Sobolev and Hardy-Littlewood-Sobolev inequalities

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Abstract

This paper is devoted to improvements of Sobolev and Onofri inequalities. The additional terms involve the dual counterparts, *i.e.* Hardy-Littlewood-Sobolev type inequalities. We focus our attention on optimal constants, that can be achieved either by completion of the square methods or by using nonlinear flows, and provide various new estimates.

Keywords: Sobolev spaces, Sobolev inequality, Hardy-Littlewood-Sobolev inequality, logarithmic Hardy-Littlewood-Sobolev inequality, Onofri's inequality, Caffarelli-Kohn-Nirenberg inequalities, extremal functions, duality, best constants, stereographic projection, fast diffusion equation
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1. Introduction

E. Carlen, J.A. Carrillo and M. Loss noticed in [1] that Hardy-Littlewood-Sobolev inequalities in dimension $d \geq 3$ can be deduced from some special Gagliardo-Nirenberg inequalities using a fast diffusion equation. Sobolev's inequalities and Hardy-Littlewood-Sobolev inequalities are dual. A fundamental reference for this issue is E.H. Lieb's paper [2]. This duality has also been investigated using a fast diffusion flow in [3]. Although [1] has motivated [3], the two approaches are so far unrelated. Actually [3] is closely connected with the approach by Legendre's duality developed in [2]. We shall take advantage of this fact in the present paper.

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For any $d \geq 3$, the space $\mathcal{D}^{1,2}(\mathbb{R}^d)$ is defined as the completion of smooth solutions with compact support w.r.t. the norm

$$w \mapsto \|w\| := \left(\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right)^{1/2} ,$$

where $2^* := \frac{2\,d}{d-2}$. The Sobolev inequality in $\mathbb{R}^d$ is

$$\mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq 0 \quad \forall\, u \in \mathcal{D}^{1,2}(\mathbb{R}^d) \,, \tag{1}$$

where the best constant, or Aubin-Talenti constant, is given by

$$\mathsf{S}_d = \frac{1}{\pi\, d\, (d-2)} \left(\frac{\Gamma(d)}{\Gamma(\frac{d}{2})} \right)^{\frac{2}{d}}$$

(see Appendix A for details). The optimal Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} \, v \; dx \geq 0 \quad \forall\, v \in \mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d) \tag{2}$$

involves the same best constant S_d, as a result of the duality method of [2]. When $d \geq 5$, using a well chosen flow, it has been established in [3] that the l.h.s. in (1) is actually bounded from below by the l.h.s. in (2), multiplied by some positive constant. Our first result is based on an elementary use of the duality method – in fact a simple completion of the square method – which provides an optimal proportionality constant in any dimension.

Theorem 1. *For any $d \geq 3$, if $q = \frac{d+2}{d-2}$ the inequality*

$$\mathsf{S}_d \, \|u^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} u^q \, (-\Delta)^{-1} \, u^q \; dx \\ \leq \mathsf{C}_d \, \|u\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right] \tag{3}$$

holds for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ with $\mathsf{C}_d = \mathsf{S}_d$, which is moreover the optimal constant.

As we shall see in Section 3, the constant $\mathsf{C}_d = \mathsf{S}_d$ is the optimal proportionality constant relating the two sides of the above inequality. This result is achieved by expanding both sides of the inequality around the Aubin-Talenti functions, which are optimal for Sobolev's and Hardy-Littlewood-Sobolev inequalities (see Section 2 for more details). The computation based on the flow as was done in [3] can be improved in order to provide the proportionality constant S_d. Moreover it can be combined with the result of Theorem 1 to give an improved inequality: this will also be studied in Section 4 (see Theorem 7).

In dimension $d = 2$, consider the probability measure $d\mu$ defined by

$$d\mu(x) := \mu(x)\,dx \quad \text{with} \quad \mu(x) := \frac{1}{\pi\,(1+|x|^2)^2} \quad \forall\, x \in \mathbb{R}^2.$$

The Euclidean version of Onofri's inequality [4]

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla f|^2\,dx - \log\left(\int_{\mathbb{R}^2} e^f\,d\mu\right) + \int_{\mathbb{R}^2} f\,d\mu \ge 0 \quad \forall\, f \in \mathcal{D}(\mathbb{R}^2) \tag{4}$$

plays the role of Sobolev's inequality in higher dimensions. Here the inequality is written for smooth and compactly supported functions in $\mathcal{D}(\mathbb{R}^2)$, but can be extended to the appropriate Orlicz space which corresponds to functions such that both sides of the inequality are finite.

This inequality is dual of the logarithmic Hardy-Littlewood-Sobolev inequality that can be written as follows: for any $g \in \mathrm{L}^1_+(\mathbb{R}^2)$ with $M = \int_{\mathbb{R}^2} g\,dx$, such that $g\log g$, $(1+\log|x|^2)\,g \in \mathrm{L}^1(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2} g\,\log\left(\frac{g}{M}\right)dx - \frac{4\,\pi}{M}\int_{\mathbb{R}^2} g\,(-\Delta)^{-1}\,g\,dx + M\,\log(\pi\,e) \ge 0\,. \tag{5}$$

with

$$\int_{\mathbb{R}^2} g\,(-\Delta)^{-1}\,g\,dx = -\frac{1}{2\,\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} g(x)\,g(y)\,\log|x-y|\,dx\,dy\,.$$

Then, in dimension $d = 2$, we have an analogue of Theorem 1, which goes as follows.

Theorem 2. *Assume that $d = 2$. The inequality*

$$\left(\int_{\mathbb{R}^2} e^f \, d\mu\right)^2 - 4\pi \int_{\mathbb{R}^d} e^f \mu \, (-\Delta)^{-1} \, e^f \mu \, dx \\ \le \left(\int_{\mathbb{R}^2} e^f \, d\mu\right)^2 \left[\frac{1}{16\,\pi} \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \int_{\mathbb{R}^2} f \, d\mu - \log\left(\int_{\mathbb{R}^2} e^f \, d\mu\right)\right] \quad (6)$$

holds for any function $f \in \mathcal{D}(\mathbb{R}^2)$.

Using for instance [5] or [6, Lemma 2] (also see [7, chapter 3–4]), it is known that optimality is achieved in (1), (2), (4) or (5) when the problem is reduced to radially symmetric functions. However, no such result applies when considering a difference of the terms in two such inequalities, like in (3) or (6). Optimality therefore requires a special treatment. In Section 2, we shall use the *completion of the square method* to establish the inequalities (without optimality), under an assumption of radial symmetry in case of Theorem 2. For radial functions, Theorem 1 can indeed be written with $d > 2$ considered as a real parameter and Theorem 2 corresponds, in this setting, to the limit case as $d \to 2_+$. Section 3 is devoted to linearization results and an estimate of the proportionality constant which completes the proof of Theorem 1. In Section 4, we explore some consequences of our results for a flow that was at the core of the results in [3]. Combined with linearization results this flow shows complements the result on the optimality of the proportionality constant in Theorem 1. In Section 5, we extend the results established for Sobolev inequalities to weighted spaces and obtain an improved version of the Caffarelli-Kohn-Nirenberg inequalities (see Theorem 11). Playing with weights is equivalent to varying d or taking limits with respect to d, except that no symmetry assumption is required. This allows to complete the proof of Theorem 2. Technical results regarding the computation of the constants, a weighted Poincaré inequality and the stereographic projection, and symmetry results for Caffarelli-Kohn-Nirenberg inequalities have been collected in Appendix A, Appendix B and Appendix C respectively.

At this point, we emphasize that Theorems 11 and 12, which are used as intermediate steps in the proof of Theorem 2 are slightly more general than, respectively, Theorems 1 and 2.

Let us conclude this introduction by a brief review of the literature. Our approach is based on a completion of the square method which accounts for duality issues, on linearization and on estimates based on a nonlinear flow.

Although some of these methods have been widely used in the literature, for instance in the context of Hardy inequalities (see [8] and references therein), it seems that they have not been fully exploited yet in the case of the functional inequalities considered in this paper. The main tool in [3] is a flow of fast diffusion type, which has been considered earlier in [9]. In dimension $d = 2$, we may refer to various papers (see for instance [10, 11, 12]) in connection with Ricci's flow.

Many papers have been devoted to the asymptotic behaviour near extinction of the solutions of nonlinear flows, in bounded domains (see for instance [13, 14, 15, 16]) or in the whole space (see [17, 18, 19] and references therein). In particular, the Cauchy-Schwarz inequality has been repeatedly used, for instance in [13, 15], and turns out to be a key tool in the main result of [3], as well as the solution *with separation of variables,* which is related to the Aubin-Talenti optimal function for (1).

Getting improved versions of Sobolev's inequality is a question which has attracted lots of attention. See [22] in the bounded domain case and [21] for an earlier related paper. However, in [22], H. Brezis and E. Lieb also raised the question of measuring the distance to the manifold of optimal functions in the case of the Euclidean space. A few years later, G. Bianchi and H. Egnell gave an answer in [20] using the concentration-compactness method, with no explicit value of the constant. Since then, considerable efforts have been devoted to obtain quantitative improvements of Sobolev's inequality. On the whole Euclidean space, nice estimates based on rearrangements have been obtained in [23] and we refer to [24] for an interesting review of various related results. The method there is in some sense constructive, but it hard to figure what is the practical value of the constant that can be deducted. As in [3] our approach involves much weaker notions of distances to optimal functions, but on the other hand offers clear-cut estimates. Moreover, it provides an interesting way of obtaining global estimates based on a linearization around Aubin-Talenti optimal functions.

