text{, such that} \\ a(u_h, v_h) = \langle F, v_h \rangle, \quad \forall v_h \in V_h, \end{cases}$$

with $a(\cdot,\cdot)$ and $\langle F, \cdot \rangle$ same as before, and

$$V_h = \{v \in S_h^1(0,1) \colon v(0) = 0\}.$$

Using the finite element isomorphism

$$v_h(x) = \sum_{k=0}^{N} v_k \varphi_k(x) \in S_h^0(0,1) \quad \leftrightarrow \quad \vec{v} = (v_0, \dots, v_N)^\top \in \mathbb{R}^{N+1}$$

we can formulate this using a linear system of equations:

$$\begin{cases} \text{Find } \vec{u} \in \mathbb{R}^{N+1} \text{, such that} \\ K\vec{u} = \vec{f}, \end{cases}$$

with

$$\begin{aligned} K_{ij} &= \int_0^1 \varphi_j'(x)\varphi_i'(x) + a\varphi_j(x)\varphi_i(x)dx + \alpha\varphi_j(1)\varphi_i(1), \\ f_i &= \int_0^1 f(x)\varphi_i(x)dx + \alpha g_b \varphi_i(1). \end{aligned}$$

and the restriction $u_0 = 0$.

Now we compute the elements of the stiffness matrix $K \in \mathbb{R}^{(N+1)\times(N+1)}$ and the right hand side vector $f \in \mathbb{R}^{N+1}$.

We first compute the local element matrices of K by integrating the basis functions over one element x_i.

For $i = 0, \dots, N-1$ we have

$$\begin{aligned}
K_{11}^{(i)} &= \int_{x_i}^{x_{i+1}} (\varphi_i'(x))^2 + a(\varphi_i(x))^2 dx + \alpha(\varphi_i(1))^2 = \\
&= \int_{x_{i-1}}^{x_i} \frac{1}{h^2} + a\frac{1}{h^2}(x - x_{i-1})^2 dx = \frac{1}{h} + \frac{ah}{3}, \\
K_{12}^{(i)} &= \int_{x_i}^{x_{i+1}} \varphi_{i+1}'(x)\varphi_i'(x) + a\varphi_{i+1}(x)\varphi_i(x)dx + \alpha\varphi_{i+1}(1)\varphi_i(1) = \\
&= \int_{x_i}^{x_{i+1}} -\frac{1}{h^2} + a\frac{1}{h^2}(x - x_i)(x_{i+1} - x)dx = -\frac{1}{h} + \frac{ah}{6}, \\
K_{21}^{(i)} &= K_{12}^{(i)} = -\frac{1}{h} + \frac{ah}{6}, \\
K_{22}^{(i)} &= \int_{x_i}^{x_{i+1}} (\varphi_{i+1}'(x))^2 + a(\varphi_{i+1}(x))^2 dx + \alpha(\varphi_{i+1}(1))^2 = \\
&= \int_{x_i}^{x_{i+1}} \frac{1}{h^2} + a\frac{1}{h^2}(x_{i+1} - x)^2 dx + +\alpha(\varphi_{i+1}(1))^2 = \\
&= \frac{1}{h} + \frac{ah}{3} + \begin{cases} \alpha, & \text{for} \quad i = N, \\ 0, & \text{else.} \end{cases}
\end{aligned}$$

This results in the stiffness matrix

$$K = \begin{pmatrix}
\frac{1}{h} + \frac{ah}{3} & -\frac{1}{h} + \frac{ah}{6} & & & \\
-\frac{1}{h} + \frac{ah}{6} & \frac{2}{h} + \frac{2ah}{3} & -\frac{1}{h} + \frac{ah}{6} & & \\
& \ddots & \ddots & \ddots & \\
& & -\frac{1}{h} + \frac{ah}{6} & \frac{2}{h} + \frac{2ah}{3} & -\frac{1}{h} + \frac{ah}{6} \\
& & & -\frac{1}{h} + \frac{ah}{6} & \frac{1}{h} + \frac{ah}{3} + \alpha
\end{pmatrix}.$$

To compute the entries of $\vec{f}$, we have

$$f_i = \int_{x_{i-1}}^{x_i} f(x)\frac{1}{h}(x - x_{i-1})dx + \int_{x_i}^{x_{i+1}} f(x)\frac{1}{h}(x_{i+1} - x)dx + \alpha g_b \varphi_i(1).$$

By choosing $f(x) = c$ as a constant for example, this results in

$$\vec{f} = \begin{pmatrix} ch \\ ch \\ \vdots \\ ch \\ ch + \alpha g_b \end{pmatrix}.$$

The Dirichlet boundary conditions are taken care of in the code.

