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On the arithmetic of Krull monoids with infinite cyclic class group[☆]

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ABSTRACT

Let H be a Krull monoid with infinite cyclic class group G and let $G_P \subset G$ denote the set of classes containing prime divisors. We study under which conditions on G_P some of the main finiteness properties of factorization theory – such as local tameness, the finiteness and rationality of the elasticity, the structure theorem for sets of lengths, the finiteness of the catenary degree, and the existence of monotone and near monotone chains of factorizations – hold in H. In many cases, we derive explicit characterizations.

1. Introduction

By an atomic monoid, we mean a commutative cancellative semigroup with unit element such that every non-unit has a factorization as a finite product of atoms (irreducible elements). The multiplicative monoid consisting of the nonzero elements from a noetherian domain is such a monoid. Let H be an atomic monoid. Then every non-unit has a unique factorization into atoms if and only if H is a Krull monoid with trivial class group. The first objective of factorization theory is to describe the various phenomena related to the non-uniqueness of factorizations. This is done by studying a variety of arithmetical invariants such as sets of lengths, elasticities and the catenary and tame degrees of the monoids. The second main objective is to then characterize the finiteness (or even to find the precise values) of these arithmetical invariants in terms of classical algebraic invariants of the objects under investigation. We illustrate this in the next paragraph. To be able to do so, recall that the elasticity $\rho(H)$ of H is the supremum over all k/l for which there is an equation $u_1 \cdot \ldots \cdot u_k = v_1 \cdot \ldots \cdot v_l$, where $u_1, \ldots, u_k, v_1, \ldots, v_l$ are atoms of H.

The following result by Carlitz (achieved in 1960) is considered as a starting point of factorization theory: the ring of integers $\mathfrak{o}_K$ of an algebraic number field has elasticity $\rho(\mathfrak{o}_K) = 1$ if and only if its class group has at most two elements. A non-principal order $\mathfrak{o}$ in an algebraic number field has finite elasticity if and only if, for every prime ideal $\mathfrak{p}$ containing the conductor, there is precisely one prime ideal $\overline{\mathfrak{p}}$ in the principal order $\overline{\mathfrak{o}}$ such that $\overline{\mathfrak{p}} \cap \mathfrak{o} = \mathfrak{p}$. This result (achieved by Halter-Koch in 1995) has far reaching generalizations (achieved by Kainrath) to finitely generated domains and to various classes of Mori domains satisfying natural finiteness conditions (see [3,35,39,38]).

This paper is concerned with Krull monoids. Their arithmetic is completely determined by the class group and the distribution of prime divisors in the classes. We outline this in greater detail. First, recall that an integral domain is a Krull domain if and only if its multiplicative monoid of nonzero elements is a Krull monoid, and a noetherian domain is Krull if and only if it is integrally closed. A reduced Krull monoid is uniquely determined by its class group and by the distribution of prime divisors in the classes (see Lemma 3.3 for a precise statement). Suppose H is a Krull monoid with class group G and let $G_P \subset G$ denote the set of classes containing prime divisors. Suppose that $G_P = G$. In that case, it is comparatively

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easy to show that any of the arithmetical invariants under discussion is finite if and only if G is finite (the precise values of arithmetical invariants – when G is finite – are studied by methods of Additive and Combinatorial Number Theory; see [28, Chapter 6] or [25] for a survey on this direction). However, very little is known about the arithmetic of H when G is infinite and G_P is a proper subset of G.

The present paper provides an in-depth study of the arithmetic of Krull monoids having an infinite cyclic class group. This situation was studied first by Anderson, Chapman and Smith [1], then by Hassler [37], and the most recent progress (again due to Chapman et al.) was achieved in [2]. We continue this work, thus studying monoids with class group $G \cong \mathbb{Z}$ but having the set of classes containing primes G_P being a proper subset. The arithmetical properties under investigation are discussed in Section 2 and at the beginning of Section 5. The required material on Krull monoids, together with a list of relevant examples, is summarized in Section 3. Our main results are Theorems 4.2, 5.2 and 6.4 and Corollary 7.4.

In Theorem 4.2, we give a lengthy list of factorization invariants, the finiteness of any one of which is shown to be equivalent to a natural finiteness condition on G_P, thus characterizing when these 'weak' invariants, including the local tame degree, the catenary degree, the set of distances, and several other invariants, are each finite (several previously known equivalences are included for completeness). Perhaps more importantly, Theorem 4.2 shows these conditions to be equivalent to the elasticity invariant being a rational number, which was an open problem first proposed in 1994 and which is here resolved.

One of the crowning results of early Factorization Theory was the establishment, for a large class of monoids, of a structure theorem describing the number of atoms possible in a factorization of a given element, often now known as the Structure Theorem for Sets of Lengths. The Structure Theorem for Sets of Lengths holding for a monoid is a much 'stronger' invariant than those found in Theorem 4.2, as it tells us a great deal more about the structure of possible factorizations. Two other such 'stronger' invariants are the monotone catenary degree and the successive distance (see later sections for full details and definitions), which concern whether there is always a well-behaved sequence of factorizations slowly transforming one factorization of a given element into another. In Theorem 5.2, we establish several implications involving these stronger invariants and, in Theorem 6.4, we characterize when the Structure Theorem for Sets of Lengths holds for class group $G \cong \mathbb{Z}$ under some additional restrictions on G_P. Counterexamples to these invariants always being finite are also furnished. Finally, Corollary 7.4 (and the other results of Section 7) show that while the monotone catenary may unfortunately be infinite, there is nonetheless still a nice chain of factorizations between any two factorizations which are not overly pathological (in a technical sense made explicit in the section).

Along the way, several new methods are introduced, particularly for the proofs of Proposition 4.8 and Theorem 7.3. More detailed discussion of the main results is shifted to the relevant sections where we have the required terminology at our disposal.

2. Preliminaries

Our notation and terminology are consistent with [28]. We briefly gather some key notions. We denote by $\mathbb{N}$ the set of positive integers, and we put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$. For a subset X of (possibly negative) integers, we use $\gcd X$ and $\operatorname{lcm} X$ to denote the greatest common divisor and least common multiple respectively, and their values are always chosen to be nonnegative regardless of the sign of the inputs.

Let $L, L' \subset \mathbb{Z}$. We set $-L = \{-a \mid a \in L\}$, $L^+ = L \cap \mathbb{N}$ and $L^- = L \cap (-\mathbb{N})$. We denote by $L + L' = \{a + b \mid a \in L,\ b \in L'\}$ their *sumset*. If $\emptyset \neq L \subset \mathbb{N}$, we call

$$\rho(L) = \sup\left\{\frac{m}{n} \;\middle|\; m, n \in L\right\} = \frac{\sup L}{\min L} \in \mathbb{Q}_{\ge 1} \cup \{\infty\}$$

the *elasticity* of L, and we set $\rho(\{0\}) = 1$. Distinct elements $k, l \in L$ are called *adjacent* if $L \cap [\min\{k, l\}, \max\{k, l\}] = \{k, l\}$. A positive integer $d \in \mathbb{N}$ is called a *distance* of L if there exist adjacent elements $k, l \in L$ with $d = |k - l|$. We denote by $\Delta(L)$ the *set of distances* of L. Note that $\Delta(L) = \emptyset$ if and only if $|L| \le 1$, and that L is an arithmetical progression with difference $d \in \mathbb{N}$ if and only if $\Delta(L) \subset \{d\}$. We need the following generalization of an arithmetical progression.

Let $d \in \mathbb{N}$, $M \in \mathbb{N}_0$ and $\{0, d\} \subset \mathcal{D} \subset [0, d]$. Then L is called an *almost arithmetical multiprogression* (AAMP for short) with *difference* d, *period* $\mathcal{D}$, and *bound* M, if

$$L = y + (L' \cup L^* \cup L'') \subset y + \mathcal{D} + d\mathbb{Z}$$

where

- L^* is finite and nonempty with $\min L^* = 0$ and $L^* = (\mathcal{D} + d\mathbb{Z}) \cap [0, \max L^*]$
- $L' \subset [-M, -1]$ and $L'' \subset \max L^* + [1, M]$
- $y \in \mathbb{Z}$.

Note that an AAMP is finite and nonempty. An AAMP with period $\{0, d\}$ is called an *almost arithmetical progression* (AAP for short).

By a *monoid*, we mean a commutative, cancellative semigroup with unit element; we denote the unit element by **1**. Let H be a monoid and let $a, b \in H$. We call a a *divisor* of b and write $a \mid b$ (or, more precisely, $a \mid_H b$) if $b \in aH$. We call a and b *associated* (in symbols, $a \simeq b$) if $aH = bH$ (or, equivalently, if $aH^\times = bH^\times$). An element $u \in H$ is called

- *invertible* if there is an element $v \in H$ with $uv = 1$.

- *irreducible* (or an *atom*) if u is not invertible and, for all $a, b \in H$, $u = ab$ implies a is invertible or b is invertible.
- *prime* if u is not invertible and, for all $a, b \in H$, $u \mid ab$ implies $u \mid a$ or $u \mid b$.

We denote by $\mathcal{A}(H)$ the set of atoms of H, by $H^\times$ the group of invertible elements, and by $H_{\mathrm{red}} = \{aH^\times \mid a \in H\}$ the associated reduced monoid of H. We say that H is reduced if $|H^\times| = 1$. We denote by $\mathsf{q}(H)$ a quotient group of H with $H \subset \mathsf{q}(H)$, and for a prime element $p \in H$, let $\mathsf{v}_p \colon \mathsf{q}(H) \to \mathbb{Z}$ be the p-adic valuation. For a subset $H_0 \subset H$, we denote by $[H_0] \subset H$ the submonoid generated by H_0 and by $\langle H_0 \rangle \subset \mathsf{q}(H)$ the subgroup generated by H_0.

For a set P, we denote by $\mathcal{F}(P)$ the *free* (*abelian*) *monoid* with basis P. Then every $a \in \mathcal{F}(P)$ has a unique representation in the form

$$a = \prod_{p \in P} p^{\mathsf{v}_p(a)} \quad \text{with } \mathsf{v}_p(a) \in \mathbb{N}_0 \text{ and } \mathsf{v}_p(a) = 0 \text{ for almost all } p \in P.$$

We call $|a| = \sum_{p \in P} \mathsf{v}_p(a)$ the *length* of a.

The free monoid $\mathsf{Z}(H) = \mathcal{F}\big(\mathcal{A}(H_{\mathrm{red}})\big)$ is called the *factorization monoid* of H, and the unique homomorphism

$$\pi \colon \mathsf{Z}(H) \to H_{\mathrm{red}} \quad \text{satisfying} \quad \pi(u) = u \quad \text{for each } u \in \mathcal{A}(H_{\mathrm{red}})$$

is called the *factorization homomorphism* of H. For $a \in H$ and $k \in \mathbb{N}$, the set

$$\begin{aligned}
&\mathsf{Z}_H(a) = \mathsf{Z}(a) = \pi^{-1}(aH^\times) \subset \mathsf{Z}(H) \quad \text{is the } \textit{set of factorizations} \text{ of } a,\\
&\mathsf{Z}_k(a) = \{z \in \mathsf{Z}(a) \mid |z| = k\} \quad \text{is the } \textit{set of factorizations} \text{ of } a \text{ of length } k, \quad \text{and}\\
&\mathsf{L}_H(a) = \mathsf{L}(a) = \big\{|z| \;\big|\; z \in \mathsf{Z}(a)\big\} \subset \mathbb{N}_0 \quad \text{is the } \textit{set of lengths} \text{ of } a.
\end{aligned}$$

By definition, we have $\mathsf{Z}(a) = \{\mathbf{1}\}$ and $\mathsf{L}(a) = \{0\}$ for all $a \in H^\times$. The monoid H is called

- *atomic* if $\mathsf{Z}(a) \ne \emptyset$ for all $a \in H$,
- a BF-*monoid* (a bounded factorization monoid) if $\mathsf{L}(a)$ is finite and nonempty for all $a \in H$,
- *factorial* if $|\mathsf{Z}(a)| = 1$ for all $a \in H$,
- *half-factorial* if $|\mathsf{L}(a)| = 1$ for all $a \in H$.

We now introduce the arithmetical concepts which are used throughout the whole paper. Some more specific notions will be recalled at the beginning of Section 5. Let H be atomic and $a \in H$. Then $\rho(a) = \rho\big(\mathsf{L}(a)\big)$ is called the *elasticity* of a, and the *elasticity* of H is defined as

$$\rho(H) = \sup\{\rho(b) \mid b \in H\} \in \mathbb{R}_{\ge 1} \cup \{\infty\}.$$

We say that H has *accepted elasticity* if there exists some $b \in H$ with $\rho(b) = \rho(H)$.

Let $k \in \mathbb{N}$. If $H \ne H^\times$, then

$$\mathcal{V}_k(H) = \bigcup_{k \in \mathsf{L}(a),\, a \in H} \mathsf{L}(a)$$

is the union of all sets of lengths containing k. When $H^\times = H$, we set $\mathcal{V}_k(H) = \{k\}$. In both cases, we define $\rho_k(H) = \sup \mathcal{V}_k(H)$ and $\lambda_k(H) = \min \mathcal{V}_k(H)$. Clearly, we have $\mathcal{V}_1(H) = \{1\}$ and $k \in \mathcal{V}_k(H)$. By its definition, H is half-factorial if and only if $\mathcal{V}_k(H) = \{k\}$ for each $k \in \mathbb{N}$.

We denote by

$$\Delta(H) = \bigcup_{b \in H} \Delta\big(\mathsf{L}(b)\big) \subset \mathbb{N}$$

the *set of distances* of H, and by $\mathcal{L}(H) = \{\mathsf{L}(b) \mid b \in H\}$ the *system of sets of lengths* of H.

Let $z, z' \in \mathsf{Z}(H)$. Then we can write

$$z = u_1 \cdot \ldots \cdot u_l v_1 \cdot \ldots \cdot v_m \quad \text{and} \quad z' = u_1 \cdot \ldots \cdot u_l w_1 \cdot \ldots \cdot w_n,$$

where $l, m, n \in \mathbb{N}_0$ and $u_1, \ldots, u_l, v_1, \ldots, v_m, w_1, \ldots, w_n \in \mathcal{A}(H_{\mathrm{red}})$ are such that

$$\{v_1, \ldots, v_m\} \cap \{w_1, \ldots, w_n\} = \emptyset.$$

Then $\gcd(z, z') = u_1 \cdot \ldots \cdot u_l$, and we call

$$\mathsf{d}(z, z') = \max\{m, n\} = \max\{|z \gcd(z, z')^{-1}|, |z' \gcd(z, z')^{-1}|\} \in \mathbb{N}_0$$

the *distance* between z and z'. If $\pi(z) = \pi(z')$ and $z \ne z'$, then

$$2 + \big||z| - |z'|\big| \le \mathsf{d}(z, z') \tag{2.1}$$

by [28, Lemma 1.6.2]. For subsets $X, Y \subset \mathsf{Z}(H)$, we set

$$\mathsf{d}(X, Y) = \min\{\mathsf{d}(x, y) \mid x \in X,\ y \in Y\},$$

and thus $X \cap Y \ne \emptyset$ if and only if $\mathsf{d}(X, Y) = 0$.

We recall the concepts of the (monotone) catenary and tame degrees (see also the beginning of Section 7). The *catenary degree* $\mathsf{c}(a)$ of the element a is the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ such that, for any two factorizations z, z' of a, there exists a finite sequence $z = z_0, z_1, \ldots, z_k = z'$ of factorizations of a such that $\mathsf{d}(z_{i-1}, z_i) \le N$ for all $i \in [1, k]$. The *monotone catenary degree* $\mathsf{c}_{\mathrm{mon}}(a)$ is defined in the same way with the additional restriction that $|z_0| \le \cdots \le |z_k|$ or $|z_0| \ge \cdots \ge |z_k|$.

We say that the two factorizations z and z' can be concatenated by a (monotone) N-chain if a sequence fulfilling the above conditions exists. Moreover,

$$\mathsf{c}(H) = \sup\{\mathsf{c}(b) \mid b \in H\} \in \mathbb{N}_0 \cup \{\infty\} \quad \text{and} \quad \mathsf{c}_{\mathrm{mon}}(H) = \sup\{\mathsf{c}_{\mathrm{mon}}(b) \mid b \in H\} \in \mathbb{N}_0 \cup \{\infty\}$$

denote the *catenary degree* and the *monotone catenary degree* of H. Clearly, we have $\mathsf{c}(a) \le \mathsf{c}_{\mathrm{mon}}(a)$ for all $a \in H$, as well as $\mathsf{c}(H) \le \mathsf{c}_{\mathrm{mon}}(H)$, and (2.1) implies that $2 + \sup \Delta(H) \le \mathsf{c}(H)$.

For $x \in \mathsf{Z}(H)$, let $\mathsf{t}(a, x) \in \mathbb{N}_0 \cup \{\infty\}$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

If $\mathsf{Z}(a) \cap x\mathsf{Z}(H) \ne \emptyset$ and $z \in \mathsf{Z}(a)$, then there exists $z' \in \mathsf{Z}(a) \cap x\mathsf{Z}(H)$ such that $\mathsf{d}(z, z') \le N$.

For subsets $H' \subset H$ and $X \subset \mathsf{Z}(H)$, we define

$$\mathsf{t}(H', X) = \sup\left\{\mathsf{t}(b, x) \,\middle|\, b \in H', x \in X\right\} \in \mathbb{N}_0 \cup \{\infty\}.$$

H is called *locally tame* if $\mathsf{t}(H, u) < \infty$ for all $u \in \mathcal{A}(H_{\mathrm{red}})$ (see the beginning of Section 4 and Definition 6.1).

3. Krull monoids: basic properties and examples

The theory of Krull monoids is presented in detail in the monographs [36,33,28]. Here we first gather the required terminology. After that, we recall some facts concerning transfer homomorphisms, since the arithmetic of Krull monoids is studied via such homomorphisms. In particular, we deal with block homomorphisms (which are transfer homomorphisms; see Lemma 3.3 and the comment thereafter) from Krull monoids into the associated block monoids. At the end of this section, we discuss examples of Krull monoids with infinite cyclic class group.

Krull monoids. Let H and D be monoids. A monoid homomorphism $\varphi\colon H \to D$ is called

- a *divisor homomorphism* if $\varphi(a) \mid \varphi(b)$ implies that $a \mid b$ for all $a, b \in H$
- *cofinal* if for every $a \in D$ there exists some $u \in H$ such that $a \mid \varphi(u)$
- a *divisor theory* (for H) if $D = \mathcal{F}(P)$ for some set P, φ is a divisor homomorphism, and for every $p \in P$ (equivalently for every $a \in \mathcal{F}(P)$), there exists a finite subset $\emptyset \ne X \subset H$ satisfying $p = \gcd\big(\varphi(X)\big)$.

Note that, by definition, every divisor theory is cofinal. We call $\mathcal{C}(\varphi) = \mathsf{q}(D)/\mathsf{q}(\varphi(H))$ the class group of φ and use additive notation for this group. For $a \in \mathsf{q}(D)$, we denote by $[a] = [a]_\varphi = a\,\mathsf{q}(\varphi(H)) \in \mathsf{q}(D)/\mathsf{q}(\varphi(H))$ the class containing a. We recall that φ is cofinal if and only if $\mathcal{C}(\varphi) = \{[a] \mid a \in D\}$, and if φ is a divisor homomorphism, then $\varphi(H) = \{a \in D \mid [a] = [1]\}$. If $\varphi\colon H \to \mathcal{F}(P)$ is a cofinal divisor homomorphism, then

$$G_P = \{[p] = p\mathsf{q}(\varphi(H)) \mid p \in P\} \subset \mathcal{C}(\varphi)$$

is called the *set of classes containing prime divisors*, and we have $[G_P] = \mathcal{C}(\varphi)$ (for a converse, see Lemma 3.4). If $H \subset D$ is a submonoid, then H is called *cofinal* (*saturated*, resp.) in D if the imbedding $H \hookrightarrow D$ is cofinal (a divisor homomorphism, resp.).

The monoid H is called a *Krull monoid* if it satisfies one of the following equivalent conditions ([28, Theorem 2.4.8]; see [41] for recent progress):

- H is v-noetherian and completely integrally closed.
- H has a divisor theory.
- H_{red} is a saturated submonoid of a free monoid.

In particular, H is a Krull monoid if and only if H_{red} is a Krull monoid. Let H be a Krull monoid. Then a divisor theory $\varphi\colon H \to \mathcal{F}(P)$ is unique up to unique isomorphism. In particular, the class group $\mathcal{C}(\varphi)$ defined via a divisor theory of H and the subset of classes containing prime divisors depend only on H. Thus it is called the *class group* of H and is denoted by $\mathcal{C}(H)$. If H is a Krull monoid, then $\mathcal{I}_v^*(H)$ denotes the monoid of v-invertible v-ideals of H, which is a free monoid with basis $\mathfrak{X}(H)$. In such case, the map $\delta : H \to \mathcal{I}_v^*(H)$ given by $a \mapsto aH$ is a divisor theory, and thus $\mathcal{C}(H)$ is the v-class group of H (up to isomorphism).

Transfer homomorphisms. We recall some of the main properties which are needed in the following sections (details can be found in [28, Section 3.2]).

Definition 3.1. A monoid homomorphism $\theta\colon H \to B$ is called a *transfer homomorphism* if it has the following properties:

(T1) $B = \theta(H)B^\times$ and $\theta^{-1}(B^\times) = H^\times$.
(T2) If $u \in H$, $b, c \in B$ and $\theta(u) = bc$, then there exist $v, w \in H$ such that $u = vw$, $\theta(v) \simeq b$ and $\theta(w) \simeq c$.

Every transfer homomorphism θ gives rise to a unique extension $\overline{\theta}\colon \mathsf{Z}(H) \to \mathsf{Z}(B)$ satisfying

$$\overline{\theta}(uH^\times) = \theta(u)B^\times \quad \text{for each } u \in \mathcal{A}(H).$$

For $a \in H$, we denote by $\mathsf{c}(a, \theta)$ the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

If $z,\ z' \in \mathsf{Z}_H(a)$ and $\overline{\theta}(z) = \overline{\theta}(z')$, then there exist some $k \in \mathbb{N}_0$ and factorizations $z = z_0, \ldots, z_k = z' \in \mathsf{Z}_H(a)$ such that $\overline{\theta}(z_i) = \overline{\theta}(z)$ and $\mathsf{d}(z_{i-1}, z_i) \le N$ for all $i \in [1, k]$ (that is, z and z' can be concatenated by an N-chain in the fiber $\mathsf{Z}_H(a) \cap \overline{\theta}^{-1}(\overline{\theta}(z))$).

Then

$$\mathsf{c}(H, \theta) = \sup\{\mathsf{c}(a, \theta) \mid a \in H\} \in \mathbb{N}_0 \cup \{\infty\}$$

denotes the *catenary degree in the fibres*.

Lemma 3.2. *Let $\theta : H \to B$ and $\theta' : B \to B'$ be transfer homomorphisms of atomic monoids.*

1. *For every $a \in H$, we have $\overline{\theta}(\mathsf{Z}_H(a)) = \mathsf{Z}_B(\theta(a))$ and $\mathsf{L}_H(a) = \mathsf{L}_B(\theta(a))$.*
2. $\mathsf{c}(B) \le \mathsf{c}(H) \le \max\{\mathsf{c}(B), \mathsf{c}(H, \theta)\}$, $\mathsf{c}_{\text{mon}}(B) \le \mathsf{c}_{\text{mon}}(H) \le \max\{\mathsf{c}_{\text{mon}}(B), \mathsf{c}(H, \theta)\}$ *and* $\delta(B) = \delta(H)$.
3. *For every $a \in H$ and all $k, l \in \mathsf{L}(a)$, we have* $\mathsf{d}\big(\mathsf{Z}_k(a), \mathsf{Z}_l(a)\big) = \mathsf{d}\big(\mathsf{Z}_k\big(\theta(a)\big), \mathsf{Z}_l\big(\theta(a)\big)\big)$.
4. *For every $a \in H$, we have* $\mathsf{c}(a, \theta' \circ \theta) \le \max\{\mathsf{c}(a, \theta), \mathsf{c}(\theta(a), \theta')\}$.
   In particular, $\mathsf{c}(H, \theta' \circ \theta) \le \max\{\mathsf{c}(H, \theta), \mathsf{c}(B, \theta')\}$.

Proof. 1. This follows from [28, Proposition 3.2.3].

