Study of two crowd motion and herding models

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Introduction

We study the stationary states of two models for crowd motion and herding. Both systems involve two nonlinear parabolic evolution equations, one for the density and one for the mean field potential. They resemble the Keller-Segel model for chemotaxis in its "prevention of overcrowding" variant. We investigate plateau-like solutions found by **?**. On balls we are able to prove multiplicity and qualitative properties of radial solutions, and also provide a numerical study of the stability of these solutions.

The models

The dynamical system reads as follow, models (I) and (II) differ by the source term in the equation for D.
On a bounded, open set $\Omega \subset \mathbb{R}^n$ consider:

$$\begin{cases} \partial_t \rho = \Delta\rho - \nabla\cdot(\rho(1-\rho)\nabla D)\,, \\ \partial_t D = \kappa\Delta D - \delta D + \begin{cases}\rho(1-\rho) & \text{(I)} \\ \rho & \text{(II)}\end{cases}, \end{cases}$$

with no flux and homogeneous Neumann boundary conditions for ρ and D respectively, so that mass $M = \int \rho$ is conserved.
The density ρ undergoes diffusion and drift along the potential D. The coefficient $\rho(1-\rho)$ in the drift term accounts for saturation and can be seen as "prevention of overcrowding".
D is a field potential and corresponds to the density of chemoattractant in the Keller-Segel model. It undergoes diffusion and damping, and has a source term depending on ρ.

- In model (I), D increases only if the density is not too high. This model is derived from a cellular automaton by **?**.
- Model (II) is more standard in the setting of chemotaxis, and has already been partially studied by **??**.

In can of model (II), we have the following Lyapunov functional:

$$\mathcal{L}[\rho, D] := \int \rho\,\log\rho + (1-\rho)\,\log(1-\rho) - \rho\,D + \frac{\kappa}{2}\int|\nabla D|^2 + \frac{\delta}{2}\int D^2\,.$$

Stationary solutions

We are interested in radially symmetric stationary solutions:

$$\begin{cases} -\Delta\rho - \nabla\cdot(\rho(1-\rho)\nabla D) = 0\,, \\ -\kappa\Delta D + \delta D = \begin{cases}\rho(1-\rho) & \text{(I)} \\ \rho & \text{(II)}\end{cases}. \end{cases}$$

By solving for ρ in the first equation this reduces to:

$$\begin{cases} \rho = \dfrac{1}{1+e^{-\phi}}\,, \\ -\kappa\Delta\phi + \delta(\phi + \bar{D}) - F'(\phi) = 0\,, \end{cases}$$

where $\phi = D - \bar{D}$ and $\bar{D} \in \mathbb{R}$ is chosen to meet the mass constraint $\int_\Omega \rho = M$ and

$$F(\phi) = \begin{cases} \frac{1}{1+e^{-\phi}} & \text{(I)} \\ \log(1+e^{\phi}) & \text{(II)} \end{cases}.$$

Solutions to this equation are studied as critical points of the following functional:

$$\mathcal{E}[\phi] := \frac{\kappa}{2}\int|\nabla\phi|^2 + \frac{\delta}{2}\int\phi^2 + \delta\bar{D}\int\phi - F(\phi).$$

References

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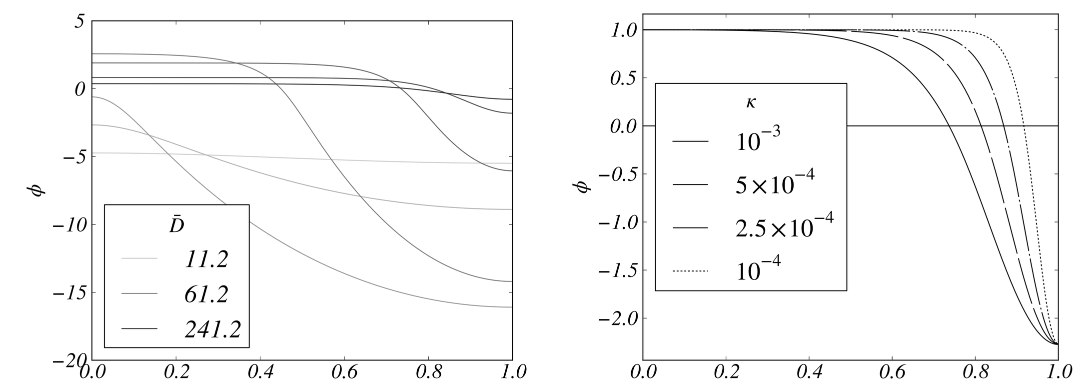
Main result

If Ω is bounded with $C^{1,\alpha}$ boundary for some $\alpha > 0$, then, **for any $\bar{D} \in \mathbb{R}$** there exists a minimizer of the energy $\mathcal{E}$ in $H^1(\Omega)$. Minimisers are radially symmetric and monotone when the domain is a ball.

