SCF UE - Sheet 4

Ⓐ Given: $-u''(x) + a\,u(x) = f(x) \qquad x \in \Omega = (0,1)$

$$u(0) = 0$$

$$u'(1) = \frac{\partial u(1)}{\partial n} = \alpha\,(g_b - u(1)) \qquad a, \alpha, g_b \in \mathbb{R} \text{ constant}$$

1.) Variational formulation:

$$\overset{\int_0^1 \cdot v\,dx}{\rightsquigarrow} \int_0^1 f v\,dx = \underbrace{\int_0^1 -u'' v\,dx}_{} + \int_0^1 a u v\,dx$$

$$= \underbrace{-u'v\Big|_0^1}_{\overset{v(0)=0}{=} -u'(1)v(1) \overset{\text{Neumann}}{=} -\alpha(g_b - u(1))v(1)} + \int_0^1 u'v'\,dx + a\int_0^1 uv\,dx$$

$$= -\alpha g_b v(1) + \alpha u(1)v(1) + \int_0^1 u'v'\,dx + a\int_0^1 uv\,dx$$

$$\rightsquigarrow \boxed{\begin{aligned} &u \in H^1(0,1):\ a(u,v) = \langle F, v\rangle \qquad \forall v \in V_0 = \{w \in H^1(0,1) \mid w(0) = 0\} \\ &\text{where } a(u,v) = \int_0^1 u'v'\,dx + a\int_0^1 uv\,dx + \alpha u(1)v(1) \\ &\qquad \langle F, v\rangle = \int_0^1 fv\,dx + \alpha g_b v(1) \end{aligned}}$$

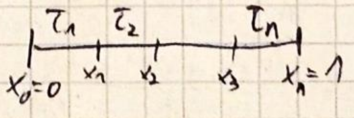
✓

2.) FEM representation:

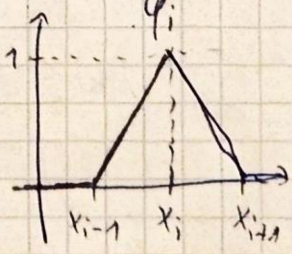
$\Omega = (0,1), \quad \overline{\Omega_h} = \bigcup_j \overline{\tau_j} \qquad h = x_j - x_{j-1} = \frac{1}{n}$

$x_i = \frac{i}{n} \qquad i = 0, \dots, n$

$\tau_j = (x_{j-1}, x_j) \qquad j = 1, \dots, n$



linear shape functions: $\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h} & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{h} & x \in (x_i, x_{i+1}) \\ 0 & x \notin [x_{i-1}, x_{i+1}] \end{cases}$



✓

$$\varphi_i(x_j) = \delta_{ij}$$

Approximate $u \approx u_h = \sum_{j=1}^{n} u_j \varphi_j(x)$ [redacted] , $v \approx v_h = \sum_{j=1}^{n} v_j \varphi_j(x)$

[redacted]

$$\leadsto \sum_i v_i \left[\sum_j u_j \underbrace{\left(\int_0^1 \varphi_j'(x)\varphi_i'(x)\,dx + a\int_0^1 \varphi_j(x)\varphi_i(x)\,dx + \alpha\varphi_j(1)\varphi_i(1) \right)}_{K_{ij}} \right] = \sum_i v_i \left[\underbrace{\int_0^1 f\varphi_i\,dx + \alpha g_b \varphi_i(1)}_{F_i} \right]$$

$v = e_i$ for $i = 1,..,n+1$

$$\leadsto \boxed{\begin{aligned} &Ku = F \qquad\qquad i,j = 0,\dots,n \\ &\text{where } u = (\overset{0}{u_0},\dots,u_n)^T \\ &K_{ij} = \int_0^1 \varphi_j'(x)\varphi_i'(x)\,dx + a\int_0^1 \varphi_j(x)\varphi_i(x)\,dx + \overbrace{\alpha\varphi_j(1)\varphi_i(1)}^{\neq 0 \text{ for } i=j=n} \\ &F_i = \int_0^1 f\varphi_i\,dx + \alpha g_b \varphi_i(1) \end{aligned}}$$ [redacted]

3.) Stiffness matrix K:

