SCF UE - Sheet 4

(A) Given:
$$-u''(x) + a\,u(x) = f(x) \qquad x \in \Omega = (0,1)$$
$$u(0) = 0$$
$$u'(1) = \frac{\partial u(1)}{\partial n} = \alpha\,(g_b - u(1)) \qquad a, \alpha, g_b \in \mathbb{R} \text{ constant}$$

1.) Variational formulation:

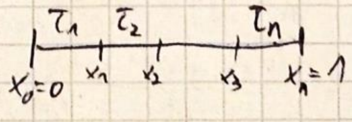
$$\int_0^1 \cdot\, v\,dx \rightsquigarrow \int_0^1 f v\,dx = \int_0^1 -u'' v\,dx + \int_0^1 a u v\,dx$$
$$= \underbrace{-u'v\Big|_0^1}_{\substack{v(0)=0 \\ = -u'(1)v(1) \overset{\text{Neumann}}{=} -\alpha(g_b - u(1))v(1)}} + \int_0^1 u'v'\,dx + a\int_0^1 uv\,dx$$
$$= -\alpha g_b v(1) + \alpha u(1)v(1) + \int_0^1 u'v'\,dx + a\int_0^1 uv\,dx$$

$$\rightsquigarrow \boxed{\begin{array}{l} u \in H^1(0,1):\quad a(u,v) = \langle F, v\rangle \qquad \forall v \in V_0 = \{w \in H^1(0,1)\,|\,w(0)=0\} \\ \text{where}\quad a(u,v) = \int_0^1 u'v'\,dx + a\int_0^1 uv\,dx + \alpha u(1)v(1) \\ \qquad\qquad \langle F, v\rangle = \int_0^1 f v\,dx + \alpha g_b v(1) \end{array}}$$

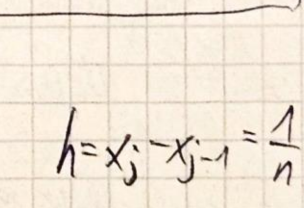
2.) FEM representation:

$$\Omega = (0,1), \quad \overline{\Omega_h} = \bigcup_j \overline{\tau_j} \qquad h = x_j - x_{j-1} = \frac{1}{n}$$

$$x_i = \frac{i}{n} \qquad i = 0, \dots, n$$
$$\tau_j = (x_{j-1}, x_j) \qquad j = 1, \dots, n$$



linear shape functions:
$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{h} & x \in (x_i, x_{i+1}) \\ 0 & x \notin [x_{i-1}, x_{i+1}] \end{cases}$$



$$\varphi_i(x_j) = \delta_{ij}$$

Approximate $u \approx u_h = \sum_{j=1}^{n} u_j \varphi_j(x)$, $v \approx v_h = \sum_{i=1}^{n} v_i \varphi_i(x)$

$$\rightsquigarrow \sum_i v_i \Big[\sum_j u_j \underbrace{\Big(\int_0^1 \varphi_j'(x)\varphi_i'(x)\,dx + a\int_0^1 \varphi_j(x)\varphi_i(x)\,dx + \alpha\, \varphi_j(1)\varphi_i(1)\Big)}_{K_{ij}} \Big] = \sum_i v_i \Big[\underbrace{\int_0^1 f\varphi_i\,dx + \alpha\, g_b\, \varphi_i(1)}_{F_i}\Big]$$

$v = e_i$ for $i = 1, \dots, n+1$

$\rightsquigarrow$

$$\boxed{\begin{aligned} &Ku = F \qquad\qquad i,j = 0,\dots,n \\ &\text{where } u = (\overset{0}{\overset{\shortparallel}{u_0}}, \dots, u_n)^T \\ &K_{ij} = \int_0^1 \varphi_j'(x)\varphi_i'(x)\,dx + a\int_0^1 \varphi_j(x)\varphi_i(x)\,dx + \alpha \overbrace{\varphi_j(1)\varphi_i(1)}^{\neq 0 \text{ for } i=j=n} \\ &F_i = \int_0^1 f\varphi_i\,dx + \alpha\, g_b\, \varphi_i(1) \end{aligned}}$$

3.) Stiffness matrix K:

$K = \sum_{k=1}^{n} K^{(\tau_k)}$ sum of local stiffness matrices where, for $\tau_k = (x_k, x_{k+1})$:

$$K_{ij}^{(\tau_k)} = \overbrace{\int_{x_k}^{x_{k+1}} \varphi_j'(x)\varphi_i'(x)\,dx}^{A_{ij}^{(\tau_k)}} + a\overbrace{\int_{x_k}^{x_{k+1}} \varphi_j(x)\varphi_i(x)\,dx}^{B_{ij}^{(\tau_k)}} + \alpha\,\varphi_j(1)\varphi_i(1)$$

$i,j = k, k+1$

else $K_{ij}^{(\tau_k)} =$

$$\int_{x_k}^{x_{k+1}} \varphi_k'\varphi_k'\,dx = \frac{x}{h^2}\Big|_{x_k}^{x_{k+1}} = \frac{1}{h} = A_{kk}^{(\tau_k)} = A_{k+1,k+1}^{(\tau_k)}$$

$$\int_{x_k}^{x_{k+1}} \varphi_k'\varphi_{k+1}'\,dx = \dots = -\frac{1}{h} = A_{k,k+1}^{(\tau_k)} = A_{k+1,k}^{(\tau_k)}$$

$$\int_{x_k}^{x_{k+1}} \varphi_k\varphi_k\,dx = \int_{x_k}^{x_{k+1}} \frac{(x_{k+1}-x)^2}{h^2}\,dx = \left| \begin{matrix} t := x_{k+1}-x \\ dt = -dx \end{matrix} \right| = \int_h^0 \frac{t^2}{h^2}(-dt) = \int_0^h \frac{t^2}{h^2}\,dx = \frac{1}{h^2}\frac{1}{3}h^3 = \frac{h}{3} = \frac{1}{a}B_{kk\,/\,k+1,k+1}^{(\tau_k)}$$

$$\int_{x_k}^{x_{k+1}} \varphi_k\varphi_{k+1}\,dx = \int_{x_k}^{x_{k+1}} \frac{(x_{k+1}-x)(x-x_k)}{h^2}\,dx = \left| \begin{matrix} t = x-x_k \to x_{k+1}-x = h-t \\ dt = dx \end{matrix} \right| = \frac{1}{h^2}\int_0^h (h-t)t\,dx = \frac{1}{h^2}\left(\frac{h^3}{2} - \frac{h^3}{3}\right) = \frac{h}{6} = \frac{1}{a}B_{k,k+1\,/\,k+1,k}^{(\tau_k)}$$

$$\rightsquigarrow K^{(\tau_k)} = \begin{array}{c} \\ \left[\begin{matrix} \ddots & 0 & 0 & \\ 0 & \frac{1}{h} + \frac{ah}{3} & -\frac{1}{h} + \frac{ah}{6} & 0 \\ 0 & -\frac{1}{h} + \frac{ah}{6} & \frac{1}{h} + \frac{ah}{3} & 0 \\ & 0 & 0 & \ddots \end{matrix}\right] \\ \quad j=k \qquad\quad j=k+1 \end{array} \begin{array}{l} \\ \leftarrow i=k \\ \leftarrow i=k+1 \\ \\ \end{array}$$

$$K^{(\tau_n)} = \left[\begin{matrix} 0 & & 0 & 0 \\ & \ddots & \frac{1}{h} + \frac{ah}{3} & -\frac{1}{h} + \frac{ah}{6} \\ 0 & \dots & -\frac{1}{h} + \frac{ah}{6} & \frac{1}{h} + \frac{ah}{3} + \alpha \end{matrix}\right] \begin{array}{l} \\ \leftarrow n-1 \\ \leftarrow n \end{array}$$