2. A completion of the square and consequences

Before proving the main results of this paper, let us explain in which sense Sobolev's inequality and the Hardy-Littlewood-Sobolev inequality, or Onofri's inequality and the logarithmic Hardy-Littlewood-Sobolev inequality, for instance, are *dual inequalities.*

To a convex functional F, we may associate the functional F^* defined by Legendre's duality as

$$F^*[v] := \sup \left(\int_{\mathbb{R}^d} u\, v\; dx - F[u] \right) .$$

For instance, to $F_1[u] = \frac{1}{2} \|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}$ defined on $\mathrm{L}^p(\mathbb{R}^d)$, we henceforth associate $F_1^*[v] = \frac{1}{2} \|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}$ on $\mathrm{L}^q(\mathbb{R}^d)$ where p and q are Hölder conjugate exponents: $1/p + 1/q = 1$. The supremum can be taken for instance on all functions in $\mathrm{L}^p(\mathbb{R}^d)$, or, by density, on the smaller space of the functions $u \in \mathrm{L}^p(\mathbb{R}^d)$ such that $\nabla u \in \mathrm{L}^2(\mathbb{R}^d)$. Similarly, to $F_2[u] = \frac{1}{2}\,\mathsf{S}_d\, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, we associate $F_2^*[v] = \frac{1}{2}\,\mathsf{S}_d^{-1} \int_{\mathbb{R}^d} v\,(-\Delta)^{-1}\, v\;\; dx$ where $(-\Delta)^{-1}\, v = G_d * v$ with $G_d(x) = \frac{1}{d-2}\,|\mathbb{S}^{d-1}|^{-1}\,|x|^{2-d}$, when $d \geq 3$, and $G_2(x) = -\frac{1}{2\pi}\,\log|x|$. As a straightforward consequence of Legendre's duality, if we have a functional inequality of the form $F_1[u] \leq F_2[u]$, then we have the *dual inequality* $F_1^*[v] \geq F_2^*[v]$. In this sense, (1) and (2) are dual of each other, as it has been noticed in [2]. Also notice that Inequality (2) is a consequence of Inequality (1).

In this paper, we go one step further and establish that

$$F_1^*[u] - F_2^*[u] \leq \mathsf{C}\,(F_2[u] - F_1[u]) \tag{7}$$

for some positive constant C, at least under some normalization condition (or up to a multiplicative term which is required for simple homogeneity reasons). Such an inequality has been established in [3, Theorem 1.2] when $d \geq 5$. Here we extend it to any $d \geq 3$ and get and improved value for the constant C.

It turns out that the proof can be reduced to the completion of a square. Let us explain how the method applies in case of Theorem 1, and how Theorem 2 can be seen as a limit of Theorem 1 in case of radial functions.

Proof of Theorem 1, part 1 : the completion of a square. Integrations by parts show that

$$\int_{\mathbb{R}^d} |\nabla(-\Delta)^{-1}\, v|^2\; dx = \int_{\mathbb{R}^d} v\,(-\Delta)^{-1}\, v\; dx$$

and, if $v=u^q$ with $q=\frac{d+2}{d-2}$,

$$\int_{\mathbb{R}^d} \nabla u\cdot\nabla(-\Delta)^{-1}\,v\;dx=\int_{\mathbb{R}^d} u\,v\;dx=\int_{\mathbb{R}^d} u^{2^*}\;dx\,.$$

Hence the expansion of the square

$$0\le\int_{\mathbb{R}^d}\left|\mathsf{S}_d\,\|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{4}{d-2}}\,\nabla u-\nabla(-\Delta)^{-1}\,v\right|^2\,dx$$

shows that

$$0\le\mathsf{S}_d\,\|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{8}{d-2}}\left[\mathsf{S}_d\,\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2-\|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^2\right]\\ -\left[\mathsf{S}_d\,\|u^q\|_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}^2-\int_{\mathbb{R}^d} u^q\,(-\Delta)^{-1}\,u^q\;dx\right].$$

Equality is achieved if and only if

$$\mathsf{S}_d\,\|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{4}{d-2}}\,u=(-\Delta)^{-1}\,v=(-\Delta)^{-1}\,u^q\,,$$

that is, if and only if u solves

$$-\,\Delta u=\frac{1}{\mathsf{S}_d}\,\|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{-\frac{4}{d-2}}\,u^q\,,$$

which means that u is an Aubin-Talenti function, optimal for (1). This completes the proof of Theorem 1, up to the optimality of the proportionality constant. Incidentally, this also proves that v is optimal for (2). □

As a first step towards the proof of Theorem 2, let us start with a result for radial functions. If d is a positive integer, we can define

$$\mathsf{s}_d:=\mathsf{S}_d\,|\mathbb{S}^{d-1}|^{\frac{2}{d}}$$

and get

$$\mathsf{s}_d=\frac{4}{d\,(d-2)}\left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d}{2}\right)}\right)^{\frac{2}{d}}\,. \tag{8}$$

Using this last expression allows us to consider d as a real parameter.

Lemma 3. *Assume that $d \in \mathbb{R}$ and $d > 2$. Then*

$$0 \le \mathsf{s}_d \left(\int_0^\infty u^{\frac{2\,d}{d-2}}\, r^{d-1}\, dr \right)^{1+\frac{2}{d}} - \int_0^\infty u^{\frac{d+2}{d-2}} \left((-\Delta)^{-1} u^{\frac{d+2}{d-2}} \right) r^{d-1}\, dr$$
$$\le \mathsf{C}_d \left(\int_0^\infty u^{\frac{2\,d}{d-2}}\, r^{d-1}\, dr \right)^{\frac{4}{d}} \left[\mathsf{s}_d \int_0^\infty |u'|^2\, r^{d-1}\, dr - \left(\int_0^\infty u^{\frac{2\,d}{d-2}}\, r^{d-1}\, dr \right)^{\frac{d-2}{d}} \right]$$

holds for any radial function $u \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ with optimal constant $\mathsf{C}_d = \mathsf{s}_d$.

Here we use the notation $(-\Delta)^{-1}\, v = w$ to express the fact that w is the solution to $w'' + \frac{d-1}{r}\, w' + v = 0$, that is,

$$(-\Delta)^{-1}\, v\,(r) = -\int_r^\infty s^{1-d} \int_0^s v(t)\, t^{d-1}\, dt\, ds \quad \forall\, r > 0\,. \tag{9}$$

Proof. In the case of a radially symmetric function u, and with the standard abuse of notations that amounts to identify $u(x)$ with $u(r)$, $r = |x|$, Inequality (1) can be written as

$$\mathsf{s}_d \int_0^\infty |u'|^2\, r^{d-1}\, dr \ge \left(\int_0^\infty |u|^{\frac{2\,d}{d-2}}\, r^{d-1}\, dr \right)^{1-\frac{2}{d}}\,. \tag{10}$$

However, if u is considered as a function of one real variable r, then the inequality also holds for any *real parameter* $d \in (2, \infty)$ and is equivalent to the one-dimensional Gagliardo-Nirenberg inequality

$$\mathsf{s}_d \left(\int_{\mathbb{R}} |w'|^2\, dt + \tfrac{1}{4}\,(d-2)^2 \int_{\mathbb{R}} |w|^2\, dt \right) \ge \left(\int_{\mathbb{R}} |w|^{\frac{2\,d}{d-2}}\, dt \right)^{1-\frac{2}{d}}$$

as can be shown using the Emden-Fowler transformation

$$u(r) = (2\,r)^{-\frac{d-2}{2}}\, w(t)\,, \quad t = -\log r\,. \tag{11}$$

The corresponding optimal function is, up to a multiplication by a constant, given by

$$w_\star(t) = (\cosh t)^{-\frac{d-2}{2}} \quad \forall\, t \in \mathbb{R}\,,$$

which solves the Euler-Lagrange equation

$$-(p-2)^2\,w''+4\,w-\;2\,p\,|w|^{p-2}\,w=0\,.$$

for any real number $d>2$ and the optimal function for (10) is

$$u_\star(r)=(2\,r)^{-\frac{d-2}{2}}\,w_\star(-\log r)=\big(1+r^2\big)^{-\frac{d-2}{2}}$$

up to translations, multiplication by a constant and scalings. This establishes (8). See Appendix A for details on the computation of s_d. The reader is in particular invited to check that the expression of s_d is consistent with the one of S_d given in the introduction.

Next we apply Legendre's transform to (10) and get a Hardy-Littlewood-Sobolev inequality that reads

$$\int_0^\infty v\,(-\Delta)^{-1}\,v\,r^{d-1}\,dr\le\mathsf{s}_d\left(\int_0^\infty v^{\frac{2\,d}{d+2}}\,r^{d-1}\,dr\right)^{1+\frac{d}{2}}\tag{12}$$

for any $d>2$. Inequality (12) holds on the functional space which is obtained by completion of the space of smooth compactly supported radial functions with respect to the norm defined by the r.h.s. in (12). Inequality (12) is the first inequality of Lemma 3.

Finally, we apply the completion of the square method. By expanding

$$0\le\int_0^\infty\big|\,a\,u'-\big((-\Delta)^{-1}v\big)'\,\big|^2\,r^{d-1}\,dr$$

with $a=\mathsf{s}_d\,\Big(\int_0^\infty u^{\frac{2\,d}{d-2}}\,r^{d-1}\,dr\Big)^{\frac{2}{d}}$ and $v=u^{\frac{d-2}{d+2}}$, we establish the second inequality of Lemma 3. □

Now let us turn our attention to the case $d=2$ and to Theorem 2. Using the fact that d in Lemma 3 is a real parameter, we can simply consider the limit of the inequalities as $d\to2_+$.

Corollary 4. *For any function $f\in\mathrm{L}^1(\mathbb{R}^+;\,r\,dr)$ such that $f'\in\mathrm{L}^2(\mathbb{R}^+;\,r\,dr)$,* *we have the inequality*

$$\begin{aligned} 0 \;&\le\; \left(\int_0^\infty e^f \,\frac{2\,r\,dr}{(1+r^2)^2}\right)^2 - \int_0^\infty \frac{e^f}{(1+r^2)^2}\,(-\Delta)^{-1}\left(\frac{e^f}{(1+r^2)^2}\right)\,r\,dr \\ &\le\; \left(\int_0^\infty e^f \,\frac{2\,r\,dr}{(1+r^2)^2}\right)^2 \\ &\qquad \cdot \left[\frac{1}{8}\int_0^\infty |f'|^2\,r\,dr + \int_0^\infty f\,\frac{2\,r\,dr}{(1+r^2)^2} - \log\left(\int_0^\infty e^f\,\frac{2\,r\,dr}{(1+r^2)^2}\right)\right]. \end{aligned}$$

Here again $(-\Delta)^{-1}$ is defined by (9), but it coincides with the inverse of $-\Delta$ acting on radial functions.