$K = \sum_{k=1}^{n} K^{(\tau_k)}$ sum of local stiffness matrices where (for $\tau_k = (x_k, x_{k+1})$)

$$K_{ij}^{(\tau_k)} = \overbrace{\int_{x_k}^{x_{k+1}} \varphi_j'(x)\varphi_i'(x)\,dx}^{A_{ij}^{(\tau_k)}} + a\overbrace{\int_{x_k}^{x_{k+1}} \varphi_j(x)\varphi_i(x)\,dx}^{B_{ij}^{(\tau_k)}} + \alpha\varphi_j(1)\varphi_i(1)$$

$i,j = k, k+1$

else $K_{ij}^{(\tau_k)} =$ [redacted]

$$\int_{x_k}^{x_{k+1}} \varphi_k'\varphi_k'\,dx = \frac{x}{h^2}\Big|_{x_k}^{x_{k+1}} = \frac{1}{h} = A_{kk}^{(\tau_k)} = A_{k+1,k+1}^{(\tau_k)}$$

$$\int_{x_k}^{x_{k+1}} \varphi_k'\varphi_{k+1}'\,dx = \;--- = -\frac{1}{h} = A_{k,k+1}^{(\tau_k)} = A_{k+1,k}^{(\tau_k)}$$

$$\int_{x_k}^{x_{k+1}} \varphi_k\varphi_k\,dx = \int_{x_k}^{x_{k+1}} \frac{(x_{k+1}-x)^2}{h^2}\,dx = \left| \begin{aligned} t &:= x_{k+1}-x \\ dt &= -dx \end{aligned} \right| = \int_h^0 \frac{t^2}{h^2}(-dt) = \int_0^h \frac{t^2}{h^2}\,dx = \frac{1}{h^2}\frac{1}{3}h^3 = \frac{h}{3} = \frac{1}{a}B_{kk\,/\,k+1,k+1}^{(\tau_k)}$$

$$\int_{x_k}^{x_{k+1}} \varphi_k\varphi_{k+1}\,dx = \int_{x_k}^{x_{k+1}} \frac{(x_{k+1}-x)(x-x_k)}{h^2}\,dx = \left| \begin{aligned} t &= x-x_k \to x_{k+1}-x = h-t \\ dt &= dx \end{aligned} \right| = \frac{1}{h^2}\int_0^h (h-t)t\,dx = \frac{1}{h^2}\left(h\frac{h^2}{2} - \frac{h^3}{3}\right) = \frac{h}{6} = \frac{1}{a}B_{\substack{k,k+1\\k+1,k}}^{(\tau_k)}$$

$$\leadsto K^{(\tau_k)} = \begin{array}{cl} \begin{bmatrix} \ddots & \vdots & \vdots & \\ 0 \cdots & \frac{1}{h}+\frac{ah}{3} & -\frac{1}{h}+\frac{ah}{6} & \cdots 0 \\ 0 \cdots & -\frac{1}{h}+\frac{ah}{6} & \frac{1}{h}+\frac{ah}{3} & \cdots 0 \\ & \vdots & \vdots & \ddots \end{bmatrix} & \begin{array}{l} \\ \leftarrow i=k \\ \leftarrow i=k+1 \\ \\ \end{array} \\ \qquad\qquad\uparrow j=k \qquad \uparrow j=k+1 & \end{array}$$

$$K^{(\tau_n)} = \begin{array}{cl} \begin{bmatrix} 0 & \ddots & 0 & 0 \\ & & \frac{1}{h}+\frac{ah}{3} & -\frac{1}{h}+\frac{ah}{6} \\ 0 & \cdots & -\frac{1}{h}+\frac{ah}{6} & \frac{1}{h}+\frac{ah}{3}+\alpha \end{bmatrix} & \begin{array}{l} \\ \leftarrow n-1 \\ \leftarrow n \end{array} \\ \qquad\qquad\uparrow n-1 \qquad \uparrow n & \end{array}$$

$\alpha \to$ Neumann bc

$$K = \begin{bmatrix} \frac{1}{h}+\frac{ah}{3} & -\frac{1}{h}+\frac{ah}{6} & & & 0 \\ -\frac{1}{h}+\frac{ah}{6} & \frac{2}{h}+\frac{2ah}{3} & & & \\ & & \ddots & & \\ & & & \frac{2}{h}+\frac{2ah}{3} & -\frac{1}{h}+\frac{ah}{6} \\ 0 & & & -\frac{1}{h}+\frac{ah}{6} & \frac{1}{h}+\frac{ah}{3}+\alpha \end{bmatrix}$$