$\qquad\qquad\qquad\qquad\quad n-1 \qquad\quad n \qquad \to$ Neumann bc

$$K = \left[\begin{matrix} \frac{1}{h} + \frac{ah}{3} & -\frac{1}{h} + \frac{ah}{6} & & & \\ -\frac{1}{h} + \frac{ah}{6} & \frac{2}{h} + \frac{2ah}{3} & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \frac{2}{h} + \frac{2ah}{3} & -\frac{1}{h} + \frac{ah}{6} \\ 0 & & & -\frac{1}{h} + \frac{ah}{6} & \frac{1}{h} + \frac{ah}{3} + \alpha \end{matrix}\right]$$

counted twice

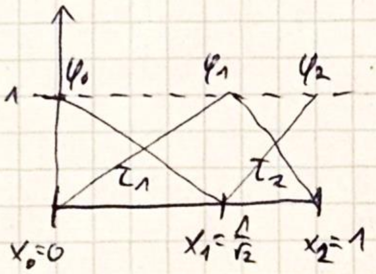
(B) $$-(\lambda(x)\cdot u'(x))' = 0 \qquad x\in\Omega=(0,1)$$
$$u(0) = 0$$
$$u(1) = 1 \qquad \text{where } \lambda(x)=\begin{cases}1 & x\in(0,\frac{1}{\sqrt{2}})\\ 10 & x\in(\frac{1}{\sqrt{2}},1)\end{cases}$$

1.) Varform:

$$\overset{\int_\Omega \cdot v\,dx}{\rightsquigarrow}\; 0 = -\int_0^1 (\lambda(x)u'(x))'\,v(x)\,dx \qquad \forall v\in V_0=\{w\in H^1(\Omega)\,|\,w(0)=w(1)=0\}$$
$$= \underbrace{-\lambda(x)u'(x)v(x)\Big|_0^1}_{=0 \text{ since } v(1)=v(0)=0} + \int_0^1 \lambda(x)u'(x)v'(x)\,dx$$

$$\rightsquigarrow \boxed{u\in V=\{u\in H^1(\Omega)\,|\,u(0)=0,\,u(1)=1\}:\quad \int_0^{\frac{1}{\sqrt{2}}} u'v'\,dx + \int_{\frac{1}{\sqrt{2}}}^1 10\,u'v'\,dx = 0 \qquad \forall v\in V_0}$$

2.) FEM: (2 elements, already exact)



$x_0 = 0$
$x_1 = \frac{1}{\sqrt{2}}$ $\qquad h_1 = x_1 - x_0 = \frac{1}{\sqrt{2}}$
$x_2 = 1$ $\qquad h_2 = x_2 - x_1 = 1-\frac{1}{\sqrt{2}}$

$$\rightsquigarrow K^{(\tau_1)} = \begin{bmatrix} \int_0^{\frac{1}{\sqrt{2}}} \varphi_0'\varphi_0'\,dx & \int_0^{\frac{1}{\sqrt{2}}} \varphi_1'\varphi_0'\,dx & 0 \\ \int_0^{\frac{1}{\sqrt{2}}} \varphi_0'\varphi_1'\,dx & \int_0^{\frac{1}{\sqrt{2}}} \varphi_1'\varphi_1'\,dx & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{h_1} & -\frac{1}{h_1} & 0 \\ -\frac{1}{h_1} & \frac{1}{h_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$K^{(\tau_2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 10\cdot\int_{\frac{1}{\sqrt{2}}}^1 \varphi_1'\varphi_1'\,dx & 10\cdot\int_{\frac{1}{\sqrt{2}}}^1 \varphi_2'\varphi_1'\,dx \\ 0 & 10\cdot\int_{\frac{1}{\sqrt{2}}}^1 \varphi_1'\varphi_2'\,dx & 10\cdot\int_{\frac{1}{\sqrt{2}}}^1 \varphi_2'\varphi_2'\,dx \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{10}{h_2} & -\frac{10}{h_2} \\ 0 & -\frac{10}{h_2} & \frac{10}{h_2} \end{bmatrix}$$

$$\int_0^{\frac{1}{\sqrt{2}}} [\varphi_0']^2\,dx = \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{h_1^2}\,dx = \frac{1}{h_1} \qquad \int_0^{\frac{1}{\sqrt{2}}} \varphi_0'\varphi_1'\,dx = -\frac{1}{h_1} \qquad \int_{\frac{1}{\sqrt{2}}}^1 [\varphi_2']^2\,dx = \frac{1}{h_2}$$
$$\int_0^{\frac{1}{\sqrt{2}}} [\varphi_1']^2\,dx = \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{h_1^2}\,dx = \frac{1}{h_1} \qquad \int_{\frac{1}{\sqrt{2}}}^1 \varphi_1'\varphi_2'\,dx = -\frac{1}{h_2}$$