2. The first statement follows from Theorem 3.2.5.4, the second from Lemma 3.2.6 in [28], and the third from [26, Theorem 3.14].

3. Let $a \in H$ and $k, l \in \mathsf{L}(a)$. If $z, z' \in \mathsf{Z}(a)$ with $|z| = k$ and $|z'| = l$, then $|\overline{\theta}(z)| = k$, $|\overline{\theta}(z')| = l$ and $\mathsf{d}\big(\overline{\theta}(z), \overline{\theta}(z')\big) \le \mathsf{d}(z, z')$, which implies that $\mathsf{d}\big(\mathsf{Z}_k\big(\theta(a)\big), \mathsf{Z}_l\big(\theta(a)\big)\big) \le \mathsf{d}\big(\mathsf{Z}_k(a), \mathsf{Z}_l(a)\big)$. To verify the reverse inequality, let $\overline{z}_1, \overline{z}_2 \in \mathsf{Z}(\theta(a))$ be given. We pick any $z_1 \in \mathsf{Z}(a)$ with $\overline{\theta}(z_1) = \overline{z}_1$. By [28, Proposition 3.2.3.3(c)], there exists a factorization $z_2 \in \mathsf{Z}(a)$ such that $\overline{\theta}(z_2) = \overline{z}_2$ and $\mathsf{d}(z_1, z_2) = \mathsf{d}(\overline{z}_1, \overline{z}_2)$. Since $|z_i| = |\overline{z}_i|$ for $i \in \{1, 2\}$, it follows that $\mathsf{d}\big(\mathsf{Z}_k(a), \mathsf{Z}_l(a)\big) \le \mathsf{d}\big(\mathsf{Z}_k\big(\theta(a)\big), \mathsf{Z}_l\big(\theta(a)\big)\big)$.

4. We recall that $\theta' \circ \theta$ is a transfer homomorphism (see the paragraph after [28, Definition 3.2.1]). Let $a \in H$. Let $z, z' \in \mathsf{Z}_H(a)$ with $\overline{\theta' \circ \theta}(z) = \overline{\theta' \circ \theta}(z')$. Let $\overline{z} = \overline{\theta}(z)$ and $\overline{z'} = \overline{\theta}(z')$. We have $\overline{z}, \overline{z'} \in \mathsf{Z}_B(\theta(a))$ and $\overline{\theta'}(\overline{z}) = \overline{\theta'}(\overline{z'})$. Thus, by the definition of $\mathsf{c}(\theta(a), \theta')$, there exist some $k \in \mathbb{N}_0$ and $\overline{z} = \overline{z_0}, \ldots, \overline{z_k} = \overline{z'} \in \mathsf{Z}_B(\theta(a))$ such that $\overline{\theta'}(\overline{z_i}) = \overline{\theta'}(\overline{z})$ and $\mathsf{d}(\overline{z_{i-1}}, \overline{z_i}) \le \mathsf{c}(\theta(a), \theta')$ for each $i \in [1, k]$. Let $z_0 = z$. Again, by [28, Proposition 3.2.3.3(c)], for each $i < k$, there exists some factorization $z_{i+1} \in \mathsf{Z}_H(a)$ such that $\overline{\theta}(z_{i+1}) = \overline{z_{i+1}}$ and $\mathsf{d}(z_i, z_{i+1}) = \mathsf{d}(\overline{z_i}, \overline{z_{i+1}})$.

Now, we have $\overline{\theta}(z_k) = \overline{z'} = \overline{\theta}(z')$. Thus, by the definition of $\mathsf{c}(a, \theta)$, there exist some $l \in \mathbb{N}_0$ and $z_k = y_0, \ldots, y_l = z' \in \mathsf{Z}_H(a)$ such that $\overline{\theta}(y_i) = \overline{\theta}(z')$ and $\mathsf{d}(y_{i-1}, y_i) \le \mathsf{c}(a, \theta)$ for each $i \in [1, l]$. Since $\overline{\theta}(y_i) = \overline{\theta}(z')$ clearly implies $\overline{\theta' \circ \theta}(y_i) = \overline{\theta' \circ \theta}(z')$, we get that the $\max\{\mathsf{c}(\theta(a), \theta'), \mathsf{c}(a, \theta)\}$-chain $z = z_0, \ldots, z_k = y_0, \ldots, y_l = z'$ has the required properties. $\square$

Monoids of zero-sum sequences. Let G be an additive abelian group, $G_0 \subset G$ a subset and $\mathcal{F}(G_0)$ the free monoid with basis G_0. According to the tradition of combinatorial number theory, the elements of $\mathcal{F}(G_0)$ are called *sequences* over G_0. Thus a sequence $S \in \mathcal{F}(G_0)$ will be written in the form

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G_0} g^{\mathsf{v}_g(S)},$$

and we use all the notions (such as the length) as in general free monoids. Again using traditional language, we refer to $\mathsf{v}_g(S)$ as the *multiplicity* of g in S and refer to a divisor of S as a *subsequence*. If $T|S$, then $T^{-1}S$ denotes the subsequence of S obtained by removing the terms of T. We call the set $\operatorname{supp}(S) = \{g_1, \ldots, g_l\} \subset G_0$ the *support* of S, $\sigma(S) = g_1 + \cdots + g_l \in G$ the *sum* of S, and define

$$\Sigma(S) = \Big\{\sum_{i \in I} g_i \mid \emptyset \ne I \subset [1, l]\Big\} \subset G \quad \text{and, for } k \in \mathbb{N},$$

$$\Sigma_k(S) = \Big\{\sum_{i \in I} g_i \mid I \subset [1, l],\ |I| = k\Big\} \subset G.$$

We set $-S = (-g_1) \cdot \ldots \cdot (-g_l)$. If $G = \mathbb{Z}$, then we define

$$S^+ = \prod_{g \in G_0^+} g^{\mathsf{v}_g(S)} \quad \text{and} \quad S^- = \prod_{g \in G_0^-} g^{\mathsf{v}_g(S)},$$

and thus we have $S = S^+ S^- 0^{\mathsf{v}_0(S)}$. The monoid

$$\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) \mid \sigma(S) = 0\}$$

is called the *monoid of zero-sum sequences* over G_0, and its elements are called *zero-sum sequences* over G_0. A sequence $S \in \mathcal{F}(G_0)$ is zero-sum free if it has no proper, nontrivial zero-sum subsequence (note the trivial/empty sequence is defined to have sum zero). For every arithmetical invariant $*(H)$ defined for a monoid H, we write $*(G_0)$ instead of $*(\mathcal{B}(G_0))$. In particular, we set $\mathcal{A}(G_0) = \mathcal{A}(\mathcal{B}(G_0))$. We denote the *Davenport constant* of G_0 by

$$\mathsf{D}(G_0) = \sup\{|U| \mid U \in \mathcal{A}(G_0)\} \in \mathbb{N}_0 \cup \{\infty\},$$

which is a central invariant in zero-sum theory (see [20], and also [25] for its relevance in factorization theory).

Clearly, $\mathcal{B}(G_0) \subset \mathcal{F}(G_0)$ is saturated, and hence $\mathcal{B}(G_0)$ is a Krull monoid. We note that $\mathcal{B}(G_0) \subset \mathcal{F}(G_0)$ is cofinal if and only if for each $g \in G_0$ there is a $B \in \mathcal{B}(G_0)$ with $\mathsf{v}_g(B) > 0$ (see [28, Proposition 2.5.6]); if this is the case, then the set G_0 is called *condensed*. For a condensed set G_0, the class group of $\mathcal{B}(G_0) \hookrightarrow \mathcal{F}(G_0)$ is $\langle G_0 \rangle$, and the subset of classes containing prime divisors is G_0.

For $G_0 \subset \mathbb{Z}$, we have that G_0 is condensed if and only if either $G_0^+ \neq \emptyset$ and $G_0^- \neq \emptyset$ or $G_0 \subset \{0\}$. The latter case, which in our context can be disregarded (see Lemma 3.3), is frequently automatically excluded by some of the conditions we impose in our results; if not, we impose the extra condition $|G_0| \geq 2$.

Block monoids associated to Krull monoids. We will make substantial use of the following result [28, Section 3.4].

Lemma 3.3. *Let H be a Krull monoid, $\varphi \colon H \to F = \mathcal{F}(P)$ a cofinal divisor homomorphism, $G = \mathcal{C}(\varphi)$ its class group, and $G_P \subset G$ the set of classes containing prime divisors. Let $\widetilde{\boldsymbol{\beta}} \colon F \to \mathcal{F}(G_P)$ denote the unique homomorphism defined by $\widetilde{\boldsymbol{\beta}}(p) = [p]$ for all $p \in P$.*

1. *The homomorphism $\boldsymbol{\beta} = \widetilde{\boldsymbol{\beta}} \circ \varphi \colon H \to \mathcal{B}(G_P)$ is a transfer homomorphism with $\mathsf{c}(H, \boldsymbol{\beta}) \leq 2$. In particular, it has all the properties mentioned in Lemma 3.2.*
2. *$\mathcal{B}(G_P) \subset \mathcal{F}(G_P)$ is saturated and cofinal. If G is infinite cyclic, then $G_P \subset G$ is a condensed set and $|G_P| \geq 2$.*

The homomorphism $\boldsymbol{\beta}$ in Lemma 3.3 is called the *block homomorphism*, and $\mathcal{B}(G_P)$ is called the *block monoid* associated to φ. If φ is a divisor theory, then $\mathcal{B}(G_P)$ is called the block monoid associated to H.

A lemma and four examples. The following lemma highlights the strong connection between the algebraic structure of a Krull monoid and its class group and provides a realization result (see [28, Theorem 2.5.4]). Let G be an abelian group and $(m_g)_{g \in G}$ a family of cardinal numbers. We say H has *characteristic* $(G, (m_g)_{g\in G})$ if there is a group isomorphism $\Phi \colon G \xrightarrow{\sim} \mathcal{C}(H)$ such that $\operatorname{card}(P \cap \Phi(g)) = m_g$ for every $g \in G$.

Lemma 3.4. *Let G be an abelian group, $(m_g)_{g\in G}$ a family of cardinal numbers and $G_0 = \{g \in G \mid m_g \neq 0\}$.*

1. *The following statements are equivalent:*
   (a) *There exists a Krull monoid H and a group isomorphism $\Phi \colon G \to \mathcal{C}(H)$ such that $\operatorname{card}(P \cap \Phi(g)) = m_g$ for every $g \in G$.*
   (b) *$G = [G_0]$, and $G = [G_0 \setminus \{g\}]$ for every $g \in G_0$ with $m_g = 1$.*
2. *Two Krull monoids H and H' have the same characteristic if and only if $H_{\mathrm{red}} \cong H'_{\mathrm{red}}$.*

Apart from the above abstract realization result, there are many concrete and naturally occurring instances of Krull monoids having infinite cyclic class group. We list a few such specific examples below.

Examples 3.5. 1. Domains. A domain R is a Krull domain if and only if its multiplicative monoid of nonzero elements is a Krull monoid. As a special case of Claborn's Realization Theorem, there is the following result: For every subset $G_0 \subset \mathbb{Z}$ with $[G_0] = \mathbb{Z}$, there is a Dedekind domain R and an isomorphism $\Phi \colon G \to \mathcal{C}(R)$ such that $\Phi(G_0) = \{g \in \mathcal{C}(R) \mid g \cap \mathfrak{X}(R) \neq \emptyset\}$ ([28, Theorem 3.7.8]. More results of this flavor are discussed in [28, Section 3.7] and [27, Section 5].

Let R be a domain and H a monoid such that the monoid domain $R[H]$ is a Krull domain. There are a variety of results on the class group of $R[H]$, which provide many explicit monoid domains having infinite cyclic class group ([32, Section 16], see also [40]). Generalized power series domains that are Krull are studied in [42].

2. Zero-sum sequences. Let $G_0 \subset \mathbb{Z}$ be a subset such that $[G_0 \setminus \{g\}] = \mathbb{Z}$ for all $g \in G_0$. Then the monoid of zero-sum sequences $\mathcal{B}(G_0)$ is a Krull monoid with class group isomorphic to $\mathbb{Z}$, and G_0 corresponds to the set of classes containing prime divisors [28, Proposition 2.5.6].

3. Module theory. Let R be a (not necessarily commutative) ring and $\mathcal{C}$ a class of (right) R-modules – closed under finite direct sums, direct summands and isomorphisms – such that $\mathcal{C}$ has a set $V(\mathcal{C})$ of representatives (that is, every module $M \in \mathcal{C}$ is isomorphic to a unique $[M] \in V(\mathcal{C})$). Then $V(\mathcal{C})$ becomes a commutative semigroup under the operation $[M] + [N] = [M \oplus N]$, which carries detailed information about the direct-sum behavior of modules in $\mathcal{C}$, e.g., whether or not the Krull–Remak–Azumaya–Schmidt Theorem holds, and, when it does not, how badly it fails. If every module $M \in \mathcal{C}$ has a semilocal endomorphism ring, then $\mathcal{V}(C)$ is a Krull monoid [10]. For situations where this condition is satisfied and when the class group of $\mathcal{V}(C)$ is cyclic, we refer to recent work of Facchini, Hassler, Wiegand et al. (see, for example, [46,12,11,13]).

4. Diophantine monoids. A Diophantine monoid is a monoid which consists of the set of solutions in nonnegative integers to a system of linear Diophantine equations. In more technical terms, if $m, n \in \mathbb{N}$ and $A \in M_{m,n}(\mathbb{Z})$, then $H = \{\boldsymbol{x} \in \mathbb{N}_0^n \mid A\boldsymbol{x} = \boldsymbol{0}\}$ is a Diophantine monoid. Moreover, H is a Krull monoid, and if $m = 1$, then its class group is cyclic and there is a characterization of when it is infinite ([7, Theorem 1.3], [8, Proposition 4.3]; see also [28, Theorem 2.7.14] and [33, Chapter II.8]).

4. Arithmetical properties equivalent to the finiteness of G_P^+ or G_P^-

Before we formulate our main characterization result, Theorem 4.2, we recall a recent characterization of tameness, which is in contrast with our present results. Let H be an atomic monoid. For an element $b \in H$, let $\omega(H, b)$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

For all $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in H$, if $b \mid a_1 \cdot \ldots \cdot a_n$, then there exists a subset $\Omega \subset [1, n]$ such that $|\Omega| \le N$ and

$$b \,\Big|\, \prod_{\nu \in \Omega} a_\nu .$$

Clearly, $b \in H$ is a prime if and only if $\omega(H, b) = 1$, and so the $\omega(H, \cdot)$ values measure how far away atoms are from being primes. The invariant $\omega(H, \cdot)$ is closely related to the local tame degrees $\mathsf t(H, \cdot)$. A detailed study of their relationship can be found in [30, Section 3], but here we mention only two simple facts (to simplify the formulation, we suppose that H is reduced):

- $\omega(H, u) \le \mathsf t(H, u)$ for all $\mathbf 1 \ne u \in H$ which are not prime (this follows from the definition).
- $\sup\{\mathsf t(H, u) \mid u \in \mathcal A(H)\} < \infty$ if and only if $\sup\{\omega(H, u) \mid u \in \mathcal A(H)\} < \infty$ [31, Proposition 3.5].

The monoid H is said to be *tame* if the above suprema are finite. Note that the finiteness in Proposition 4.1.1 holds without any assumption on G_P. Indeed, it holds for all v-noetherian monoids [30, Theorem 4.2]. In particular, one should compare Proposition 4.1.1, 4.1.2(c) and Theorem 4.2(b).

Proposition 4.1. *Let H be a Krull monoid and $\varphi\colon H \to \mathcal F(P)$ a cofinal divisor homomorphism into a free monoid such that the class group $G = \mathcal C(\varphi)$ is an infinite cyclic group that we identify with $\mathbb Z$. Let $G_P \subset G$ denote the set of classes containing prime divisors.*

1. *$\omega(H, u) < \infty$ for all $u \in \mathcal A(H)$.*
2. *If φ is a divisor theory, then the following statements are equivalent:*
   (a) *G_P is finite.*
   (b) *$\mathsf D(G_P) < \infty$.*
   (c) *H is tame.*

The equivalence of the three properties is a special case of [31, Theorem 4.2]. It is essential that the imbedding is a divisor theory and not only a cofinal divisor homomorphism. Indeed, if $G_0 = \{-1\} \cup \mathbb N$, then $\mathcal B(G_0) \hookrightarrow \mathcal F(G_0)$ is a cofinal divisor homomorphism and $\mathsf D(G_0) = \infty$, yet $\mathcal B(G_0)$ is factorial and hence tame (see also Lemmas 3.4 and 5.3).

Theorem 4.2. *Let H be a Krull monoid and $\varphi\colon H \to \mathcal F(P)$ a cofinal divisor homomorphism into a free monoid such that the class group $G = \mathcal C(\varphi)$ is an infinite cyclic group that we identify with $\mathbb Z$. Let $G_P \subset G$ denote the set of classes containing prime divisors. The following statements are equivalent:*

(a) *G_P^+ or G_P^- is finite.*
(b) *H is locally tame, i.e., $\mathsf t(H, u) < \infty$ for all $u \in \mathcal A(H_{\mathrm{red}})$.*
(c) *The catenary degree $\mathsf c(H)$ is finite.*
(d) *The set of distances $\Delta(H)$ is finite.*
(e) *The elasticity $\rho(H)$ is a rational number.*
(f) *$\rho_2(H)$ is finite.*
(g) *There exists some $M \in \mathbb N$ such that, for each $k \in \mathbb N$, we have $\rho_{k+1}(H) - \rho_k(H) \le M$.*
(h) *There exists some $M \in \mathbb N$ such that, for each $k \in \mathbb N$, the set $\mathcal V_k(H)$ is an* AAP *with difference* $\min \Delta(H)$ *and bound M.*

We point out the crucial implications in the above result. Suppose that (a) holds. Then (b), (c), (e), (g) and (h) are strong statements on the arithmetic of H. The conditions (d) and (f) are very weak arithmetical statements (indeed, the implications (e) ⇒ (f), (g) ⇒ (f) and (h) ⇒ (f) hold trivially in any atomic monoid). The crucial point is that (d) and (f) both imply (a). In [1], it was first proved that (in the setting of Krull domains) (a) is equivalent to the finiteness of the elasticity $\rho(H)$, and the problem was put forward whether or not $\rho(H)$ would always be rational; part (e) shows that this is indeed so. In [2], it was recently shown that (a) is equivalent to (c) as well as to (d) (also in the setting of Krull monoids). We will give a complete proof of all implications, not only because our setting is slightly more general – being valid for any divisor *homomorphism* rather than divisor *theory* – but also because we need all the required tools regardless (in particular, for the monotone catenary degree in Section 5), and thus little could be saved by not doing so.

Note, if the equivalent conditions of Theorem 4.2 hold, then [21, Theorem 4.2] implies that

$$\lim_{k\to\infty} \frac{|\mathcal V_k(H)|}{k} = \frac{1}{\min \Delta(H)}\Big(\rho(H) - \frac{1}{\rho(H)}\Big).$$

Under a certain additional assumption, the sets $\mathcal V_k(H)$ are even arithmetical progressions and not only AAPs [18, Theorem 3.1]; for more on the sets $\mathcal V_k(H)$, see [25, Theorem 3.1.3].

As mentioned in the introduction, there are characterizations of arithmetical properties in various algebraic settings. In most of them, the finiteness of the elasticity is equivalent to the finiteness of all $\rho_k(H)$ (though this does not hold in all atomic monoids). But in none of these settings is the finiteness of the elasticity equivalent to the finiteness of the catenary degree. The reader may want to compare Proposition 4.1 and Theorem 4.2 with [28, Corollary 3.7.2], [38, Theorem 4.5] or [30, Theorem 4.4].

The remainder of this section is devoted to the proof of Theorem 4.2. We start with the necessary preparations.

Lemma 4.3. *Let $G_0 \subset \mathbb{Z}$ be a condensed subset. Then*

$$|U^+| \le |\inf G_0| \quad \text{for each atom } U \in \mathcal{A}(G_0).$$

If in particular G_0 is finite, then $\mathsf{D}(G_0) \le \max G_0 + |\min G_0|$.

Proof. This is due to Lambert (see [43]); for a proof in the present terminology, see [2, Theorem 3.2]. □

Lemma 4.4. *Let $G_0 \subset \mathbb{Z}$ be a condensed subset such that G_0^+ is infinite. For each $S \in \mathcal{F}(G_0^-)$, there exists some $U \in \mathcal{A}(G_0)$ with $S \mid U$.*

Proof. Let $d = \gcd(G_0^-)$. Then $[G_0^-] \subset -d\mathbb{N}$ and there exists some $g \in \mathbb{N}$ such that $-gd - d\mathbb{N} \subset [G_0^-]$. Since G_0^+ is infinite, let $b \in G_0^+$ with $b > |\sigma(S)| + gd$, and let $\beta \in [1, d]$ be minimal such that $\beta b \in d\mathbb{N}$. By the definition of g, there exists some $S' \in \mathcal{F}(G_0^-)$ such that $\sigma(S') = -(\beta b - |\sigma(S)|) = -(\beta b + \sigma(S))$. Thus, $\sigma(b^\beta S S') = 0$ and, by the minimality of β, it follows that $b^\beta S S'$ is an atom. □

The next lemma uses ideas from the proof of Theorem 3.1 in [2]. It will be used for the investigation of the catenary degree as well as for the monotone catenary degree (Proposition 5.8).

Lemma 4.5. *Let $G_0 \subset \mathbb{Z}$ be a condensed subset such that G_0^- is finite and nonempty. Let $A \in \mathcal{B}(G_0)$ be nontrivial and $z, \bar{z} \in \mathsf{Z}(A)$ with $|z| \le |\bar{z}|$. Then there exists a $U \in \mathcal{A}(G_0)$ with $U \,|\, \bar{z}$ and a factorization $\widehat{z} \in \mathsf{Z}(A) \cap U\mathsf{Z}(G_0)$ such that $\mathsf{d}(z, \widehat{z}) \le \big(|\min G_0| + |G_0^-|^2\big)\, |\min G_0|$.*

Proof. Let $\bar{z} = U_1 \cdot \ldots \cdot U_m$ and $z = V_1 \cdot \ldots \cdot V_l$ where $l, m \in \mathbb{N}$ and $U_1, \ldots, U_m, V_1, \ldots, V_l \in \mathcal{A}(G_0)$. We proceed in two steps. Note that we may assume $0 \nmid A$, else the lemma is trivial taking $U = 0$ and $\hat{z} = z$.

1. We assert that there is an $i \in [1, m]$ and a set $I \subset [1, l]$ such that

$$|I| \le |\min G_0| + |G_0^-|^2 \quad \text{and} \quad U_i \,\Big|\, \prod_{\nu \in I} V_\nu .$$

We assume $l > |G_0^-|$, since otherwise the claim is obvious. Since

$$\sum_{i=1}^m \max\Big\{ \frac{\mathsf{v}_g(U_i)}{\mathsf{v}_g(A)} \mid g \in G_0^- \Big\} \le \sum_{i=1}^m \sum_{g \in G_0^-} \frac{\mathsf{v}_g(U_i)}{\mathsf{v}_g(A)} = \sum_{g \in G_0^-} \Big(\frac{1}{\mathsf{v}_g(A)} \sum_{i=1}^m \mathsf{v}_g(U_i) \Big) = |G_0^-|,$$

there exists an $i \in [1, m]$ such that

$$\frac{\mathsf{v}_g(U_i)}{\mathsf{v}_g(A)} \le \frac{|G_0^-|}{m}. \tag{4.1}$$

For each $g \in G_0^-$, there is an $I_g \subset [1, l]$ with $|I_g| = |G_0^-|$ such that

$$\mathsf{v}_g\Big(\prod_{\nu \in I_g} V_\nu \Big) \ge \frac{|G_0^-| \mathsf{v}_g(A)}{l}.$$

Hence, since $l \le m$, it follows by (4.1) that

$$\mathsf{v}_g\Big(\prod_{\nu \in I_g} V_\nu \Big) \ge \frac{|G_0^-| \mathsf{v}_g(A)}{l} \ge \frac{m \mathsf{v}_g(U_i)}{\mathsf{v}_g(A)} \frac{\mathsf{v}_g(A)}{l} \ge \mathsf{v}_g(U_i).$$

Since by Lemma 4.3 we have $|U_i^+| \le |\min G_0|$, there is an $I_0 \subset [1, l]$ with $|I_0| \le |\min G_0|$ such that

$$\mathsf{v}_g(U_i) \le \mathsf{v}_g\Big(\prod_{\nu \in I_0} V_\nu \Big) \quad \text{for all } g \in G_0^+.$$

Then, for $I = I_0 \cup \bigcup_{g \in G_0^-} I_g$, we get $\mathsf{v}_g(U_i) \le \mathsf{v}_g\big(\prod_{\nu \in I} V_\nu\big)$ for each $g \in G_0$, i.e., $U_i \mid \prod_{\nu \in I} V_\nu$. Noting that $|I| \le |\min G_0| + |G_0^-|^2$, the argument is complete.