Counted twice

✓

Ⓑ

$$-(\lambda(x)\cdot u'(x))' = 0 \qquad x\in\Omega=(0,1)$$
$$u(0) = 0$$
$$u(1) = 1 \qquad \text{where } \lambda(x)=\begin{cases}1 & x\in(0,\frac{1}{\sqrt2}) \\ 10 & x\in(\frac{1}{\sqrt2},1)\end{cases}$$

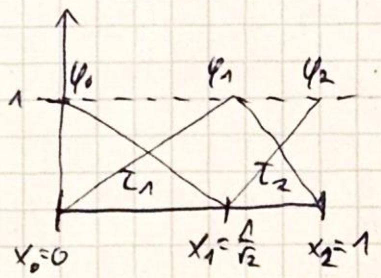
1.) Varform:

$$\overset{\int_\Omega \cdot v\,dx}{\rightsquigarrow} 0 = -\int_0^1 (\lambda(x)\,u'(x))'\,v(x)\,dx \qquad \forall v\in V_0=\{w\in H^1(\Omega)\,|\,w(0)=w(1)=0\}$$

$$= \underbrace{-\lambda(x)u'(x)v(x)\Big|_0^1}_{=0 \text{ since } v(1)=v(0)=0} + \int_0^1 \lambda(x)u'(x)v'(x)\,dx$$

$$\rightsquigarrow \boxed{\begin{array}{l} u\in V=\{u\in H^1(\Omega)\,|\,u(0)=0,\ u(1)=1\}: \\ \displaystyle\int_0^{\frac{1}{\sqrt2}} u'v'\,dx + \int_{\frac{1}{\sqrt2}}^1 10\,u'v'\,dx = 0 \qquad \forall v\in V_0 \end{array}}$$

2.) FEM: (2 elements, already exact)



$x_0=0$

$x_1=\frac{1}{\sqrt2}$ $\qquad h_1=x_1-x_0=\frac{1}{\sqrt2}$

$x_2=1$ $\qquad h_2=x_2-x_1=1-\frac{1}{\sqrt2}$

$$\rightsquigarrow K^{(\tau_1)} = \begin{bmatrix} \int_0^{\frac{1}{\sqrt2}} \varphi_0'\varphi_0'\,dx & \int_0^{\frac{1}{\sqrt2}} \varphi_1'\varphi_0'\,dx & 0 \\ \int_0^{\frac{1}{\sqrt2}} \varphi_0'\varphi_1'\,dx & \int_0^{\frac{1}{\sqrt2}} \varphi_1'\varphi_1'\,dx & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{h_1} & -\frac{1}{h_1} & 0 \\ -\frac{1}{h_1} & \frac{1}{h_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underbrace{K^{(\tau_2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 10\cdot\int_{\frac{1}{\sqrt2}}^1 \varphi_1'\varphi_1'\,dx & 10\cdot\int_{\frac{1}{\sqrt2}}^1 \varphi_2'\varphi_1'\,dx \\ 0 & 10\cdot\int_{\frac{1}{\sqrt2}}^1 \varphi_1'\varphi_2'\,dx & 10\cdot\int_{\frac{1}{\sqrt2}}^1 \varphi_2'\varphi_2'\,dx \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{10}{h_2} & -\frac{10}{h_2} \\ 0 & -\frac{10}{h_2} & \frac{10}{h_2} \end{bmatrix}}$$

$$\int_0^{\frac{1}{\sqrt2}} [\varphi_0']^2\,dx = \int_0^{\frac{1}{\sqrt2}} \frac{1}{h_1^2}\,dx = \frac{1}{h_1} \qquad \int_0^{\frac{1}{\sqrt2}} \varphi_0'\varphi_1'\,dx = -\frac{1}{h_1} \qquad \int_{\frac{1}{\sqrt2}}^1 [\varphi_2']^2\,dx = \frac{1}{h_2}$$

$$\int_0^{\frac{1}{\sqrt2}} [\varphi_1']^2\,dx = \int_0^{\frac{1}{\sqrt2}} \frac{1}{h_1^2}\,dx = \frac{1}{h_1} \qquad \int_{\frac{1}{\sqrt2}}^1 \varphi_1'\varphi_2'\,dx = -\frac{1}{h_2}$$

