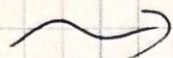


SCF - U.E. Sheet 2

(A) stationary heat in 1D $\Omega = (0,1)$ (no con-duction):

$$-\frac{d}{dx}(\lambda(x) \frac{du}{dx}) = f(x) \quad x \in (0,1)$$

heat conductivity: λ and
heating: $f(x)=c$



$$\begin{cases} -\lambda u''(x) = c \\ u(0) = 0 \\ u(1) = 1 \end{cases} \quad x \in (0,1)$$

• Function spaces:

classical ($\Omega=(0,1)$)
 $u(x) \in C^2([0,1])$
 $\lambda(x) \in C^1([0,1])$
 $f(x) \in C^0([0,1])$

general
 $u \in H^2(\Omega)$
 $\lambda \in L^\infty(\Omega)$ (bounded)
 $f \in L^2(\Omega)$

• $u(x) = u_h(x) + u_p(x)$

$$\left. \begin{array}{l} 1.) \quad \begin{cases} -\lambda u_h'' = 0 \\ u_h(0) = 0 \\ u_h(1) = 1 \end{cases} \\ 2.) \quad \begin{cases} -\lambda u_p'' = f \\ u_p(0) = u_p(1) = 0 \end{cases} \end{array} \right\} \rightarrow \begin{cases} -\lambda u'' = -\lambda(u_h + u_p)'' = -\lambda u_h'' - \lambda u_p'' = f \\ u(0) = u_h(0) + u_p(0) = 0 \\ u(1) = u_h(1) + u_p(1) = 1 \end{cases}$$

$$\left. \begin{array}{l} 1.) \quad \begin{cases} u_h = Ax + B \\ u_h(0) = B \stackrel{!}{=} 0 \\ u_h(1) = A \stackrel{!}{=} 1 \end{cases} \end{array} \right\} \rightarrow u_h(x) = x \rightarrow$$

$$\begin{cases} u(x) = x + \frac{c}{2\lambda}(x-x^2) \\ u''(x) = -\frac{c}{\lambda} \quad \checkmark \\ u(0) = 0 \quad \checkmark \\ u(1) = 1 \quad \checkmark \end{cases} \quad \text{probe } \checkmark$$

$$\left. \begin{array}{l} 2.) \quad \begin{cases} u_p'' = -\frac{c}{\lambda} \\ u_p(x) = -\frac{c}{2\lambda}x^2 + Cx + D \\ u_p(0) = D \stackrel{!}{=} 0 \\ u_p(1) = -\frac{c}{2\lambda} + C \stackrel{!}{=} 0 \rightarrow C = \frac{c}{2\lambda} \end{cases} \end{array} \right\} \rightarrow u_p(x) = \frac{c}{2\lambda}(x-x^2)$$

(B)
$$\begin{cases} \text{P.D.E.} & -(\lambda(x) u'(x))' = 0 \\ & u(0) = 0 \\ & u(1) = 1 \\ \text{I.C.} & (*) \quad u(0.5^-) = u(0.5^+) \\ \text{interface} & (**) \quad \lambda(0.5^-) u'(0.5^-) = \lambda(0.5^+) u'(0.5^+) \\ \text{conditions} & \end{cases}$$

$x \in (0, 1)$ where $\lambda(x) = \begin{cases} 1 & x \in (0, \frac{1}{2}) \\ 10 & x \in (\frac{1}{2}, 1) \end{cases}$

• Function spaces: $u \in C^2((0, \frac{1}{2})) \cap C^2((\frac{1}{2}, 1)) \cap C([0, 1])$
 $\lambda \in C^1((0, \frac{1}{2})) \cap C^1((\frac{1}{2}, 1))$ (is piecewise constant actually)
 $f \in C^0([0, 1])$

• Solve on each subdomain

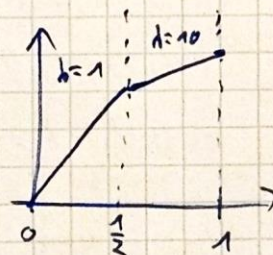
$0 < x < \frac{1}{2}$: $\lambda = 1$: $u_1''(x) = 0 \rightarrow u_1 = Ax + B$
 $u(0) = u_1(0) = 0 \rightarrow \underline{u_1(x) = Ax}$

$\frac{1}{2} < x < 1$: $\lambda = 10$: $10 u_2''(x) = 0 \rightarrow u_2 = Cx + D$
 $u(1) = u_2(1) = 1 \rightarrow \underline{u_2(x) = Cx + (1 - C)}$

I.C.: $(*) \quad u_1(\frac{1}{2}) = u_2(\frac{1}{2}) \Rightarrow \frac{A}{2} = \frac{C}{2} + 1 - C$
 $(\Rightarrow) \quad A + C = 2$

$(**) \quad 1 \cdot u_1'(\frac{1}{2}) = 10 u_2'(\frac{1}{2}) \Rightarrow A = 10C$ $\left. \begin{matrix} A + C = 2 \\ A = 10C \end{matrix} \right\} \begin{matrix} C = \frac{2}{11} \\ A = \frac{20}{11} \end{matrix}$

$\Rightarrow \boxed{u(x) = \begin{cases} \frac{20}{11} x & x \in (0, \frac{1}{2}) \\ \frac{2}{11} x + \frac{9}{11} & x \in (\frac{1}{2}, 1) \end{cases}}$



③ Lösung problem:

$$\begin{aligned} -u''(x) + p u'(x) &= 0 & x \in (0, 1) & \quad p \in \mathbb{R} \\ u(0) &= 0 \\ u(1) &= 1 \end{aligned}$$

substitute: $z(x) = u'(x)$

$$\rightarrow -z'(x) + p z(x) = 0 \quad \Leftrightarrow p z(x) = z'(x) = \frac{dz}{dx}$$

$$\Leftrightarrow \int p dx = \int \frac{1}{z} dz$$

$$\Leftrightarrow px + c = \ln|z|$$

$$\Leftrightarrow z(x) = A e^{px}$$

$$\begin{aligned} p=0 \rightarrow u(x) &= kx + d \\ u(0) &= 0 \\ u(1) &= 1 \end{aligned} \rightarrow u(x) = x$$

$$\begin{aligned} p \neq 0 \rightarrow u(x) &= \frac{A}{p} e^{px} + B \\ u(0) &= \frac{A}{p} + B \stackrel{!}{=} 0 \Leftrightarrow B = -\frac{A}{p} \\ u(1) &= \frac{A}{p} e^p - \frac{A}{p} \stackrel{!}{=} 1 \Leftrightarrow \frac{A}{p} = \frac{1}{e^p - 1} \end{aligned}$$

$$\rightarrow \boxed{u(x) = \frac{e^{px} - 1}{e^p - 1} \quad p \neq 0}$$