2. By part 1, we may suppose without restriction that $U_1 \mid \prod_{\nu=1}^{k} V_\nu$ with $k \le \big(|\min G_0| + |G_0^-|^2\big)$. We consider a factorization $V_1 \cdot \ldots \cdot V_k = W_1 W_2 \cdot \ldots \cdot W_n$, where $U_1 = W_1, W_2, \ldots, W_n \in \mathcal{A}(G_0)$, and by Lemma 4.3,

$$\begin{aligned} n &\le |(W_1 \cdot \ldots \cdot W_n)^+| = |(V_1 \cdot \ldots \cdot V_k)^+| \\ &\le k\,|\min G_0| \le \big(|\min G_0| + |G_0^-|^2\big)\,|\min G_0|. \end{aligned}$$

Now we set $\widehat{z} = W_1 \cdot \ldots \cdot W_n V_{k+1} \cdot \ldots \cdot V_l$ and get

$$\mathsf{d}(z, \widehat{z}) \le \max\{k, n\} \le \big(|\min G_0| + |G_0^-|^2\big)\,|\min G_0|. \quad \square$$

Lemma 4.6. *Let $G_0 \subset \mathbb{Z}$ be a condensed set such that G_0^- is finite and nonempty.*

1. *There exists some $M \in \mathbb{N}$ such that $\rho_{k+1}(G) \le 1 + kM$ for each $k \in \mathbb{N}_0$. More precisely,*
   (a) *if G_0 is infinite, then for each $k \in \mathbb{N}$,*
   $$1 \le \rho_{k+1}(G_0) - \rho_k(G_0) \le 2\,|\min G_0|.$$
   (b) *if G_0 is finite, then for each $k \in \mathbb{N}$,*
   $$1 \le \rho_{k+1}(G_0) - \rho_k(G_0) \le \mathsf{D}(G_0) - 1.$$
2. *For each $k \in \mathbb{N}$,*
   $$-1 \le \lambda_k(G_0) - \lambda_{k+1}(G_0) < \big(|\min G_0| + |G_0^-|^2\big)\,|\min G_0|.$$

Proof. 1. We recall that $\rho_1(G_0) = 1$. Let $k \in \mathbb{N}$ and $U_1, \ldots, U_k, V_1, \ldots, V_l \in \mathcal{A}(G_0)$ with $U_1 \cdot \ldots \cdot U_k = V_1 \cdot \ldots \cdot V_l$. By Lemma 4.3, it follows that $l \le |(U_1 \cdot \ldots \cdot U_k)^+| = \sum_{i=1}^{k} |U_i^+| \le k \cdot |\min G_0^-|$, and thus $\rho_k(G_0) \le k \cdot |\min G_0^-| < \infty$.
1(a) The left inequality is trivial and it remains to verify the right inequality. Let $m = |\min G_0|$. Let $l \in \mathbb{N}$, and let $A_1, \ldots, A_{k+1}, U_1, \ldots, U_l \in \mathcal{A}(G_0)$ be such that

$$A_1 \cdot \ldots \cdot A_{k+1} = U_1 \cdot \ldots \cdot U_l\,.$$

We claim that $l \le \rho_k(G_0) + 2m$; then we have $\rho_{2k+1}(G_0) \le \rho_k(G_0) + 2m$. By Lemma 4.3, we know that $|A^+| \le m$ for each $A \in \mathcal{A}(G_0)$. Thus, we may assume that $(A_k A_{k+1})^+ \mid U_1 \cdot \ldots \cdot U_{2m}$. Then $(\prod_{j=2m+1}^{l} U_j)^+ \mid \prod_{i=1}^{k-1} A_i$. Let $S = (\prod_{j=2m+1}^{l} U_j)^-$. By Lemma 4.4, there exists some $A_k' \in \mathcal{A}(G_P)$ with $S \mid A_k'$. We consider $B = (\prod_{i=1}^{k-1} A_i)A_k'$, which is a product of k atoms. We observe that $\prod_{j=2m+1}^{l} U_j \mid B$. Thus, $\max \mathsf{L}(B) \ge l - 2m$, establishing the claim.
1(b) This follows from [31, Proposition 3.6] (see also Lemma 4.3 in that paper and note that $\mathsf{D}(G_0) \ge 2$).
2. The left inequality is trivial and it remains to verify the right inequality. Let $s = \lambda_{k+1}(G_0)$ and let $U_1, \ldots, U_s, A_1, \ldots, A_{k+1} \in \mathcal{A}(G_0)$ be such that

$$U_1 \cdot \ldots \cdot U_s = A_1 \cdot \ldots \cdot A_{k+1}.$$

After renumbering if necessary, Lemma 4.5 implies that $A_1 \mid U_1 \cdot \ldots \cdot U_j$ and $U_1 \cdot \ldots \cdot U_j = A_1 W_2 \cdot \ldots \cdot W_i$ with $W_1, \ldots, W_i \in \mathcal{A}(G_0)$ and $i \le \big(|\min G_0| + |G_0^-|^2\big)|\min G_0| = M_2$ (note that in order to apply Lemma 4.5, we used that $s \le k+1$). Then

$$W_2 \cdot \ldots \cdot W_i U_{j+1} \cdot \ldots \cdot U_s = A_2 \cdot \ldots \cdot A_{k+1},$$

and hence

$$\begin{aligned} \lambda_k(G_0) &\le \min \mathsf{L}(A_2 \cdot \ldots \cdot A_{k+1}) \le \min \mathsf{L}(U_{j+1} \cdot \ldots \cdot U_s) + \min \mathsf{L}(W_2 \cdot \ldots \cdot W_i) \\ &\le s - j + i - 1 \le \lambda_{k+1}(G_0) + (M_2 - 1). \quad \square \end{aligned}$$

We continue with a lemma that is used when investigating the sets of distances and local tameness. To simplify the formulation, we introduce the following notation. For $a \in -\mathbb{N}$ and $b \in \mathbb{N}$, let $V_{a,b}$ denote the unique atom with support $\{a, b\}$, that is $V_{a,b} = a^\alpha b^\beta$ with $\alpha = \mathrm{lcm}(a, b)/|a|$ and $\beta = \mathrm{lcm}(a, b)/b$.

Lemma 4.7. *Let $G_0 \subset \mathbb{Z}$ and let $v \in \mathbb{N}$. Suppose there exist distinct $a, a_2 \in G_0^-$ and $b, b_1 \in G_0^+$ that satisfy $b_1 \ge b|a|$ and $|a_2| \ge (vb_1 + b)|a|$. For a given $z \in \mathsf{Z}((V_{a,b_1} V_{a_2,b})^v)$, let z_0 be the (unique) minimal divisor of z such that $\mathsf{v}_{a_2}(\pi(z_0^{-1} z)) = 0$, and let $t(z) = \mathsf{v}_{b_1}(\pi(z_0))$. Then,*

$$|z| \in \left[\frac{b_1}{\mathrm{lcm}(a, b)}\, t(z) - D,\ \frac{b_1}{\mathrm{lcm}(a, b)}\, t(z) + D\right] \quad \text{where } D = v(b + |a|)\gcd(a, b).$$

Moreover, if $t(z) = 0$, then $z = V_{a,b_1}^{v} \cdot V_{a_2,b}^{v}$.

Since it is relevant in applications of this lemma, we point out that D depends neither on a_2 nor on b_1.

Proof. To simplify notation without suppressing the information on the origin of certain quantities, we set $\alpha = \mathsf{v}_a(V_{a,b})$, $\alpha_1 = \mathsf{v}_a(V_{a,b_1})$, and $\alpha_2 = \mathsf{v}_{a_2}(V_{a_2,b})$. Likewise, we set $\beta = \mathsf{v}_b(V_{a,b})$, $\beta_1 = \mathsf{v}_{b_1}(V_{a,b_1})$, and $\beta_2 = \mathsf{v}_b(V_{a_2,b})$.
From the explicit descriptions given or by applying Lemma 4.3, we get $\beta, \beta_1 \in [1, |a|]$ and $\alpha, \alpha_2 \in [1, b]$.

Let $z = U_1 \cdot \ldots \cdot U_m$, where $U_1, \ldots, U_m \in \mathcal{A}(G_0)$, and $k, l \in [1, m]$ with $k \le l$ be such that

- $a_2 \mid U_\nu$ for each $\nu \in [1, k]$,
- $a_2 \nmid U_\nu$ and $b_1 \mid U_\nu$ for each $\nu \in [k+1, l]$, and
- $a_2 \nmid U_\nu$ and $b_1 \nmid U_\nu$ for each $\nu \in [l+1, m]$;

in particular, $z_0 = U_1 \cdot \ldots \cdot U_k \in \mathsf{Z}(G_0)$. Also note that $U_\nu = V_{a,b}$ for each $\nu \in [l+1, m]$.

For $\nu \in [1, k]$, we have

$$U_\nu = a_2^{\alpha_{\nu,2}} a^{\alpha_{\nu,1}} b_1^{\beta_{\nu,1}} b^{\beta_{\nu,2}},$$

where $\alpha_{\nu,2} \in \mathbb{N}$ and $\alpha_{\nu,1}, \beta_{\nu,1}, \beta_{\nu,2} \in \mathbb{N}_0$. By the assumption on $|a_2|$ and since $\beta, \beta_1 \in [1, |a|]$, we have $|a_2| \ge v\beta_1 b_1 + \beta b$. Thus, in view of $\mathsf{v}_{b_1}(\pi(z)) = \beta_1 v$, it follows that $\beta_{\nu,2} \ge \beta$. Hence $\alpha_{\nu,1} \le \alpha - 1$, since otherwise $V_{a,b} \mid U_\nu$, which is impossible (as $a_2 | U_\nu$).

Let $\alpha_2' = \mathsf{v}_a(\pi(z_0))$ and $\beta_2' = \mathsf{v}_b(\pi(z_0))$. In view of $\alpha_{\nu,1} \le \alpha - 1$, $k \le v\alpha_2$ and $\alpha, \alpha_2 \in [1, b]$, we have $0 \le \alpha_2' \le vb^2$.

We note that $\sigma(\pi(z_0)^-) = v\alpha_2 a_2 + \alpha_2' a$, and thus

$$t(z)b_1 + \beta_2' b = v\alpha_2|a_2| + \alpha_2'|a|,$$

i.e., $\beta_2' = b^{-1}(v\alpha_2|a_2| + \alpha_2'|a| - t(z)b_1)$. In particular, note that if $t(z) = 0$, then, since

$$\sigma(b^{\mathsf{v}_b(\pi(z))}) = v \cdot \sigma(b^{\mathsf{v}_b(V_{a_2,b})}) = -v \cdot \sigma(a_2^{\mathsf{v}_{a_2}(V_{a_2,b})})$$

implies $\mathsf{v}_b((V_{a,b_1}V_{a_2,b})^v) = b^{-1}(v\alpha_2|a_2|)$, it follows that $\alpha_2' = 0$ and $z_0 = V_{a_2,b}^v$; this establishes the "moreover"-statement.

Consequently,

$$b^{-1}(v\alpha_2|a_2| - t(z)b_1) \le \beta_2' \le b^{-1}(v\alpha_2|a_2| + vb^2|a| - t(z)b_1). \tag{4.2}$$

For $\nu \in [k+1, l]$, we have

$$U_\nu = b_1^{\beta''_{\nu,1}} b^{\beta''_{\nu,2}} a^{\alpha''_{\nu,1}},$$

with $\beta''_{\nu,1} \in \mathbb{N}$ and $\alpha''_{\nu,1}, \beta''_{\nu,2} \in \mathbb{N}_0$. We have $\alpha''_{\nu,1}|a| \ge b_1$. Thus, by the assumption on b_1 and since $\alpha \in [1, b]$, we get $\alpha''_{\nu,1} \ge \alpha$, and hence $\beta''_{\nu,2} \le \beta - 1$ (as otherwise $U_\nu = V_{a,b}$ with $b_1 | U_\nu$ but $b_1 \nmid V_{a,b}$, a contradiction).

Let $\beta_2'' = \mathsf{v}_b(\prod_{\nu=k+1}^{l} U_\nu)$. We note that $l - k \le \mathsf{v}_{b_1}((V_{a,b_1}V_{a_2,b})^v) - t(z) = v\beta_1 - t(z) \le v|a| - t(z) \le v|a|$. Thus, we obtain that

$$0 \le \beta_2'' \le (l-k)(\beta - 1) \le v|a|(\beta - 1) \le v|a|^2. \tag{4.3}$$

Let $\beta_2''' = \mathsf{v}_b(\prod_{\nu=l+1}^{m} U_\nu)$. We have

$$\beta_2''' = \mathsf{v}_b((V_{a,b_1}V_{a_2,b})^v) - \beta_2' - \beta_2'' = v\beta_2 - \beta_2' - \beta_2''.$$

In combination with (4.2) and (4.3), we get that

$$v\beta_2 - b^{-1}\big(v\alpha_2|a_2| + vb^2|a| - t(z)b_1\big) - v|a|^2 \le \beta_2''' \le v\beta_2 - b^{-1}\big(v\alpha_2|a_2| - t(z)b_1\big).$$

Thus, since $\beta_2 = b^{-1}\alpha_2|a_2|$ (in view of $V_{a_2,b} = a_2^{\alpha_2} b^{\beta_2}$), it follows that

$$\beta_2''' \in \frac{b_1}{b} t(z) + [-vb|a| - v|a|^2, 0]. \tag{4.4}$$

Since $U_\nu = V_{a,b}$ for each $\nu \in [l+1, m]$, it follows that $\beta_2''' = (m-l)\beta$. Since $k \in [0, vb]$ and $l - k \in [0, v|a|]$, we get that $m \in (m-l) + [0, v(b + |a|)]$. Combining with $\beta_2''' = (m-l)\beta$ and (4.4) then yields

$$m \in \left[\frac{b_1}{b\beta} t(z) - \frac{vb|a| + v|a|^2}{\beta}, \frac{b_1}{b\beta} t(z) + v(b + |a|)\right],$$

and, since $\beta \le |a|$, we have $v(b + |a|) \le v(b + |a|)|a|/\beta$. Substituting the explicit value of β, the claim follows. □

The following proposition is a major portion of Theorem 4.2.

Proposition 4.8. *Let $G_0 \subset \mathbb{Z}$ be a condensed set such that G_0^- is finite and nonempty. Then $\rho(G_0)$ is a rational number.*

To prove this result, we need the concept of factorizations with respect to a (not necessarily minimal) generating set. This idea is also used in the recent paper [6], where a generalized set of distances is studied for numerical monoids.

Let H be a monoid and $S \subset H_{\text{red}} \setminus \{\mathbf{1}\}$ a subset. We call $\mathsf{Z}^S(H) = \mathcal{F}(S)$ the factorization monoid of H with respect to S. The homomorphism $\pi_H^S = \pi^S \colon \mathsf{Z}^S(H) \to H_{\text{red}}$ defined by $\pi^S(z) = \prod_{u \in S} u^{\mathsf{v}_u(z)}$ is called the factorization homomorphism of H with respect to S. For $a \in H$, we set $\mathsf{Z}_H^S(a) = \mathsf{Z}^S(a) = (\pi^S)^{-1}(aH^\times)$; we call this the set of factorizations in S of a. The set $\mathsf{L}^S(a) = \{|z| \mid z \in \mathsf{Z}^S(a)\}$ is called the set of lengths of a with respect to S.

We note that $\mathsf{Z}^S(a) \neq \emptyset$ for each $a \in H$ if and only if S generates H_{red} (as a monoid). If S generates H_{red}, then $\mathcal{A}(H_{\text{red}}) \subset S$ by [28, Proposition 1.1.7]. If $S = \mathcal{A}(H_{\text{red}})$, then $\mathsf{Z}^S(a) = \mathsf{Z}(a)$, and all other notions coincide with the usual ones. Suppose that $S \subset H_{\text{red}}$ is a generating set. For $a \in H$, let $\rho^S(a) = \rho(\mathsf{L}^S(a))$ denote the elasticity of a with respect to S, and $\rho^S(H) = \sup\{\rho^S(a) \mid a \in H\}$ the elasticity of H with respect to S; note that $0 \in \mathsf{L}^S(a)$ if and only if $\mathsf{L}^S(a) = \{0\}$, i.e., $a \in H^\times$. We say that the elasticity of H with respect to S is accepted if there exists some $a \in H$ with $\rho^S(a) = \rho^S(H)$.

The proof of the following result is a direct modification of the one for the (usual) elasticity of finitely generated monoids ([28, Theorem 3.1.4]) and contains it as the special case $S = \mathcal{A}(H_{\text{red}})$.

Lemma 4.9. *Let H be a monoid and $S \subset H_{\text{red}} \setminus \{\mathbf{1}\}$ a finite generating set of H_{red}. Then $\rho^S(H)$ is finite, accepted and, in particular, rational.*

Proof. By construction, $\mathsf{Z}^S(H) \times \mathsf{Z}^S(H)$ is a finitely generated free monoid. Obviously, $Z = \{(x, y) \in \mathsf{Z}^S(H) \times \mathsf{Z}^S(H) \mid \pi^S(x) = \pi^S(y)\}$ is a saturated submonoid, thus finitely generated by [28, Proposition 2.7.5]. Let $Z^\bullet = Z \setminus Z^\times$; clearly $|Z^\times| = 1$ and, for each $(x, y) \in Z^\bullet$, we have that both $|x| \neq 0$ and $|y| \neq 0$. We note that $\rho^S(H) = \sup\{|x|/|y| \mid (x, y) \in Z^\bullet\}$. We assert that $\sup\{|x|/|y| \mid (x, y) \in Z^\bullet\} = \sup\{|x|/|y| \mid (x, y) \in \mathcal{A}(Z)\}$. Since $\mathcal{A}(Z)$ is finite, this implies the result.

Let $s = (x_s, y_s) \in Z^\bullet$ and let $s = t_1 \cdot \ldots \cdot t_l$ with $t_i = (x_i, y_i) \in \mathcal{A}(Z)$ be a factorization of s in the monoid Z. We have, using the standard inequality for the mediant,

$$\frac{|x_s|}{|y_s|} = \frac{\sum_{i=1}^{l} |x_i|}{\sum_{i=1}^{l} |y_i|} \leq \max\left\{\frac{|x_i|}{|y_i|} \mid i \in [1, l]\right\},$$

showing that $\sup\{|x|/|y| \mid (x, y) \in Z^\bullet\} \leq \sup\{|x|/|y| \mid (x, y) \in \mathcal{A}(Z)\}$. The other inequality being trivial, the claim follows. □

For a condensed set $G_0 \subset \mathbb{Z}$ with $|G_0| \geq 2$, we define

$$\mathcal{B}(G_0)^+ = \{B^+ \mid B \in \mathcal{B}(G_0)\} \text{ and } \mathcal{A}(G_0)^+ = \{A^+ \mid A \in \mathcal{A}(G_0)\}.$$

Lemma 4.10. *Let $G_0 \subset \mathbb{Z}$ be a condensed set with $|G_0| \geq 2$.*

1. *$\mathcal{B}(G_0)^+ \subset \mathcal{F}(G_0^+)$ is a submonoid.*
2. *$\mathcal{A}(G_0)^+$ is a generating set of $\mathcal{B}(G_0)^+$.*
3. *$|F| \leq |\inf G_0^-|$ for each $F \in \mathcal{A}(G_0)^+$.*

Proof. The first two claims are immediate, and the last one is a direct consequence of Lemma 4.3. □

Clearly, $\mathcal{A}(G_0)^+$ contains $\mathcal{A}(\mathcal{B}(G_0)^+)$, the set of atoms of $\mathcal{B}(G_0)^+$, yet it is in general not equal to this set. By definition, we have that $F \in \mathcal{A}(G_0)^+$ if and only if there exists some $A \in \mathcal{A}(G_0)$ such that $F = A^+$, yet $F \in \mathcal{A}(\mathcal{B}(G_0)^+)$ if and only if we have $B \in \mathcal{A}(G_0)$ for each $B \in \mathcal{B}(G_0)$ with $F = B^+$. Moreover, $\mathcal{B}(G_0)^+$ is in general not a saturated submonoid of $\mathcal{F}(G_0^+)$.

The following technical result is used to partition $\mathcal{A}(G_0)$ into finitely many classes.

Lemma 4.11. *Let $G_0 \subset \mathbb{Z}$ be a condensed set such that G_0^- is finite and nonempty. Let $F \in \mathcal{F}(G_0^+)$, $g \in \operatorname{supp}(F)$ with $g \geq |G_0^-|\,|\min G_0^-|\, \operatorname{lcm}(G_0^-)$, and let $g' = g + k \operatorname{lcm}(G_0^-) \in G_0^+$ where $k \in \mathbb{N}$. Then $F \in \mathcal{A}(G_0)^+$ if and only if $g' g^{-1} F \in \mathcal{A}(G_0)^+$.*

Proof. We set $T = g' g^{-1} F \in \mathcal{F}(G_0^+)$. Suppose $F \in \mathcal{A}(G_0)^+$. Let $R \in \mathcal{F}(G_0^-)$ such that $FR \in \mathcal{A}(G_0)$. Since $\sigma(F) \geq g \geq |G_0^-|\,|\min G_0^-|\, \operatorname{lcm}(G_0^-)$, there exists some $a \in G_0^-$ such that $\mathsf{v}_a(R) \geq \operatorname{lcm}(G_0^-)$. Let $R_1 = R a^{k \operatorname{lcm}(G_0^-)/|a|}$. Then $TR_1 \in \mathcal{B}(G_0)$. Assume to the contrary that TR_1 is not an atom, say $TR_1 = (T'R_1')(T''R_1'')$, where $g' \mid T'$, $T = T'T''$ and $R_1 = R_1'R_1''$. Let $l' \in \mathbb{N}_0$ be maximal such that $a^{l' \operatorname{lcm}(G_0^-)/|a|} \mid R_1'$ and let $l = \min\{l', k\}$. We note that $a^{-l \operatorname{lcm}(G_0^-)/|a|} R_1' \mid R$. Moreover, since

$$\begin{aligned} |\sigma(a^{-l \operatorname{lcm}(G_0^-)/|a|} R_1')| &\geq g' - l \operatorname{lcm}(G_0^-) \geq (k - l) \operatorname{lcm}(G_0^-) + |G_0^-|\,|\min G_0^-|\, \operatorname{lcm}(G_0^-) \\ &\geq (k - l) \cdot \operatorname{lcm}(G_0^-) + \sum_{x \in G_0^-} |x| \left(\frac{\operatorname{lcm}(G_0^-)}{|x|} - 1\right), \end{aligned}$$

there exists a subsequence $R_2' \mid a^{-l \operatorname{lcm}(G_0^-)/|a|} R_1'$ such that $\sigma(R_2') = -(k - l) \operatorname{lcm}(G_0^-)$. We set $R_0 = R_2'^{-1} a^{-l \operatorname{lcm}(G_0^-)/|a|} R_1'$. Then $\sigma(R_0) = \sigma(R_1') + k \operatorname{lcm}(G_0^-)$. Thus $\sigma(g g'^{-1} T' R_0) = 0$, yet $g g'^{-1} T' R_0 \mid FR$, contradicting that TR_1 is not an atom.