Actually, we know that minimizers are constant if we look at the problem for fixed $\bar{D}$. But for fixed mass, we cannot conclude. Other solutions can be found numerically using a shooting method.

Profile of monotone solutions

In the one dimensional unit ball:

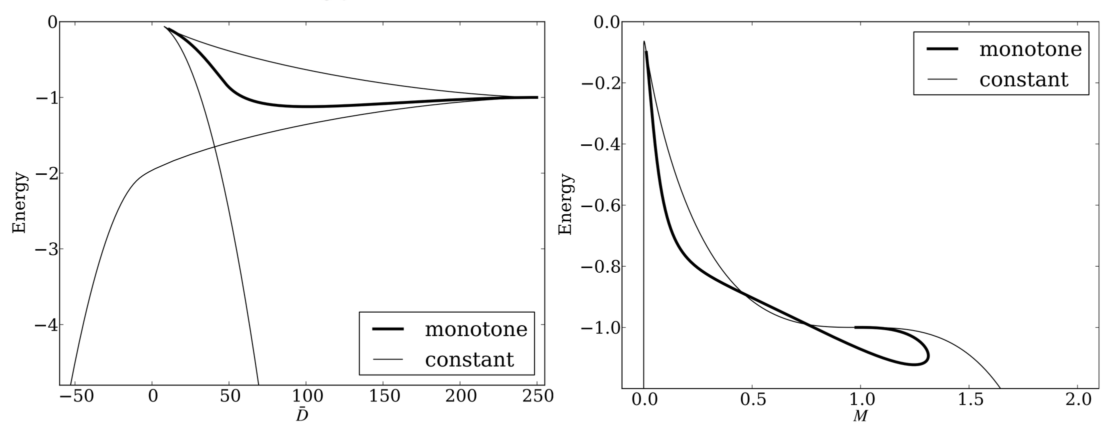


- On the left: evolution of the profile along the branch, *i.e.* as $\bar{D}$ varies
- On the right: steepening of the profile as κ decreases for a given initial condition.

In 1D, reflections with respect to $x = \frac{1}{2}$ are also solutions because the equation for ϕ is autonomous.
The parameters used here —and elsewhere when not specified— are $\delta = 10^{-3}$, $\kappa = 5\times10^{-4}$. We use these values to get interesting behaviour (existence of plateaus), but this also results in a relatively sharp numerical behaviour.

Energy comparison

At fixed $\bar{D}$, on the left, constant solutions always have lower energy.



But for fixed mass, on the right, the energies are not ordered.

Stability

For stationary solutions, stability can be defined in two ways:

- A solution is *variationally stable* if it is a local minimizer for $\mathcal{E}$.
- It is *dynamically stable* if small perturbations decay exponentially in time. This can be checked by looking at the spectrum of the linearised evolution operator around this solution.

Since that for the variational problem we have the additional constraint

$$\rho = \frac{1}{1+e^{-\phi}},$$

dynamical stability implies variational stability.
Interestingly, it seems that both definitions coincide for model (II) but not for model (I).

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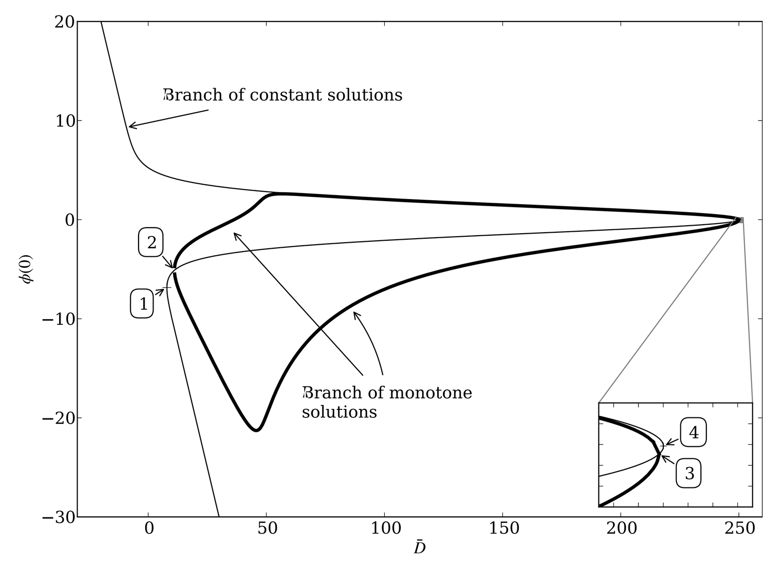
Numerics