Dirichlet bc

$$\rightsquigarrow \; Ku = f \; \rightsquigarrow \; \begin{bmatrix} \frac{1}{h_1} & -\frac{1}{h_1} & 0 \\ -\frac{1}{h_1} & \frac{1}{h_1}+\frac{10}{h_2} & -\frac{10}{h_2} \\ 0 & -\frac{10}{h_2} & \frac{10}{h_2} \end{bmatrix} \begin{bmatrix} 0 \\ u_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Dirichlet bc

2nd row

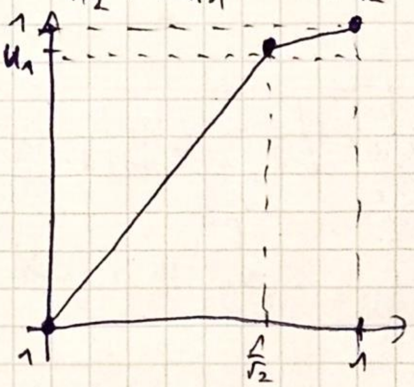
$$\rightsquigarrow \left(\frac{1}{h_1}+\frac{10}{h_2}\right)u_1 - \frac{10}{h_2} = 0$$

$$\Leftrightarrow \quad u_1 = \frac{\frac{10}{h_2}}{\frac{1}{h_1}+\frac{10}{h_2}} = \frac{10h_1}{h_2+10h_1} = \frac{\frac{10}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}+\frac{10}{\sqrt{2}}} = \frac{10}{\sqrt{2}-1+10}$$

$$\underline{\underline{u_1 \approx 0{,}96022}}$$

$u_0 = 0$

$u_2 = 1$



(C) Péclet problem: $-u''(x) + p\,u'(x) = 0 \quad x \in \Omega = (0,1)$

$$u(0) = 0$$
$$u(1) = 1 \qquad p \in \mathbb{R}$$

1.) Varformij:

$$\overset{\int \cdot v\,dx}{\rightsquigarrow} \quad 0 = \int -u'' v\,dx + p\int u' v\,dx \qquad \forall v \in V_0 = \{ w \in H^1(\Omega) \mid w(0) = w(1) = 0\}$$
$$= \underbrace{-u'v\Big|_0^1}_{=0 \text{ since } v(0)=v(1)=0} + \int u'v'\,dx + p\int u'v\,dx$$

$$\rightsquigarrow \boxed{u \in V = \{u \in H^1(\Omega) \mid u(0)=0,\ u(1)=1\}: \quad \int_0^1 u'v'\,dx + p\int_0^1 u'v\,dx = 0 \qquad \forall v \in V_0}$$

2.) FEM: $Ku = f = 0$

$$\rightsquigarrow K_{ij} = \underbrace{\int_0^1 \varphi_j' \varphi_i' \,dx}_{A_{ij}} + \underbrace{p\int_0^1 \varphi_j' \varphi_i\,dx}_{B_{ij}}$$

not sym.

ex(A)

$$\rightarrow A^{(\tau_k)} = \begin{bmatrix} \frac{1}{h} & -\frac{1}{h} \\ -\frac{1}{h} & \frac{1}{h} \end{bmatrix}$$

$$\int_{x_k}^{x_{k+1}} \varphi_k' \varphi_k\,dx = \int_{x_k}^{x_{k+1}} \left(-\frac{1}{h}\right)\left(\frac{x_{k+1}-x}{h}\right)dx = \left|\begin{matrix} t = x_{k+1}-x \\ dt = -dx \end{matrix}\right| = \left(-\frac{1}{h^2}\right)\int_h^0 t\,(-dt) = -\frac{1}{2}$$

$$\int_{x_k}^{x_{k+1}} \varphi_{k+1}' \varphi_{k+1}\,dx = \int_{x_k}^{x_{k+1}} \left(\frac{1}{h}\right)\left(\frac{x-x_k}{h}\right)dx = \left|\begin{matrix} t = x-x_k \\ dt = dx \end{matrix}\right| = \left(\frac{1}{h^2}\right)\int_0^h t\,dt = \frac{1}{2}$$

