

SCIENTIFIC COMPUTING & FEM

SHEET 4 / A

$$-u''(x) + a \cdot u(x) = f(x) \quad , \quad x \in (0,1) =: \Omega$$

$$u(0) = 0$$

$$\frac{\partial u(1)}{\partial n} = \alpha (g_0 - u(1))$$

with $a, \alpha, g_0 \in \mathbb{R}$ (constant)

•) variational formulation (multiply with $v(x)$ and integrate)
 $\hookrightarrow$ with $v(0) = 0$

$$\begin{aligned} & \int_0^1 -u''(x) v(x) dx + \int_0^1 a u(x) v(x) dx \\ &= \int_0^1 u'(x) v'(x) dx - [u'(x) v(x)]_0^1 + a \int_0^1 u(x) v(x) dx \\ &= \int_0^1 u'(x) v'(x) dx - [u'(1) v(1) - \underbrace{u'(0) v(0)}_{=0}] + a \int_0^1 u(x) v(x) dx \\ &= \int_0^1 u'(x) v'(x) dx - \alpha (g_0 - u(1)) v(1) + a \int_0^1 u(x) v(x) dx = \int_0^1 f v dx \end{aligned}$$

$\Rightarrow$ find $u \in V_0 = \{v \in H^1(0,1) : v(0) = 0\}$ such that

$$a(u, v) = F(v) \quad \text{for all } v \in V_0$$

$$\text{with } a(u, v) = \int_0^1 u'(x) v'(x) dx + a \int_0^1 u(x) v(x) dx + \alpha u(1) v(1)$$

$$F(v) = \int_0^1 f(x) v(x) dx + \alpha g_0 v(1)$$

•) 1D FEM representation N elements

$$x_i = \frac{i}{N} \quad \text{for } i=0, \dots, N \quad (\text{nodes})$$

$$\tau_i = (x_{i-1}, x_i) \quad \text{for } i=1, \dots, N$$

our hat functions are

$$\varphi_i(x) = \begin{cases} (x - x_{i-1}) N & x \in \tau_i = (x_{i-1}, x_i) \\ (x_{i+1} - x) N & x \in \tau_{i+1} = (x_i, x_{i+1}) \\ 0 & \text{else} \end{cases}$$

for $i=0, \dots, N$

we now approximate u by u_h and v by v_h :

$$u(x) = \sum_{i=0}^N u_i \varphi_i(x)$$

by choosing the canonical basis vector for v we get the following discrete representation

$$K u = f$$

with $K_{ij} = a(\varphi_i, \varphi_j)$, $f_i = \langle F, \varphi_i \rangle$

to insert boundary condition $u(0) = 0 \Rightarrow K_{00}^{\text{new}} = K_{00} \cdot (1 + 10^6)$
(the rest remains the same)

•) calculate K (stiffness matrix) (symmetric)

$$K_{ij} = 0 \quad \text{for } |i-j| > 1 \quad (\text{support of functions})$$

$$\begin{aligned} \underline{K_{ii}} &= \int_0^1 \varphi_i' \varphi_i' dx + a \int_0^1 \varphi_i^2 dx + \alpha \varphi_i^2(1) \\ &= \int_{x_{i-1}}^{x_i} N^2 dx + \int_{x_i}^{x_{i+1}} (-N)^2 dx + a \int_{x_{i-1}}^{x_i} (x-x_i)^2 N^2 dx + a \int_{x_i}^{x_{i+1}} (x_{i+1}-x)^2 N^2 dx + \alpha \sin \\ &= N^2 (x_{i+1} - x_{i-1}) + a N^2 \left. \frac{(x-x_i)^3}{3} \right|_{x_{i-1}}^{x_i} - a N^2 \left. \frac{(x_{i+1}-x)^3}{3} \right|_{x_i}^{x_{i+1}} + \alpha \sin \\ &= N^2 \left(\frac{i+1-i-1}{N} \right) - a N^2 \frac{(x_{i-1}-x_i)^3}{3} + a N^2 \frac{(x_{i+1}-x_i)^3}{3} + \alpha \sin \\ &= 2N - a N^2 \left(\frac{-1}{N} \right)^3 / 3 + a N^2 \left(\frac{1}{N} \right)^3 / 3 + \alpha \sin \\ &= 2N + a \frac{2}{3} \cdot \frac{1}{N} + \alpha \sin \end{aligned}$$