Proof of Theorem 2: a passage to the limit in the radial case. We may now pass to the limit in (10) written in terms of

$$u(r) = u_\star(r)\,(1 + \tfrac{d-2}{2\,d}\,f)$$

to get the radial version of Onofri's inequality for f. By expanding the expression of $|u'|^2$ we get

$$u'^2 = u_\star'^2 + \frac{d-2}{d}\,u_\star'\,(u_\star\,f)' + \left(\frac{d-2}{2\,d}\right)^2 (u_\star'\,f + u_\star\,f')^2\;.$$

Using the fact that $\lim_{d\to 2_+}(d-2)\,\mathsf{s}_d = 1$,

$$\mathsf{s}_d = \frac{1}{d-2} + \frac{1}{2} - \frac{1}{2}\,\log 2 + o(1) \quad \text{as} \quad d \to 2_+\,,$$

and

$$\lim_{d\to 2_+} \frac{1}{d-2}\int_0^\infty |u_\star'|^2\,r^{d-1}\,dr = 1\,,$$

$$\frac{1}{d-2}\int_0^\infty |u_\star'|^2\,r^{d-1}\,dr - 1 \sim -\frac{1}{2}\,(d-2)\,,$$

$$\lim_{d\to 2_+} \frac{1}{d-2}\int_0^\infty u_\star'\,(u_\star\,f)'\,r^{d-1}\,dr = \int_0^\infty f\,\frac{2\,r\,dr}{(1+r^2)^2}\,,$$

$$\lim_{d\to 2_+} \frac{1}{4\,d^2}\int_0^\infty |f'|^2\,u_\star^2\,r^{d-1}\,dr = \frac{1}{16}\int_0^\infty |f'|^2\,r\,dr\,,$$

and finally

$$\lim_{d\to 2_+}\int_0^\infty |u_\star\,(1+\tfrac{d-2}{2\,d}\,f)|^{\frac{2\,d}{d-2}}\,r^{d-1}\,dr=\int_0^\infty e^f\,\frac{r\,dr}{(1+r^2)^2}\,,$$

so that, as $d\to 2_+$,

$$\left(\int_0^\infty |u_\star\,(1+\tfrac{d-2}{2\,d}\,f)|^{\frac{2\,d}{d-2}}\,r^{d-1}\,dr\right)^{\frac{d-2}{d}}-1$$
$$\sim\frac{1}{2}\,(d-2)\,\log\left(\int_0^\infty e^f\,\frac{r\,dr}{(1+r^2)^2}\right)\,.$$

By keeping only the highest order terms, which are of the order of $(d-2)$, and passing to the limit as $d\to 2_+$ in (10), we obtain that

$$\frac{1}{8}\int_0^\infty |f'|^2\,r\,dr+\int_0^\infty f\,\frac{2\,r\,dr}{(1+r^2)^2}\geq\log\left(\int_0^\infty e^f\,\frac{2\,r\,dr}{(1+r^2)^2}\right)\,,$$

which is Onofri's inequality written for radial functions.

Similarly, we can pass to the limit as $d\to 2_+$ in (12), with

$$v=u^{\frac{d+2}{d-2}}=(1+r^2)^{-\frac{d+2}{2}}\,(1+\tfrac{d-2}{2\,d}\,f)^{\frac{d+2}{d-2}}$$

By doing so, we find that

$$\int_0^\infty\frac{e^f}{(1+r^2)^2}\,(-\Delta)^{-1}\left(\frac{e^f}{(1+r^2)^2}\right)\,r\,dr\leq\left(\int_0^\infty e^f\,\frac{2\,r\,dr}{(1+r^2)^2}\right)^2\,.$$

Passing to the limit in the inequalities of Lemma 3 concludes the proof of Corollary 4.

The proof in the non-radial case will be provided at the end of Section 5. □

3. Linearization

In the previous section, we have proved that the optimal constant C_d in (3) is such that $\mathsf{C}_d\leq\mathsf{S}_d$. To complete the proof the Theorem 1 we have to show that this is indeed an inequality, and that is the goal of this section. Let $\mathcal{F}$ and $\mathcal{G}$ be the positive integral quantities associated with, respectively, the Sobolev and Hardy-Littlewood-Sobolev inequalities:

$$\mathcal{F}[u] := \mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \,,$$

$$\mathcal{G}[v] := \mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} \, v \, dx \,.$$

Since that, for the Aubin-Talenti extremal function $u_\star$, we have $\mathcal{F}[u_\star] = \mathcal{G}[u_\star^q] = 0$, so that $u_\star$ gives a case of equality for (3), a natural question to ask is whether the infimum of $\mathcal{F}[u]/\mathcal{G}[u^q]$, under an appropriate normalization of $\|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$, is achieved as a perturbation of the $u_\star$.

Recall that $u_\star$ is the Aubin-Talenti extremal function

$$u_\star(x) := (1 + |x|^2)^{-\frac{d-2}{2}} \quad \forall \, x \in \mathbb{R}^d \,.$$

With a slight abuse of notations, we use the same notation as in Section 2. We may notice that $u_\star$ solves

$$-\Delta u_\star = d \, (d-2) \, u_\star^{\frac{d+2}{d-2}}$$

which allows to compute the optimal Sobolev constant as

$$\mathsf{S}_d = \frac{1}{d \, (d-2)} \left(\int_{\mathbb{R}^d} u_\star^{2^*} \, dx \right)^{-\frac{2}{d}} \tag{13}$$

using (11). See Appendix A for details. This shows that

$$\frac{1}{\mathsf{S}_d} \, \mathcal{F}[u] = \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \, d \, (d-2) \left(\int_{\mathbb{R}^d} u^{2^*} \, dx \right)^{1-\frac{2}{d}} \left(\int_{\mathbb{R}^d} u_\star^{2^*} \, dx \right)^{\frac{2}{d}} .$$

The goal of this section is to perform a linearization. By expanding $\mathcal{F}[u_\varepsilon]$ with $u_\varepsilon = u_\star + \varepsilon \, f$, for some f such that $\int_{\mathbb{R}^d} \frac{f \, u_\star}{(1+|x|^2)^2} \, dx = 0$ at order two in terms of ε, we get that

$$\frac{1}{\mathsf{S}_d} \, \mathcal{F}[u_\varepsilon] = \varepsilon^2 \, \mathsf{F}[f] + o(\varepsilon^2)$$

where

$$\mathsf{F}[f] := \int_{\mathbb{R}^d} |\nabla f|^2 \, dx - \, d \, (d+2) \int_{\mathbb{R}^d} \frac{|f|^2}{(1+|x|^2)^2} \, dx \,.$$

According to Lemma 13 (see Appendix B), we know that

$$\mathsf{F}[f] \geq 4\,(d+2) \int_{\mathbb{R}^d} \frac{|f|^2}{(1+|x|^2)^2}\;dx$$

for any $f \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \frac{f\,f_i}{(1+|x|^2)^2}\;dx = 0 \quad \forall\, i = 0\,,\; 1\,,\; 2\,,\; \ldots d+1\,, \tag{14}$$

where

$$f_0 := u_\star\,, \quad f_i(x) = \frac{x_i}{1+|x|^2}\,u_\star(x) \quad \text{and} \quad f_{d+1}(x) := \frac{1-|x|^2}{1+|x|^2}\,u_\star(x)\,.$$

Notice for later use that

$$-\,\Delta f_0 = d\,(d-2)\,\frac{f_0}{(1+|x|^2)^2}$$

and

$$-\,\Delta f_i = d\,(d+2)\,\frac{f_i}{(1+|x|^2)^2} \quad \forall\, i = 1\,,\; 2\,,\; \ldots d+1\,.$$

Also notice that

$$\int_{\mathbb{R}^d} \frac{f_i\,f_j}{(1+|x|^2)^2}\;dx = 0$$

for any i, $j = 0$, 1, $\ldots d+1$, $j \neq i$.

Similarly, we can consider the functional associated with the Hardy-Littlewood-Sobolev inequality and given by

$$\mathcal{G}[v] := \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} v\,(-\Delta)^{-1}\,v\;dx \quad \forall\; v \in \mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)$$

and whose minimum $\mathcal{G}[v_\star] = 0$ is achieved by $v_\star := u_\star^q$, $q = \frac{d+2}{d-2}$. Consistently with the above computations, let $v_\varepsilon := (u_\star + \varepsilon\,f)^q = v_\star\,\big(1 + \varepsilon\,\frac{f}{u_\star}\big)^q$ where f is again such that $\int_{\mathbb{R}^d} \frac{f\,f_0}{(1+|x|^2)^2}\;dx = 0$. By expanding $\mathcal{G}[v_\varepsilon]$ at order two in terms of ε, we get that

$$\mathcal{G}[v_\varepsilon] = \varepsilon^2\,\left(\frac{d+2}{d-2}\right)^2\,\mathsf{G}[f] + o(\varepsilon^2)$$

where

$$
\mathsf{G}[f] := \frac{1}{d\,(d+2)} \int_{\mathbb{R}^d} \frac{|f|^2}{(1+|x|^2)^2}\,dx \\
- \int_{\mathbb{R}^d} \frac{f}{(1+|x|^2)^2}\,(-\Delta)^{-1}\left(\frac{f}{(1+|x|^2)^2}\right)dx\,.
$$

Lemma 5. $\mathrm{Ker}(\mathsf{F}) = \mathrm{Ker}(\mathsf{G})$.