Suppose $T \in \mathcal{A}(G_0)^+$. Let $R' \in \mathcal{F}(G_0^-)$ be such that $TR' \in \mathcal{A}(G_0)$. Since

$$-\sigma(R_1) = \sigma(T) \geq g' \geq k \cdot \operatorname{lcm}(G_0^-) + |G_0^-|\,|\min G_0^-|\, \operatorname{lcm}(G_0^-) \geq k \cdot \operatorname{lcm}(G_0^-) + \sum_{x \in G_0^-} |x| \left(\frac{\operatorname{lcm}(G_0^-)}{|x|} - 1\right),$$

there exists a subsequence $R_1' \mid R'$ with $\sigma(R_1') = -k \cdot \operatorname{lcm}(G_0^-)$. Let $R = R_1'^{-1} R'$. Then FR is a zero-sum sequence. Assume FR is not an atom, say $FR = (F'R_2')(F''R_2'')$, where $g \mid F'$, $F = F'F''$ and $R = R_2'R_2''$. Then $g' g^{-1} F' R_2' R_1' \mid TR'$ and it is a zero-sum sequence, contradicting that FR is not an atom. □

Let $G_0 \subset \mathbb{Z} \setminus \{0\}$ be a condensed set such that G_0^- is finite and nonempty. In view of Lemma 4.11, we introduce the following relation on G_0^+. For $g, h \in G_0^+$, we say that g is equivalent to h if $g = h$ or if $g, h \ge |G_0^-|\,|\min G_0^-|\ \mathrm{lcm}(G_0^-)$ and $g \equiv h \mod \mathrm{lcm}(G_0^-)$. This relation is an equivalence relation and it partitions G_0^+ into finitely many – namely, less than $|G_0^-|\,|\min G_0^-|\ \mathrm{lcm}(G_0^-) + \mathrm{lcm}(G_0^-)$ – equivalence classes; we denote the equivalence class of g by $\kappa(g)$ and also use κ to denote the extension of this map to $\mathcal{F}(G_0^+)$.

We note that $\kappa(\mathcal{A}(G_0)^+)$ is a finite set, since it consists of sequences over the finite set $\kappa(G_0^+)$ and the length of each sequence is at most $|\min G_0^-|$ by Lemma 4.10. Moreover, it is a generating set of the monoid $\kappa(\mathcal{B}(G_0)^+)$.

In order to study factorizations, we extend κ to $\mathsf{Z}(G_0)$ via

$$\kappa(A_1 \cdot \ldots \cdot A_l) = \kappa(A_1^+) \cdot \ldots \cdot \kappa(A_l^+).$$

This is an element of $\mathcal{F}(\kappa(\mathcal{A}(G_0)^+))$, i.e., $\mathsf{Z}^{\kappa(\mathcal{A}(G_0)^+)}(\kappa(\mathcal{B}(G_0)^+))$; for brevity, we denote this factorization monoid by Z^κ. Likewise, for $F \in \kappa(\mathcal{B}(G_0)^+)$, we denote $\mathsf{Z}^{\kappa(\mathcal{A}(G_0)^+)}(F)$ by $\mathsf{Z}^\kappa(F)$; $\pi^{\kappa(\mathcal{A}(G_0)^+)}$ by π^κ; and $\rho^{\kappa(\mathcal{A}(G_0)^+)}$ by ρ^κ. The homomorphism $\kappa : \mathsf{Z}(G_0) \to \mathsf{Z}^\kappa$ is epimorphic.

We note that, for $B \in \mathcal{B}(G_0)$, we have that $\kappa(\mathsf{Z}(B)) \subset (\pi^\kappa)^{-1}(\kappa(B^+))$, and in general, this is a proper inclusion. However, we have, for each $F \in \mathcal{B}(G_0)^+$, by Lemma 4.11,

$$(\pi^\kappa)^{-1}(\kappa(F)) = \bigcup_{B \in \mathcal{B}(G_0),\, B^+ = F} \kappa(\mathsf{Z}(B)), \tag{4.5}$$

whenever $G_0 \subset \mathbb{Z} \setminus \{0\}$ is condensed with G_0^- finite and nonempty.

Lemma 4.12. *Let $G_0 \subset \mathbb{Z} \setminus \{0\}$ be a condensed set such that G_0^- is finite and nonempty.*

1. *For each $B \in \mathcal{B}(G_0)$, we have $\rho(B) \le \rho^\kappa(\kappa(B^+))$. In particular, $\rho(G_0) \le \rho^\kappa(\kappa(\mathcal{B}(G_0)^+))$.*
2. *If G_0 is infinite, then $\rho(G_0) = \rho^\kappa(\kappa(\mathcal{B}(G_0)^+))$.*

Proof. 1. Let $B \in \mathcal{B}(G_0) \setminus \{1\}$, $x, y \in \mathsf{Z}(B)$ with $|x| = \max \mathsf{L}(B)$ and $|y| = \min \mathsf{L}(B)$. Since $\kappa(x), \kappa(y) \in \mathsf{Z}^\kappa(\kappa(B^+))$, we have that $\rho(B) = |x|/|y| = |\kappa(x)|/|\kappa(y)| \le \rho^\kappa(\kappa(B^+))$. The additional claim is clear.

2. By part 1, it remains to show that $\rho(G_0) \ge \rho^\kappa(\kappa(\mathcal{B}(G_0)^+))$.

By Proposition 4.9 and since $\kappa(\mathcal{A}(G_0)^+)$ is finite, we know that $\rho^\kappa(\kappa(\mathcal{B}(G_0)^+))$ is accepted. Let $B_\kappa \in \kappa(\mathcal{B}(G_0)^+)$ be such that $\rho^\kappa(B_\kappa) = \rho^\kappa(\kappa(\mathcal{B}(G_0)^+))$, and let $x_\kappa, y_\kappa \in \mathsf{Z}^\kappa(B_\kappa)$ be such that $|x_\kappa|/|y_\kappa| = \rho^\kappa(B_\kappa)$. By (4.5), we know that there exist $B_x, B_y \in \mathcal{B}(G_0)$ with $B_x^+ = B_y^+$, $x \in \mathsf{Z}(B_x)$ with $\kappa(x) = x_\kappa$, and $y \in \mathsf{Z}(B_y)$ with $\kappa(y) = y_\kappa$; in particular, we have $\kappa(B_x^+) = \kappa(B_y^+) = B_\kappa$.

Let $n \in \mathbb{N}$. Since G_0^+ is infinite, Lemma 4.4 yields some $U_n \in \mathcal{A}(G_0)$ with $(B_x^n)^- \mid U_n$. We set $D_n = B_y^n U_n$ and note that, since $(B_x^n)^+ = (B_y^n)^+$ and $(B_x^n)^- | U_n^-$, the sequence B_x^n is a proper subsequence of D_n. Thus,

$$\min \mathsf{L}(D_n) \le |y^n| + 1 = n|y_\kappa| + 1 \quad \text{and} \quad \max \mathsf{L}(D_n) \ge |x^n| + 1 = n|x_\kappa| + 1.$$

So we get

$$\rho(D_n) \ge \frac{n|x_\kappa| + 1}{n|y_\kappa| + 1}.$$

Thus, for each $n \in \mathbb{N}$,

$$\rho(G_0) \ge \frac{n|x_\kappa| + 1}{n|y_\kappa| + 1},$$

and letting $n \to \infty$, we have

$$\rho(G_0) \ge \frac{|x_\kappa|}{|y_\kappa|} = \rho^\kappa(\kappa(\mathcal{B}(G_0)^+)). \quad \square$$

Proof of Proposition 4.8. Since $\rho(G_0) = \rho(G_0 \setminus \{0\})$, we may assume that $0 \notin G_0$.

If G_0 is finite, then $\mathcal{B}(G_0)$ is finitely generated [28, Theorem 3.4.2.1], and thus the elasticity is rational by Lemma 4.9 (applied with $S = \mathcal{A}(H_{\mathrm{red}})$). Suppose G_0 is infinite. By Lemma 4.12, we have that $\rho(G_0) = \rho^\kappa(\kappa(\mathcal{B}(G_0)^+))$, and by Lemma 4.9, we know that $\rho^\kappa(\kappa(\mathcal{B}(G_0)^+))$ is rational. $\square$

Proof of Theorem 4.2. (a) $\Rightarrow$ (b) Since $\mathcal{B}(G_P)$ and $\mathcal{B}(-G_P)$ are isomorphic, we may without restriction suppose that G_P^- is finite. Let $u \in \mathcal{A}(H_{\mathrm{red}})$. We have to show that $\mathsf{t}(H, u) < \infty$. If u is prime, then $\mathsf{t}(H, u) = 0$. Suppose that u is not prime. Let $a \in H$ and $a' = aH^\times$ be such that $u \mid a'$. Let $z = v_1 \cdot \ldots \cdot v_n \in \mathsf{Z}(a)$. There is a minimal subset $\Omega \subset [1, n]$, say $\Omega = [1, k]$, such that $u \mid v_1 \cdot \ldots \cdot v_k$ and $k \le |\varphi_{\mathrm{red}}(u)|$. We consider any factorization of $v_1 \cdot \ldots \cdot v_k$ containing u, say $v_1 \cdot \ldots \cdot v_k = u_1 \cdot \ldots \cdot u_l$, where $u = u_1, \ldots, u_l \in \mathcal{A}(H_{\mathrm{red}})$.

For $i \in [1, k]$ and $j \in [1, l]$, we set $V_i = \boldsymbol{\beta}(v_i)$ and $U_j = \boldsymbol{\beta}(u_j)$. Then $U_1, \ldots, U_l, V_1, \ldots, V_k \in \mathcal{A}(G_P)$. Since u is not a prime and Ω is minimal, it follows that $0 \nmid V_1 \cdot \ldots \cdot V_k$. Hence, for every $j \in [1, l]$, U_j contains an element from G_P^+, and Lemma 4.3 implies that

$$l \le |(U_1 \cdot \ldots \cdot U_l)^+| = |(V_1 \cdot \ldots \cdot V_k)^+| \le k\,|\min G_P^-| \le |\varphi_{\mathrm{red}}(u)|\,|\min G_P^-|.$$

Setting $z' = u_1 \cdot \ldots \cdot u_l v_{k+1} \cdot \ldots \cdot v_n$, we infer that $\mathsf{d}(z, z') \le \max\{k, l\} \le |\varphi_{\mathrm{red}}(u)|\,|\min G_P^-|$, and hence $\mathsf{t}(H, u) \le |\varphi_{\mathrm{red}}(u)|\,|\min G_P^-|$.

(a) ⇒ (c) Without restriction, we may suppose that G_P^- is finite. By Lemma 3.3, it suffices to show that $\mathsf{c}(G_P) < \infty$. We set $M = \big(|\min G_P| + |G_P^-|^2\big)\,|\min G_P|$, and assert that $\mathsf{c}(A) \le M$ for all $A \in \mathcal{B}(G_P)$. To do so, we proceed by induction on $\max \mathsf{L}(A)$. If $A \in \mathcal{B}(G_P)$ with $\max \mathsf{L}(A) \le M$, then $\mathsf{c}(A) \le \max \mathsf{L}(A) \le M$. Let $A \in \mathcal{B}(G_P)$, let $z, \overline{z} \in \mathsf{Z}(A)$ with $|z| \le |\overline{z}|$, and suppose that $\mathsf{c}(B) \le M$ for all $B \in \mathcal{B}(G_P)$ with $\max \mathsf{L}(B) < \max \mathsf{L}(A)$. By Lemma 4.5, there is a $U \in \mathcal{A}(G_P)$ and a factorization $\widehat{z} \in \mathsf{Z}(A) \cap U\mathsf{Z}(G_P)$ such that $U \,|\, \overline{z}$ and $\mathsf{d}(z, \widehat{z}) \le M$, say $\widehat{z} = U\widehat{y}$ and $\overline{z} = U\overline{y}$ with $\widehat{y}, \overline{y} \in \mathsf{Z}(B)$ and $B = U^{-1}A$. Since $\max \mathsf{L}(B) < \max \mathsf{L}(A)$, there is an M-chain $\widehat{y} = y_0, \ldots, y_k = \overline{y}$ of factorizations of B, and hence $z, \widehat{z} = Uy_0, Uy_1, \ldots, Uy_k = U\overline{y} = \overline{z}$ is an M-chain of factorizations concatenating z and $\overline{z}$.

(a) ⇒ (e) Without restriction, we may suppose that G_P^- is finite. The claim follows by Proposition 4.8 and Lemma 3.3.

(c) ⇒ (d) and (e) ⇒ (f) hold for all atomic monoids [28, Proposition 1.4.2 and Theorem 1.6.3].

(b) ⇒ (a), (d) ⇒ (a), and (f) ⇒ (a) Assume to the contrary that G_P^+ and G_P^- are both infinite. We show that $\mathcal{B}(G_P)$ is not locally tame, which implies that H is not locally tame [28, Theorem 3.4.10.6]. Along the way, we show that $\rho_2(G_P) = \infty$ and that $\Delta(G_P)$ is infinite, which by Lemma 3.3 implies the according statements for H.

We set $a = \max G_P^-$ and $b = \min G_P^+$. Using the notation of Lemma 4.7, let $U = V_{a,b} = a^\alpha b^\beta \in \mathcal{A}(G_P)$. We pick an arbitrary $N \in \mathbb{N}_{\ge 2}$ and show that $\mathsf{t}(G_P, U) \ge N$, which implies the assertion.

We intend to apply Lemma 4.7 with $v = 1$. Thus, let $D = |a|(b + |a|)\gcd(a, b)$, let $b_1 \in G_P^+$ be such that

$$\frac{b_1}{\operatorname{lcm}(a, b)} \ge N + D,$$

and let $a_2 \in G_P^-$ be such that $|a_2| \ge (b_1 + b)|a|$. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}$ be such that $V_{a,b_1} = a^{\alpha_1} b_1^{\beta_1}$ and $V_{a_2,b} = a_2^{\alpha_2} b^{\beta_2}$ are elements of $\mathcal{A}(G_P)$.

We note that all conditions of Lemma 4.7 with $v = 1$ are fulfilled. Since $\alpha \le b \le \alpha_1$ and $\beta \le |a| \le \beta_2$, we have $U \,|\, V_{a,b_1} V_{a_2,b}$, and therefore $\mathsf{Z}(V_{a,b_1} V_{a_2,b}) \cap U\mathsf{Z}(G_P) \ne \emptyset$. Let $z \in \mathsf{Z}(V_{a,b_1} V_{a_2,b}) \setminus \{V_{a,b_1} \cdot V_{a_2,b_1}\}$, which exists in view of $U | V_{a,b_1} V_{a_2,b}$. By Lemma 4.7, we get that $t(z) \ne 0$, and thus that

$$|z| \ge \frac{b_1}{\operatorname{lcm}(a, b)} - D \ge N.$$

This shows that $\max \Delta\big(\mathsf{L}(V_{a,b_1} V_{a_2,b})\big) \ge N - 2$, $\mathsf{t}(G_p, U) \ge N$ and

$$\rho_2(G_P) \ge \max \mathsf{L}(V_{a,b_1} V_{a_2,b}) \ge N.$$

(a) ⇒ (g) This follows from Lemma 4.6.

(g) ⇒ (f) We have $\rho_2(H) \le M + \rho_1(H) = M + 1$, where M is as given by (g).

(a) ⇒ (h) If (a) holds, then (d) and (g) hold. Thus all assumptions of [21, Theorem 4.2] are fulfilled, and (h) follows.

(h) ⇒ (f) We have $\rho_2(H) = \sup \mathcal{V}_2(H) < \infty$. □

5. Arithmetical Properties stronger than the finiteness of G_P^+ or G_P^-

Let H be a Krull monoid and $G_P \subset G$ as always (see Theorem 5.2). In this section, we discuss arithmetical properties which are finite if G_P is finite or $\min\{|G_P^+|, |G_P^-|\} = 1$, and whose finiteness implies that G_P^+ or G_P^- is finite. However, it will turn out that none of the implications can be reversed (with the possible exception (c) ⇒ (b4), which remains open), and that the finiteness of these properties cannot be characterized by the size of G_P^+ and G_P^- but also depends on the structure of these sets. We start with some definitions and then formulate the main result.

Definition 5.1. Let H be an atomic monoid and $\pi : \mathsf{Z}(H) \to H_{\mathrm{red}}$ the factorization homomorphism.

1. For $z \in \mathsf{Z}(H)$, we denote by $\delta(z)$ the smallest $N \in \mathbb{N}_0$ with the following property: if $k \in \mathbb{N}$ is such that k and $|z|$ are adjacent lengths of $\mathsf{L}\big(\pi(z)\big)$, then

$$\mathsf{d}(z, \mathsf{Z}_k(a)) \le N.$$

Globally, we define

$$\delta(H) = \sup\{\,\delta(z) \mid z \in \mathsf{Z}(H)\} \in \mathbb{N}_0 \cup \{\infty\},$$

and we call $\delta(H)$ the *successive distance* of H.

2. We say that the *Structure Theorem for Sets of Lengths* holds (for the monoid H) if H is atomic and there exist some $M \in \mathbb{N}_0$ and a finite, nonempty set $\Delta^* \subset \mathbb{N}$ such that, for every $a \in H$, the set of lengths $\mathsf{L}(a)$ is an AAMP with some difference $d \in \Delta^*$ and bound M. In that case, we say more precisely that the Structure Theorem for Sets of Lengths holds with set Δ^* and bound M.

Theorem 5.2. *Let H be a Krull monoid and $\varphi \colon H \to \mathcal{F}(P)$ a cofinal divisor homomorphism into a free monoid such that the class group $G = \mathcal{C}(\varphi)$ is an infinite cyclic group that we identify with $\mathbb{Z}$. We denote by $G_P \subset G$ the set of classes containing prime divisors and consider the following conditions:*

(a) *G_P is finite or $\min\{|G_P^+|, |G_P^-|\} = 1$.*

(b1) *The Structure Theorem for Sets of Lengths holds for H with set $\Delta(G_P)$.*
(b2) *The successive distance $\delta(H)$ is finite.*
(b3) *The monotone catenary degree $\mathsf{c}_{\mathrm{mon}}(H)$ is finite.*
(b4) *There is an $M \in \mathbb{N}$ such that, for all $a \in H$ and for each two adjacent lengths $k, l \in \mathsf{L}(a) \cap [\min \mathsf{L}(a) + M, \max \mathsf{L}(a) - M]$, we have $\mathsf{d}\big(\mathsf{Z}_k(a), \mathsf{Z}_l(a)\big) \le M$.*

(c) *G_P^+ or G_P^- is finite.*

Then we have

1. *Condition* (a) *implies each of the conditions* (b1) *to* (b4)*.*
2. *Each of the conditions* (b1) *to* (b4) *implies* (c)*.*
3. (b2) $\Rightarrow$ (b3) $\Rightarrow$ (b4)*.*

We briefly discuss the newly introduced arithmetical properties and point out the trivial implications in the above result. The successive distance of H was introduced by Foroutan in [14] in order to study the monotone catenary degree. For Krull monoids with finite class group, an explicit upper bound for the successive distance was recently given in [19, Theorem 6.5]. Note that, by definition, $\delta(H) < \infty$ implies that $\Delta(H)$ is finite. The significance of the Structure Theorem for Sets of Lengths will be discussed at the beginning of Section 6. Note that, if it holds for a monoid H, then H is a BF-monoid with finite set of distances $\Delta(H)$. Moreover, if $G_P = \mathbb{Z}$, then the Structure Theorem fails badly: indeed, then every finite subset $L \subset \mathbb{N}_{\ge 2}$ occurs as a set of lengths by Kainrath's Theorem [28, Theorem 7.4.1]; for recent progress in this direction see [9]. The implications (b2) $\Rightarrow$ (b4) and (b3) $\Rightarrow$ (b4) follow from the definitions. A condition implying (b1) as well as (b4) is given in Proposition 6.2. The bound M in (b4) reflects the fact that in many settings, factorizations z of an element $a \in H$ show more unusual phenomena if their length $|z|$ is close either to $\max \mathsf{L}(a)$ or to $\min \mathsf{L}(a)$ (the reader may want to consult [28, Theorem 4.9.2], [16, Theorem 3.1], [17, Theorem 3.1] and the associated examples showing the relevance of the bound M).

In Sections 6 and 7, we obtain results showing that even under the more restrictive assumption that φ is a divisor theory, the Conditions (b1) to (b4) do not imply (a) (Proposition 6.9), and (c) does not imply (b1) to (b3) (Theorem 6.4, Propositions 6.9, 6.10 and 7.1). Proposition 6.10 shows that (b3) does not imply (b2). Moreover, (b1), (b2) and (b3) may hold as well as may fail even if $\min\{|G_P^+|, |G_P^-|\} = 2$. Most of the observed phenomena (around the non-reversibility of implications) have not been pointed out before in any v-noetherian monoid, and in particular not in any Krull monoid. Finally, by Theorem 5.2, a Krull monoid H satisfies strong arithmetical properties both when G_P is finite and when $\min\{|G_P^+|, |G_P^-|\} = 1$. Note that an arithmetical difference between these two cases was pointed out in Proposition 4.1.

The remainder of this section is devoted to the proof of Theorem 5.2, which heavily uses Theorem 4.2. We start with the necessary preparations. To show that (a) implies each of the Conditions (b1) to (b4), we will construct transfer homomorphisms to finitely generated monoids.

Lemma 5.3. *Let $G_0 \subset \mathbb{Z}$ be a condensed set with $\min\{|G_0^+|, |G_0^-|\} = 1$, say $G_0^- = \{-n\}$. The map*

$$\varphi\colon \begin{cases} \mathcal{B}(G_0) \to \mathcal{F}(G_0 \setminus \{-n\}) \\ B \mapsto (-n)^{-\mathsf{v}_{-n}(B)} B \end{cases}$$

is a cofinal divisor homomorphism. Its class group $\mathcal{C}(\varphi)$ is isomorphic to a subgroup of $\mathbb{Z}/n\mathbb{Z}$, and the set of classes containing prime divisors corresponds to $\{b + n\mathbb{Z} \mid b \in G_0 \setminus \{-n\}\}$. In particular, the class group of the Krull monoid $\mathcal{B}(G_0)$ is a finite cyclic group.

Proof. Clearly, φ is a cofinal monoid homomorphism. In order to show that φ is a divisor homomorphism, let $A, B \in \mathcal{B}(G_0)$ be such that $\varphi(A) \mid \varphi(B)$. We have to verify that $A \mid B$, and for that it suffices to check that $\mathsf{v}_{-n}(A) \le \mathsf{v}_{-n}(B)$. For each $C \in \mathcal{B}(G_0)$, we have $\mathsf{v}_{-n}(C) = \sigma(C^+)/n$ and $\sigma(C^+) = \sigma(\varphi(C))$. Since $\varphi(A) \mid \varphi(B)$, we have $\sigma(\varphi(A)) \le \sigma(\varphi(B))$, and thus $\mathsf{v}_{-n}(A) \le \mathsf{v}_{-n}(B)$ follows.

Now, we show that, for $F_1, F_2 \in \mathcal{F}(G_0 \setminus \{-n\})$, we have $F_1 \in F_2\mathsf{q}(\varphi(\mathcal{B}(G_0)))$ if and only if $\sigma(F_1) \equiv \sigma(F_2) \mod n$. This establishes the results regarding $\mathcal{C}(\varphi)$ and the set of classes containing prime divisors.

First, suppose that $\sigma(F_1) \equiv \sigma(F_2) \mod n$. We note that $F_i F_j^{n-1} (-n)^{(\sigma(F_i) + (n-1)\sigma(F_j))/n} \in \mathcal{B}(G_0)$, for $i, j \in \{1, 2\}$. Thus, F_j^n and $F_i F_j^{n-1}$ are elements of $\varphi(\mathcal{B}(G_0))$ for $i, j \in \{1, 2\}$. Since $F_1 = F_2 (F_1 F_2^{n-1})(F_2^{-n})$, the claim follows. Since $\sigma(\varphi(C)) \equiv 0 \mod n$ for each $C \in \mathcal{B}(G_0)$, the converse claim follows.

By [28, Theorem 2.4.7], the class group of $\mathcal{B}(G_0)$ is an epimorphic image of a subgroup of $\mathcal{C}(\varphi)$, and thus it is a finite cyclic group. □

The following example shows that $\mathcal{C}(\varphi)$ can be a proper subgroup of $\mathbb{Z}/n\mathbb{Z}$ and that $\mathcal{C}(\varphi)$ can be distinct from the class group of $\mathcal{B}(G_0)$. However, if $[G_0] = \mathbb{Z}$, then $\mathcal{C}(\varphi) = \mathbb{Z}/n\mathbb{Z}$; and, applying [44, Theorem 3.1], there is a simple and explicit method to determine the class group of $\mathcal{B}(G_0)$ from $\mathcal{C}(\varphi)$ as well as the subset of classes containing prime divisors (note that $\mathcal{C}(\varphi)$ is a torsion group).