$$\begin{aligned} \underline{K_{i,i+1}} &= \int_0^1 \varphi_i' \varphi_{i+1}' dx + a \int_0^1 \varphi_i \varphi_{i+1} dx + \underbrace{\alpha \varphi_i(1) \varphi_{i+1}(1)}_{=0} \\ &\stackrel{\text{support}}{\rightarrow} = \int_{x_i}^{x_{i+1}} -N^2 dx + a \int_{x_i}^{x_{i+1}} (x_{i+1}-x) N (x-x_i) N dx \\ &= -N \left. \frac{1}{2} (x-x_i)^2 \right|_{x_i}^{x_{i+1}} + a N^2 \left. \frac{(x_{i+1}-x)^3}{6} \right|_{x_i}^{x_{i+1}} \\ &= -N + a N^2 \frac{1}{N^3} \cdot \frac{1}{6} \\ &= -N + a \cdot \frac{1}{6} \cdot \frac{1}{N} \end{aligned}$$

$$K = \left(2N + \frac{2a}{3N} \right) (1 + 10^6) - N + \frac{a}{6N}$$

$$-N + \frac{a}{6N} \quad 2N + \frac{2a}{3N} \quad -N + \frac{a}{6N}$$

$$-N + \frac{a}{6N} \quad 2N + \frac{2a}{3N}$$

$$-N + \frac{a}{6N} \quad 2N + \frac{2a}{3N}$$

$$-N + \frac{a}{6N} \quad 2N + \frac{2a}{3N} + 2$$

SCIENTIFIC COMPUTING & FEM

SHEET 4 / B

$$-(\lambda(x) u'(x))' = 0 \quad x \in (0,1) = \Omega$$

$$u(0) = 0$$

$$u(1) = 1$$

$$\text{with } \lambda(x) = \begin{cases} 1 & x \in (0, \frac{1}{\sqrt{2}}) \\ 10 & x \in (\frac{1}{\sqrt{2}}, 1) \end{cases}$$

Remark: $-(\lambda(x) u'(x))' = -\lambda(x) u''(x)$

•) variational formulation: let $v(0) = 0 = v(1)$

$$\text{take } u \in V_{0,1} = \{v \in H^1(\Omega) : v(0) = 0 \text{ and } v(1) = 1\}$$

$$v \in V_0 = \{\tilde{v} \in H^1(\Omega) : \tilde{v}(0) = 0 = \tilde{v}(1)\}$$

$$\begin{aligned} \Rightarrow - \int_0^1 (\lambda(x) u'(x))' v(x) dx &= \int_0^1 \lambda(x) u'(x) v'(x) dx - \lambda(x) u'(x) v(x) \Big|_0^1 \\ &= \int_0^1 \lambda(x) u'(x) v'(x) dx = \int_0^1 0 v(x) dx = 0 \end{aligned}$$

↑
partial int.
for H^1

$\Rightarrow$ find $u \in V_{0,1}$ such that $a(u, v) = 0 \quad \forall v \in V_0$

$$\text{with } a(u, v) = \int_0^1 \lambda(x) u'(x) v'(x) dx$$

•) discrete representation for FEM

approximation of u and v by sums of hat functions φ_i

$$N \text{ elements, } N+1 \text{ nodes} \quad [u \leftrightarrow u_h = \sum_{i=0}^N u_i \varphi_i(x)]$$

$$\Rightarrow K u = f \quad \text{with } K \in \mathbb{R}^{(N+1) \times (N+1)}, \quad u, f \in \mathbb{R}^{N+1}$$

$$\text{with matrix entries: } K_{ij} = a(\varphi_i, \varphi_j) = \int_0^1 \lambda(x) \varphi_i'(x) \varphi_j'(x) dx$$

$$\text{and load vector } f = [0 \dots 0]$$

we need to adapt our systems of equations to fulfill the
Dirichlet Boundary conditions $u(0)=0$ and $u(1)=1$

$$\bullet) \quad K_{00}^{\text{new}} = K_{00}(1+10^6)$$

$$f_0^{\text{new}} = f_0 = 0$$

$$\bullet) \quad K_{NN}^{\text{new}} = K_{NN}(1+10^6)$$

$$f_N^{\text{new}} = K_{NN} \cdot 10^6$$

SCIENTIFIC COMPUTING & FEM

SHEET 4 / C

Poet problem

$$-u''(x) + pu'(x) = 0 \quad x \in (0,1) =: \Omega$$

$$u(0) = 0$$

$$u(1) = 1$$

with $p \in \mathbb{R}$

•) variational formulation:

$$\text{let } u \in V_{0,1} = \{v \in H^1(\Omega) : v(0) = 0 \text{ and } v(1) = 1\}$$

$$\text{and take } v \in V_0 = \{\tilde{v} \in H^1(\Omega) : \tilde{v}(0) = 0 = \tilde{v}(1)\}$$

$$\begin{aligned} & - \int_0^1 u''(x) v(x) dx + p \int_0^1 u'(x) v(x) dx \\ &= \int_0^1 u'(x) v'(x) dx - u'(x) v(x) \Big|_0^1 + p \int_0^1 u'(x) v(x) dx \\ &= \int_0^1 u'(x) v'(x) dx + p \int_0^1 u'(x) v(x) dx = \int_0^1 0 \cdot v(x) dx = 0 \end{aligned}$$