It is straightforward to check that the kernel is generated by f_i with $i = 1$, 2, $\ldots d$, $d+1$. Details are left to the reader. Next, by Legendre duality we find that

$$
\frac{1}{2} \int_{\mathbb{R}^d} \frac{|g|^2}{(1+|x|^2)^2}\,dx = \sup_f \left(\int_{\mathbb{R}^d} \frac{f\,g}{(1+|x|^2)^2}\,dx - \frac{1}{2} \int_{\mathbb{R}^d} \frac{|f|^2}{(1+|x|^2)^2}\,dx \right),
$$

$$
\frac{1}{2} \int_{\mathbb{R}^d} \frac{g}{(1+|x|^2)^2}\,(-\Delta)^{-1}\left(\frac{g}{(1+|x|^2)^2}\right)dx \\
= \sup_f \left(\int_{\mathbb{R}^d} \frac{f\,g}{(1+|x|^2)^2}\,dx - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla f|^2\,dx \right).
$$

Here the supremum is taken for all f satisfying the orthogonality conditions (14). It is then straightforward to see that duality holds if g is restricted to functions satisfying (14) as well. Consider indeed an optimal function f subject to (14). There are Lagrange multipliers $\mu_i \in \mathbb{R}$ such that

$$
g - f - \sum_{i=0}^{d+1} \mu_i\, f_i = 0
$$

and after multiplying by $f\,(1+|x|^2)^{-2}$, an integration shows that

$$
\int_{\mathbb{R}^d} \frac{f\,g}{(1+|x|^2)^2}\,dx = \int_{\mathbb{R}^d} \frac{|f|^2}{(1+|x|^2)^2}\,dx
$$

using the fact that f satisfies (14). On the other hand, if g satisfies (14), after multiplying by $g\,(1+|x|^2)^{-2}$, an integration gives

$$\int_{\mathbb{R}^d} \frac{|g|^2}{(1+|x|^2)^2}\;dx = \int_{\mathbb{R}^d} \frac{f\,g}{(1+|x|^2)^2}\;dx\,,$$

which establishes the first identity of duality. As for the second identity, the optimal function satisfies the Euler-Lagrange equation

$$\frac{g}{(1+|x|^2)^2} + \Delta\, f = \sum_{i=0}^{d+1} \mu_i\,\frac{f_i}{(1+|x|^2)^2}$$

for some Lagrange multipliers that we again denote by μ_i. By multiplying by f and $(-\Delta)^{-1}\big(g\,(1+|x|^2)^{-2}\big)$, we find that

$$\int_{\mathbb{R}^d} \frac{f\,g}{(1+|x|^2)^2}\;dx = \int_{\mathbb{R}^d} |\nabla f|^2\;dx$$
$$\int_{\mathbb{R}^d} \frac{g}{(1+|x|^2)^2}\,(-\Delta)^{-1}\left(\frac{g}{(1+|x|^2)^2}\right)dx = \int_{\mathbb{R}^d} \frac{f\,g}{(1+|x|^2)^2}\;dx$$

where we have used the fact that

$$\int_{\mathbb{R}^d} \frac{f_i}{(1+|x|^2)^2}\,(-\Delta)^{-1}\left(\frac{g}{(1+|x|^2)^2}\right)dx = \int_{\mathbb{R}^d} \frac{g}{(1+|x|^2)^2}\,(-\Delta)^{-1}\left(\frac{f_i}{(1+|x|^2)^2}\right)dx = 0$$

because $(-\Delta)^{-1}\big(f_i\,(1+|x|^2)^{-2}\big)$ is proportional to f_i. As a straightforward consequence, the dual form of Lemma 13 then reads as follows.

Corollary 6. *For any g satisfying the orthogonality conditions* (14), *we have*

$$\int_{\mathbb{R}^d} \frac{g}{(1+|x|^2)^2}\,(-\Delta)^{-1}\left(\frac{g}{(1+|x|^2)^2}\right)dx \le \frac{1}{(d+2)\,(d+4)}\int_{\mathbb{R}^d} \frac{g^2}{(1+|x|^2)^2}\;dx\,.$$

Moreover, if f obeys to (14), *then we have*

$$\frac{4}{d\,(d+4)(d+2)}\int_{\mathbb{R}^d} \frac{f^2}{(1+|x|^2)^2}\;dx \le \mathsf{G}[f] \le \frac{1}{d^2\,(d+2)^2}\,\mathsf{F}[f]$$

and equalities are achieved in $\mathrm{L}^2(\mathbb{R}^d,(1+|x|^2)^{-2}\,dx)$.

Proof. The first inequality follows from the above considerations on duality and the second one from the definition of G, using

$$\frac{4}{d\,(d+4)(d+2)}=\frac{1}{d\,(d+2)}-\frac{1}{(d+2)\,(d+4)}\,.$$

Next we do an expansion of the square as in Section 2. With $a=d\,(d+2)$, let us compute

$$\begin{aligned}0\le&\int_{\mathbb{R}^d}\left|\nabla f-\,a\,\nabla\,(-\Delta)^{-1}\left(\frac{f}{(1+|x|^2)^2}\right)\right|^2\,dx\\&=\int_{\mathbb{R}^d}|\nabla f|^2\,dx-\,2a\int_{\mathbb{R}^d}\frac{|f|^2}{(1+|x|^2)^2}\,dx\\&\quad+a^2\int_{\mathbb{R}^d}\frac{f}{(1+|x|^2)^2}\,(-\Delta)^{-1}\left(\frac{f}{(1+|x|^2)^2}\right)dx=\mathsf{F}[f]-a^2\,\mathsf{G}[f]\,.\end{aligned}$$

Equality is then achieved by the stereographic projection of any spherical harmonic function associated with $\lambda_2=2\,(d{+}1)$ (see Appendix B for details). □

As a consequence of Corollary 6 and (13), we have found that

$$\frac{1}{\mathsf{C}_d}=\inf_{\mathcal{G}[u^q]\neq0}\frac{\|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{8}{d-2}}\,\mathcal{F}[u]}{\mathcal{G}[u^q]}\le\frac{\|u_\star\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{8}{d-2}}\,\mathsf{S}_d}{d^2\,(d+2)^2}\;\inf_f\frac{\mathsf{F}[f]}{\mathsf{G}[f]}=\frac{1}{\mathsf{S}_d}\,,\qquad(15)$$

where the last infimum is taken on the set of all non-trivial functions in $\mathrm{L}^2(\mathbb{R}^d,(1{+}|x|^2)^{-2}\,dx)$ satisfying (14). This completes the proof of Theorem 1.

4. Improved inequalities and nonlinear flows

In Section 3, the basic strategy was based on the completion of a square. The initial approach for the improvement of Sobolev inequalities in [3] was based on a fast diffusion flow. Let us give some details and explain how even better results can be obtained using a combination of the two approaches.

Let us start with a summary of the method of [3]. It will be convenient to define the functionals

$$\mathsf{J}_d[v]:=\int_{\mathbb{R}^d}v^{\frac{2\,d}{d+2}}\,dx\quad\text{and}\quad\mathsf{H}_d[v]:=\int_{\mathbb{R}^d}v\,(-\Delta)^{-1}v\,dx-\mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}\,.$$

Consider a positive solution v of the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d\,, \quad m = \frac{d-2}{d+2} \tag{16}$$

and define the functions

$$\mathsf{J}(t) := \mathsf{J}_d[v(t,\cdot)] \quad \text{and} \quad \mathsf{H}(t) := \mathsf{H}_d[v(t,\cdot)]\,.$$

We shall denote by J_0 and H_0 the corresponding initial values. Elementary computations show that

$$\mathsf{J}' = -\,(m+1)\,\|\nabla v^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le -\,\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{1-\frac{2}{d}} = -\,\frac{2\,d}{d+2}\,\frac{1}{\mathsf{S}_d}\,\mathsf{J}^{1-\frac{2}{d}}\,, \tag{17}$$

where the inequality is a consequence of Sobolev's inequality. Hence v has a finite extinction time $T > 0$ and since

$$\mathsf{J}(t)^{\frac{2}{d}} \le \mathsf{J}_0^{\frac{2}{d}} - \frac{4}{d+2}\,\frac{t}{\mathsf{S}_d}\,,$$

we find that

$$T \le \frac{d+2}{4}\,\mathsf{S}_d\,\mathsf{J}_0^{\frac{2}{d}}\,.$$

We notice that H is nonpositive because of the Hardy-Littlewood-Sobolev inequality and by applying the flow of (16), we get that

$$\frac{1}{2}\,\mathsf{J}^{-\frac{2}{d}}\,\mathsf{H}' = \mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \text{with} \quad u = v^{\frac{d-2}{d+2}}\,.$$

The right hand side is nonnegative because of Sobolev's inequality. One more derivation with respect to t gives that

$$\mathsf{H}'' = \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' - 4\,m\,\mathsf{S}_d\,\mathsf{J}^{\frac{2}{d}}\,\mathsf{K}$$

where $\mathsf{K} := \int_{\mathbb{R}^d} v^{m-1}\,|\Delta v^m + \Lambda\,v|^2\;dx$ and $\Lambda := \frac{d+2}{2\,d}\,\frac{\mathsf{J}'}{\mathsf{J}}$. This identity makes sense in dimension $d \ge 5$, because, close to the extinction time, v behaves like the Aubin-Talenti functions. The reader is invited to check that all terms are finite when expanding the square in K. A straightforward consequence is the fact that

$$\frac{\mathsf{H}''}{\mathsf{H}'} \le \frac{\mathsf{J}'}{\mathsf{J}} \le -\,\kappa \quad \text{with} \quad \kappa := \frac{2\,d}{d+2}\,\frac{\mathsf{J}_0^{-\frac{2}{d}}}{\mathsf{S}_d}$$

where the last inequality is a consequence of (17). Two integrations with respect to t show that

$$-\,\mathsf{H}_0 \le \frac{1}{\kappa}\,\mathsf{H}_0'\,(1-e^{-\kappa T}) \le \frac{1}{2}\,\mathcal{C}\,\mathsf{S}_d\,\mathsf{J}_0^{\frac{2}{d}}\,\mathsf{H}_0' \quad \text{with} \quad \mathcal{C} = \frac{d+2}{d}\,(1-e^{-d/2})\,,$$

which is the main result of [3], namely

$$-\,\mathsf{H}_0 \le \mathcal{C}\,\mathsf{S}_d\,\mathsf{J}_0^{\frac{4}{d}}\,\left[\mathsf{S}_d\,\|\nabla u_0\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u_0\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right] \quad \text{with} \quad u_0 = v_0^{\frac{d-2}{d+2}}\,.$$