Exercise (B)
We are given the PDE

$$\begin{aligned} -(\lambda(x)u'(x))' = 0 \quad \text{for} \quad x \in (0,1), \\ u(0) = 0, \\ u(1) = 1. \end{aligned}$$

with

$$\lambda(x) = \begin{cases} 1, & x \in (1, \frac{1}{\sqrt{2}}), \\ 10, & x \in (\frac{1}{\sqrt{2}}, 1). \end{cases}$$

Solution (B)
For the variational formulation, integrate by parts and choose the test functions to be in $H_0^1(0,1)$. Then we get

$$\begin{cases} \text{Find } u \in \{H^1(0,1)\colon u(0) = 0, u(1) = 1\}, \text{ such that} \\ a(u,v) = 0, \quad \forall v \in H_0^1(0,1), \end{cases}$$

with

$$a(u,v) = \int_0^1 \lambda(x)u'(x)v'(x)dx.$$

We separate the domain $(0,1)$ into two intervals $(1, \frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, 1)$. Since $\lambda(x)$ is constant on each subdomain respectively, we get for $i = 0, \ldots, N-1$

$$\begin{aligned} K_{11}^{(i)} &= \int_{x_i}^{x_{i+1}} \lambda(x)(\varphi_i'(x))^2 dx = \int_{x_i}^{x_{i+1}} \lambda(x)\frac{1}{h^2}dx = \frac{\lambda}{h}, \\ K_{12}^{(i)} &= \int_{x_i}^{x_{i+1}} \lambda(x)\varphi_{i+1}'(x)\varphi_i'(x)dx = -\int_{x_i}^{x_{i+1}} \lambda(x)\frac{1}{h^2}dx = -\frac{\lambda}{h}, \\ K_{21}^{(i)} &= K_{12}^{(i)} = -\frac{\lambda}{h}, \\ K_{22}^{(i)} &= K_{11}^{(i)} = \frac{\lambda}{h}. \end{aligned}$$

Hence the global stiffness matrix takes the form (illustrated for $N+1 = 3$)

$$K = \begin{pmatrix} \frac{1}{h} & -\frac{1}{h} & \\ -\frac{1}{h} & \frac{11}{h} & -\frac{10}{h} \\ & -\frac{10}{h} & \frac{10}{h} \end{pmatrix}.$$

This results in the linear system of equations

$$\begin{cases} \text{Find } \vec{u} \in \mathbb{R}^{N+1}, \text{ such that} \\ K\vec{u} = \vec{0}, \end{cases}$$

with the restrictions $u_0 = 0$ and $u_N = 1$. The Dirichlet boundary conditions are taken care of in the code.

Exercise (C)

We are given the PDE (Péclet problem)

$$\begin{aligned} -u''(x) + pu'(x) &= 0, \quad x \in (0,1) \\ u(0) &= 0 \\ u(1) &= 1 \end{aligned}$$

with $p \in \mathbb{R}$.

Solution (C)

For the variational formulation, integrate by parts and choose the test functions to be in $H_0^1(0,1)$. Then we get

$$\begin{cases} \text{Find } u \in \{H^1(0,1) \colon u(0) = 0, u(1) = 1\}, \text{ such that} \\ a(u,v) = 0, \quad \forall v \in H_0^1(0,1), \end{cases}$$

with

$$a(u,v) = \int_0^1 u'(x)v'(x) + pu'(x)v(x)dx.$$

For $i = 0, \dots, N-1$ we have

$$K_{11}^{(i)} = \int_{x_i}^{x_{i+1}} (\varphi_i'(x))^2 + p\varphi_i'(x)\varphi_i(x)dx = \int_{x_i}^{x_{i+1}} \frac{1}{h^2} - p\frac{1}{h^2}(x_{i+1} - x) = \frac{1}{h} - \frac{p}{2}$$

$$K_{12}^{(i)} = \int_{x_i}^{x_{i+1}} \varphi_{i+1}'(x)\varphi_i'(x) + p\varphi_{i+1}'(x)\varphi_i(x)dx = \int_{x_i}^{x_{i+1}} -\frac{1}{h^2} + p\frac{1}{h^2}(x_{i+1} - x)dx = -\frac{1}{h} + \frac{p}{2}$$

$$K_{21}^{(i)} = \int_{x_i}^{x_{i+1}} \varphi_i'(x)\varphi_{i+1}'(x) + p\varphi_i'(x)\varphi_{i+1}(x)dx = \int_{x_i}^{x_{i+1}} -\frac{1}{h^2} - p\frac{1}{h^2}(x - x_i)dx = -\frac{1}{h} - \frac{p}{2},$$

$$K_{22}^{(i)} = \int_{x_i}^{x_{i+1}} (\varphi_{i+1}'(x))^2 + p\varphi_{i+1}'(x)\varphi_{i+1}(x)dx = \int_{x_i}^{x_{i+1}} \frac{1}{h^2} + p\frac{1}{h^2}(x - x_i)dx = \frac{1}{h} + \frac{p}{2}$$