Example 5.4. Let $d_1, d_2 \in \mathbb{N}_{\ge 2}$, $n = d_1 d_2$ and $G_0 = \{-n, d_1\}$. Then G_0 fulfils all assumptions of Lemma 5.3, and with φ as in Lemma 5.3, we get that $\mathcal{C}(\varphi) = \langle d_1 + n\mathbb{Z}\rangle \subsetneq \mathbb{Z}/n\mathbb{Z}$. However, $\mathcal{B}(G_0)$ is factorial, and thus its class group is trivial.

Proposition 5.5. *Let H be a Krull monoid and $\varphi\colon H \to \mathcal{F}(P)$ a cofinal divisor homomorphism into a free monoid such that the class group $G = \mathcal{C}(\varphi)$ is an infinite cyclic group that we identify with $\mathbb{Z}$. Let $G_P \subset G$ denote the set of classes containing prime divisors. Suppose that G_P is finite or that $\min\{|G_P^+|, |G_P^-|\} = 1$. Then there exists a transfer homomorphism $\theta\colon H \to H_0$ into a finitely generated monoid H_0 such that $\mathsf{c}(H, \theta) \le 2$. Moreover, the following statements hold.*

1. *$\mathcal{L}(H) = \mathcal{L}(H_0)$, in particular, the Structure Theorem for Sets of Lengths holds for H with $\Delta(H) = \Delta(H_0)$ and some bound M, and $\rho(H) = \rho(H_0)$ is finite and accepted.*
2. *$\delta(H) = \delta(H_0) < \infty$.*
3. *$\mathsf{c}_{\mathrm{mon}}(H) \le \max\{\mathsf{c}_{\mathrm{mon}}(H_0), 2\} < \infty$.*

Proof. First we show the existence of the required transfer homomorphism. For this, we recall that a monoid of zero-sum sequences over a finite set is finitely generated ([28, Theorem 3.4.2]). If G_P is finite, then $\boldsymbol{\beta}\colon H \to \mathcal{B}(G_P)$ has the desired properties by Lemma 3.3. Now suppose that $\min\{|G_P^+|, |G_P^-|\} = 1$, say $G_P^- = \{-n\}$, and set $G_0 = \{b + n\mathbb{Z} \mid b \in G_P^+\} \subset \mathbb{Z}/n\mathbb{Z}$. Using Lemmas 3.3 and 5.3, we have block homomorphisms $\boldsymbol{\beta}\colon H \to \mathcal{B}(G_P)$ and $\boldsymbol{\beta}'\colon \mathcal{B}(G_P) \to \mathcal{B}(G_0)$. By Lemma 3.2, the composition $\theta = \boldsymbol{\beta}' \circ \boldsymbol{\beta}\colon H \to \mathcal{B}(G_0)$ still has the required properties.

Again, by Lemmas 3.2 and 3.3, it suffices to verify the additional statements for finitely generated monoids: we refer to [28, Theorem 4.4.11] for the Structure Theorem, to [28, Theorem 3.1.4] for the elasticity and the successive distance, and to [14, Theorem 5.1] for the monotone catenary degree. □

Lemma 5.6. *Let H be an atomic monoid, $a \in H$ and $z, z' \in \mathsf{Z}(a)$ and $l = \big||z| - |z'|\big|$. Then there exists some $z'' \in \mathsf{Z}(a)$ such that $|z''| = |z'|$ and $\mathsf{d}(z, z'') \le l\delta(H)$.*

Proof. See [28, Lemma 3.1.3]. □

Lemma 5.7. *Let H be an atomic monoid with $\delta(H) < \infty$. Let $M \in \mathbb{N}$, $a \in H$, $u \in \mathcal{A}(H_{\mathrm{red}})$ and $z, \widehat{z}, \overline{z} \in \mathsf{Z}(a)$ be such that*

$$|z| \le |\overline{z}|,\ u \mid \overline{z},\ u \mid \widehat{z} \quad \text{and} \quad \mathsf{d}(z, \widehat{z}) \le M.$$

Then there is a $z' \in \mathsf{Z}(a) \cap u\mathsf{Z}(H)$ such that $|z| \le |z'| \le |\overline{z}|$ and $\mathsf{d}(z, z') \le M + \Big(M + \max \Delta(H)\Big)\delta(H)$.

Proof. Let $v \in H$ be such that $vH^\times = u$. We set $b = v^{-1}a$, $\overline{z} = u\overline{y}$ and $\widehat{z} = u\widehat{y}$, where $\overline{y}, \widehat{y} \in \mathsf{Z}(b)$. If $|z| \le |\widehat{z}| \le |\overline{z}|$, then $z' = \widehat{z}$ fulfills the requirements. If not, then either $|\widehat{z}| < |z|$ or $|\overline{z}| < |\widehat{z}|$, and we decide these two cases separately.

Case 1: $|\widehat{z}| < |z|$.
Since $|\widehat{y}| = |\widehat{z}| - 1 \in \mathsf{L}(b)$ and $|\overline{y}| = |\overline{z}| - 1 \in \mathsf{L}(b)$, there is a

$$k \in \mathsf{L}(b) \cap [|z| - 1, |\overline{z}| - 1] \quad \text{with } k \le |z| - 1 + \max \Delta(H).$$

Let $y'' \in \mathsf{Z}(b)$ with $|y''| = k$. Then

$$\begin{aligned} |y''| - |\widehat{y}| &= k - |\widehat{z}| + 1 \le |z| - 1 + \max \Delta(H) - |\widehat{z}| + 1 \\ &\le \mathsf{d}(z, \widehat{z}) + \max \Delta(H) \le M + \max \Delta(H). \end{aligned}$$

Thus, by Lemma 5.6, there is a $y' \in \mathsf{Z}(b)$ with $|y'| = |y''|$ and $\mathsf{d}(\widehat{y}, y') \le \Big(M + \max \Delta(H)\Big)\delta(H)$. Then $z' = uy' \in \mathsf{Z}(a) \cap u\mathsf{Z}(H)$ with $|z'| = 1 + k \in [|z|, |\overline{z}|]$ and

$$\mathsf{d}(z, z') \le \mathsf{d}(z, \widehat{z}) + \mathsf{d}(u\widehat{y}, uy') \le M + \Big(M + \max \Delta(H)\Big)\delta(H).$$

Case 2: $|\overline{z}| < |\widehat{z}|$.
By Lemma 5.6, there is a $y' \in \mathsf{Z}(b)$ with $|y'| = |\overline{y}|$ and

$$\begin{aligned} \mathsf{d}(\widehat{y}, y') &\le \Big(|\widehat{y}| - |\overline{y}|\Big)\delta(H) = \Big(|\widehat{z}| - |\overline{z}|\Big)\delta(H) \\ &\le \Big(|\widehat{z}| - |z|\Big)\delta(H) \le \mathsf{d}(\widehat{z}, z)\delta(H) \le M\delta(H). \end{aligned}$$

Then $z' = uy' \in \mathsf{Z}(a) \cap u\mathsf{Z}(H)$ with $|z'| = |\overline{z}|$ and

$$\mathsf{d}(z, z') \le \mathsf{d}(z, \widehat{z}) + \mathsf{d}(u\widehat{y}, uy') \le M + M\delta(H). \quad \square$$

Proposition 5.8. *Let H be a Krull monoid and $\varphi\colon H \to \mathcal{F}(P)$ a cofinal divisor homomorphism into a free monoid with infinite cyclic class group $\mathcal{C}(\varphi)$. If the successive distance $\delta(H)$ is finite, then the monotone catenary degree $\mathsf{c}_{\mathrm{mon}}(H)$ is finite.*

Proof. We set $G = \mathcal{C}(\varphi)$, identify G with $\mathbb{Z}$ and denote by $G_P \subset G$ the set of classes containing prime divisors. Suppose that $\delta(H) < \infty$. By Lemma 3.3, $\boldsymbol{\beta}$ is a transfer homomorphism with $\mathsf{c}(H, \boldsymbol{\beta}) \le 2$. Thus Lemma 3.2 implies that $\delta(H) = \delta(G_P)$ and $\mathsf{c}_{\mathrm{mon}}(H) \le \max\{\mathsf{c}_{\mathrm{mon}}(G_P), \mathsf{c}(H, \boldsymbol{\beta})\} \le \max\{\mathsf{c}_{\mathrm{mon}}(G_P), 2\}$. Thus it suffices to verify that $\mathsf{c}_{\mathrm{mon}}(G_P) < \infty$. Note that $\Delta(G_P)$ is finite (since $\delta(G_P) < \infty$), and thus by Theorem 4.2 we get that (say) G_P^- is finite.

We set $M = \big(|\min G_P| + |G_P^-|^2\big)\, |\min G_P|$, $M^* = M + \Big(M + \max \Delta(H)\Big)\delta(H)$, and assert that

$$\mathsf{c}_{\mathrm{mon}}(G_P) \le M^*.$$

For this, we have to show that $\mathsf{c}_{\mathrm{mon}}(A) \le M^*$ for all $A \in \mathcal{B}(G_P)$, and we proceed by induction on $\max \mathsf{L}(A)$.

If $A \in \mathcal{B}(G_P)$ with $\max \mathsf{L}(A) = 1$, then $A \in \mathcal{A}(G_P)$ and $\mathsf{c}_{\mathrm{mon}}(A) = 0$. Now let $A \in \mathcal{B}(G_P)$ with $\max \mathsf{L}(A) > 1$ and suppose that $\mathsf{c}_{\mathrm{mon}}(B) \le M^*$ for all $B \in \mathcal{B}(G_P)$ with $\max \mathsf{L}(B) < \max \mathsf{L}(A)$.

We pick $z, \overline{z} \in \mathsf{Z}(A)$ with $|z| \le |\overline{z}|$ and must find a monotone M^*-chain of factorizations from z to $\overline{z}$.

By Lemma 4.5 there is a $U \mid \overline{z}$ with $U \in \mathcal{A}(G_P)$ and a $\widehat{z} \in \mathsf{Z}(A) \cap U\mathsf{Z}(G_P)$ such that $\mathsf{d}(z, \widehat{z}) \le M$. By Lemma 5.7, there is a $z' \in \mathsf{Z}(A) \cap U\mathsf{Z}(G_P)$ such that $|z| \le |z'| \le |\overline{z}|$ and $\mathsf{d}(z, z') \le M^*$. Now we set

$$B = U^{-1}A, \quad \overline{z} = U\overline{y} \quad \text{and} \quad z' = Uy',$$

where $\overline{y}, y' \in \mathsf{Z}(B)$. Since $\max \mathsf{L}(B) < \max \mathsf{L}(A)$, the induction hypothesis gives a monotone M^*-chain $y' = y_1, \ldots, y_k = \overline{y}$ of factorizations of B from y' to $\overline{y}$. Therefore

$$z, z' = Uy' = Uy_1, Uy_2, \ldots, Uy_k = U\overline{y} = \overline{z}$$

is a monotone M^*-chain of factorizations of A from z to $\overline{z}$. □

Proof of Theorem 5.2. 3. The implication $(b3) \Rightarrow (b4)$ follows since, for $a \in H$ and each two adjacent lengths $k, l \in \mathsf{L}(a)$, we have, by definition, $\mathsf{d}\big(\mathsf{Z}_k(a), \mathsf{Z}_l(a)\big) \le \mathsf{c}_{\mathrm{mon}}(H)$. The implication $(b2) \Rightarrow (b3)$ is Proposition 5.8.

1. By Proposition 5.5, we know that (a) implies (b1), (b2), and (b3); and, by part 3, we know that (b3) implies (b4).

2. By definition, each of (b1), (b2) and (b3) implies the finiteness of $\Delta(H)$. Thus, Theorem 4.2 implies the assertion. It remains to show that (b4) implies (c).

Suppose that (b4) holds with some $M \in \mathbb{N}$ and assume to the contrary that (c) does not hold, i.e., G_P^+ and G_P^- are both infinite. We proceed as in the proof of Theorem 4.2, part (b) ⇒ (a).

We set $a = \max G_P^-$ and $b = \min G_P^+$ and let $\alpha \in [1, b]$ and $\beta \in [1, |a|]$ be such that $V_{a,b} = a^\alpha b^\beta \in \mathcal{A}(G_P)$. We intend to apply Lemma 4.7 with $v = 3$. Thus, let $D = 3|a|(b + |a|)\gcd(a, b)$, let $b_1 \in G_P^+$ with

$$\frac{b_1}{\mathrm{lcm}(a, b)} \ge 2D + M,$$

and let $a_2 \in G_P^-$ with $|a_2| \ge (3b_1 + b)|a|$. Let $\alpha_1, \alpha_2, \beta_2, \beta_2 \in \mathbb{N}$ be such that $V_{a,b_1} = a^{\alpha_1} b_1^{\beta_1}$ and $V_{a_2,b} = a_2^{\alpha_2} b^{\beta_2}$ are elements of $\mathcal{A}(G_P)$.

First, we assert that there exist $z_0, z_1, z_2, z_3 \in \mathsf{Z}((V_{a,b_1}V_{a_2,b})^3)$ with, where $t(\cdot)$ is defined as in Lemma 4.7,

$$t(z_0) < t(z_1) < t(z_2) < t(z_3).$$

We note that $V_{a,b} \mid V_{a,b_1}V_{a_2,b}$ (by the same reasoning used in the proof of Theorem 4.2), and thus there exists some $y \in \mathsf{Z}(V_{a,b_1}V_{a_2,b})$ with $t(y) \ne 0$. For $i \in [0, 3]$, we set $z_i = y^i (V_{a,b_1} \cdot V_{a_2,b})^{3-i}$. Then we have $t(z_i) = it(y)$, establishing the claim.

Let $z_0', z_1', z_2', z_3' \in \mathsf{Z}((V_{a,b_1}V_{a_2,b})^3)$ be such that $t(z_0') < t(z_1') < t(z_2') < t(z_3')$ and such that there exists no $z \in \mathsf{Z}((V_{a,b_1}V_{a_2,b})^3)$ with $t(z_1') < t(z) < t(z_2')$.

By Lemma 4.7, we get, for $i \in [0, 2]$, that

$$|z_{i+1}'| - |z_i'| \ge \frac{b_1}{\mathrm{lcm}(a, b)}\big(t(z_{i+1}') - t(z_i')\big) - 2D \ge M.$$

Since $\min \mathsf{L}((V_{a,b_1}V_{a_2,b})^3) \le |z_0'| < |z_1'| < |z_2'| < |z_3'| \le \max \mathsf{L}((V_{a,b_1}V_{a_2,b})^3)$, we get that

$$|z_1'|, |z_2'| \in \big[\min \mathsf{L}\big((V_{a,b_1}V_{a_2,b})^3\big) + M, \max \mathsf{L}\big((V_{a,b_1}V_{a_2,b})^3\big) - M\big].$$

Let

$$k = \max\left(\mathsf{L}\big((V_{a,b_1}V_{a_2,b})^3\big) \cap \left[\frac{b_1}{\mathrm{lcm}(a, b)}\, t(z_1') - D, \frac{b_1}{\mathrm{lcm}(a, b)}\, t(z_1') + D\right]\right)$$

and

$$l = \min\left(\mathsf{L}\big((V_{a,b_1}V_{a_2,b})^3\big) \cap \left[\frac{b_1}{\mathrm{lcm}(a, b)}\, t(z_2') - D, \frac{b_1}{\mathrm{lcm}(a, b)}\, t(z_2') + D\right]\right);$$

note that, by Lemma 4.7, $|z_1'|$ is an element from the set used above to define k while $|z_2'|$ is an element of the set used above to define l; also note that these two intervals are disjoint. In particular, we have $|z_1'| \le k < l \le |z_2'|$. Since there exists no $z \in \mathsf{Z}((V_{a,b_1}V_{a_2,b})^3)$ with $t(z_1') < t(z) < t(z_2')$, it follows by Lemma 4.7 that k and l are adjacent lengths. Since $k - l \ge \frac{b_1}{\mathrm{lcm}(a,b)} - 2D \ge M$ and by (2.1), we have $\mathsf{d}\big(\mathsf{Z}_k(a), \mathsf{Z}_l(a)\big) \ge M + 2$, a contradiction to the assumption that (b4) holds with M. □

6. The structure theorem for sets of lengths

The Structure Theorem for Sets of Lengths is a central finiteness result in factorization theory. Apart from Krull monoids – which will be discussed below – the Structure Theorem holds, among others, for weakly Krull domains with finite v-class group and for Mori domains A with complete integral closure $\widehat{A} = R$ for which the conductor $\mathfrak{f} = (A\!:\!R) \ne \{0\}$ and $\mathcal{C}(R)$ and $R/\mathfrak{f}$ are both finite (see [28, Section 4.7] for an overview, and [26,31] for recent progress). Moreover, it was recently shown that the Structure Theorem is sharp for Krull monoids with finite class group [45].

Let H be a Krull monoid and $G_P \subset G$ as always. By Theorem 5.2, it suffices to consider the situation when G_P^+ is finite and $2 \le |G_P^-| < \infty$. Essentially, all results so far which establish the Structure Theorem for some class of monoids use the machinery of pattern ideals and tame generating sets (presented in detail in [28, Section 4.3]). First, we repeat these concepts and outline their significance for the Structure Theorem. However, Proposition 6.3 shows that in our situation this approach is not applicable in general. The main result of this section, Theorem 6.4, provides a full characterization of when the Structure Theorem holds. Although the setting is special, it shows that, in Theorem 5.2, condition (b1) does not imply condition (a) and provides, together with Proposition 6.3, the first example of any Krull monoid for which the Structure Theorem holds without tame generation of pattern ideals. Furthermore, note by Lemma 3.4 that, for the sets G_P considered in Theorem 5.2, there actually exists a Krull monoid such that G_P is the set of classes containing prime divisors with respect to a divisor theory of H.

Likewise, all previous examples of monoids H with finite monotone catenary degree $\mathsf{c}_{\mathrm{mon}}(H)$ have been achieved by using that $\delta(H)$ is finite. However, in Proposition 6.10, we give the first example of a monoid H with $\mathsf{c}_{\mathrm{mon}}(H) < \infty$ but $\delta(H) = \infty$.

Definition 6.1. Let H be an atomic monoid, let $\mathfrak{a} \subset H$ and let $A \subset \mathbb{Z}$ be a finite nonempty subset.

1. We say that a subset $L \subset \mathbb{Z}$ *contains the pattern* A if there exists some $y \in \mathbb{Z}$ such that $y + A \subset L$. We denote by $\Phi(A) = \Phi_H(A)$ the set of all $a \in H$ for which $\mathsf{L}(a)$ contains the pattern A.
2. Now $\mathfrak{a}$ is called a *pattern ideal* if $\mathfrak{a} = \Phi(B)$ for some finite nonempty subset $B \subset \mathbb{Z}$.
3. A subset $E \subset H$ is called a *tame generating set* of $\mathfrak{a}$ if $E \subset \mathfrak{a}$ and there exists some $N \in \mathbb{N}$ with the following property: for every $a \in \mathfrak{a}$, there exists some $e \in E$ such that

$$e \,|\, a, \quad \sup \mathsf{L}(e) \le N \quad \text{and} \quad \mathsf{t}(a, \mathsf{Z}(e)) \le N.$$

In this case, we call E a *tame generating set with bound* N, and we say that $\mathfrak{a}$ is *tamely generated*.

The significance of tamely generated pattern ideals stems from the following result.

Proposition 6.2. *Let H be a* BF*-monoid with finite nonempty set of distances $\Delta(H)$ and suppose that all pattern ideals of H are tamely generated. Then there exists a constant $M \in \mathbb{N}_0$ such that the following properties are satisfied:*

(a) *The Structure Theorem for Sets of Lengths holds with $\Delta(H)$ and bound M.*
(b) *For all $a \in H$ and for each two adjacent lengths $k, l \in \mathsf{L}(a) \cap [\min \mathsf{L}(a) + M, \max \mathsf{L}(a) - M]$, we have $\mathsf{d}\big(\mathsf{Z}_k(a), \mathsf{Z}_l(a)\big) \le M$.*

Proof. The first statement follows from [28, Theorem 4.3.11] and the second from [31, Proposition 5.4]. □

Proposition 6.3. *Let H be a Krull monoid and $\varphi\colon H \to \mathcal{F}(P)$ a cofinal divisor homomorphism into a free monoid such that the class group $G = \mathcal{C}(\varphi)$ is an infinite cyclic group that we identify with $\mathbb{Z}$. Let $G_P \subset G$ denote the set of classes containing prime divisors. Suppose that*

- *G_P^+ is infinite and*
- *there are $a_1, a_2 \in G_P^-$ and $b \in G_P^+$ such that*

$$a_1\frac{\gcd(a_2,b)}{\gcd(a_1,a_2,b)} \equiv a_2\frac{\gcd(a_1,b)}{\gcd(a_1,a_2,b)} \mod b \quad \textit{but } a_1\frac{\gcd(a_2,b)}{\gcd(a_1,a_2,b)} \ne a_2\frac{\gcd(a_1,b)}{\gcd(a_1,a_2,b)}.$$

Then both H and $\mathcal{B}(G_P)$ have a pattern ideal which is not tamely generated.

Proof. By [26, Proposition 3.14], it suffices to show that $\mathcal{B}(G_P)$ has a pattern ideal which is not tamely generated.

First we show that $\mathcal{B}(\{a_1, a_2, b\})$ is half-factorial. By Lemma 5.3, it suffices to show that $\mathcal{B}(\{a_1 + b\mathbb{Z}, a_2 + b\mathbb{Z}\})$ is half-factorial. By [22, Proposition 5], this follows by (indeed, it is equivalent to) the assumed congruence on a_1, a_2, and b.

We set $\alpha_1 = b/\gcd(a_1, b)$, $\beta_1 = |a_1|/\gcd(a_1, b)$, $\alpha_2 = b/\gcd(a_2, b)$, $\beta_2 = |a_2|/\gcd(a_2, b)$ and observe that our assumption $a_1\frac{\gcd(a_2,b)}{\gcd(a_1,a_2,b)} \ne a_2\frac{\gcd(a_1,b)}{\gcd(a_1,a_2,b)}$ implies $d = a_1\alpha_1 - a_2\alpha_2 \ne 0$, say $d > 0$. Noting that $\alpha_1 a_1 = \mathrm{lcm}(a, b)$ and $\alpha_2 a_2 = \mathrm{lcm}(a_2, b)$, we can consider the two atoms

$$U_1 = a_1^{\alpha_1} b^{\beta_1} \quad \text{and} \quad U_2 = a_2{}^{\alpha_2} b^{\beta_2} \in \mathcal{A}(G_P).$$

Since G_P^+ is infinite, it contains arbitrarily large elements. Let $N \in G_P^+ \setminus \{b\}$. We define

$$\gamma = \min\{\mathsf v_N(U) \mid U \in \mathcal A(\{a_1, a_2, b, N\}) \text{ with } N \mid U\}.$$

Since $N^{|a_1|}a_1^N \in \mathcal B(G_P)$, it follows that $\gamma \in [1, |a_1|]$. Now we pick an atom $U_N \in \mathcal A(\{a_1, a_2, b, N\})$ with $\gamma = \mathsf v_N(U_N)$ for which $\mathsf v_b(U_N)$ is minimal, say

$$U_N = N^\gamma b^\beta a_1^{M_1} a_2^{M_2} \in \mathcal A(G_P), \quad \text{where } \beta, \gamma, M_1, M_2 \in \mathbb N_0 \text{ depend on } N.$$

If $M_2 \ge |a_1|$, then $U_N' = U_N a_1^{|a_2|} a_2^{a_1}$ has sum zero, and by the minimality of $\mathsf v_N(U_N)$ and $\mathsf v_b(U_N)$, it is an atom (as each atom must have at least one positive element). Thus, we may additionally choose U_N such that $M_2 < |a_1|$, which implies (recall $a_2 < 0$)

$$M_1 = \frac{1}{|a_1|}\Big(\gamma N + \beta b + a_2 M_2\Big) \ge \frac{1}{|a_1|}\Big(\gamma N + a_2|a_1|\Big) \ge \frac{N}{|a_1|} + a_2. \tag{6.1}$$

In view of this inequality, we may suppose that N is sufficiently large to guarantee that $M_1 \ge |a_2|\alpha_1\alpha_2$. Note that, since U_N is an atom and $M_1 \ge |a_2|\alpha_1\alpha_2 \ge \alpha_1$, we have $\beta < \beta_1$. We consider the element

$$A_N = U_N U_2^{M_1} \in \mathcal B(G_P).$$

Let $k \in \Big[0, \big\lfloor \frac{M_1}{|a_2|\alpha_1\alpha_2} \big\rfloor\Big]$. Then we have

$$U_{N,k} = N^\gamma b^\beta a_1^{M_1 + (a_2\alpha_1\alpha_2)k} a_2^{M_2 + (|a_1|\alpha_1\alpha_2)k} \in \mathcal B(G_P),$$

and by the minimality of γ and β, it follows that $U_{N,k} \in \mathcal A(G_P)$. Clearly, we get

$$z_{N,k} = U_{N,k} U_1^{-a_2\alpha_2 k} U_2^{M_1 + a_1\alpha_1 k} \in \mathsf Z(A_N).$$

This shows that

$$\mathsf L(A_N) \supset \left\{ M_1 + 1 + dk \;\middle|\; k \in \left[0, \left\lfloor \frac{M_1}{|a_2|\alpha_1\alpha_2} \right\rfloor\right]\right\}. \tag{6.2}$$

Thus, we have $A_N \in \Phi(\{0, d\})$ for each sufficiently large $N \in G_P^+$.