Since this inequality holds for any initial datum $u_0 = u$, we have indeed shown that

$$-\;\mathsf{H}_d[v] \le \mathcal{C}\,\mathsf{S}_d\,\mathsf{J}_d[v]^{\frac{4}{d}}\,\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right]$$
$$\forall\, u \in \mathcal{D}^{1,2}(\mathbb{R}^d)\,,\; v = u^{\frac{d+2}{d-2}}\,.$$

It is straightforward to check that our result of Theorem 1 is an improvement, not only because the restriction $d \ge 5$ is removed, but also because the inequality holds with $\mathcal{C} = 1 < \frac{d+2}{d}\,(1-e^{-d/2})$. In other words, the result of Theorem 1 is equivalent to

$$-\;\mathsf{H}_0 \le \frac{1}{2}\,\mathsf{S}_d\,\mathsf{J}_0^{\frac{2}{d}}\,\mathsf{H}_0'\,. \tag{18}$$

Now let us reinject in the flow method described above our improved inequality of Theorem 1, which can also be written as

$$\mathsf{S}_d\,\mathsf{J}^{\frac{4}{d}}\,\left[\frac{d+2}{2\,d}\,\mathsf{S}_d\,\mathsf{J}' + \mathsf{J}^{1-\frac{2}{d}}\right] - \mathsf{H} \le 0 \tag{19}$$

if v is still a positive solution of (16). From the inequality

$$\frac{\mathsf{H}''}{\mathsf{H}'} \le \frac{\mathsf{J}'}{\mathsf{J}}\,,$$

we deduce that

$$\mathsf{H}' \le \kappa_0\,\mathsf{J} \quad \text{with} \quad \kappa_0 := \frac{\mathsf{H}_0'}{\mathsf{J}_0}\,.$$

Since $t \mapsto \mathsf{J}(t)$ is monotone decreasing, there exists a function Y such that

$$\mathsf{H}(t) = -\,\mathsf{Y}(\mathsf{J}(t)) \quad \forall\, t \in [0,T)\,.$$

Differentiating with respect to t, we find that

$$-\,\mathsf{Y}'(\mathsf{J})\,\mathsf{J}' = \mathsf{H}' \le \kappa_0\,\mathsf{J}$$

and, by inserting this expression in (19), we arrive at

$$-\,\frac{d+2}{2\,d}\,\kappa_0\,\mathsf{S}_d^2\,\frac{\mathsf{J}^{1+\frac{4}{d}}}{\mathsf{Y}'} + \mathsf{S}_d\,\mathsf{J}^{1+\frac{2}{d}} + \mathsf{Y} \le 0\,.$$

Summarizing, we end up by considering the differential inequality

$$\mathsf{Y}'\left(\mathsf{S}_d\,s^{1+\frac{2}{d}} + \mathsf{Y}\right) \le \frac{d+2}{2\,d}\,\kappa_0\,\mathsf{S}_d^2\,s^{1+\frac{4}{d}}\,, \quad \mathsf{Y}(0) = 0\,, \quad \mathsf{Y}(\mathsf{J}_0) = -\,\mathsf{H}_0$$

on the interval $[0,\mathsf{J}_0] \ni s$. It is then possible to obtain estimates as follows. On the one hand we know that

$$\mathsf{Y}' \le \frac{d+2}{2\,d}\,\kappa_0\,\mathsf{S}_d\,s^{\frac{2}{d}}$$

and, hence,

$$\mathsf{Y}(s) \le \frac{1}{2}\,\kappa_0\,\mathsf{S}_d\,s^{1+\frac{2}{d}} \quad \forall\, s \in [0,\mathsf{J}_0]\,.$$

On the other hand, after integrating by parts on the interval $[0,\mathsf{J}_0]$, we get

$$\frac{1}{2}\,\mathsf{H}_0^2 - \,\mathsf{S}_d\,\mathsf{J}_0^{1+\frac{2}{d}}\,\mathsf{H}_0 \le \frac{1}{4}\,\kappa_0\,\mathsf{S}_d^2\,\mathsf{J}_0^{2+\frac{4}{d}} + \frac{d+2}{d}\,\mathsf{S}_d \int_0^{\mathsf{J}_0} s^{\frac{2}{d}}\,\mathsf{Y}(s)\,ds\,.$$

Using the above estimate, we find that

$$\frac{d+2}{d}\,\mathsf{S}_d \int_0^{\mathsf{J}_0} s^{\frac{2}{d}}\,\mathsf{Y}(s)\,ds \le \frac{1}{4}\,\mathsf{J}_0^{2+\frac{4}{d}}\,,$$

and finally

$$\frac{1}{2}\,\mathsf{H}_0^2-\,\mathsf{S}_d\,\mathsf{J}_0^{1+\frac{2}{d}}\,\mathsf{H}_0\le\frac{1}{2}\,\kappa_0\,\mathsf{S}_d^2\,\mathsf{J}_0^{2+\frac{4}{d}}\,.$$

This is a strict improvement of (18) since (18) is equivalent to $-\,\mathsf{S}_d\,\mathsf{J}_0^{1+\frac{2}{d}}\,\mathsf{H}_0\le\frac{1}{2}\,\kappa_0\,\mathsf{S}_d^2\,\mathsf{J}_0^{2+\frac{4}{d}}$. Altogether, we have shown an improved inequality that can be stated as follows.

Theorem 7. *Assume that $d\ge5$. Then we have*

$$0\le\mathsf{H}_d[v]+\mathsf{S}_d\,\mathsf{J}_d[v]^{1+\frac{2}{d}}\,\varphi\left(\mathsf{J}_d[v]^{\frac{2}{d}-1}\,\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}-\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right]\right)\\ \forall\,u\in\mathcal{D}^{1,2}(\mathbb{R}^d)\,,\;v=u^{\frac{d+2}{d-2}}$$

where $\varphi(x):=\sqrt{1+2\,x}-1$ for any $x\ge0$.

Proof. We have shown that $x^2-\,2\,x-\,\kappa_0\le0$ with $x=-\,\mathsf{H}_0/(\mathsf{S}_d\,\mathsf{J}_0^{1+\frac{2}{d}})\ge0$. This proves that $x\le\sqrt{1+\kappa_0}-1$, which proves that

$$-\,\mathsf{H}_0\le\mathsf{S}_d\,\mathsf{J}_0^{1+\frac{2}{d}}\,\left(\sqrt{1+\kappa_0}-1\right)$$

after recalling that

$$\kappa_0=\mathsf{H}_0'/\mathsf{J}_0=2\,\mathsf{J}_d[v_0]^{\frac{2}{d}-1}\,\left[\mathsf{S}_d\,\|\nabla u_0\|^2_{\mathrm{L}^2(\mathbb{R}^d)}-\|u_0\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right]\,.$$

□

We may observe that $x\mapsto x-\varphi(x)$ is a convex nonnegative function which is equal to 0 if and only if $x=0$. Moreover, we have

$$\varphi(x)\le x\quad\forall\,x\ge0$$

with equality if and only if $x=0$.

Corollary 8. *With the notations of Theorem 1, if $d\ge5$, there is no function u such that equality holds in Inequality (3) unless $u\in\mathcal{D}^{1,2}(\mathbb{R}^d)$ is proportional to $u_\star$ up to a multiplication by a constant, scalings or translations. As a consequence*

$$\frac{1}{\mathsf{S}_d}=\inf_{\mathcal{G}[u^q]\neq0}\frac{\|u\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\,\mathcal{F}[u]}{\mathcal{G}[u^q]}$$

is not achieved, and any minimizing sequence $(u_n)_{n\in\mathbb{N}}$ *of* $\mathcal{F}/\mathcal{G}$ *such that* $\|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} = 1$ *is such that* $\lim_{n\to\infty}\mathcal{F}[u_n] = \lim_{n\to\infty}\mathcal{G}[u_n^q] = 0$.

Proof. Here we use the notations of Section 3. Let us consider a minimizing sequence $(u_n)_{n\in\mathbb{N}}$ for the functional $u \mapsto \frac{\mathcal{F}[u]}{\mathcal{G}[u^q]}$ but assume that $\mathsf{J}_d[u_n^q] = \mathsf{J}_d[u_\star^q] =: \mathsf{J}_\star$ for any $n \in \mathbb{N}$. This condition is not restrictive because of the homogeneity of the inequality. It implies that $(\mathcal{G}[u_n^q])_{n\in\mathbb{N}}$ is bounded. Working up to the extraction of subsequences, we can therefore distinguish two cases:

(i) If $\lim_{n\to\infty}\mathcal{G}[u_n^q] > 0$, then we also have $\mathsf{L} := \lim_{n\to\infty}\mathcal{F}[u_n] > 0$. As a consequence we find that

$$\begin{aligned} 0 = \lim_{n\to\infty}\left(\mathsf{S}_d\,\mathsf{J}_\star^{\frac{4}{d}}\,\mathcal{F}[u_n] - \mathcal{G}[u_n^q]\right) \\ = \mathsf{S}_d \lim_{n\to\infty}\left[\mathsf{J}_\star^{\frac{4}{d}}\,\mathcal{F}[u_n] - \mathsf{J}_\star^{1+\frac{2}{d}}\,\varphi\left(\mathsf{J}_\star^{\frac{2}{d}-1}\,\mathcal{F}[u_n]\right)\right] \\ + \lim_{n\to\infty}\left[\mathsf{S}_d\,\mathsf{J}_\star^{1+\frac{2}{d}}\,\varphi\left(\mathsf{J}_\star^{\frac{2}{d}-1}\,\mathcal{F}[u_n]\right) - \mathcal{G}[u_n^q]\right], \end{aligned}$$

a contradiction since

$$\mathsf{S}_d\,\mathsf{J}_\star^{1+\frac{2}{d}}\,\varphi\left(\mathsf{J}_\star^{\frac{2}{d}-1}\,\mathcal{F}[u_n]\right) - \mathcal{G}[u_n^q]$$

is nonnegative by Theorem 7 and

$$\mathsf{J}_\star^{\frac{4}{d}}\,\mathsf{L} - \mathsf{J}_\star^{1+\frac{2}{d}}\,\varphi\left(\mathsf{J}_\star^{\frac{2}{d}-1}\,\mathsf{L}\right)$$

is positive unless $\mathsf{L} = 0$.