$$\int_{x_k}^{x_{k+1}} \varphi_k' \varphi_{k+1}\,dx = \int \left(-\frac{1}{h}\right)\left(\frac{x-x_k}{h}\right)dx = \left|\begin{matrix} t = x-x_k \\ dt = dx \end{matrix}\right| = \left(-\frac{1}{h^2}\right)\int_0^h t\,dt = -\frac{1}{2}$$

$$\int_{x_k}^{x_{k+1}} \varphi_{k+1}' \varphi_k\,dx = \int_{x_k}^{x_{k+1}} \left(\frac{1}{h}\right)\left(\frac{x_{k+1}-x}{h}\right)dx = \left|\begin{matrix} t = x_{k+1}-x \\ dt = -dx \end{matrix}\right| = \left(\frac{1}{h^2}\right)\int_h^0 t\,(-dt) = \frac{1}{2}$$

$$\rightarrow B^{(\tau_k)} = \begin{bmatrix} -\frac{p}{2} & \frac{p}{2} \\ -\frac{p}{2} & \frac{p}{2} \end{bmatrix} \qquad \rightarrow K^{(\tau_k)} = \begin{bmatrix} \frac{1}{h} - \frac{p}{2} & -\frac{1}{h} + \frac{p}{2} \\ -\frac{1}{h} - \frac{p}{2} & \frac{1}{h} + \frac{p}{2} \end{bmatrix}$$

$$\rightsquigarrow\quad \underbrace{\begin{bmatrix} \frac{1}{h}-\frac{f}{2} & -\frac{1}{h}+\frac{f}{2} & & & 0 \\ -\frac{1}{h}-\frac{f}{2} & \frac{2}{h} & -\frac{1}{h}+\frac{f}{2} & & \\ & -\frac{1}{h}-\frac{f}{2} & \frac{2}{h} & \ddots & \\ & & \ddots & \ddots & \\ & & & \frac{2}{h} & -\frac{1}{h}+\frac{f}{2} \\ 0 & & & -\frac{1}{h}-\frac{f}{2} & \frac{1}{h}+\frac{f}{2} \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} u_0 = 0 \\ u_1 \\ \vdots \\ \\ u_{n-1} \\ u_n = 1 \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ \\ \\ \\ 0 \end{bmatrix}}_{f}$$

↳ leave out known Dirichlet nodes and move Dirichlet contributions to rhs

$$\begin{bmatrix} \frac{2}{h} & -\frac{1}{h}+\frac{f}{2} & & 0 \\ -\frac{1}{h}-\frac{f}{2} & \frac{2}{h} & \ddots & \\ & \ddots & \frac{2}{h} & -\frac{1}{h}+\frac{f}{2} \\ 0 & & -\frac{1}{h}-\frac{f}{2} & \frac{2}{h} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} -\left(-\frac{1}{h}-\frac{f}{2}\right)\overset{0}{\overset{\shortparallel}{u_0}} = 0 \\ 0 \\ \vdots \\ 0 \\ -\left(-\frac{1}{h}+\frac{f}{2}\right)\underset{1}{\underset{\shortparallel}{u_n}} = \frac{1}{h}-\frac{f}{2} \end{bmatrix}$$

↳ $(n-1)\times(n-1)$

↳ Solve system, see git.

Alternative: reorder rows/columns/variables in $Ku=f$ such that

$$\left[\begin{array}{c|c} K_{FF} & K_{FD} \\ \hline K_{DF} & K_{DD} \end{array}\right] \left[\begin{array}{c} u_F \\ \hline u_D \end{array}\right] = \begin{bmatrix} f_F \\ f_D \end{bmatrix}$$

u_F ... unknown, free, nonDirichlet

u_D ... known from Dirichlet bc

$$\rightsquigarrow\quad K_{FF}u_F + K_{FD}u_D = f_F$$

$$\rightsquigarrow\quad \underline{\underline{u_F = K_{FF}^{-1}\left(f_F - K_{FD}u_D\right)}}$$