SCIENTIFIC COMPUTING & FEM

SHEET 4 / A

$$-u''(x) + a \cdot u(x) = f(x) \quad , \quad x \in (0,1) =: \Omega$$

$$u(0) = 0$$

$$\frac{\partial u(1)}{\partial \vec{n}} = \alpha \left(g_B - u(1) \right)$$

with $a, \alpha, g_B \in \mathbb{R}$ (constant)

•) variational formulation (multiply with $v(x)$ and integrate)
↳ with $v(0)=0$

$$\int_0^1 -u''(x) v(x)\, dx + \int_0^1 a\, u(x) v(x)\, dx$$

$$= \int_0^1 u'(x) v'(x)\, dx - \left[u'(x) v(x) \right]_0^1 + a \int_0^1 u(x) v(x)\, dx$$

$$= \int_0^1 u'(x) v'(x)\, dx - \left[u'(1) v(1) - u'(0) \underbrace{v(0)}_{=0} \right] + a \int_0^1 u(x) v(x)\, dx$$

$$= \int_0^1 u'(x) v'(x)\, dx - \alpha \left(g_B - u(1) \right) v(1) + a \int_0^1 u(x) v(x)\, dx = \int_0^1 f v\, dx$$

$\Rightarrow$ find $u \in V_0 = \{ v \in H^1(0,1) : v(0) = 0 \}$ such that

$$a(u,v) = F(v) \quad \text{for all } v \in V_0$$

with $a(u,v) = \int_0^1 u'(x) v'(x)\, dx + a \int_0^1 u(x) v(x)\, dx + \alpha\, u(1) v(1)$

$$F(v) = \int_0^1 f(x) v(x)\, dx + \alpha\, g_B\, v(1)$$

✓

•) to FEM representation N elements

$x_i = \frac{i}{N}$ for $i = 0, \dots, N$ (nodes)

$\tau_i = (x_{i-1}, x_i)$ for $i = 1, \dots, N$

Korrekt für äquidistante Diskretisierung.

our hat functions are

$$\varphi_i(x) = \begin{cases} (x - x_{i-1}) N & x \in \tau_i = (x_{i-1}, x_i) \\ (x_{i+1} - x) N & x \in \tau_{i+1} = (x_i, x_{i+1}) \\ 0 & \text{else} \end{cases}$$

for $i = 0, \dots, N$

we now approximate u by u_h and v by v_h:

$$u(x) = \sum_{i=0}^{N} u_i \varphi_i(x)$$

by choosing the canonical basis vectors for v we get the following discrete representation

$$K \underline{u} = \underline{f}$$

with $K_{ij} = a(\varphi_i, \varphi_j)$, $f_i = \langle F, \varphi_i \rangle$

to insert boundary condition $u(0) = 0$ $\Rightarrow K_{00}^{new} = K_{00} \cdot (1 + 10^6)$

(the rest remains the same)

·) calculate K (stiffness matrix) (symmetric)

$K_{ij} = 0$ for $|i-j| > 1$ (support of functions)

$$\underline{\underline{K_{ii}}} = \int_0^1 \varphi_i' \varphi_i' dx + a \int_0^1 \varphi_i^2 dx + \alpha \varphi_i^2(1)$$

$$= \int_{x_{i-1}}^{x_i} N^2 dx + \int_{x_i}^{x_{i+1}} (-N)^2 dx + a \int_{x_{i-1}}^{x_i} (x - x_i)^2 N^2 dx + a \int_{x_i}^{x_{i+1}} (x_{i+1} - x)^2 N^2 dx + \alpha \delta_{iN}$$

$$= N^2 (x_{i+1} - x_{i-1}) + a N^2 \left. \frac{(x - x_i)^3}{3} \right|_{x_{i-1}}^{x_i} - a N^2 \left. \frac{(x_{i+1} - x)^3}{3} \right|_{x_i}^{x_{i+1}} + \alpha \delta_{iN}$$

$$= N^2 \left(\frac{i+1-i+1}{N} \right) - a N^2 \frac{(x_{i-1} - x_i)^3}{3} + a N^2 \frac{(x_{i+1} - x_i)^3}{3} + \alpha \delta_{iN}$$

$$= 2N - a N^2 \left(-\frac{1}{N} \right)^3 / 3 + a N^2 (1/N)^3 / 3 + \alpha \delta_{iN}$$

$$\underline{\underline{= 2N + a \frac{2}{3} \cdot \frac{1}{N} + \alpha \delta_{iN}}}$$