Let $E_N \in \Phi(\{0, d\})$ with $E_N \mid A_N$. Since $\{a_1, a_2, b\}$, is half-factorial, it follows that $N \mid E_N$. By the definition of γ, there is a $U_N' \in \mathcal A(G_P)$ with $N^\gamma \mid U_N' \mid E_N$. Note that [28, Lemma 1.6.5.6] shows that $\mathsf t(A_N, U_N') \le \mathsf t(A_N, \mathsf Z(E_N))$.

Let $A_N = U_N' W_N$ with $W_N \in \mathcal B(G_P)$. Then $\operatorname{supp}(W_N) = \{a_1, a_2, b\}$ and hence $|\mathsf L(W_N)| = 1$. Thus all factorizations in $\mathsf Z(A_N) \cap U_N'\mathsf Z(G_P)$ have the same length. We pick some factorization $z_N \in \mathsf Z(A_N) \cap U_N'\mathsf Z(G_P)$. Clearly, there is a factorization $z_N^* \in \mathsf Z(A_N)$ such that (in view of (6.2))

$$\big||z_N| - |z_N^*|\big| \ge \frac{\max \mathsf L(A_N) - \min \mathsf L(A_N)}{2} \ge \frac{d}{2}\left\lfloor \frac{M_1}{|a_2|\alpha_1\alpha_2} \right\rfloor.$$

This implies that

$$\begin{aligned} \mathsf t(A_N, \mathsf Z(E_N)) \ge \mathsf t(A_N, U_N') &\ge \min\{\mathsf d(z_N^*, y_N) \mid y_N \in \mathsf Z(A_N) \cap U_N'\mathsf Z(G_P)\} \\ &\ge \min\Big\{\big||z_N^*| - |y_N|\big| \mid y_N \in \mathsf Z(A_N) \cap U_N'\mathsf Z(G_P)\Big\} \\ &\ge \big||z_N| - |z_N^*|\big| \ge \tfrac{d}{2}\left\lfloor \tfrac{M_1}{|a_2|\alpha_1\alpha_2} \right\rfloor. \end{aligned}$$

Since N can be arbitrarily large and by (6.1), we get that $\Phi(\{0, d\})$ is not tamely generated. □

We will frequently make use of the following simple observation. Let G be an abelian group and $G_1 \subset G_0 \subset G$ subsets. Then $\mathcal B(G_1) \subset \mathcal B(G_0)$ is a divisor-closed submonoid (this means if $A \in \mathcal B(G_1)$ and $B \in \mathcal B(G_0)$ with $B \mid A$, then $B \in \mathcal B(G_1)$), and hence $\mathcal L(G_1) \subset \mathcal L(G_0)$. Therefore, if the Structure Theorem holds for $\mathcal B(G_0)$, then it holds for $\mathcal B(G_1)$. In particular, if condition (b) holds, then the Structure Theorem holds for all $\mathcal B(G_0)$ with $G_0 \subset G_P$, and if (b) fails, then the Structure Theorem fails for all $\mathcal B(G_0)$ with $G_P \subset G_0$—where G_P is as in Theorem 6.4.

Theorem 6.4. *Let H be a Krull monoid and $\varphi \colon H \to \mathcal F(P)$ a cofinal divisor homomorphism into a free monoid such that the class group $G = \mathcal C(\varphi)$ is an infinite cyclic group that we identify with $\mathbb Z$. Let $G_P \subset G$ denote the set of classes containing prime divisors. Suppose that $1 \in G_P^+$ and $G_P^- = \{-d, -1\}$ for some $d \in \mathbb N$. Then the following statements are equivalent:*

(a) *The Structure Theorem for Sets of Lengths holds for H.*
(b) *$G_P^+ \setminus d\mathbb Z$ is finite or a subset of $1 + d\mathbb Z$.*

The remainder of this section is devoted to the proof of Theorem 6.4.

Lemma 6.5. *Let H be an atomic monoid. Suppose that there exists some $e \in \mathbb N$ such that, for each $N \in \mathbb N$, there exists some $a \in H$ such that $\mathsf L(a) \cap [\min \mathsf L(a), \min \mathsf L(a) + N] \subset \min \mathsf L(a) + e\mathbb Z$, yet $\mathsf L(a) \not\subset \min \mathsf L(a) + e\mathbb Z$. Then the Structure Theorem does not hold for H.*

Proof. We assume to the contrary that there exists some finite nonempty set $\Delta^* \subset \mathbb{N}$ and some $M \in \mathbb{N}$ such that, for each $b \in H$, the set $\mathsf{L}(b)$ is an AAMP with difference $d \in \Delta^*$ and bound M.

Let $D = 2\ \mathrm{lcm}(\Delta^*)$. Let $N \geq 2M+D$ and let $a \in H$ with the properties from the statement of the lemma. Let $l_1 = \min \mathsf{L}(a)$ and $l_2 = \max \mathsf{L}(a)$. Note that $l_2 \geq l_1 + N$ (by the property assumed for a). By assumption, we get that $\mathsf{L}(a)$ is an AAMP, i.e.,

$$\mathsf{L}(a) = y + (L' \cup L^* \cup L'') \subset y + \mathcal{D} + d\mathbb{Z}$$

where $d \in \Delta^*$, $\{0, d\} \subset \mathcal{D} \subset [0, d]$, L^* is finite nonempty with $\min L^* = 0$ and $L^* = (\mathcal{D} + d\mathbb{Z}) \cap [0, \max L^*]$, $L' \subset [-M, -1]$ and $L'' \subset \max L^* + [1, M]$, and $y \in \mathbb{N}$.

Since

$$l_2 \geq l_1 + N \geq l_1 + 2M + D \geq l_1 - \min L' + M + D,$$

it follows that $[l_1 - \min L', l_1 - \min L' + D - 1] \cap \mathsf{L}(a) \subset L^*$, and thus

$$[l_1 - \min L', l_1 - \min L' + D - 1] \cap \mathsf{L}(a) = [l_1 - \min L', l_1 - \min L' + D - 1] \cap (y + \mathcal{D} + d\mathbb{Z}).$$

On the other hand, by the property assumed for a, and since $N \geq 2M + D \geq -\min L' + D$, we have

$$[l_1 - \min L', l_1 - \min L' + D - 1] \cap \mathsf{L}(a) \subset l_1 + e\mathbb{Z}.$$

Thus

$$A = [-\min L', -\min L' + D - 1] \cap (y - l_1 + \mathcal{D} + d\mathbb{Z}) \subset e\mathbb{Z}.$$

Since $D \geq 2d$, it follows that for each $d' \in \mathcal{D}$ there exists some $k \in \mathbb{Z}$ and $\epsilon \in \{-1, 1\}$ such that $y - l_1 + d' + kd$, $y - l_1 + d' + (k + \epsilon)d \in A$. Thus $e \mid d$ and, furthermore, $e \mid y - l_1 + d'$. Consequently, $y + \mathcal{D} + d\mathbb{Z} \subset l_1 + e\mathbb{Z}$. This yields a contradiction, since $\mathsf{L}(a) \subset y + \mathcal{D} + d\mathbb{Z}$, yet $\mathsf{L}(a) \not\subset l_1 + e\mathbb{Z}$ by hypothesis. $\square$

Lemma 6.6. *Let $d \in \mathbb{N}$, $e \in [2, d-1]$ with $\gcd(e, d) > 1$ and $G_0 \subset \mathbb{Z}$. If $\{-d, -1, 1\} \subset G_0$ and $G_0^+ \cap (e + d\mathbb{Z})$ is infinite, then the Structure Theorem does not hold for $\mathcal{B}(G_0)$.*

Proof. We may assume $d \geq 4$, since otherwise there exists no $e \in [2, d-1]$ with $\gcd(e, d) > 1$. Let $k \in \mathbb{N}$ such that $e + dk \in G_0$; by assumption, we know that arbitrarily large k with this property exist, and we thus may impose that $k \geq 10$. Let $f \in \mathbb{N}$ be minimal such that $ef \in d\mathbb{N}$, say $ef = du$. Since $\gcd(e, d) > 1$, we see that $f \in [2, d/2]$ and $u \leq e/2 \leq d/2$. We consider the sequence

$$B = (e + dk)^f (-d)^{u+fk} (-1)^{d(u+fk)} 1^{d(u+fk)}.$$

Since $ef = du$, we have $B \in \mathcal{B}(G_0)$. First, we consider two specific factorizations of B. Then, we investigate the length of all factorizations of B of small length. Let

$$z_1 = ((e + dk)^f (-d)^{u+fk}) \cdot ((-1)1)^{d(u+fk)}$$

and

$$z_2 = ((e + dk)(-1)^{e+dk})^f \cdot ((-d)1^d)^{u+fk}.$$

We note that $z_1, z_2 \in \mathsf{Z}(B)$ and that $|z_1| = 1 + d(u + fk)$ and $|z_2| = f + (u + fk)$. Since $f - 1 \notin (d-1)\mathbb{Z}$ (as $f \in [2, d/2]$), we have $|z_1| - |z_2| \notin (d-1)\mathbb{Z}$.

We claim that there exists an absolute positive constant c such that, for each $z \in \mathsf{Z}(B)$ with

$$|z| \leq |z_2| + c(d-1)k,$$

we have

$$|z| - |z_2| \in (d-1)\mathbb{N}_0.$$

By Lemma 6.5 and since k can be arbitrarily large, this implies that the Structure Theorem does not hold. Thus, it suffices to establish this claim. For definiteness, we set $c = 1/6$ (it is apparent from the subsequent argument that it only has to be less than $1/2$). Let

$$z = A_1 \cdot \ldots \cdot A_s U_1 \cdot \ldots \cdot U_t \in \mathsf{Z}(B)$$

with $A_i, U_j \in \mathcal{A}(G_0)$, and $(e + dk) \mid A_i$ and $(e + dk) \nmid U_j$ for all i, j. We proceed to show that $\mathsf{v}_{e+dk}(A_i) = 1$ for each i, i.e., $s = f$. Clearly, $\mathsf{v}_{(-1)1}(z) \leq |z|$, and thus we have

$$\begin{aligned}
\mathsf{v}_{-1}(\pi(A_1 \cdot \ldots \cdot A_s)) &\geq d(u + fk) - |z| \\
&\geq d(u + fk) - (f + u + fk + c(d-1)k) \\
&= (f-2)(e + dk) + 2(e + dk) - (f + u + fk + c(d-1)k) \\
&\geq (f-2)(e + dk) + dk - (d/2 + d + dk/2 + cdk) \\
&> (f-2)(e + dk) + d(k - 3/2 - k/2 - ck).
\end{aligned}$$

Since $c = 1/6$ and $k \geq 10$, we have $k(1/2 - c) - 3/2 \geq 1$. So we have

$$\mathsf{v}_{-1}(\pi(A_1 \cdot \ldots \cdot A_s)) \geq (f-2)(e+dk) + d. \tag{6.3}$$

If $s \leq f-1$, then, since $\mathsf{v}_{-1}(A_i) \leq e + dk$ for each i, we conclude from (6.3) that $\mathsf{v}_{-1}(A_i) \geq d$ for each i, implying (since $\operatorname{supp}(A_i^-) \subset \{-1, -d\}$) that $\mathsf{v}_{e+dk}(A_i) = 1$ for each i, contradicting $s \leq f-1$. Thus $s = f$. We have $U_j \in \{(-1)1, ((-d)1^d)\}$ for each j. Thus

$$z = A_1 \cdot \ldots \cdot A_f ((-1)1)^a ((-d)1^d)^b$$

where $a = d(u+fk) - \mathsf{v}_{-1}(\pi(A_1 \cdot \ldots \cdot A_f))$ and $b = u + fk - \mathsf{v}_{-d}(\pi(A_1 \cdot \ldots \cdot A_f))$. We have

$$|z| = f + (u+fk)(d+1) - (\mathsf{v}_{-1}(\pi(A_1 \cdot \ldots \cdot A_f)) + \mathsf{v}_{-d}(\pi(A_1 \cdot \ldots \cdot A_f)))$$

and, since

$$d \cdot \mathsf{v}_{-d}(\pi(A_1 \cdot \ldots \cdot A_f)) + \mathsf{v}_{-1}(\pi(A_1 \cdot \ldots \cdot A_f)) = (u+fk)d,$$

this implies

$$|z| = f + u + fk + (d-1)\mathsf{v}_{-d}(\pi(A_1 \cdot \ldots \cdot A_f)),$$

establishing $|z| - |z_2| \in (d-1)\mathbb{N}_0$. $\square$

Lemma 6.7. *Let $d \in \mathbb{N}$, $e \in [1, d-1]$ with $\gcd(e,d) = 1$ and $G_0 \subset \mathbb{Z}$. If $\{-d, -1, 1\} \subset G_0$, $G_0^+ \cap (e + d\mathbb{Z})$ is infinite and $G_0^+ \setminus ((e + d\mathbb{Z}) \cup d\mathbb{Z})$ is nonempty, then the Structure Theorem does not hold for $\mathcal{B}(G_0)$.*

Proof. We may assume $d \geq 3$, as the hypotheses are null otherwise. Since $G_0^+ \setminus ((e + d\mathbb{Z}) \cup d\mathbb{Z})$ is nonempty, let $f \in [1, d-1] \setminus \{e\}$ and $\ell \in \mathbb{N}_0$ be such that $f + d\ell \in G_0^+$. Since $\{-d, -1, 1\} \subset G_0$, $G_0^+ \cap (e + d\mathbb{Z})$ is infinite, let $k \in \mathbb{N}$ be such that $e + dk \in G_0^+$ and $e + dk \geq f + d\ell$. Since $\gcd(e, d) = 1$, let $x \in [1, d-1]$ be the integer such that $f + xe \in d\mathbb{Z}$, say $f + xe = ud$. Since $f \in [1, d-1] \setminus \{e\}$, we have $x \neq d-1$ and $u \leq d-1$.

We proceed similarly to Lemma 6.7. We consider the following element of $\mathcal{B}(G_0)$:

$$B = (f + d\ell)(e + dk)^x (-d)^{u+xk+\ell} (-1)^{d(u+xk+\ell)} 1^{d(u+xk+\ell)}.$$

Again, we first consider two specific factorizations of B, namely

$$z_1 = ((f + d\ell)(e + dk)^x (-d)^{u+xk+\ell}) \cdot ((-1)1)^{d(u+xk+\ell)}$$

and

$$z_2 = ((f + d\ell)(-1)^{f+d\ell}) \cdot ((e + dk)(-1)^{e+dk})^x \cdot ((-d)1^d)^{u+xk+\ell}.$$

The respective lengths of these factorizations are $1 + d(u + xk + \ell)$ and $1 + x + (u + xk + \ell)$. Thus, $|z_1| - |z_2| \notin (d-1)\mathbb{Z}$.

As in Lemma 6.6, we show that there exists a positive c, now depending on d (but not on k), such that, for each $z \in \mathsf{Z}(B)$ with

$$|z| \leq |z_2| + c(d-1)k,$$

we have

$$|z| - |z_2| \in (d-1)\mathbb{N}_0,$$

which again completes the proof by Lemma 6.5. We set $c = 1/(d-1)$ (this choice is not optimal). Let

$$z = A_1 \cdot \ldots \cdot A_s ((-1)1)^a ((-d)1^d)^b$$

where $A_i \notin \{(-1)1, (-d)1^d\}$. We proceed to show that $|A_i^+| = 1$ for each i. From the definition of B, we have $s \leq x+1$. Again, $\mathsf{v}_{(-1)1}(z) \leq |z|$, and thus

$$\begin{aligned}
\mathsf{v}_{-1}(\pi(A_1 \cdot \ldots \cdot A_s)) &\geq d(u + xk + \ell) - |z| \\
&\geq d(u + xk + \ell) - (1 + x + (u + xk + \ell) + c(d-1)k) \\
&= (x-1)(e+dk) + (f + d\ell) + (e + dk) - (1 + x + (u + xk + \ell) + c(d-1)k) \\
&\geq (x-1)(e+dk) + (f + d\ell) + (e + dk) - (d - 1 + (d - 1 + (d-2)k + \ell) + c(d-1)k) \\
&\geq (x-1)(e+dk) + d + 2k - 3d - c(d-1)k.
\end{aligned}$$

Since $c = 1/(d-1)$, we have, for $k \geq 3d$,

$$\mathsf{v}_{-1}(\pi(A_1 \cdot \ldots \cdot A_s)) \geq (x-1)(e+dk) + d.$$

If $s = x+1$, the claim is obvious. Thus, assume $s \leq x$. Since $\mathsf{v}_{-1}(A_i) \leq e + dk$ for each i (recall that $e + dk \geq f + d\ell$), we get that $\mathsf{v}_{-1}(A_i) \geq d$ for each i, establishing the claim (since $\operatorname{supp}(A_i^-) \subset \{-1, -d\}$).

Thus

$$z = A_1 \cdot \ldots \cdot A_s((-1)1)^a((-d)1^d)^b,$$

where $a = d(u + xk + \ell) - \mathsf{v}_{-1}(\pi(A_1 \cdot \ldots \cdot A_s))$ and $b = (u + xk + \ell) - \mathsf{v}_{-d}(\pi(A_1 \cdot \ldots \cdot A_s))$. We have

$$|z| = s + (d + 1)(u + xk + \ell) - (\mathsf{v}_{-1}(\pi(A_1 \cdot \ldots \cdot A_s)) + \mathsf{v}_{-d}(\pi(A_1 \cdot \ldots \cdot A_s))).$$

We note that if $f + \ell d \neq 1$, then $s = 1 + x$, and if $f + \ell d = 1$, then $s = x$. Moreover, if the former holds true, then

$$d \cdot \mathsf{v}_{-d}(\pi(A_1 \cdot \ldots \cdot A_f)) + \mathsf{v}_{-1}(\pi(A_1 \cdot \ldots \cdot A_f)) = d(u + xk + \ell),$$

whereas if the latter holds true, then

$$d \cdot \mathsf{v}_{-d}(\pi(A_1 \cdot \ldots \cdot A_f)) + \mathsf{v}_{-1}(\pi(A_1 \cdot \ldots \cdot A_f)) = d(u + xk + \ell) - 1.$$

In both cases, this implies

$$|z| = 1 + x + (u + xk + \ell) + (d - 1)\mathsf{v}_{-d}(\pi(A_1 \cdot \ldots \cdot A_s))$$

establishing $|z| - |z_2| \in (d - 1)\mathbb{N}_0$, as claimed. □

Proposition 6.8. *Let $\{-1, 1\} \subset G_0 \subset \mathbb{Z}$ with G_0^- finite such that the Structure Theorem holds for $\mathcal{B}(G_0)$. For each $-d \in G_0^-$, at least one of the following statements holds:*

(a) $|G_0^+ \setminus d\mathbb{Z}| < \infty$.
(b) $G_0^+ \setminus d\mathbb{Z} \subset 1 + d\mathbb{Z}$.

Proof. The claim is trivial for $d \leq 2$. Suppose $d \geq 3$. Let $E \subset [0, d - 1]$ be such that $G_0^+ \cap (e + d\mathbb{Z})$ is infinite for each $e \in E$. If there exists some $e \in E \setminus \{0\}$ with $\gcd(e, d) > 1$, Lemma 6.6 yields a contradiction. Thus, $\gcd(e, d) = 1$ for each $e \in E \setminus \{0\}$. By Lemma 6.7 we get that if $\gcd(e, d) = 1$, then $e = 1$ (note that $1 \in G_0^+$), and moreover, in this case we have $G_0^+ \subset ((1 + d\mathbb{Z}) \cup d\mathbb{Z})$. □

Now, we show that the Structure Theorem indeed holds for monoids of zero-sum sequences over sets of the form considered in Theorem 6.4 not covered by Lemma 6.5 – Proposition 6.8. Moreover, we investigate the finiteness of the successive distance for these sets. Again, note that the set $F_0 \cup d\mathbb{N}$ in the result below does not fulfil condition (a) of Theorem 5.2, yet by Lemma 3.4 it can occur as the subset of classes containing prime divisors of a Krull monoid, even with respect to a divisor theory, showing that the conditions (b1), (b2), and (b3) do not imply (a), not even combined.

Proposition 6.9. *Let $d \in \mathbb{N}_{\geq 2}$ and $F_0 \subset \mathbb{Z}$ with $F_0^- = \{-d, -1\}$.*

1. *The Structure Theorem holds for $\mathcal{B}(F_0 \cup d\mathbb{N})$ if and only if it holds for $\mathcal{B}(F_0 \cup \{d\})$. More precisely, for each $L \in \mathcal{L}(F_0 \cup d\mathbb{N})$, there exists some $y \in \mathbb{N}_0$ such that $y + L \in \mathcal{L}(F_0 \cup \{d\})$.*
2. *$\delta(F_0 \cup d\mathbb{N}) = \delta(F_0 \cup \{d\})$.*
3. *There is a map $\psi : \mathcal{B}(F_0 \cup d\mathbb{N}) \to \mathcal{B}(F_0 \cup \{d\})$ such that, for each $B \in \mathcal{B}(F_0 \cup d\mathbb{N})$ and adjacent lengths k and l of $\mathsf{L}(B)$, we have $\mathsf{d}(\mathsf{Z}_k(B), \mathsf{Z}_l(B)) \leq \mathsf{d}(\mathsf{Z}_{k'}(\psi(B)), \mathsf{Z}_{l'}(\psi(B)))$ with k', l' adjacent lengths of $\mathsf{L}(\psi(B))$. More precisely, we can choose $k' = k + y$ and $l' = l + y$ with y such that $y + \mathsf{L}(B) = \mathsf{L}(\psi(B))$.*

In particular, if F_0 is finite, then the Structure Theorem holds for $\mathcal{B}(F_0 \cup d\mathbb{N})$, and both $\delta(F_0 \cup d\mathbb{N})$ and $\mathsf{c}_{\mathrm{mon}}(F_0 \cup d\mathbb{N})$ are finite.

Proof. Let $G_0 = F_0 \cup d\mathbb{N}$ and $G_1 = F_0 \cup \{d\}$.

1. Since $G_1 \subset G_0$, one implication is clear and it remains to show that if the Structure Theorem holds for $\mathcal{B}(G_1)$, then it holds for $\mathcal{B}(G_0)$. Indeed, the more precise assertion we establish shows that the Structure Theorem holds with the same bound and the same set of differences.

Let $\psi : \mathcal{F}(G_0) \to \mathcal{F}(G_1)$ denote the monoid homomorphism defined via $\psi(g) = g$ for $g \notin d\mathbb{N}$ and $\psi(kd) = d^k$ for $kd \in d\mathbb{N}$. We note that $\sigma(S) = \sigma(\psi(S))$ for each $S \in \mathcal{F}(G_0)$; thus ψ yields a homomorphism, and indeed an epimorphism, from $\mathcal{B}(G_0)$ to $\mathcal{B}(G_1)$.

Moreover, we observe that if $A \in \mathcal{A}(G_0)$ with $kd \mid A$, for some $k \in \mathbb{N}$, then $A^+ = kd$. This implies that, for such an atom, $\psi(A) = d^k(-1)^{d\ell}(-d)^{k-\ell}$ and $(d(-1)^d)^{\ell} \cdot (d(-d))^{k-\ell} \in \mathsf{Z}(\psi(A))$ is the unique factorization of $\psi(A)$. We denote this factorization by $\overline{\psi}(A)$ and we note that $|\overline{\psi}(A)| = \sigma(A^+)/d$. Setting $\overline{\psi}(A) = A$ for each atom not of this form, i.e., $A \in \mathcal{A}(G_0)$ with $\operatorname{supp}(A) \cap d\mathbb{N} = \emptyset$, and extending this map to $\mathsf{Z}(G_0)$, we get a homomorphism, indeed an epimorphism, $\overline{\psi} : \mathsf{Z}(G_0) \to \mathsf{Z}(G_1)$.