Hence the global stiffness matrix takes the form

$$K = \begin{pmatrix} \frac{1}{h} - \frac{p}{2} & -\frac{1}{h} + \frac{p}{2} & & & \\ -\frac{1}{h} - \frac{p}{2} & \frac{2}{h} & -\frac{1}{h} + \frac{p}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{h} - \frac{p}{2} & \frac{2}{h} & -\frac{1}{h} + \frac{p}{2} \\ & & & -\frac{1}{h} - \frac{p}{2} & \frac{1}{h} + \frac{p}{2} \end{pmatrix}.$$

This results in the linear system of equations

$$\begin{cases} \text{Find } \vec{u} \in \mathbb{R}^{N+1}, \text{ such that} \\ K\vec{u} = \vec{0}, \end{cases}$$

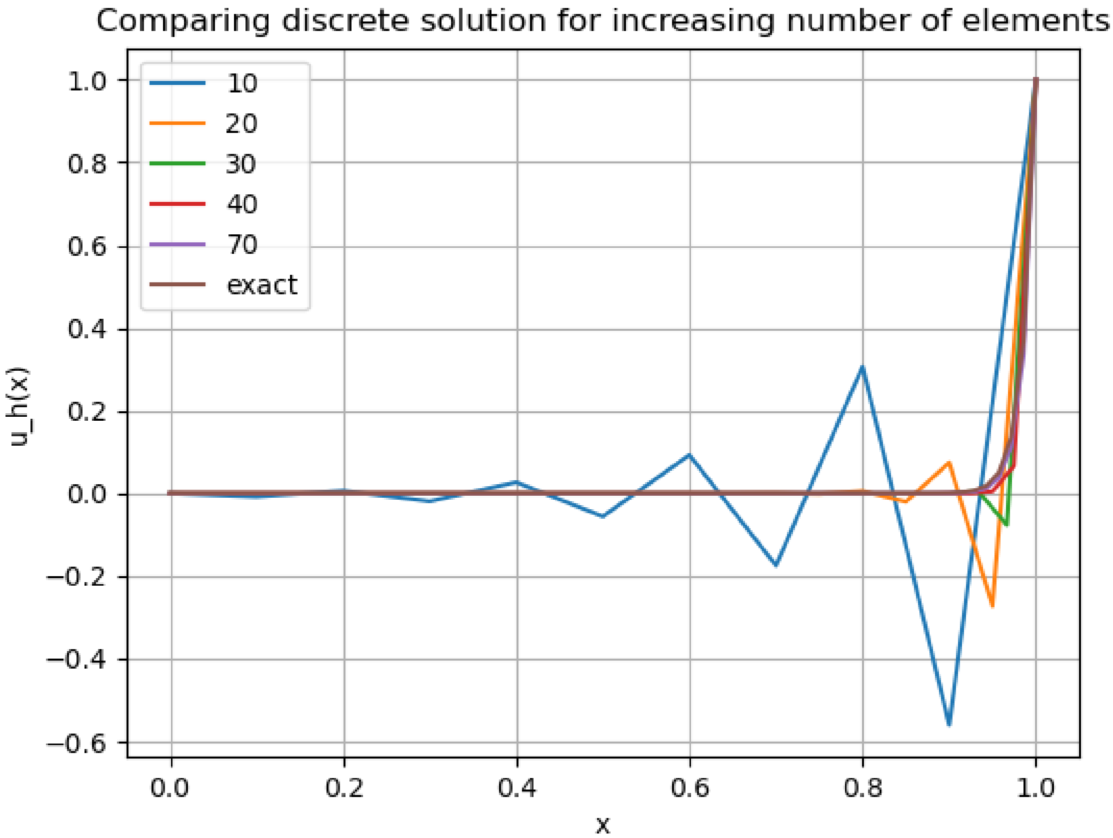

Figure 1: Approximation of the solution in python

with the restrictions $u_0 = 0$ and $u_N = 1$. The Dirichlet boundary conditions are taken care of in the code.

In figure 1, we can see that for small element numbers, there are instabilities in the discrete solution. For an increasing number of elements, the calculation becomes more stable. This is due to the steep curve of the exact solution near the point 1.