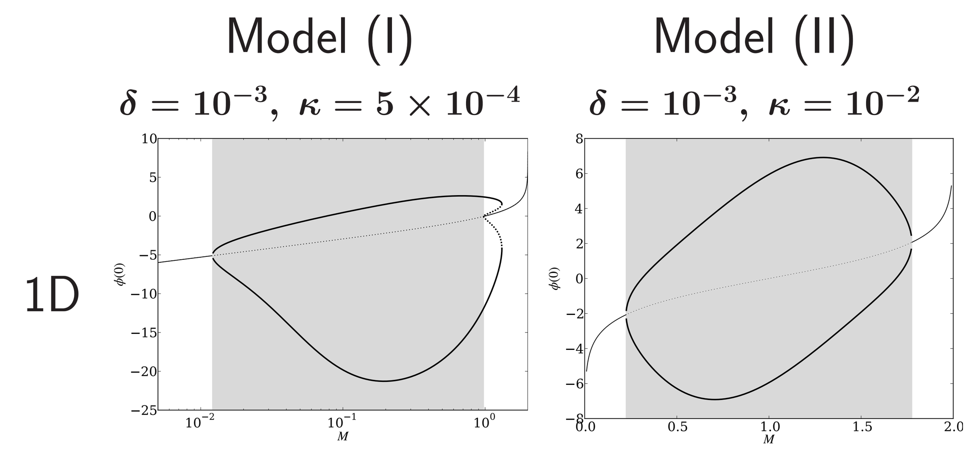
Computations were done using the NumPy/SciPy packages. They are based on a simple shooting method that enables us to find all radially symmetric solutions.

- Parametrization by $\bar{D}$:

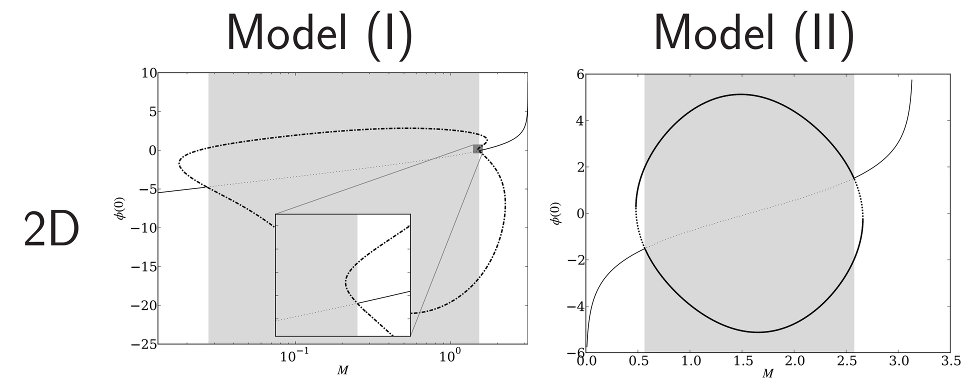


1. First bifurcation point, two constant solutions appear.
2. Second bifurcation from the unstable constant, two monotone solutions appear.
3. Non-constant solutions disappear.
4. Two constant solutions collapse, leaving only one constant solution.

- Multiplicity and dynamical stability, parametrization by mass:



In the two cases we have the branch of constant solutions (thin lines) and the branch of plateau-like solutions (bold lines). Dotted lines show the part of the branch where solutions are *dynamically* unstable. The shaded area shows the region of instability for constant solutions.
In model (I) there is an interval of mass for which both constant solutions and non constant solutions are stable. In model (II), stability is exclusive to either branch.



In 2 dimensions, only variational stability is shown for model (II). The figure on the right shows coexistence of stable constant and non-constant solutions for model (II).

Perspectives

- Understand the relation between parametrizations by M and $\bar{D}$.
- Better link variational and dynamical stability
- Complete the stability analysis with non-radial perturbations
- Study existence and stability of non-radial stationary solutions

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