

SciComp and FEM in WS25

Exercise 4: PDE in 1D and FEM.

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Status:

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The goal of this exercise consists in an analytic description of FEM for the given problems and its numerical solution by your own code afterwards.

You can use whatever programming language you prefer, I did my hack in Matlab but python should be similarly easy, especially in solving the system of equations to determine the discrete solution.

In case of using C++ you can use the LAPACK library [task 7 from exercise 3] for solving the resulting system of equation or the simple Jacobi iteration scheme (good enough for the simple problems).

(A) Consider the PDE

[3 pts]

$$\begin{aligned} -u''(x) + a \cdot u(x) &= f(x) & x \in (0, 1) &=: \Omega \\ u(0) &= 0 \\ \frac{\partial u(1)}{\partial \vec{n}} &= \alpha (g_b - u(1)) \end{aligned}$$

with constant quantities $a, \alpha, g_B \in \mathbb{R}$.

- Write the variational formulation of that PDE. Define $a(\cdot, \cdot)$ and $\langle F, \cdot \rangle$.
- Write down the FEM representation using an equidistant discretization of the computational domain Ω and linear shapefunction in each of the n elements.
- Compute the elements of the stiffness matrix [JL, §3.3], [LB, §2.4] .
- Solve the system of equations.

(B) Consider the PDE

[2 pts]

$$\begin{aligned} -(\lambda(x) \cdot u'(x))' &= 0 & x \in (0, 1) &=: \Omega \\ u(0) &= 0 \\ u(1) &= 1 \end{aligned}$$

$$\text{with } \lambda(x) = \begin{cases} 1 & x \in \left(0, \frac{1}{\sqrt{2}}\right) \\ 10 & x \in \left(\frac{1}{\sqrt{2}}, 1\right) \end{cases}.$$

- Write the variational formulation and the FEM system.
- Solve the system of equations.

(C) Solve the Péclet¹ problem

[3 pts]

$$\begin{aligned} -u''(x) + pu'(x) &= 0 & x \in (0, 1) \\ u(0) &= 0 \\ u(1) &= 1 \end{aligned}$$

with FEM for a constant $p \in \mathbb{R}$, see also [JL, §3.10 (example 2)]

- Write the variational formulation and the FEM system.
- Solve the system of equations with $p = 70$ and $n \in \{10, 20, 30, 40, 70\}$.
Explain the behavior of the discrete solution u_h .

Task (C) might be extended.

Literatur

- [LB] Larson/Bengzon: “The Finite Element Method“, Springer, TSCE 10, 2013 (e-book Uni Graz)
- [DHL] Douglas/Haase/Langer: “A Tutorial on Elliptic PDE Solvers and their Parallelization“, SIAM, 2003 (e-book)
- [JL] Jung/Langer: “Methode der finiten Elemente für Ingenieure“, Springer, 2013 (e-Book Uni Graz)

¹https://en.wikipedia.org/wiki/Jean-Claude_Eug%C3%A8ne_P%C3%A9clet