(ii) If $\lim_{n\to\infty}\mathcal{G}[u_n^q] = 0$, then we also have $\lim_{n\to\infty}\mathcal{F}[u_n] = 0$ because the quotient has a finite limit. Standard tools of the concentration-compactness method (see for instance [25]) allow to prove that up to translations and scalings, the sequence sequence $(u_n)_{n\in\mathbb{N}}$ converges to $u_\star$. Arguing as in [20] and with $f^n = u_n - u_\star$ for any $n \in \mathbb{N}$, we get that

$$\frac{1}{\mathsf{S}_d\,\mathsf{J}_\star^{\frac{4}{d}}} = \lim_{n\to\infty}\frac{\mathcal{F}[u_n]}{\mathcal{G}[u_n^q]} = \frac{1}{d^2\,(d+2)^2}\,\frac{1}{\mathsf{S}_d}\,\lim_{n\to\infty}\frac{\mathsf{F}[f^n]}{\mathsf{G}[f^n]}\,.$$

□

The reader is invited to notice that, up to the extraction of a subsequence,

$f^n = \varepsilon_n\, f(1+o(1))$ as $n \to \infty$, where $\varepsilon_n \neq 0$ is a proportionality constant such that $\lim_{n\to\infty} \varepsilon_n = 0$ and $f \in \mathrm{Vect}(\{f_i\}_{1\le i\le d+1})$. This precisely describes the asymptotic behavior of the minimizing sequence $(u_n)_{n\in\mathbb{N}}$ at leading order.

5. Caffarelli-Kohn-Nirenberg inequalities and duality

Let $2^* := \infty$ if $d = 1$ or 2, $2^* := 2\,d/(d-2)$ if $d \ge 3$ and $a_c := (d-2)/2$. Consider the space $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$ obtained by completion of $\mathcal{D}(\mathbb{R}^d \setminus \{0\})$ with respect to the norm $u \mapsto \|\,|x|^{-a}\,\nabla u\,\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$. In this section, we shall consider the Caffarelli-Kohn-Nirenberg inequalities

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}}\,dx\right)^{\frac{2}{p}} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx \tag{20}$$

These inequalities generalize to $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$ the Sobolev inequality (1) and in particular the exponent p is given in terms of a and b by

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

as can be checked by a simple scaling argument. A precise statements on the range of validity of (20) goes as follows.

Lemma 9. [26] *Let $d \ge 1$. For any $p \in [2, 2^*]$ if $d \ge 3$ or $p \in [2, 2^*)$ if $d = 1$ or 2, there exists a positive constant $\mathsf{C}_{a,b}$ such that (20) holds if a, b and p are related by $b = a - a_c + d/p$, with the restrictions $a < a_c$, $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$ and $a + 1/2 < b \le a+1$ if $d = 1$.*

At least for radial solutions in $\mathbb{R}^d$, weights can be used to work as in Section 2 as if the dimension d was replaced by the *dimension* $(d-2a)$. We will apply this heuristic idea to the case $d = 2$ and $a < 0$, $a \to 0$ in order to prove Theorem 2. See Appendix C for symmetry results for optimal functions in (20).

On $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$, let us define the functionals

$$\mathsf{F}_1[u] := \frac{1}{2}\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}}\,dx\right)^{\frac{2}{p}} \quad \text{and} \quad \mathsf{F}_2[u] := \frac{1}{2}\,\mathsf{C}_{a,b}\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx$$

so that Inequality (20) amounts to $\mathsf{F}_1[u] \le \mathsf{F}_2[u]$. Assume that $\langle \cdot, \cdot \rangle$ denotes the natural scalar product on $\mathrm{L}^2\big(\mathbb{R}^d, |x|^{-2a}\,dx\big)$, that is,

$$\langle u, v \rangle := \int_{\mathbb{R}^d} \frac{u\,v}{|x|^{2a}}\,dx$$

and denote by $\|u\| = \langle u, u\rangle^{1/2}$ the corresponding norm. Consider the operators

$$\mathsf{A}_a\,u := \nabla u\,, \quad \mathsf{A}_a^*\,w := -\nabla\cdot w + 2a\,\frac{x}{|x|^2}\cdot w$$
$$\text{and} \quad \mathsf{L}_a\,u := \mathsf{A}_a^*\,\mathsf{A}_a\,u = -\,\Delta u + 2a\,\frac{x}{|x|^2}\cdot\nabla u$$

defined for u and w respectively in $\mathrm{L}^2\big(\mathbb{R}^d, |x|^{-2a}\,dx\big)$ and $\mathrm{L}^2\big(\mathbb{R}^d, |x|^{-2a}\,dx\big)^d$. Elementary integrations by parts show that

$$\langle u, \mathsf{L}_a\,u\rangle = \langle \mathsf{A}_a\,u, \mathsf{A}_a\,u\rangle = \|\mathsf{A}_a\,u\|^2 = \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx\,.$$

If we define the Legendre dual of F_i by $\mathsf{F}_i^*[v] = \sup_{u\in\mathcal{D}_a^{1,2}(\mathbb{R}^d)}\big(\langle u, v\rangle - \mathsf{F}_i[u]\big)$, then it is clear that we formally have the inequality $\mathsf{F}_2^*[v] \le \mathsf{F}_1^*[v]$ for any $v \in \mathrm{L}^q(\mathbb{R}^d, |x|^{-(2a-\,b)\,q}\,dx) \cap \mathsf{L}_a(\mathcal{D}_a^{1,2}(\mathbb{R}^d))$, where q is Hölder's conjugate of p, *i.e.*

$$\frac{1}{p} + \frac{1}{q} = 1\,.$$

Using the invertibility of L_a, we indeed observe that

$$\mathsf{F}_2^*[v] = \langle u, v\rangle - \mathsf{F}_2[u] \quad \text{with} \quad v = \mathsf{C}_{a,b}\,\mathsf{L}_a\,u \iff u = \frac{1}{\mathsf{C}_{a,b}}\,\mathsf{L}_a^{-1}\,v\,,$$

hence proving that

$$\mathsf{F}_2^*[v] = \frac{1}{2\,\mathsf{C}_{a,b}}\,\langle v, \mathsf{L}_a^{-1}\,v\rangle\,.$$

Similarly, we get that $\mathsf{F}_1^*[v] = \langle u, v\rangle - \mathsf{F}_1[u]$ with

$$|x|^{-\,2a}\,v = \kappa^{2-p}\,|x|^{-\,bp}\,u^{p-1} \tag{21}$$

and

$$\kappa = \left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}}\,dx \right)^{\frac{1}{p}} = \langle u, v \rangle = \left(\int_{\mathbb{R}^d} \frac{|v|^q}{|x|^{(2a-\,b)\,q}}\,dx \right)^{\frac{1}{q}} ,$$

that is

$$\mathsf{F}_1^*[v] = \frac{1}{2} \left(\int_{\mathbb{R}^d} \frac{|v|^q}{|x|^{(2a-\,b)\,q}}\,dx \right)^{\frac{2}{q}} .$$

This proves the following result.

Lemma 10. *With the above notations and under the same assumptions as in Lemma 9, we have*

$$\frac{1}{\mathsf{C}_{a,b}} \langle v, \mathsf{L}_a^{-1}\, v \rangle \le \left(\int_{\mathbb{R}^d} \frac{|v|^q}{|x|^{(2a-\,b)\,q}}\,dx \right)^{\frac{2}{q}} \quad \forall\, v \in \mathrm{L}^q(\mathbb{R}^d, |x|^{-(2a-\,b)\,q}\,dx) \cap \mathsf{L}_a(\mathcal{D}_a^{1,2}(\mathbb{R}^d))\,.$$

The next step is based on the completion of the square. Let us compute

$$\begin{aligned} &\|\mathsf{A}_a\, u - \lambda\, \mathsf{A}_a\, \mathsf{L}_a^{-1}\, v\|^2 \\ &= \|\mathsf{A}_a\, u\|^2 - \,2\,\lambda\, \langle \mathsf{A}_a\, u, \mathsf{A}_a\, \mathsf{L}_a^{-1}\, v\rangle + \,\lambda^2\, \langle \mathsf{A}_a\, \mathsf{L}_a^{-1}\, v, \mathsf{A}_a\, \mathsf{L}_a^{-1}\, v\rangle \\ &= \|\mathsf{A}_a\, u\|^2 - \,2\,\lambda\, \langle u, v\rangle + \,\lambda^2\, \langle v, \mathsf{L}_a^{-1}\, v\rangle\,. \end{aligned}$$

With the choice $\lambda = 1/\mathsf{C}_{a,b}$ and v given by (21), we have proved the following

Theorem 11. *Under the assumptions of Lemma 9 and with the above notations, for any $u \in \mathcal{D}_a^{1,2}(\mathbb{R}^d)$ and any $v \in \mathrm{L}^q(\mathbb{R}^d, |x|^{-(2a-\,b)\,q}\,dx) \cap \mathsf{L}_a(\mathcal{D}_a^{1,2}(\mathbb{R}^d))$ we have*

$$0 \le \left(\int_{\mathbb{R}^d} \frac{|v|^q}{|x|^{(2a-\,b)\,q}}\,dx \right)^{\frac{2}{q}} - \frac{1}{\mathsf{C}_{a,b}} \langle v, \mathsf{L}_a^{-1}\, v \rangle \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx - \left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}}\,dx \right)^{\frac{2}{p}}$$

if u and v are related by (21), if a, b and p are such that $b = a - a_c + d/p$ and verify the conditions of Lemma 9, and if $q = p/(p-1)$.