Since $\pi(\overline{\psi}(z)) = \psi(\pi(z))$, we see that $\overline{\psi}(\mathsf{Z}(B)) \subset \mathsf{Z}(\psi(B))$ for each $B \in \mathcal{B}(G_0)$. Moreover, for $B \in \mathcal{B}(G_0)$ and $z \in \mathsf{Z}(B)$, we have, denoting $F = \prod_{g \in d\mathbb{N}} g^{\mathsf{v}_g(B)}$, that $|\overline{\psi}(z)| = |z| + (\sigma(F)/d - |F|)$. In particular, the value of $|\overline{\psi}(z)| - |z|$ is the same for each $z \in \mathsf{Z}(B)$.

Thus, to establish our claim on sets of lengths, it suffices to show that $\overline{\psi}(\mathsf{Z}(B)) = \mathsf{Z}(\psi(B))$ for each $B \in \mathcal{B}(G_0)$. Let $B \in \mathcal{B}(G_0)$ and again let $F = \prod_{g \in d\mathbb{N}} g^{\mathsf{v}_g(B)} = \prod_{i=1}^{|F|}(k_i d)$, where $k_i \in \mathbb{N}$. Let $z' \in \mathsf{Z}(\psi(B))$. There exists a unique decomposition $z' = z_1' z_2'$ such that z_1' is minimal with $d^{\sigma(F)/d} \mid \pi(z_1')$ (note that $\mathsf{v}_d(\psi(B)) = \sigma(F)/d$). We have $|z_1'| = \sigma(F)/d$. Write $z_1' = \prod_{i=1}^{|F|} y_i'$ such that each factor $y_i' \in \mathsf{Z}(\psi(B))$ contains exactly $|y_i'| = k_i$ atoms. Then letting $A_i = (k_i d)d^{-k_i}\pi(y_i')$, we have $A_i \in \mathcal{A}(G_0)$, and so $z = A_1 \cdot \ldots \cdot A_{|F|} z_2'$ is a factorization of B with $\overline{\psi}(z) = \psi(A_1) \cdot \ldots \cdot \psi(A_{|F|}) z_2' = y_1' \cdot \ldots \cdot y_s' z_2' = z'$, establishing our claim.

2. Since $\delta(G_1) \le \delta(G_0)$ is obvious, we only have to show that $\delta(G_0) \le \delta(G_1)$. We show the following slightly stronger result. Let $B \in \mathcal{B}(G_0)$ and $z \in \mathsf{Z}(B)$. Then $\delta(z) \le \delta(\overline{\psi}(z))$.

Let F and $z = z_1 z_2$ be defined as above, and let $z_1 = \prod_{i=1}^{|F|} A_i$ and let $A_i^+ = k_i d$, where $k_i \in \mathbb{N}$. Moreover, let $z' = \overline{\psi}(z)$ and let $z' = z_1' z_2'$ with $z_1' = \overline{\psi}(z_1)$ and $z_2' = \overline{\psi}(z_2) = z_2$. Additionally, let $y_i' = \overline{\psi}(A_i)$ for each i. Let $j \in \mathbb{Z}$ be such that $|z|$ and $|z| + j$ are adjacent lengths of $\mathsf{L}(B)$. By the already established result for sets of lengths, it follows that $|\overline{\psi}(z)|$ and $|\overline{\psi}(z)| + j$ are adjacent lengths of $\mathsf{L}(\psi(B))$. Thus, by definition, there exists some factorization $x' \in \mathsf{Z}(\psi(B))$ with $|x'| = |\overline{\psi}(z)| + j$ and $d(x', \overline{\psi}(z)) \le \delta(\overline{\psi}(z))$. Let $x' = x_1' x_2'$ with x_1' minimal such that $d^{\sigma(F)/d} \mid \pi(x_1')$. We note that

$$\mathsf{d}(z', x') = \mathsf{d}(z_1', x_1') + \mathsf{d}(z_2', x_2'). \tag{6.4}$$

Thus, by re-indexing appropriately, we find a

$$t \le \mathsf{d}(z_1', x_1') \tag{6.5}$$

such that $\prod_{i=t+1}^{|F|} y_i' \mid x_1'$.

Let $x_1'' = x_1' \left(\prod_{i=t+1}^{|F|} y_i'\right)^{-1}$. As we argued at the end of part 1, there exists, for $i \le t$, factorizations $y_i'' \in \mathsf{Z}(\psi(B))$, each containing exactly $|y_i'| = k_i$ atoms, such that $\prod_{i=1}^{t} y_i'' = x_1''$. For $i \le t$, let $A_i' = d^{-k_i}(k_i d)\pi(y_i'')$, and for $i \in [t+1, |F|]$, let $A_i' = A_i$. Then, with $x_1 = \prod_{i=1}^{|F|} A_i'$ and $x_2 = x_2'$, we have that $x = x_1 x_2$ is a factorization of B, and since $\overline{\psi}(x) = x_1''(\prod_{i=t+1}^{|F|} y_i')x_2' = x_1' x_2'$, we get that $|x| - |z| = |\overline{\psi}(x)| - |\overline{\psi}(z)| = |x'| - |\overline{\psi}(z)| = j$. Finally, using (6.4) and (6.5), we have

$$\mathsf{d}(z, x) \le \mathsf{d}(z_1, x_1) + \mathsf{d}(z_2, x_2) \le t + \mathsf{d}(z_2, x_2) \le \mathsf{d}(z_1', x_1') + \mathsf{d}(z_2', x_2') = \mathsf{d}(z', x'),$$

establishing the claim.

3. We assert that the already defined map ψ has the claimed properties. Let $B \in \mathcal{B}(G_0)$ and let $k, l \in \mathsf{L}(B)$ be adjacent lengths. By the proof of 1, we know that there exists some y such that $y + \mathsf{L}(B) = \mathsf{L}(\psi(B))$. Let $k' = k + y$ and $l' = l + y$; in particular, k' and l' are adjacent lengths of $\mathsf{L}(\psi(B))$. Let $z' \in \mathsf{Z}_{k'}(\psi(B))$ and $x' \in \mathsf{Z}_{l'}(\psi(B))$ with $\mathsf{d}(z', x') \le \mathsf{d}(\mathsf{Z}_{k'}(\psi(B)), \mathsf{Z}_{l'}(\psi(B)))$. Similarly to the argument in 2, we can construct $z \in \mathsf{Z}_k(B)$ and $x \in \mathsf{Z}_l(B)$ with $\mathsf{d}(z, x) \le \mathsf{d}(z', x')$.

We now address the additional statements. Suppose that F_0 is finite. By Proposition 5.5, we know that the Structure Theorem holds for $\mathcal{B}(F_0 \cup \{d\})$ and that $\delta(F_0 \cup \{d\})$ is finite. Thus, by parts 1 and 2, we get that the Structure Theorem holds for $\mathcal{B}(F_0 \cup d\mathbb{N})$ and that $\delta(F_0 \cup d\mathbb{N})$ is finite. Since $\delta(F_0 \cup d\mathbb{N})$ is finite, Proposition 5.8 implies that $\mathsf{c}_{\mathrm{mon}}(F_0 \cup d\mathbb{N})$ is finite. □

The systems of sets of lengths of $\mathcal{B}(F_0 \cup d\mathbb{N})$ and $\mathcal{B}(F_0 \cup \{d\})$ are very closely related, but they are different in general. For finite F_0, the elasticity of $\mathcal{B}(F_0 \cup \{d\})$ is accepted (Proposition 5.5), yet we will see in Corollary 6.11 that this is, in general, not the case for $\mathcal{B}(F_0 \cup d\mathbb{N})$.

Proposition 6.10. *Let $d \in \mathbb{N}_{\ge 2}$ and $G_0 = \{-d, -1\} \cup (1 + d\mathbb{N}_0) \cup d\mathbb{N}_0$.*

1. *The Structure Theorem holds for $\mathcal{B}(G_0)$. More precisely, each $L \in \mathcal{L}(G_0)$ is an arithmetical progression with difference $d - 1$.*
2. *For each $B \in \mathcal{B}(G_0)$ and adjacent lengths k and l of $\mathsf{L}(B)$, we have $\mathsf{d}(\mathsf{Z}_k(B), \mathsf{Z}_l(B)) = d + 1$.*
3. *$\delta(G_0) = \infty$.*
4. *$\mathsf{c}_{\mathrm{mon}}(G_0) = d + 1$.*

Proof. Before we start the argument for the individual parts, we start with some general remarks. We begin by investigating $\mathcal{A}(G_0)$. Let $A \in \mathcal{A}(G_0)$. If $kd \mid A$ for some $k \in \mathbb{N}_0$, then $A = (kd)(-1)^{dl}(-d)^{k-l}$ for some $l \in [0, k]$. In particular, we have two atoms containing d, namely $U_1 = d(-1)^d$ and $U_d = d(-d)$. Suppose $\operatorname{supp}(A) \cap d\mathbb{N}_0 = \emptyset$. Then $A^+ = \prod_{i=1}^{|A^+|}(1 + k_i d)$ with $k_i \in \mathbb{N}_0$. It follows that $|A^+| \in \{1, d\}$. Moreover, if $|A^+| = d$, then $-1 \nmid A$ and therefore $A = A^+(-d)^{\sigma(A^+)/d}$. Thus, either $|A^+| = 1$ or else $|A^+| = d$ and $A = A^+(-d)^{\sigma(A^+)/d}$. Conversely, each zero-sum sequence $B \in \mathcal{B}(G_0 \setminus \{0\})$ with $B^+ = \prod_{i=1}^{d}(1 + k_i d)$, $k_i \in \mathbb{N}_0$ and $-1 \notin \operatorname{supp}(B^-)$ is an atom.

Let $B \in \mathcal{B}(G_0 \setminus \{0\})$ and let $z \in \mathsf{Z}(B)$. In view of the considerations just made, there exists a unique decomposition $z = z_1 z_d$ such that, whenever $A \mid z_1$, we have $|A^+| = 1$ and, whenever $A \mid z_d$, we have $|A^+| = d$. We denote $|z_d|$ by $t_d(z)$. Since $|B^+| = |z_1| + d|z_d|$, it follows that

$$|z| = |z_1| + |z_d| = |B^+| - (d-1)|z_d| = |B^+| - (d-1)t_d(z), \tag{6.6}$$

i.e., $|z|$ is determined by B^+ and $t_d(z)$.

By Proposition 6.9, and since 0 is a prime, it suffices to consider the set $G_1 = \{-d, -1\} \cup (1 + d\mathbb{N}_0) \cup \{d\}$ for the proofs of parts 1 and 3.

1. Let $B \in \mathcal{B}(G_1)$. Let $z \in \mathsf{Z}(B)$ and let $z = z_1 z_d$ be defined as above. Since $\mathsf{v}_{-1}(A) \ge 1$ for each A that neither fulfils $|A^+| = d$ nor equals U_d, it follows that

$$t_d(z) \ge \frac{\left(|B^+| - \mathsf{v}_d(B)\right) - \mathsf{v}_{-1}(B)}{d}. \tag{6.7}$$

By (6.6), we get that $\mathsf{L}(B)$ is contained in an arithmetical progression with difference $(d-1)$. In view of this, it suffices to establish the following claim.

Claim 1: If $|z| < \max \mathsf{L}(B)$, then there exists some $z' \in \mathsf{Z}(B)$ with $|z'| = |z| + (d-1)$ and $\mathsf{d}(z, z') = d+1$; in particular, $t_d(z') = t_d(z) - 1$. Moreover, $\mathsf{d}(z, z') \le d+1$.

To prove this, we first investigate the case $|z| = \max \mathsf{L}(B)$.

Claim 2: If $t_d(z) = 0$ or $\mathsf{v}_{-1}(A) \le 1$ for each $A \mid z$, then $|z| = \max \mathsf{L}(B)$.

Proof of Claim 2. If $t_d(z) = 0$, the claim is clear by (6.6). Thus, assume $\mathsf{v}_{-1}(A) \le 1$ for each $A \mid z$. In view of the characterization of atoms, it follows that $z_1 = z_1' U_d^{\mathsf{v}_d(B)}$ and $\mathsf{v}_{-1}(A) = 1$ for each atom $A \mid z_1'$. In particular, we have $|z_1| = \mathsf{v}_{-1}(B) + \mathsf{v}_d(B)$. In view of $d \cdot t_d(z) = |B^+| - |z_1|$, this implies

$$t_d(z) = \frac{\big(|B^+| - \mathsf{v}_d(B)\big) - \mathsf{v}_{-1}(B)}{d}.$$

Thus equality holds in (6.7), which by (6.6) implies that $|z|$ is maximal.

Proof of Claim 1. Suppose $|z| < \max \mathsf{L}(B)$. By Claim 2, we know that $t_d(z) > 0$ and that there exists some atom $C \mid z$ such that $\mathsf{v}_{-1}(C) > 1$. In view of the characterization of atoms given above, we have $\mathsf{v}_{-1}(C) \ge d$ and $|C^+| = 1$. Since $t_d(z) > 0$, there exists some atom $A_d \mid z$ with $|A_d^+| = d$. Let $z = A_d C z_0$. We consider the zero-sum sequences $B_1 = (-d)^{-1} A_d (-1)^d$ and $B_2 = (-1)^{-d} C(-d)$. Clearly, $\pi(B_1 B_2 z_0) = B$. We note that B_2 is an atom as $|B_2^+| = 1$. Yet, since $|B_1^+| = d$ but $\mathsf{v}_{-1}(B_1) \ge 1$, we get that B_1 is not an atom; more precisely, $\max \mathsf{L}(B_1) = d$. Thus, replacing the two atoms A_d and C in z by the atom B_2 and any factorization of length d of B_1 completes the claim.

2. By Proposition 6.9.3 and since 0 is a prime, it suffices to consider G_1 for finding an upper bound on $\mathsf{d}(\mathsf{Z}_k(B), \mathsf{Z}_l(B))$. Thus, by Claim 2, we get that $\mathsf{d}(\mathsf{Z}_k(B), \mathsf{Z}_l(B)) \le d+1$. The reverse inequality follows by (2.1) in view of Proposition 6.9.3 and part 1.

3. We consider $B = (1+kd)^d d^{1+kd} (-d)^{1+kd} (-1)^{d(1+kd)}$. We note that $\mathsf{L}(B) = \{2+kd, 1+d+kd\}$ and $z = \big((1+kd)^d (-d)^{1+kd}\big) \cdot (d(-1)^d)^{1+kd}$ is its only factorization of length $2+kd$. The factorization $z' = \big((1+kd)(-1)^{1+kd}\big)^d \cdot \big(d(-d)\big)^{1+kd}$ has length $1+d+kd$ and $\mathsf{d}(z', z) = |z'| = 1+d+kd$, implying that $\delta(B) \ge 1+d+kd$, and the claim follows by letting $k \to \infty$.

4. By part 2 and since 0 is prime, it is sufficient to show that, for any two factorizations $z, y \in \mathsf{Z}(G_0 \setminus \{0\})$ with $\pi(z) = \pi(y)$, we have that: if $|z| = |y|$, then z and y can be concatenated by a monotone 2-chain. Clearly, in this case monotone means that each factorization in this chain has length $|z|$, i.e., we claim that z and y can be concatenated by a 2-chain in $\mathsf{Z}_{|z|}(\pi(z))$. We proceed by induction on $|z|$. Let $z, y \in \mathsf{Z}(G_0)$ with $\pi(z) = \pi(y)$ and suppose that $|z| = |y|$. If $|z| = 1$, the statement is trivial. Thus, assume $|z| \ge 2$ and that the statement is true for factorizations of length at most $|z| - 1$. We make the following claim.

Claim 3: There exist $z', y' \in \mathsf{Z}(\pi(z))$ with $|z'| = |y'| = |z|$ such that z and z', as well as y and y', can be concatenated by a 2-chain in $\mathsf{Z}_{|z|}(\pi(z))$ and $\gcd\{z', y'\} \ne 1$.

We assume this claim is true and complete the argument. Let z' and y' be factorizations with the claimed properties and let $U \in \mathcal{A}(G_0)$ with $U \mid \gcd\{z', y'\}$. We set $z'' = U^{-1} z'$ and $y'' = U^{-1} y'$. By induction hypothesis, there exists a 2-chain $z'' = z_0'', z_1'', \ldots, z_s'' = y''$ in $\mathsf{Z}_{|z''|}(\pi(U^{-1} z'))$. We note that $U \cdot z_i'' \in \mathsf{Z}_{|z|}(\pi(z))$ for each $i \in [0, s]$. Thus, z' and y' can be concatenated by a 2-chain in $\mathsf{Z}_{|z|}(\pi(z))$. Combining these three chains, the result follows.

Proof of Claim 3. If $0 \mid z$, then $0 \mid y$ and the claim is trivial. Thus, assume $0 \nmid z$.

Let $z = z_1 z_d$ and $y = y_1 y_d$ be as defined at the beginning of the proof and recall that $|z| = |y|$ is equivalent to $t_d(z) = t_d(y)$.

Before starting the actual argument, we make three subclaims.

Claim 3.1: Let $h \mid \pi(z_1)$ and $g \mid \pi(z_d)$ with $g, h \in 1 + d\mathbb{N}_0$ and $h \le g$. Then there exists a factorization x of $\pi(z)$ such that, with $x = x_1 x_d$ as above, $\pi(x_1)^+ = \pi(z_1)^+ g h^{-1}$ and $\pi(x_d)^+ = \pi(z_d)^+ h g^{-1}$ and $\mathsf{d}(z, x) \le 2$; in particular, $|x| = |z|$.

To see this, let $A_h \mid z_1$ and $A_g \mid z_d$ with $h \mid A_h$ and $g \mid A_g$. We set $A_h' = h A_g g^{-1} (-d)^{-(g-h)/d}$ and $A_g' = g A_h h^{-1} (-d)^{(g-h)/d}$. Note that this process is well-defined and that A_g' and A_h' are atoms by the above characterization of atoms. Let $x = z A_g' A_h' A_g^{-1} A_h^{-1}$. Noting that $x_1 = A_g' A_h^{-1} z_1$ and $x_d = A_h' A_g^{-1} z_d$, the claim is established.

Claim 3.2: Suppose that $t_d(z) = 0$. Then z and y can be concatenated by a 2-chain in $\mathsf{Z}_{|z|}(\pi(z))$.

Informally, each atom in z and y contains exactly one positive element, hence distinct atoms containing the same positive element can only differ in the negative part. Successively exchanging $(-1)^d$ for $-d$ and vice versa, for suitable pairs of atoms, we can construct such a chain.

To give a formal argument, we use the independent material of Section 7 which follows. Note that, in this case, $|z| = |y| = |\pi(z)^+|$ and $\mathcal{A}(\mathcal{E}(G_P^-)) = \{(-d, -d), (-1, -1), ((-1)^d, -d), (-d, (-1)^d)\}$. Thus $G' \cong \mathbb{Z}$ with $G_0' = \{0, 1, -1\}$, where G' and G_0' are as defined before Theorem 7.3, whence $\mathsf{D}(\mathcal{S}(G_P^-), \mathcal{E}(G_P^-)) = 2$ by (7.14). Hence Theorem 7.3 shows that there is a 2-chain concatenating z and y.

Claim 3.3: Suppose that $t_d(z) = |z|$. Then z and y can be concatenated by a 2-chain in $\mathsf{Z}_{|z|}(\pi(z))$.

Informally, since in this case $\operatorname{supp}(\pi(z)) = \{-d\}$, we can apply an argument similar to the one in Claim 3.1, without additional condition on the relative size of g and h.

To get a formal argument, note that in this case $\pi(z) \in \mathcal{B}(G_0 \setminus \{-1\})$. By Lemma 5.3, we get that the block monoid associated to $\mathcal{B}(G_0 \setminus \{-1\})$ is $\mathcal{B}(\{0 + d\mathbb{Z}, 1 + d\mathbb{Z}\}) \subset \mathcal{B}(\mathbb{Z}/d\mathbb{Z})$. However, $\mathcal{B}(\{0 + d\mathbb{Z}, 1 + d\mathbb{Z}\})$ is factorial, and thus its catenary degree is 0; also note that the former monoid is thus half-factorial. Since the catenary degree in the fibers of the block homomorphism is 2 (see Lemma 3.3), the claim follows.

Now, we give the actual proof of Claim 3. In view of Claim 3.2, we may assume that $t_d(z) > 0$. Hence, let $S \mid \pi(z)$ be a subsequence with $\mathrm{supp}(S) \subset 1 + d\mathbb{N}_0$ and $|S| = d$. Moreover, assume that $\sigma(S)$ is minimal among all such subsequences of $\pi(z)$. We assert that there exists some $x' \in \mathsf{Z}_{|z|}(\pi(z))$ such that $S \mid \pi(x'_d)$ and z and x' can be concatenated by a 2-chain in $\mathsf{Z}_{|z|}(\pi(z))$. Let $x' \in \mathsf{Z}_{|z|}(\pi(z))$ be a factorization such that z and x' can be concatenated by a 2-chain in $\mathsf{Z}_{|z|}(\pi(z))$ and such that $S' = \gcd\{\pi(x'_d), S\}$ is maximal. We show that $S' = S$. Assume to the contrary that $S' \neq S$. Let $h \mid \pi(x'_1)$ with $hS' \mid S$. We observe that there exists some $g \mid S'^{-1}\pi(x'_d)$ with $g \in 1 + d\mathbb{N}_0$ and $g \geq h$; otherwise, the sequence $gh^{-1}S$ would contradict the minimality of $\sigma(S)$.

We apply Claim 3.1 to x' (with these elements g and h) and denote the resulting factorization by x''. Since it can be concatenated to z by a 2-chain in $\mathsf{Z}_{|z|}(\pi(z))$ and yet $hS' \mid \gcd\{\pi(x''_d), S\}$, its existence contradicts the maximality of S' for x'. Thus $S' = S$.

Since $S \mid \pi(x'_d)$, we have that $U = S(-d)^{\sigma(S)/d} \mid \pi(x'_d)$. Let $z'_d \in \mathsf{Z}(\pi(x'_d))$ with $U \mid z'_d$. Since $t_d(\pi(x'_d)) = |x'_d|$, Claim 3.3 applied to x'_d yields that x'_d and z'_d can be concatenated by a 2-chain in $\mathsf{Z}_{|x'_d|}(\pi(x'_d))$. We set $z' = z'_d x'_1$ and observe that x' and z', and thus z and z', can be concatenated by a 2-chain in $\mathsf{Z}_{|z|}(\pi(z))$ and $U \mid z'$

In the same way, noting that S depends only on $\pi(z)$ and not on z, we get a factorization $y' \in \mathsf{Z}_{|z|}(\pi(z))$ with $U \mid y'$ such that y and y' can be concatenated by a 2-chain in $\mathsf{Z}_{|z|}(\pi(z))$. Since $U \mid \gcd\{z', y'\}$, the claim is established. □

Proof of Theorem 6.4. By Lemma 3.3, it suffices to consider $\mathcal{B}(G_P)$. The case $d = 1$ is trivial. Suppose $d \geq 2$. The implication from (a) to (b) is merely Proposition 6.8. The other one follows, when $G_P^+ \setminus d\mathbb{Z}$ is finite, by Proposition 6.9, and when G_P^+ is a subset of $1 + d\mathbb{Z}$, by Proposition 6.10. □

By [1], it is known that Krull monoids with infinite cyclic class group can have finite, non-accepted elasticity. The following result shows that, even if the Structure Theorem holds, the elasticity is not necessarily accepted.

Corollary 6.11. *Let H be a Krull monoid and $\varphi\colon H \to \mathcal{F}(P)$ a cofinal divisor homomorphism into a free monoid such that the class group $G = \mathcal{C}(\varphi)$ is an infinite cyclic group that we identify with $\mathbb{Z}$. Let $G_P \subset G$ denote the set of classes containing prime divisors. Suppose that $1 \in G_P^+$ and $G_P^- = \{-d, -1\}$ for some $d \in \mathbb{N}$. Suppose that the Structure Theorem holds for H. Then exactly one of the following two statements holds:*

(a) *H is half-factorial or G_P is finite.*
(b) *$\rho(H) = d$ and the elasticity is not accepted.*

Proof. Half-factorial monoids obviously have accepted elasticity and monoids with G_P finite also have accepted elasticity (Proposition 5.5). Thus, we assume that H is not half-factorial and that G_P is infinite, and show that under these assumptions $\rho(H) = d$ and the elasticity is not accepted. Note that since H is not half-factorial, we have $d \geq 2$.