$$\rightsquigarrow Ku = f \rightsquigarrow \begin{bmatrix} \cancel{\frac{1}{h_1}} & \cancel{-\frac{1}{h_1}} & \cancel{0} \\ -\frac{1}{h_1} & \frac{1}{h_1}+\frac{10}{h_2} & -\frac{10}{h_2} \\ \cancel{0} & \cancel{-\frac{10}{h_2}} & \cancel{\frac{10}{h_2}} \end{bmatrix} \begin{bmatrix} \cancel{\overset{\leftarrow \text{Dirichlet bc}}{0}} \\ u_1 \\ \cancel{\underset{\uparrow \text{Dirichlet bc}}{1}} \end{bmatrix} = \begin{bmatrix} \cancel{0} \\ 0 \\ \cancel{0} \end{bmatrix}$$

2nd row

$$\rightsquigarrow \left(\frac{1}{h_1}+\frac{10}{h_2}\right)u_1 - \frac{10}{h_2} = 0$$

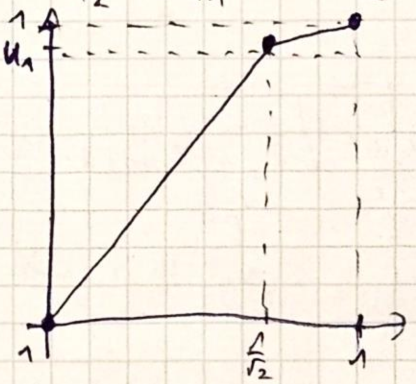
$$\Leftrightarrow \quad u_1 = \frac{\frac{10}{h_2}}{\frac{1}{h_1}+\frac{10}{h_2}} = \frac{10h_1}{h_2+10h_1} = \frac{\frac{10}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}+\frac{10}{\sqrt{2}}} = \frac{10}{\sqrt{2}-1+10}$$

$$\underline{\underline{u_1 \approx 0{,}96022}}$$

✓

$u_0 = 0$

$u_2 = 1$



Ⓒ Péclet problem:

$$\begin{aligned} -u''(x) + p\,u'(x) &= 0 \qquad x \in \Omega = (0,1) \\ u(0) &= 0 \\ u(1) &= 1 \qquad p \in \mathbb{R} \end{aligned}$$

1.) Varform:

$$\overset{\int \cdot v\,dx}{\rightsquigarrow} \quad 0 = \int -u''v\,dx + p\int u'v\,dx \qquad \forall v \in V_0 = \{w \in H^1(\Omega) \mid w(0) = w(1) = 0\}$$

$$= \underbrace{-u'v\Big|_0^1}_{=0 \text{ since } v(0)=v(1)=0} + \int u'v'\,dx + p\int u'v\,dx$$

$$\rightsquigarrow \boxed{\begin{aligned} & u \in V = \{u \in H^1(\Omega) \mid u(0) = 0,\ u(1) = 1\}: \\ & \int_0^1 u'v'\,dx + p\int_0^1 u'v\,dx = 0 \qquad \forall v \in V_0 \end{aligned}}$$

2.) FEM: $Ku = f = 0$

not sym.

$$\rightsquigarrow K_{ij} = \underbrace{\int_0^1 \varphi_j'\varphi_i'\,dx}_{A_{ij}} + \underbrace{p\int_0^1 \varphi_j'\varphi_i\,dx}_{B_{ij}}$$

ex Ⓐ

$$\rightarrow A^{(\tau_k)} = \begin{bmatrix} \frac{1}{h} & -\frac{1}{h} \\ -\frac{1}{h} & \frac{1}{h} \end{bmatrix}$$

$$\int_{x_k}^{x_{k+1}} \varphi_k'\varphi_k\,dx = \int_{x_k}^{x_{k+1}} \left(-\frac{1}{h}\right)\left(\frac{x_{k+1}-x}{h}\right)dx = \left|\begin{matrix} t = x_{k+1}-x \\ dt = -dx \end{matrix}\right| = \left(-\frac{1}{h^2}\right)\int_h^0 t\,(-dt) = -\frac{1}{2}$$

$$\int_{x_k}^{x_{k+1}} \varphi_{k+1}'\varphi_{k+1}\,dx = \int_{x_k}^{x_{k+1}} \left(\frac{1}{h}\right)\left(\frac{x-x_k}{h}\right)dx = \left|\begin{matrix} t = x-x_k \\ dt = dx \end{matrix}\right| = \left(\frac{1}{h^2}\right)\int_0^h t\,dt = \frac{1}{2}$$