✓

$$\underline{\underline{K_{i,i+1}}} = \int_0^1 \varphi_i' \varphi_{i+1}' dx + a \int_0^1 \varphi_i \varphi_{i+1} dx + \alpha \underbrace{\varphi_i(1) \varphi_{i+1}(1)}_{=0}$$

$$= \int_{x_i}^{x_{i+1}} -N^2 dx + a \int_{x_i}^{x_{i+1}} (x_{i+1} - x) N (x - x_i) N dx$$

support

$$= -N + a N^2 \frac{(x_{i+1} - x_i)^3}{6}$$

$$= -N + a N^2 \frac{1}{N^3} \cdot \frac{1}{6}$$

$$\underline{\underline{= -N + a \cdot \frac{1}{6} \cdot \frac{1}{N}}}$$

$$K = \begin{pmatrix} \left(2N + \frac{2a}{3N}\right)(1+10^6) & -N + \frac{a}{6N} & & & & 0 \\ -N + \frac{a}{6N} & 2N + \frac{2a}{3N} & -N + \frac{a}{6N} & & & \\ & -N + \frac{a}{6N} & 2N + \frac{2a}{3N} & \ddots & & \\ & & \ddots & \ddots & -N + \frac{a}{6N} & \\ & 0 & & \ddots & 2N + \frac{2a}{3N} & -N + \frac{a}{6N} \\ & & & & -N + \frac{a}{6N} & 2N + \frac{2a}{3N} + \alpha \end{pmatrix}$$

SCIENTIFIC COMPUTING & FEM

SHEET 4 / B

$$-(\lambda(x)u'(x))' = 0 \qquad x \in (0,1) =: \Omega$$

$$u(0) = 0$$

$$u(1) = 1$$

with $\lambda(x) = \begin{cases} 1 & x \in (0, \frac{1}{\sqrt{2}}) \\ 10 & x \in (\frac{1}{\sqrt{2}}, 1) \end{cases}$

Remark: $\boxed{-(\lambda(x)u'(x))' = -\lambda(x)u''(x)}$ f Das gilt nur für $\lambda(x) = \text{const}$!

•) variational formulation: let $v(0) = 0 = v(1)$

take $u \in V_{0,1} = \{ v \in H^1(\Omega) : u(0) = 0 \text{ and } u(1) = 1 \}$
$v \in V_0 = \{ \tilde{v} \in H^1(\Omega) : \tilde{v}(0) = 0 = \tilde{v}(1) \}$

$$\Rightarrow -\int_0^1 (\lambda(x)u'(x))'v(x)\,dx \underset{\substack{\uparrow \\ \text{partial int. for } H^1}}{=} \int_0^1 \lambda(x)u'(x)v'(x)\,dx - \lambda(x)u'(x)v(x)\Big|_0^1$$

$$= \int_0^1 \lambda(x)u'(x)v'(x)\,dx = \int_0^1 0\,v(x)\,dx = 0$$

$\Rightarrow$ find $u \in V_{0,1}$ such that $a(u,v) = 0 \quad \forall v \in V_0$

with $a(u,v) = \int_0^1 \lambda(x)u'(x)v'(x)\,dx$ ✓

•) discrete representation for FEM

approximation of u and v by sums of hat functions φ_i

N elements, $N+1$ nodes $\quad [\underline{u} \leftrightarrow u_h = \sum_{i=0}^{N} u_i\varphi_i(x)]$

$\Rightarrow \quad K\underline{u} = \underline{f} \quad$ with $K \in \mathbb{R}^{(N+1)\times N+1}$, $\underline{u}, \underline{f} \in \mathbb{R}^{N+1}$

with matrix entries: $K_{ij} = a(\varphi_i, \varphi_j) = \int_0^1 \lambda(x)\varphi_i'(x)\varphi_j'(x)\,dx$

and load vector $\underline{f} = [0 \dots 0]$

we need to adapt our systems of equations to fullfill the Dirichlet boundary conditions $u(0)=0$ and $u(1)=1$

- $K_{00}^{new} = K_{00}(1+10^6)$

  $f_0^{new} = f_0 = 0$

- $K_{NN}^{new} = K_{NN}(1+10^6)$

  $f_N^{new} = K_{NN} \cdot 10^6$

✓

SCIENTIFIC COMPUTING & FEM

SHEET 4 / C

Peclet problem $\quad -u''(x) + pu'(x) = 0 \qquad x \in (0,1) =: \Omega$

$$u(0) = 0$$
$$u(1) = 1 \qquad \text{with } p \in \mathbb{R}$$

•) variational formulation:

let $u \in V_{0,1} = \{ v \in H^1(\Omega) : v(0) = 0 \text{ and } v(1) = 1 \}$

and take $v \in V_0 = \{ \tilde{v} \in H^1(\Omega) : \tilde{v}(0) = 0 = \tilde{v}(1) \}$

$$-\int_0^1 u''(x)v(x)dx + p\int_0^1 u'(x)v(x)\,dx$$
$$= \int_0^1 u'(x)v'(x)dx - u'(x)v(x)\Big|_0^1 + p\int_0^1 u'(x)\,v(x)\,dx$$
$$= \int_0^1 u'(x)v'(x)dx + p\int_0^1 u'(x)\,v(x)\,dx = \int_0^1 0\cdot v(x)dx = 0$$