find $u \in V_{0,1}$ such that $a(u, v) = 0 \quad \forall v \in V_0$

$$\text{with } a(u, v) = \int_0^1 u'(x) v'(x) dx + p \int_0^1 u'(x) v(x) dx$$

•) representation for FEM

N elements, $N+1$ nodes ($x_0=0, \dots, x_N=1$)

we approximate u and v by the sum of hat functions

$$u_R = \sum_{i=0}^N u_i \varphi_i(x) \quad \underline{u} \mapsto u_R$$

$$\varphi_i(x) = \begin{cases} (x - x_{i-1})N & x \in (x_{i-1}, x_i) =: \tilde{T}_i \\ (x_{i+1} - x)N & x \in (x_i, x_{i+1}) =: \tilde{T}_{i+1} \\ 0 & \text{else} \end{cases}$$

so we get the system

$$K \underline{u} = \underline{f} \quad \text{with } K \in \mathbb{R}^{(n+1) \times (n+1)}, \quad \underline{u}, \underline{f} \in \mathbb{R}^{n+1}$$

with matrix entries: $K_{ij} = a(\varphi_j, \varphi_i) = \int_0^1 \varphi_j'(x) \varphi_i'(x) dx + \rho \int_0^1 \varphi_j(x) \varphi_i(x) dx$

and load vector $\underline{f} = [0 \dots 0]$

→ need to adapt the system in order to fulfill the boundary conditions $u(0)=0$ and $u(1)=1$

$$\bullet) K_{00}^{\text{new}} = K_{00}(1 + 10^6)$$

$$f_0^{\text{new}} = f_0 = 0$$

$$\bullet) K_{NN}^{\text{new}} = K_{NN}(1 + 10^6)$$

$$f_N^{\text{new}} = K_{NN} 10^6$$

Remark to numerical solution:

it seems so that the system is not stable for $n < \frac{1}{2} = 35$

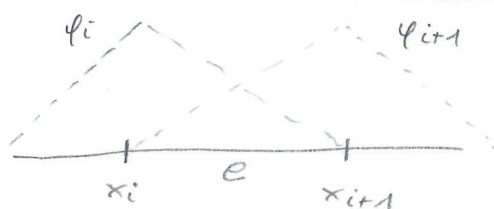
↳ oscillations and negative values (but the solution is positive)

⇒ we need a discretization which is fine enough to get a reasonable solution

Ⓟ

in Element e :

$i \rightarrow$ left node of e
 $i+1 \rightarrow$ right node of e



$$K_{ii} = a(\psi_i, \psi_i) = \int_0^1 \lambda(x) [\psi_i'(x)]^2 dx = \int_{x_i}^{x_{i+1}} \lambda(x) N^2 dx$$

$$K_{i+1, i+1} = \int_{x_i}^{x_{i+1}} \lambda(x) N^2 dx$$

$$K_{i, i+1} = a(\psi_{i+1}, \psi_i) = \int_0^1 \lambda(x) (-N) N dx \Big|_e = \int_{x_i}^{x_{i+1}} \lambda(x) (-N^2) dx$$

some notes for implementing the stiffness matrix

Ⓒ

in Element $e = (x_i, x_{i+1})$

outer Teil von $a(\cdot, \cdot)$ für $\int_0^1 \psi_i' \psi_j' dx$ bleibt gleich

$$\rightarrow ii: \quad p \int_{x_i}^{x_{i+1}} \psi_i'(x) \psi_i(x) dx = p \int_{x_i}^{x_{i+1}} -N(x_{i+1} - x) N dx = -p N^2 \int_{x_i}^{x_{i+1}} (x_{i+1} - x) dx$$

$$i+1, i+1: \quad p \int_{x_i}^{x_{i+1}} \psi_{i+1}'(x) \psi_{i+1}(x) dx = p \int_{x_i}^{x_{i+1}} N(x - x_i) N dx = p N^2 \int_{x_i}^{x_{i+1}} (x - x_i) dx$$

$$i, i+1: \quad a(\psi_{i+1}, \psi_i) = p \int_{x_i}^{x_{i+1}} \psi_{i+1}'(x) \psi_i(x) dx = p \int_{x_i}^{x_{i+1}} N(x_{i+1} - x) N dx$$

$$= p N^2 \int_{x_i}^{x_{i+1}} (x_{i+1} - x) dx$$

$$i+1, i: \quad a(\psi_i, \psi_{i+1}) = p \int_{x_i}^{x_{i+1}} \psi_i'(x) \psi_{i+1}(x) dx = p \int_{x_i}^{x_{i+1}} -N(x - x_i) N dx$$

$$= -p N^2 \int_{x_i}^{x_{i+1}} (x - x_i) dx$$