$$\int_{x_k}^{x_{k+1}} \varphi_k'\varphi_{k+1}\,dx = \int_{x_k}^{x_{k+1}} \left(-\frac{1}{h}\right)\left(\frac{x-x_k}{h}\right)dx = \left|\begin{matrix} t = x-x_k \\ dt = dx \end{matrix}\right| = \left(-\frac{1}{h^2}\right)\int_0^h t\,dt = -\frac{1}{2}$$

$$\int_{x_k}^{x_{k+1}} \varphi_{k+1}'\varphi_k\,dx = \int_{x_k}^{x_k} \left(\frac{1}{h}\right)\left(\frac{x_{k+1}-x}{h}\right)dx = \left|\begin{matrix} t = x_{k+1}-x \\ dt = -dx \end{matrix}\right| = \left(\frac{1}{h^2}\right)\int_h^0 t\,(-dt) = \frac{1}{2}$$

$$\rightarrow B^{(\tau_k)} = \begin{bmatrix} -\frac{p}{2} & \frac{p}{2} \\ -\frac{p}{2} & \frac{p}{2} \end{bmatrix} \qquad \rightarrow K^{(\tau_k)} = \begin{bmatrix} \frac{1}{h} - \frac{p}{2} & -\frac{1}{h} + \frac{p}{2} \\ -\frac{1}{h} - \frac{p}{2} & \frac{1}{h} + \frac{p}{2} \end{bmatrix}$$ ✓

$$\rightsquigarrow \quad \begin{array}{ccc} K & u & = f \\ \begin{bmatrix} \cancel{\frac{1}{h}-\frac{\ell}{2}} & \cancel{-\frac{1}{h}+\frac{\ell}{2}} & & & 0 \\ \boxed{-\frac{1}{h}-\frac{\ell}{2}} & \frac{2}{h} & -\frac{1}{h}+\frac{\ell}{2} & & \\ & -\frac{1}{h}-\frac{\ell}{2} & \frac{2}{h} & \ddots & \\ & & \ddots & \ddots & \\ 0 & & & \frac{2}{h} & \boxed{-\frac{1}{h}+\frac{\ell}{2}} \\ & & & \cancel{-\frac{1}{h}-\frac{\ell}{2}} & \cancel{\frac{1}{h}+\frac{\ell}{2}} \end{bmatrix} & \begin{bmatrix} \cancel{u_0=0} \\ u_1 \\ \vdots \\ u_{n-1} \\ \cancel{u_n=1} \end{bmatrix} & = \begin{bmatrix} \cancel{0} \\ \vdots \\ \\ \cancel{0} \end{bmatrix} \end{array}$$

↳ leave out known dirichlet nodes and move $\boxed{\text{dirichlet contributions}}$ to rhs

✓

$$\begin{bmatrix} \frac{2}{h} & -\frac{1}{h}+\frac{\ell}{2} & & 0 \\ -\frac{1}{h}-\frac{\ell}{2} & \frac{2}{h} & \ddots & \\ & \ddots & \frac{2}{h} & -\frac{1}{h}+\frac{\ell}{2} \\ 0 & & -\frac{1}{h}-\frac{\ell}{2} & \frac{2}{h} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} -\left(-\frac{1}{h}-\frac{\ell}{2}\right)\overbrace{u_0}^{0} = 0 \\ 0 \\ \vdots \\ 0 \\ -\left(-\frac{1}{h}+\frac{\ell}{2}\right)\underbrace{u_n}_{1} = +\frac{1}{h}-\frac{\ell}{2} \end{bmatrix}$$

↳ $(n-1)\times(n-1)$

↳ Solve system, see git.

<u>Alternative</u>: reorder rows/columns/variables in $Ku = f$ such that

$$\left[\begin{array}{c|c} K_{FF} & K_{FD} \\ \hline K_{DF} & K_{DD} \end{array}\right] \left[\begin{array}{c} u_F \\ \hline u_D \end{array}\right] = \begin{bmatrix} f_F \\ f_D \end{bmatrix}$$

u_F ... unknown, free, nondirichlet

u_D ... known from dirichlet bc

$$\rightsquigarrow \quad K_{FF}\,u_F + K_{FD}\,u_D = f_F$$

$$\rightsquigarrow \quad \underline{\underline{u_F = K_{FF}^{-1}\left(f_F - K_{FD}\,u_D\right)}}$$