

(B)

$$-(\lambda(x)u'(x))' = 0 \quad \text{in } \Omega = (0,1) \quad \text{with } \lambda(x) = \begin{cases} 1, & x \in (0, 0.5), \\ 10, & x \in (0.5, 1). \end{cases}$$

$$u(0) = 0, \quad u(1) = 1$$

Strong formulation with interface conditions:

$$-u''(x) = 0 \quad \text{in } (0, 0.5)$$

$$\lim_{h \rightarrow 0^-} u(0.5+h) = \lim_{h \rightarrow 0^+} u(0.5+h) \quad \text{or} \quad u(0.5-0) = u(0.5+0)$$

$$\lim_{h \rightarrow 0^-} -u'(0.5+h) = \lim_{h \rightarrow 0^+} -10u'(0.5+h) \quad \text{or} \quad -u'(0.5-0) = -10u'(0.5+0)$$

$$-10u''(x) = 0 \quad \text{in } (0.5, 1)$$

Function spaces: $u \in C^2((0, 0.5) \cup (0.5, 1)) \cap C^1(0, 1)$, $\lambda \in L^\infty(0, 1)$, $f \in L^\infty(0, 1)$

$$\text{Solution: } -u''(x) = 0 \quad \text{in } (0, 0.5) \Rightarrow u'(x) = A, \quad u(x) = Ax + B \quad \text{in } (0, 0.5)$$

$$-10u''(x) = 0 \quad \text{in } (0.5, 1) \Rightarrow u'(x) = C, \quad u(x) = Cx + D \quad \text{in } (0.5, 1)$$

$$\text{B.C.: } u(0) = B = 0 \Rightarrow B = 0$$

$$u(1) = C + D = 1 \Rightarrow 10A - \frac{9}{2}A = 1 \Rightarrow A = \frac{2}{11}$$

$$\text{Interface conditions: } \left. \begin{aligned} u(0.5) &= \frac{1}{2}A + B \\ u(0.5) &= \frac{1}{2}C + D \end{aligned} \right\} = 1 \Rightarrow \frac{1}{2}A = \frac{1}{2}C + D \Rightarrow -\frac{9}{2}A = D$$

$$\left. \begin{aligned} -u'(0.5) &= A \\ -10u'(0.5) &= C \end{aligned} \right\} = 1 \Rightarrow 10A = C \Rightarrow C = \frac{20}{11}$$

$$D = -\frac{18}{22} = -\frac{9}{11}$$

$$\Rightarrow u(x) = \frac{2}{11}x \quad \text{in } (0, 0.5)$$

$$u(x) = \frac{20}{11}x - \frac{9}{11} \quad \text{in } (0.5, 1)$$

$$\text{Check: } u(0.5) = \frac{2}{22} = \frac{20}{22} - \frac{9}{22} \quad \checkmark$$

$$u'(0.5) = \frac{2}{11} = \frac{1}{10} \cdot \frac{20}{11} \quad \checkmark$$

$$u(0) = 0 \quad \checkmark$$

$$u(1) = 1 \quad \checkmark$$