If, instead of (21), we simply require that

$$|x|^{-\,2a}\, v = |x|^{-\,bp}\, u^{p-1}\,,$$

then the inequality becomes

$$
\begin{aligned}
0 &\le \mathsf{C}_{a,b}\left(\int_{\mathbb{R}^d}\frac{|v|^q}{|x|^{(2a-b)\,q}}\,dx\right)^{\frac{2}{q}}-\langle v,\mathsf{L}_a^{-1}\,v\rangle\\
&\le \mathsf{C}_{a,b}\left(\int_{\mathbb{R}^d}\frac{|u|^p}{|x|^{bp}}\,dx\right)^{\frac{2}{p}(p-2)}\left[\mathsf{C}_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^{2a}}\,dx-\left(\int_{\mathbb{R}^d}\frac{|u|^p}{|x|^{bp}}\,dx\right)^{\frac{2}{p}}\right]
\end{aligned}
$$

Hence Theorem 11 generalizes Theorem 1, which is recovered in the special case $a=b=0$, $d\ge 3$. Because of the positivity of the l.h.s. due to Lemma 10, the inequality in Theorem 11 is an improvement of the Caffarelli-Kohn-Nirenberg inequality (20). It can also be seen as an interpolation result, namely

$$
\begin{aligned}
2\left(\int_{\mathbb{R}^d}\frac{|v|^q}{|x|^{(2a-b)\,q}}\,dx\right)^{\frac{2}{q}} &= 2\left(\int_{\mathbb{R}^d}\frac{|u|^p}{|x|^{bp}}\,dx\right)^{\frac{2}{p}}\\
&\le \mathsf{C}_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^{2a}}\,dx+\frac{1}{\mathsf{C}_{a,b}}\,\langle v,\mathsf{L}_a^{-1}\,v\rangle
\end{aligned}
$$

whenever u and v are related by (21). The explicit value of $\mathsf{C}_{a,b}$ is not known unless equality in (20) is achieved by radial functions, that is when *symmetry* holds. See Proposition 14 in Appendix C for some symmetry results. Now, as in [27], we may investigate the limit $(a,b)\to(0,0)$ with $b=\alpha\,a/(1+\alpha)$ in order to investigate the Onofri limit case. A key observation is that optimality in (20) is achieved by radial functions for any $\alpha\in(-1,0)$ and $a<0$, $|a|$ small enough. In that range $\mathsf{C}_{a,b}$ is known and given by (C.1).

Proof of Theorem 2 (continued). Theorem 2 has been established for radial functions in Section 2. Now we investigate the general case. We shall restrict our purpose to the case of dimension $d=2$. For any $\alpha\in(-1,0)$, let us denote by $d\mu_\alpha$ the probability measure on $\mathbb{R}^2$ defined by $d\mu_\alpha:=\mu_\alpha\,dx$ where

$$
\mu_\alpha:=\frac{1+\alpha}{\pi}\,\frac{|x|^{2\,\alpha}}{(1+|x|^{2\,(1+\alpha)})^2}\,.
$$

It has been established in [27] that

$$\log\left(\int_{\mathbb{R}^2} e^{\,u}\,d\mu_\alpha\right)-\int_{\mathbb{R}^2} u\,d\mu_\alpha\le\frac{1}{16\,\pi\,(1+\alpha)}\int_{\mathbb{R}^2}|\nabla u|^2\,dx\quad\forall\,u\in\mathcal{D}(\mathbb{R}^2)\,,\tag{22}$$

where $\mathcal{D}(\mathbb{R}^2)$ is the space of smooth functions with compact support. By density with respect to the natural norm defined by each of the inequalities, the result also holds on the corresponding Orlicz space.

We adopt the strategy of [27, Section 2.3] to pass to the limit in (20) as $(a,b)\to(0,0)$ with $b=\frac{\alpha}{\alpha+1}\,a$. Let

$$a_\varepsilon=-\frac{\varepsilon}{1-\varepsilon}\,(\alpha+1)\,,\quad b_\varepsilon=a_\varepsilon+\varepsilon,\quad p_\varepsilon=\frac{2}{\varepsilon}\,,$$

and

$$u_\varepsilon(x)=\big(1+|x|^{2\,(\alpha+1)}\big)^{-\frac{\varepsilon}{1-\varepsilon}}\,,$$

assuming that u_ε is an optimal function for (20), define

$$\kappa_\varepsilon=\int_{\mathbb{R}^2}\left[\frac{u_\varepsilon}{|x|^{a_\varepsilon+\varepsilon}}\right]^{2/\varepsilon}dx=\int_{\mathbb{R}^2}\frac{|x|^{2\,\alpha}}{\big(1+|x|^{2\,(1+\alpha)}\big)^2}\,\frac{u_\varepsilon^2}{|x|^{2a_\varepsilon}}\,dx=\frac{\pi}{\alpha+1}\,\frac{\Gamma\big(\frac{1}{1-\varepsilon}\big)^2}{\Gamma\big(\frac{2}{1-\varepsilon}\big)}\,,$$

$$\lambda_\varepsilon=\int_{\mathbb{R}^2}\left[\frac{|\nabla u_\varepsilon|}{|x|^{a}}\right]^2dx=4\,a_\varepsilon^2\int_{\mathbb{R}^2}\frac{|x|^{2\,(2\,\alpha+1-a_\varepsilon)}}{\big(1+|x|^{2\,(1+\alpha)}\big)^{\frac{2}{1-\varepsilon}}}\,dx=4\,\pi\,\frac{|a_\varepsilon|}{1-\varepsilon}\,\frac{\Gamma\big(\frac{1}{1-\varepsilon}\big)^2}{\Gamma\big(\frac{2}{1-\varepsilon}\big)}\,.$$

Then $w_\varepsilon=(1+\frac{1}{2}\,\varepsilon\,u)\,u_\varepsilon$ is such that

$$\lim_{\varepsilon\to0_+}\frac{1}{\kappa_\varepsilon}\int_{\mathbb{R}^2}\frac{|w_\varepsilon|^{p_\varepsilon}}{|x|^{b_\varepsilon p_\varepsilon}}\,dx=\int_{\mathbb{R}^2}e^{u}\,d\mu_\alpha\,,$$

$$\lim_{\varepsilon\to0_+}\frac{1}{\varepsilon}\left[\frac{1}{\lambda_\varepsilon}\int_{\mathbb{R}^2}\frac{|\nabla w_\varepsilon|^2}{|x|^{2a_\varepsilon}}\,dx-1\right]=\int_{\mathbb{R}^2}u\,d\mu_\alpha+\frac{1}{16\,(1+\alpha)\,\pi}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^2)}\,.$$

Hence we can recover (22) by passing to the limit in (20) as $\varepsilon\to0_+$. On the other hand, if we pass to the limit in the inequality stated in Theorem 11, we arrive at the following result, for any $\alpha\in(-1,0)$.

Theorem 12. *Let $\alpha \in (-1,0]$. With the above notations, we have*

$$0 \le \int_{\mathbb{R}^2} v \log\left(\frac{v}{\mu_\alpha}\right) dx - 4\,\pi\,(1+\alpha)\int_{\mathbb{R}^2} (v-\mu_\alpha)\,(-\Delta)^{-1}\,(v-\mu_\alpha)\,dx$$
$$\le \left(\int_{\mathbb{R}^2} e^{\,u}\,d\mu_\alpha\right)^2 \left[\frac{\int_{\mathbb{R}^2} |\nabla u|^2\,dx}{16\,\pi\,(1+\alpha)} - \log\left(\int_{\mathbb{R}^2} e^{\,u}\,d\mu_\alpha\right) - \int_{\mathbb{R}^2} u\,d\mu_\alpha\right]$$

for any $u \in \mathcal{D}$, where u and v are related by

$$v = \frac{e^u\,\mu_\alpha}{\int_{\mathbb{R}^2} e^u\,d\mu_\alpha}\,.$$

The case $\alpha = 0$ is achieved by taking the limit as $\alpha \to 0_-$. Since $-\Delta \log \mu_\alpha = 8\,\pi\,(1+\alpha)\,\mu_\alpha$ holds for any $\alpha \in (-1,0]$, the proof of Theorem 2 is now completed, with $\mu = \mu_0$. □

Appendix A. Some useful formulae

We recall that

$$f(q) := \int_{\mathbb{R}} \frac{dt}{(\cosh t)^q} = \frac{\sqrt{\pi}\,\Gamma(\frac{q}{2})}{\Gamma(\frac{q+1}{2})}$$

for any $q > 0$. An integration by parts shows that $f(q+2) = \frac{q}{q+1}\,f(q)$. The following formulae are reproduced with no change from [28] (also see [29, 30]). The function $w(t) := (\cosh t)^{-\frac{2}{p-2}}$ solves

$$-(p-2)^2\,w'' + 4\,w - 2\,p\,w^{p-1} = 0$$

and we can define

$$\mathsf{I}_q := \int_{\mathbb{R}} |w(t)|^q\,dt \quad \text{and} \quad \mathsf{J}_2 := \int_{\mathbb{R}} |w'(t)|^2\,dt\,.$$

