SCF-UE Sheet 2

(A) stationary heat in 1D $\Omega = (0,1)$ (no con-/advection):

$$-\frac{d}{dx}\left(\lambda(x)\frac{du}{dx}\right) = f(x) \qquad x \in (0,1)$$

heat conductivity: λ const
heating: $f(x) = c$

$$\boxed{\begin{aligned} -\lambda u''(x) &= c \qquad x \in (0,1) \\ u(0) &= 0 \\ u(1) &= 1 \end{aligned}}$$

✓

- Function spaces:

classical ($\Omega = (0,1)$)	general
$u(x) \in C^2([0,1])$	$u \in H^2(\Omega)$
$\lambda(x) \in C^1([0,1])$	$\lambda \in L^\infty(\Omega)$ (bounded)
$f(x) \in C^0([0,1])$	$f \in L^2(\Omega)$

- $u(x) = u_h(x) + u_f(x)$

1.) $-\lambda u_h'' = 0$, $u(0)=0$, $u(1)=1$ ✓
2.) $-\lambda u_f'' = f$, $u(0)=u(1)=0$

$\rightarrow -\lambda u'' = -\lambda(u_h + u_f)'' = -\lambda u_h'' - \lambda u_f'' = f$
$u(0) = u_h(0) + u_f(0) = 0$
$u(1) = u_h(1) + u_f(1) = 1$

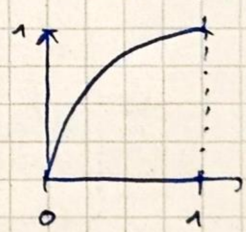
1.) $\rightsquigarrow u_h = Ax + B$
$u_h(0) = B \overset{!}{=} 0$
$u_h(1) = A \overset{!}{=} 1$
$\rightarrow u_h(x) = x \rightarrow$

✓

2.) $u_f'' = -\frac{c}{\lambda}$
$\rightsquigarrow u_f(x) = -\frac{c}{2\lambda}x^2 + Cx + D$
$u_f(0) = D \overset{!}{=} 0$
$u_f(1) = -\frac{c}{2\lambda} + C \overset{!}{=} 0 \rightarrow C = \frac{c}{2\lambda}$
$\rightarrow u_f(x) = \frac{c}{2\lambda}(x - x^2)$

$u(x) = x + \frac{c}{2\lambda}(x - x^2)$
$u''(x) = -\frac{c}{\lambda}$ ✓
$u(0) = 0$ ✓
$u(1) = 1$ ✓

Probe ✓



(B) P.D.E
$$\begin{cases} -(\lambda(x)\,u'(x))' = \overset{\text{``}f}{0} & x \in (0,1) \quad \text{where } \lambda(x) = \begin{cases} 1 & x \in (0,\frac{1}{2}) \\ 10 & x \in (\frac{1}{2},1) \end{cases} \\ u(0) = 0 \\ u(1) = 1 \end{cases}$$

I.C. interface conditions
$$\begin{cases} (*) & u(0.5^-) = u(0.5^+) \\ (**) & \lambda(0.5^-)\,u'(0.5^-) = \lambda(0.5^+)\,u'(0.5^+) \end{cases}$$

• Function spaces: $u \in C^2((0,\frac{1}{2})) \cap C^2((\frac{1}{2},1)) \cap C((0,1))$

$\lambda \in C^1((0,\frac{1}{2})) \cap C^1((\frac{1}{2},1))$ (is piecewise constant actually)

$f \in C^0((0,1))$

• Solve on each subdomain

$0<x<\frac{1}{2}$: $\lambda = 1$: $\quad u_1''(x) = 0 \rightarrow u_1 = Ax + B$

$u(0) = u_1(0) = 0 \rightarrow u_1(x) = Ax$

$\frac{1}{2}<x<1$: $\lambda = 10$: $\quad 10\,u_2''(x) = 0 \rightarrow u_2 = Cx + D$

$u(1) = u_2(1) = 1 \rightarrow u_2(x) = Cx + (1-C)$

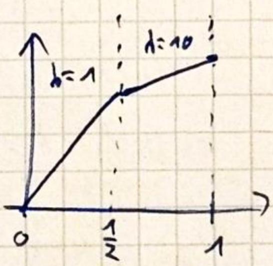
I.C: (*) $u_1(\frac{1}{2}) = u_2(\frac{1}{2}) \Rightarrow \frac{A}{2} = \frac{C}{2} + 1 - C$

$\Leftrightarrow A + C = 2$

(**) $1 \cdot u_1'(\frac{1}{2}) = 10\,u_2'(\frac{1}{2}) \Rightarrow A = 10\,C$

$\Rightarrow C = \frac{2}{11}, \quad A = \frac{20}{11}$

$$\Rightarrow \boxed{u(x) = \begin{cases} \frac{20}{11}x & x \in (0,\frac{1}{2}) \\ \frac{2}{11}x + \frac{9}{11} & x \in (\frac{1}{2},1) \end{cases}}$$



(c) Péclet problem:

$$-u''(x) + p u'(x) = 0 \qquad x \in (0,1) \qquad p \in \mathbb{R}$$
$$u(0) = 0$$
$$u(1) = 1$$

substitute: $z(x) = u'(x)$

$\rightarrow \quad -z'(x) + p z(x) = 0 \qquad \Leftrightarrow p z(x) = z'(x) = \frac{dz}{dx}$

$\Leftrightarrow \int p \, dx = \int \frac{1}{z} dz$

$\Leftrightarrow \quad px + c = \ln|z|$

$\Leftrightarrow \quad z(x) = A e^{px}$

$p=0 \rightarrow \quad u(x) = kx + d$

$\left.\begin{array}{l} u(0) = 0 \\ u(1) = 1 \end{array}\right\} \rightarrow u(x) = x$

$p \neq 0 \rightarrow \quad u(x) = \frac{A}{p} e^{px} + B$

$u(0) = \frac{A}{p} + B \overset{!}{=} 0 \quad \Leftrightarrow \quad B = -\frac{A}{p}$

$u(1) = \frac{A}{p} e^{p} - \frac{A}{p} \overset{!}{=} 1 \quad \Leftrightarrow \quad \frac{A}{p} = \frac{1}{e^{p} - 1}$

$$\rightarrow \boxed{u(x) = \frac{e^{px} - 1}{e^{p} - 1} \qquad p \neq 0}$$

✓