find $u \in V_{0,1}$ such that $a(u,v) = 0 \quad \forall v \in V_0$

with
$$a(u,v) = \int_0^1 u'(x)v'(x)dx + p\int_0^1 u'(x)v(x)\,dx$$

•) representation for FEM

N elements, $N+1$ nodes $(x_0 = 0, \dots, x_N = 1)$

we approximate u and v by the sum of hat functions

$$u_h = \sum_{i=0}^{N} u_i \varphi_i(x) \qquad \underline{u} \Leftrightarrow u_h$$

$$\varphi_i(x) = \begin{cases} (x - x_{i-1})N & x \in (x_{i-1}, x_i) = \tau_i \\ (x_{i+1} - x)N & x \in (x_i, x_{i+1}) = \tau_{i+1} \\ 0 & \text{else} \end{cases}$$

so we get the system

$$K \underline{u} = \underline{f} \qquad \text{with } K \in \mathbb{R}^{(N+1)\times(N+1)}, \quad \underline{u}, \underline{f} \in \mathbb{R}^{N+1}$$

with matrix entries: $K_{ij} = a(\varphi_j, \varphi_i) = \int_0^1 \varphi_j'(x)\,\varphi_i'(x)\,dx + p\int_0^1 \varphi_j'(x)\,\varphi_i(x)\,dx$

and load vector $\underline{f} = [0 \; \ldots \; 0]$

$\rightarrow$ need to adapt the system in order to fullfill the boundary conditions $u(0)=0$ and $u(1)=1$

- $K_{00}^{new} = K_{00}(1+10^6)$

  $f_0^{new} = f_0 = 0$

- $K_{NN}^{new} = K_{NN}(1+10^6)$

  $f_N^{new} = K_{NN}\,10^6$

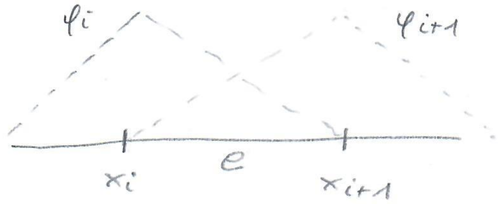
Remark to numerical solution:

it seems so that the system is not stable for $n < \frac{P}{2} = 35$

$\hookrightarrow$ oscillations and negative values (but the solution is positive)

$\Rightarrow$ we need a discretization which is fine enough to get a reasonable solution

(B)



im Element e:

$i \to$ leftnode of e
$i+1 \to$ right node of e

$$K_{ii} = a(\varphi_i, \varphi_i) = \int_0^1 \lambda(x)\left[\varphi_i'(x)\right]^2 dx = \int_{x_i}^{x_{i+1}} \lambda(x) N^2 dx$$

$$K_{i+1\,i+1} = \int_{x_i}^{x_{i+1}} \lambda(x) N^2 dx$$

$$K_{i\,i+1} = a(\varphi_{i+1}, \varphi_i) = \int_0^1 \lambda(x)(-N) N \, dx \Big|_e = \int_{x_i}^{x_{i+1}} \lambda(x)(-N^2) dx$$

some notes for implementing the stiffness matrix

(C)

im Element $e = [x_i, x_{i+1})$

erster Teil von $a(\cdot,\cdot)$ für $\int_0^1 \varphi_i' \varphi_j' dx$ bleibt gleich

$\rightsquigarrow$ ii: $$p\int_{x_i}^{x_{i+1}} \varphi_i'(x)\varphi_i(x) dx = p\int_{x_i}^{x_{i+1}} -N(x_{i+1}-x) N dx = -pN^2 \int_{x_i}^{x_{i+1}} x_{i+1} - x \, dx$$

$i+1, i+1$: $$p\int_{x_i}^{x_{i+1}} \varphi_{i+1}'(x)\varphi_{i+1}(x) dx = p\int_{x_i}^{x_{i+1}} N(x - x_i) N dx = pN^2 \int_{x_i}^{x_{i+1}} x - x_i \, dx$$

$i, i+1$: $$a(\varphi_{i+1}, \varphi_i) = p\int_{x_i}^{x_{i+1}} \varphi_{i+1}'(x)\varphi_i(x) dx = p\int_{x_i}^{x_{i+1}} N(x_{i+1} - x) N dx$$
$$= pN^2 \int_{x_i}^{x_{i+1}} x_{i+1} - x \, dx$$

$i+1, i$: $$a(\varphi_i, \varphi_{i+1}) = p\int_{x_i}^{x_{i+1}} \varphi_i'(x)\varphi_{i+1}(x) dx = p\int_{x_i}^{x_{i+1}} -N(x - x_i) N dx$$
$$= -pN^2 \int_{x_i}^{x_{i+1}} x - x_i \, dx$$