Using the function f, we can compute $\mathsf{I}_2 = f\big(\frac{4}{p-2}\big)$, $\mathsf{I}_p = f\big(\frac{2\,p}{p-2}\big) = f\big(\frac{4}{p-2}+2\big)$ and get the relations

$$\mathsf{I}_2 = \frac{\sqrt{\pi}\,\Gamma\big(\frac{2}{p-2}\big)}{\Gamma\big(\frac{p+2}{2\,(p-2)}\big)}\,, \quad \mathsf{I}_p = \frac{4\,\mathsf{I}_2}{p+2} = \frac{4\,\sqrt{\pi}\,\Gamma\left(\frac{2}{p-2}\right)}{(p+2)\,\Gamma\left(\frac{p+2}{2\,(p-2)}\right)}\,, \quad \mathsf{J}_2 = \frac{4\,\mathsf{I}_2}{(p+2)\,(p-2)}\,.$$

In particular, this establishes (8), namely

$$\mathsf{s}_d = \frac{\mathsf{I}_p^{1-\frac{2}{d}}}{\mathsf{J}_2 + \frac{1}{4}\,(d-2)^2\,\mathsf{I}_2}\,, \quad \text{with } p = \frac{2\,d}{d-2}$$

for any $d > 2$. The expression of the optimal constant in Sobolev's inequality (1): $\mathsf{S}_d = \mathsf{s}_d\,|\mathbb{S}^{d-1}|^{-2/d}$, where

$$|\mathbb{S}^{d-1}| = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}$$

denotes the volume of the unit sphere, for any integer $d \geq 3$, follows from the duplication formula

$$2^{d-1}\,\Gamma\left(\tfrac{d}{2}\right)\,\Gamma\left(\tfrac{d+1}{2}\right) = \sqrt{\pi}\,\Gamma(d)$$

according for instance to [31]. See [32, Appendix B.4] for further details.

Appendix B. Poincaré inequality and stereographic projection

On $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, consider the coordinates $\omega = (\rho\,\phi, z) \in \mathbb{R}^d \times \mathbb{R}$ such that $\rho^2 + z^2 = 1$, $z \in [-1, 1]$, $\rho \geq 0$ and $\phi \in \mathbb{S}^{d-1}$, and define the *stereographic projection* $\Sigma : \mathbb{S}^d \setminus \{\mathrm{N}\} \to \mathbb{R}^d$ by $\Sigma(\omega) = x = r\,\phi$ and

$$z = \frac{r^2-1}{r^2+1} = 1 - \frac{2}{r^2+1}\,, \quad \rho = \frac{2\,r}{r^2+1}\,.$$

The *North Pole* N corresponds to $z = 1$ (and is formally sent at infinity) while the *equator* (corresponding to $z = 0$) is sent onto the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. Now we can transform any function v on $\mathbb{S}^d$ into a function u on $\mathbb{R}^d$ using

$$v(\omega) = \big(\tfrac{r}{\rho}\big)^{\frac{d-2}{2}}\,u(x) = \big(\tfrac{r^2+1}{2}\big)^{\frac{d-2}{2}}\,u(x) = (1-z)^{-\frac{d-2}{2}}\,u(x)\,.$$

A standard computation shows that

$$\int_{\mathbb{S}^d} |\nabla v|^2 \, d\omega + \frac{1}{4} \, d\,(d-2) \int_{\mathbb{S}^d} |v|^2 \, d\omega = \int_{\mathbb{R}^d} |\nabla u|^2 \, dx$$

and

$$\int_{\mathbb{S}^d} |v|^q \, d\omega = \int_{\mathbb{R}^d} |u|^q \left(\tfrac{2}{1+|x|^2}\right)^{d-(d-2)\frac{q}{2}} \, dx \, .$$

On $\mathbb{S}^d$, the kernel of the Laplace-Beltrami operator is generated by the constants and the lowest positive eigenvalue is $\lambda_1 = d$. The corresponding eigenspace is generated by $v_0(\omega) = 1$ and $v_i(\omega) = \omega_i$, $i = 1, 2, \ldots d+1$. All eigenvalues of the Laplace-Beltrami operator are given by the formula

$$\lambda_k = k\,(k+d-1) \quad \forall\, k \in \mathbb{N}$$

according to [33]. We still denote by $u_\star$ the Aubin-Talenti extremal function

$$u_\star(x) := (1+|x|^2)^{-\frac{d-2}{2}} \quad \forall\, x \in \mathbb{R}^d \, .$$

Using the inverse stereographic projection, the reader is invited to check that Sobolev's inequality is equivalent to the inequality

$$\frac{4}{d\,(d-2)} \int_{\mathbb{S}^d} |\nabla v|^2 \, d\omega + \int_{\mathbb{S}^d} |v|^2 \, d\omega \geq |\mathbb{S}^d|^{\frac{2}{d}} \left(\int_{\mathbb{S}^d} |v|^{\frac{2\,d}{d-2}} \, d\omega \right)^{\frac{d-2}{d}}$$

so that the Aubin-Talenti extremal function is transformed into a constant function on the sphere and incidentally this shows that

$$\mathsf{S}_d = \frac{4}{d\,(d-2)} \, |\mathbb{S}^d|^{-\frac{2}{d}} \, .$$

With these preliminaries on the Laplace-Beltrami operator and the stereographic projection in hand, we can now state the counterpart on $\mathbb{R}^d$ of the Poincaré inequality on $\mathbb{S}^d$.

Lemma 13. *For any function $f \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ such that*

$$\int_{\mathbb{R}^d} f \, \frac{u_\star}{(1+|x|^2)^2} \, dx = 0 \,, \quad \int_{\mathbb{R}^d} f \, \frac{(1-|x|^2) \, u_\star}{(1+|x|^2)^3} \, dx = 0 \,,$$
$$\text{and} \quad \int_{\mathbb{R}^d} f \, \frac{x_i \, u_\star}{(1+|x|^2)^3} \, dx = 0 \quad \forall \, i = 1 \,, \; 2 \,, \; \ldots d$$

the following inequality holds

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, dx \ge (d+2) \, (d+4) \int_{\mathbb{R}^d} \frac{f^2}{(1+|x|^2)^2} \, dx \,.$$

Proof. On the sphere we know that

$$\int_{\mathbb{S}^d} |\nabla v|^2 \, d\omega + \frac{1}{4} \, d \, (d-2) \int_{\mathbb{S}^d} v^2 \, d\omega \ge \left(\lambda_2 + \frac{1}{4} \, d \, (d-2) \right) \int_{\mathbb{S}^d} v^2 \, d\omega$$
$$= \frac{1}{4} \, (d+2)(d+4) \int_{\mathbb{S}^d} v^2 \, d\omega$$

if v is orthogonal to v_i for any $i = 0, \, 1, \, \ldots d+1$. The conclusion follows from the stereographic projection. □

Appendix C. Symmetry in Caffarelli-Kohn-Nirenberg inequalities

In this Appendix, we recall some known results concerning *symmetry* and *symmetry breaking* in the Caffarelli-Kohn-Nirenberg inequalities (20).

Proposition 14. *Assume that $d \ge 2$. There exists a continuous function $\alpha : (2, 2^*) \to (-\infty, 0)$ such that $\lim_{p \to 2^*} \alpha(p) = 0$ for which the equality case in* (20) *is not achieved among radial functions if $a < \alpha(p)$ while for $a < \alpha(p)$ equality is achieved by*

$$u_\star(x) := \left(1 + |x|^{\frac{2}{\delta} \, (a_c - a)} \right)^{-\delta} \quad \forall \, x \in \mathbb{R}^d$$

where $\delta = \frac{a_c + b - a}{1 + a - b}$. Moreover the function α has the following properties

(i) For any $p \in (2, 2^*)$, $\alpha(p) \ge a_c - 2 \sqrt{\frac{d-1}{p^2-4}}$.

(ii) For any $p \in \left(2, 2 \, \frac{d^2-d+1}{d^2-3\,d+3} \right)$, $\alpha(p) \le a_c - \frac{1}{2} \sqrt{\frac{(d-1)\,(6-p)}{p-2}}$.

(iii) If $d = 2$, $\lim_{p\to 2^*} \beta(p)/\alpha(p) = 0$ where $\beta(p) := \alpha(p) - a_c + d/p$.

This result summarizes a list of partial results that have been obtained in various papers. Existence of optimal functions has been dealt with in [34], while Condition (i) in Proposition 14 has been established in [35]. See [36] for the existence of the curve $p \mapsto \alpha(p)$, [37, 38] for various results on symmetry in a larger class of inequalities, and [29] for Property (ii) in Proposition 14. Numerical computations of the branches of non-radial optimal functions and formal asymptotic expansions at the bifurcation point have been collected in [39, 40]. The paper [27] deals with the special case of dimension $d = 2$ and contains Property (iii) in Proposition 14, which can be rephrased as follows: the region of radial symmetry contains the region corresponding to $a \geq \alpha(p)$ and $b \geq \beta(p)$, and the parametric curve $p \mapsto (\alpha(p), \beta(p))$ converges to 0 as $p \to 2^* = \infty$ tangentially to the axis $b = 0$. For completeness, let us mention that [41, Theorem 3.1] covers the case $a > a_c - d/p$ also we will not use it. Finally, let us observe that in the *symmetric case*, the expression of $\mathsf{C}_{a,b}$ can be computed explicitly in terms of the Γ function as

$$\mathsf{C}_{a,b} = |\mathbb{S}^{d-1}|^{\frac{p-2}{p}} \left[\frac{(a-a_c)^2\,(p-2)^2}{p+2}\right]^{\frac{p-2}{2\,p}} \left[\frac{p+2}{2\,p\,(a-a_c)^2}\right] \left[\frac{4}{p+2}\right]^{\frac{6-p}{2\,p}} \left[\frac{\Gamma\left(\frac{2}{p-2}+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2}{p-2}\right)}\right]^{\frac{p-2}{p}} \quad \text{(C.1)}$$

where the volume of the unit sphere is given by $|\mathbb{S}^{d-1}| = 2\,\pi^{\frac{d}{2}}/\,\Gamma\left(\frac{d}{2}\right)$.

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