We recall that if $A \in \mathcal{A}(G_P)$ with $(-1) \mid A$, then $|A^+| = 1$ (as explained in the proof of Proposition 6.10).

Let $B \in \mathcal{B}(G_P)$. We show that $\rho(B) < d$. Assume to the contrary $\rho(B) \geq d$. That is, there exist $z, z' \in \mathsf{Z}(B)$ such that $|z'|/|z| \geq d$. By Lemma 4.3, we know that $|A^+| \leq d$ for each $A \in \mathcal{A}(G_P)$. Thus, we get $|z| \geq \mathsf{v}_0(z) + |B^+|/d$, whereas clearly $|z'| \leq \mathsf{v}_0(z') + |B^+|$.

Consequently, we have $\rho(B) \leq d$, and $\rho(B) = d$ is equivalent to the following: $|A^+| = d$ for each atom $A \mid z$ and $|A'^+| = 1$ for each atom $A' \mid z'$. It follows that $\mathsf{v}_{-1}(B) = 0$, i.e., $B \in \mathcal{B}(G_P \setminus \{-1\})$. By [1], or Lemma 5.3 and [28, Proposition 6.3.1], we get that $\rho(\mathcal{B}(G_P \setminus \{-1\})) \leq \rho(\mathbb{Z}/d\mathbb{Z}) = d/2 < d$, a contradiction.

It remains to show that $\rho(G_P) \geq d$. We may assume that $0 \notin G_P$. We note the existence of the two atoms $1(-1)$ and $1^d(-d)$ in $\mathcal{A}(G_P)$. Thus, 1 and 1^d are elements of $\mathcal{A}(G_P)^+$. Thus, $\rho^\kappa(1^d) \geq d$, and the claim follows by Lemma 4.12. □

Our proofs that the Structure Theorem does not hold rely on the existence of a single exceptional factorization, yet the following example illustrates that sets of lengths can deviate by more than a single element (or a globally bounded number of elements) from being an AAMP.

Example 6.12. Let $d, k, l \in \mathbb{N}$ and $e \in [1, d-1]$, and set $B = (e + kd)(-e + \ell d)1^{(k+\ell)d}(-1)^{(k+\ell)d}(-d)^{k+\ell}$. Then

$$\begin{aligned}\mathsf{L}(B) = \{1 + k + \ell + (k+\ell)(d-1)\} &\cup \{1 + e + k + \ell + i(d-1) \mid i \in [k, k+\ell-1]\} \\ &\cup \{2 - e + k + \ell + i(d-1) \mid i \in [\ell, \ell+k]\} \\ &\cup \{2 + k + \ell + i(d-1) \mid i \in [0, k+\ell-1]\}.\end{aligned}$$

7. Chains of factorizations

In a large class of monoids and domains satisfying natural (algebraic) finiteness conditions, the catenary degree is finite (see [28] for an overview and [5,29,4,38] for some recent work). However, the understanding of the structure of the concatenating chains is still very limited. Whereas, on the one hand, the finiteness of the monotone catenary degree is a rare phenomenon (inside the class of objects having finite catenary degree), the following two positive phenomena have been observed. First, in a large class of monoids, all problems with the monotonicity of concatenating chains occur only at the beginning and the end of concatenating chains ([15, Theorem 1.1], [16, Theorem 3.1]). Second, in various settings, there is a large subset consisting of 'big' elements having extremely nice concatenating chains (see [23, Theorem 4.3], [28, Theorems 7.6.9 and 9.4.11]).

Let H be a Krull monoid with infinite cyclic class group and $G_P \subset G$ as always. By Theorem 5.2, it suffices to consider the situation where G_P^+ is infinite and $2 \le |G_P^-| < \infty$. Our first result points out that, in general, the monotone catenary degree is infinite. In contrast to this, the main result (Corollary 7.4) shows that there is a constant M^* such that, for a large class of elements a, any two factorizations z and y of a with y having maximal length can be concatenated by a monotone M^*-chain of factorizations and thus, for those factorizations z and y of a neither of which need be of maximal length, there is an M^*-chain between z and y which 'changes direction' at most once.

Proposition 7.1. *Let H be a Krull monoid and $\varphi\colon H \to \mathcal{F}(P)$ a cofinal divisor homomorphism into a free monoid such that the class group $G = \mathcal{C}(\varphi)$ is an infinite cyclic group that we identify with $\mathbb{Z}$. Let $G_P \subset G$ denote the set of classes containing prime divisors. Suppose that $-d_1, -d_2, d_1d_2 \in G_P$, where $3 \le d_1 < d_2$, $\gcd(d_1, d_2) = 1$ and $d_1 - 1 \nmid d_2 - 1$, and that G_P contains infinitely many positive integers congruent to $d_1 + d_2$ modulo d_1d_2. Let $d = \gcd(d_1 - 1, d_2 - 1)$. Then, for every $M,\ N \ge 0$, there exists $a \in H$ and $z,\ z' \in \mathsf{Z}(a)$ such that*

$$|z'| = |z| + d \le |z| + d_1 - 2, \tag{7.1}$$

$$|z| \in [\min \mathsf{L}(a) + N,\ \max \mathsf{L}(a) - N], \quad \textit{and} \tag{7.2}$$

$$\mathsf{d}\Big(z, \bigcup_{i=1}^{|Z|+d_1-2} \mathsf{Z}_i(a) \setminus \{z\}\Big) > M. \tag{7.3}$$

In particular, $\mathsf{c}_{\text{mon}}(H) = \infty$ *and* $\delta(H) = \infty$

Proof. That $\mathsf{c}_{\text{mon}}(H) = \delta(H) = \infty$ follows from (7.1) and (7.3), so we need only show (7.1), (7.2) and (7.3) hold. By Lemma 3.3, it suffices to prove the assertions for $\mathcal{B}(G_P)$. We may also assume without loss of generality that

$$N \ge d_2 - 1 \quad \text{and} \quad M \ge d_1,$$

as the theorem holding for large values of M and N implies it holding for all smaller values.

In view of the hypotheses, there exists $L \in G_P$ with

$$L > d_2 M \ge d_1 d_2, \tag{7.4}$$

$$L \equiv d_1 \mod d_2 \quad \text{and} \quad L \equiv d_2 \mod d_1. \tag{7.5}$$

Let $B \in \mathcal{B}(\{d_1d_2, -d_1, -d_2, L\}) \subset \mathcal{B}(G_P)$ be the sequence

$$B = L^{2d_1d_2N}(-d_2)^{2d_1LN}(-d_1)^{2d_2LN}(d_1d_2)^{2LN}.$$

Let

$$A_1 = L^{d_1}(-d_1)^L \quad \text{and} \quad A_2 = L^{d_2}(-d_2)^L.$$

Since $\gcd(d_1, d_2) = 1$, it follows, in view of (7.5) and by reducing modulo d_1 and d_2, respectively, that A_1 and A_2 are both atoms. Also define

$$B_1 = (d_1d_2)(-d_1)^{d_2} \quad \text{and} \quad B_2 = (d_1d_2)(-d_2)^{d_1},$$

which, since they both contain exactly one positive integer, must also be atoms. In view of (7.5), define

$$A_0 = L(-d_2)^{\frac{L-d_1}{d_2}}(-d_1),$$

which is also an atom for similar reasons.

Let $z \in \mathsf{Z}(B)$ be given by

$$z = A_1^{d_2N}A_2^{d_1N}B_1^{LN}B_2^{LN}.$$

Since $d = \gcd(d_1 - 1, d_2 - 1)$, it follows that there exists an integer $l \in [1, d_2 - 1]$ such that

$$l(d_2 - d_1) \equiv -d \mod d_2 - 1.$$

Let

$$l' = \frac{l(d_2 - d_1) + d}{d_2 - 1} \in \mathbb{N}. \tag{7.6}$$

Then, since $d = \gcd(d_1 - 1, d_2 - 1) \le d_1 - 1$, it follows that $1 \le l' \le l \le d_2 - 1$. Note that we have the identities

$$\pi(A_1^{d_2}B_2^L) = \pi(A_2^{d_1}B_1^L) \quad \text{and} \quad \pi(A_2B_1) = \pi(A_0^{d_2}B_2).$$

Thus, by considering the definition of z and recalling that $N \geq d_2 - 1 \geq l \geq l'$, we see that

$$z' = A_1^{d_2N-ld_2} A_2^{d_1N+ld_1-l'} A_0^{l'd_2} B_1^{LN+lL-l'} B_2^{LN-lL+l'}$$

is another factorization $z' \in \mathsf{Z}(B)$ besides z.

Note that $|z'| - |z| = -l(d_2 - d_1) + l'(d_2 - 1) = d$. Moreover, since $d_1 - 1 \nmid d_2 - 1$, $d_1 < d_2$ and $\gcd(d_1 - 1, d_2 - 1) = d$, it follows that $d < d_1 - 1$. Thus (7.1) holds. Also, the factorizations

$$A_2^{2d_1N} B_1^{2LN} \in \mathsf{Z}(B) \quad \text{and} \quad A_1^{2d_2N} B_2^{2LN} \in \mathsf{Z}(B)$$

show that

$$\min \mathsf{L}(B) + N \leq \min \mathsf{L}(B) + (d_2 - d_1)N \leq |z| \leq \max \mathsf{L}(B) - (d_2 - d_1)N \leq \max \mathsf{L}(B) - N,$$

whence (7.2) holds. It remains to establish (7.3). We begin with the following claim.

Claim 1: If $A|B$ is an atom with $d_1d_2 \in \text{supp}(A)$, then d_1d_2 is the only positive element dividing A and $\mathsf{v}_{d_1d_2}(A) = 1$.

Suppose instead that $a \mid A(d_1d_2)^{-1}$ with $a \in \{L, d_1d_2\}$. Then we must have $\mathsf{v}_{-d_2}(A) < d_1$ and $\mathsf{v}_{-d_2} < d_1$, else $(d_1d_2)(-d_1)^{d_2}$ or $(d_1d_2)(-d_2)^{d_1}$ would be a proper, nontrivial zero-sum subsequence dividing A, contradicting that A is an atom. But now (in view of (7.4))

$$2d_1d_2 > -\sigma(A^-) = \sigma(A^+) \geq a + d_1d_2 \geq \min\{L, d_1d_2\} + d_1d_2 = 2d_1d_2,$$

a contradiction. So Claim 1 is established.

In view of Claim 1, we see that, in any factorization y of B, there will always be $2LN$ atoms A having $A(d_1d_2)^{-1}$ consisting entirely of negative terms. Thus the length of any factorization of B is determined entirely by the number of atoms containing an L. Moreover, by considering sums modulo d_i, we find (in view of (7.5) and $\gcd(d_1, d_2) = 1$) that $(d_1d_2)(-d_1)^{d_2}$ and $(d_1d_2)(-d_2)^{d_1}$ are the only atoms dividing B which contain d_1d_2. As a result, we in fact have the factorization of B completely determined by how the $2d_1d_2N$ terms equal to L are factored (that is, if $y_L|y$ is the subfactorization consisting of all atoms containing an L, then $\pi(y_L^{-1}y)$ has a unique factorization, which will always have length $2LN$). We continue with the next claim.

Claim 2: If $A|B$ is an atom with $L, -d_1, -d_2 \in \text{supp}(A)$, then $\mathsf{v}_L(A) = 1$.

Suppose instead that $L^2|A$. In view of (7.5) and (7.4), both $\frac{L-d_1}{d_2}$ and $\frac{L-d_2}{d_1}$ are positive integers. Consequently, we must have $\mathsf{v}_{-d_1}(A) < \frac{L-d_2}{d_1}$ and $\mathsf{v}_{-d_2} < \frac{L-d_1}{d_2}$, else

$$L(-d_1)^{(L-d_2)/d_1}(-d_2) \quad \text{or} \quad L(-d_2)^{(L-d_1)/d_2}(-d_1)$$

would be a proper, nontrivial zero-sum subsequence dividing A, contradicting that A is an atom. But now

$$2L - d_1 - d_2 > -\sigma(A^-) = \sigma(A^+) \geq 2L,$$

a contradiction. So Claim 2 is established.

In view of (7.5), $\gcd(d_1, d_2) = 1$ and Claim 1 and 2, we see that if $A|B$ is an atom with $L \in \text{supp}(A)$, then either

(a) $A = A_1$ and $\mathsf{v}_{-d_2}(A) = 0$,
(b) $A = A_2$ and $\mathsf{v}_{-d_1}(A) = 0$, or
(c) $\mathsf{v}_L(A) = 1$ and $\mathsf{v}_{d_1d_2}(A) = 0$.

Let $y \in \mathsf{Z}(B)$ be a factorization with $\mathsf{d}(z, y) \leq M$ and let $y_L|y$ and $z_L|z$ be the corresponding sub-factorizations consisting of all atoms which contain an L. In view of the definition of z, since $\mathsf{d}(z, y) \leq M$ and $L > d_2M$ (by (7.4)), and since $(d_1d_2)(-d_1)^{d_2}$ is the only atom containing a $-d_1$ in $z_L^{-1}z$, it follows that

$$\mathsf{v}_{-d_1}(\pi(y_L)) \leq \mathsf{v}_{-d_1}(\pi(z_L)) + Md_2 = d_2NL + Md_2 < d_2NL + L;$$

thus the multiplicity m_1 of the atom A_1 in y is at most d_2N (since each such atom A_1 requires L terms equal to $-d_1$). Likewise,

$$\mathsf{v}_{-d_2}(\pi(y_L)) \leq \mathsf{v}_{-d_2}(\pi(z_L)) + Md_1 = d_1NL + Md_1 < d_1NL + L,$$

whence the multiplicity m_2 of the atom A_2 in y is at most d_1N.

Let m_0 be the number of atoms dividing y containing exactly one term L. Since all atoms containing an L must be of one of the three previously described forms, it follows that

$$d_1m_1 + d_2m_2 + m_0 = \mathsf{v}_L(B) = 2d_1d_2N. \tag{7.7}$$

Let m'_0, m'_1 and m'_2 be analogously defined for z instead of y. Then $m'_0 = 0$, $m'_1 = d_2N$ and $m'_2 = d_1N$. In view of (7.7) and the comments after Claim 1, and since $m_1 \leq d_2N = m'_1$ and $m_2 \leq d_1N = m'_2$, it follows that

$$|y| = |z| + (m'_1 - m_1)(d_1 - 1) + (m'_2 - m_2)(d_2 - 1) \geq |z|.$$

Moreover, unless $m_1 = m_1'$ and $m_2 = m_2'$, then $|y| \ge |z| + d_1 - 1$. On the other hand, if $m_1 = m_1' = d_2N$ and $m_2 = m_2' = d_1N$, then $m_0 = 0$ (in view of (7.7)), whence $z_L = y_L$ (recalling that all atoms containing an L must be of one of the three previously described forms), from which $z = y$ follows by the comments after the proof of Claim 1. Consequently, we conclude that $\mathsf{d}(z, y) \le M$ implies either $y = z$ or $|y| \ge |z| + d_1 - 1$, which establishes (7.3), completing the proof. □

The following lemma helps describe when an atom can contain more than one positive term.

Lemma 7.2. *Let $G_0 \subset \mathbb{Z}$ be a condensed set such that G_0^- is finite and nonempty. Let $M = |\min G_0|$, let $U \in \mathcal{A}(G_0)$ and let $R|U^-$ be the subsequence consisting of all negative integers with multiplicity at least $M-1$ in U. Suppose there is some $L \in \Sigma(U^+) \setminus \{\sigma(U^+)\}$ such that*

$$|U^+| \ge 2, \quad L \ge (M-1)^2, \quad \text{and} \quad \sigma(U^+) \ge L + (M-1)^2. \tag{7.8}$$

Then the following statements hold:

1. *There is some $a \in \operatorname{supp}(U) \cap G_0^-$ with $\mathsf{v}_a(U) \ge M-1$, i.e., R is nontrivial.*
2. *For any such $a \in \operatorname{supp}(R)$, we have $(-L + a\mathbb{Z}) \cap \Sigma(U^-) = \emptyset$.*
3. *There exists a subsequence $R'|U^-$ with $R|R'$ such that $L \notin \langle \operatorname{supp}(R') \rangle = n\mathbb{Z}$ and $|R'^{-1}U^-| \le n-2$; in particular, $\operatorname{supp}(R) \subset \operatorname{supp}(R') \subset n\mathbb{Z}$ does not generate $\mathbb{Z}$.*

Proof. 1. Let $U_L|U^+$ be a proper subsequence with sum equal to L. Note that $|G_0^-| \le M$. Thus $\sigma(U^+) \ge L \ge (M-1)^2 > (M-2)|G_0^-|$, whence the pigeonhole principle implies that there is some $a \in \operatorname{supp}(U) \cap G_0^-$ with $\mathsf{v}_a(U) \ge M-1$.

2. Let $a|U^-$ with $\mathsf{v}_a(U) \ge M-1$ and let $\phi_a \colon \mathbb{Z} \to \mathbb{Z}/a\mathbb{Z}$ denote the natural homomorphism. We say that a sequence T is a zero-sum sequence (zero-sum free, resp.) modulo a if $\phi_a(T) \in \mathcal{F}(\mathbb{Z}/a\mathbb{Z})$ has the respective property. Suppose $(-L + a\mathbb{Z}) \cap \Sigma(U^-)$ is nonempty and let S be a zero-sum free modulo a subsequence $S|U^-$ (possibly trivial) with $\sigma(S) \equiv -L \mod a$. Note that any zero-sum free modulo a subsequence $T|U^-$ has length at most $\mathsf{D}(\mathbb{Z}/a\mathbb{Z}) - 1 = |a| - 1$ [28, Theorem 5.1.10], and thus

$$|\sigma(T)| \le (|a|-1) \cdot |\min((\operatorname{supp}(U) \cap G_0^-) \setminus \{a\})| \le (M-1)^2 \le L; \tag{7.9}$$

in particular, $|\sigma(S)| \le (M-1)^2 \le L$.

Now factor $S^{-1}U^- = S_0 S_1 \cdot \ldots \cdot S_t a^{\mathsf{v}_a(U^-)}$, where S_0 is zero-sum free modulo a and each S_i, for $i \ge 1$, is an atom modulo a. In view of $|\sigma(S_0)| \le (M-1)^2$ (from (7.9)) and the hypothesis $\sigma(U^+) \ge L + (M-1)^2$, we have

$$|\sigma(SS_1 \cdot \ldots \cdot S_t a^{\mathsf{v}_a(U^-)})| = |\sigma(S_0^{-1}U^-)| \ge L. \tag{7.10}$$

If $|\sigma(SS_1 \cdot \ldots \cdot S_t)| \le L$, then it follows, in view of (7.10) and the definitions of S and the S_i, that we can append on to $SS_1 \cdot \ldots \cdot S_t$ a sufficient number of terms equal to a so as to obtain a subsequence $B_L|S_0^{-1}U^-$ with $SS_1 \cdot \ldots \cdot S_t | B_L$ and $\sigma(B_L) = -L$, and now $U_L B_L | U$ is a proper, nontrivial zero-sum subsequence, contradicting that U is an atom. Therefore $|\sigma(SS_1 \cdot \ldots \cdot S_t)| > L$, and let $t' < t$ be the maximal non-negative integer such that $|\sigma(SS_1 \cdot \ldots \cdot S_{t'})| \le L$, which exists in view of $|\sigma(S)| \le (M-1)^2 \le L$. By its maximality, we have

$$|\sigma(S_1 \cdot \ldots \cdot S_{t'})| > L - |\sigma(S)| - |\sigma(S_{t'+1})| \ge L - |\sigma(S)| - |a|M, \tag{7.11}$$

where the second inequality follows by recalling that $S_{t'+1}$ is an atom modulo a and thus has length at most $\mathsf{D}(\mathbb{Z}/a\mathbb{Z}) = |a|$. From the definitions of all respective quantities, both the left and right hand side of (7.11) is divisible by a, whence

$$|\sigma(S_1 \cdot \ldots \cdot S_{t'})| \ge L - |\sigma(S)| - |a|(M-1).$$

But now we see, in view of $\mathsf{v}_a(U) \ge M-1$ and the definition of t', that we can append on to $SS_1 \cdot \ldots \cdot S_{t'}$ a sufficient number of terms equal to a so as to obtain a subsequence $B_L|S_0^{-1}U^-$ with $SS_1 \cdot \ldots \cdot S_{t'}|B_L$ and $\sigma(B_L) = -L$, once again contradicting that U is an atom. So we conclude that $(-L + a\mathbb{Z}) \cap \Sigma(U^-)$ is empty.

3. In view of part 2, we see that

$$-L \notin \langle a \rangle + \Sigma(U^-). \tag{7.12}$$

Now, if $|a^{-\mathsf{v}_a(U^-)}U^-| \le |a| - 2$, then $\operatorname{supp}(R) = \{a\}$ (recall $|a| \le M$ and $\mathsf{v}_g(R) \ge M-1$ for all $g \in \operatorname{supp}(R)$) and the final part of the lemma holds with $R' = R$ in view of (7.12). Therefore we may assume $y = |a^{-\mathsf{v}_a(U^-)}U^-| \ge |a| - 1$. Note that (7.12) implies that

$$\phi_a(-L) \notin \Sigma_y(\phi_a(a^{-\mathsf{v}_a(U^-)}U^-)0^y) = \Sigma(\phi_a(U^-)) \ne \mathbb{Z}/a\mathbb{Z}.$$

As a result, applying the Partition Theorem (see [34, Theorem 3]) to $\phi_a(a^{-\mathsf{v}_a(U^-)}U^-)0^y$, now yields part 3. To be more precise, we apply that result with sequences $S = S' = \phi_a(a^{-\mathsf{v}_a(U^-)}U^-)0^y$ and number of summands $n = y$; also note that the resulting coset from the Partition Theorem must be a subgroup in view of the high multiplicity of 0 and that $R|R'$ since $\mathsf{v}_g(R) \ge M-1 > |a| - 2$ for all $g \in \operatorname{supp}(R)$. □

Before stating the next result, we need to first introduce some notions. Let $G_0 \subset \mathbb{Z} \setminus \{0\}$ be a condensed set such that G_0^- is finite and nonempty, and let $B \in \mathcal{B}(G_0)$. If $z = A_1 \cdot \ldots \cdot A_n \in \mathsf{Z}(B)$, with $A_i \in \mathcal{A}(G_0)$, then we let

$$z^+ = A_1^+ \cdot \ldots \cdot A_n^+ \in \mathcal{F}(\mathcal{A}(G_0)^+)$$

and $\mathsf{Z}(B)^+ = \{z^+ \mid z \in \mathsf{Z}(B)\}$. We can then define a partial order on $\mathsf{Z}(B)^+$ by declaring, for $z^+, y^+ \in \mathsf{Z}(B)^+$, that $z^+ \le y^+$ when $z^+ = A_1^+ \cdot \ldots \cdot A_n^+ \in \mathsf{Z}(B)^+$, where $A_i \in \mathcal{A}(G_0)$,

$$y = (B_{1,1} \cdot \ldots \cdot B_{1,k_1}) \cdot (B_{2,1} \cdot \ldots \cdot B_{2,k_2}) \cdot \ldots \cdot (B_{n,1} \cdot \ldots \cdot B_{n,k_n})$$
$$\text{with } B_{j,i} \in \mathcal{A}(G_0) \quad \text{and} \quad A_j^+ = B_{j,1}^+ \cdot \ldots \cdot B_{j,k_j}^+ \quad \text{for } j \in [1, n] \text{ and } i \in [1, k_j].$$

We then define $\Upsilon(B)$ to be all those factorizations $z \in \mathsf{Z}(B)$ for which $z^+ \in \mathsf{Z}(B)^+$ is maximal with respect to this partial order.

Note that, if $z, y \in \mathsf{Z}(B)$ with $z^+ \lneq y^+$, then $|z| < |y|$. Thus $\Upsilon(B)$ includes all factorizations $z \in \mathsf{Z}(B)$ of maximal length $|z| = \max \mathsf{L}(B)$, and equality holds, namely

$$\Upsilon(B) = \{z \in \mathsf{Z}(B) \mid |z| = |B^+|\}, \tag{7.13}$$

when $\max \mathsf{L}(B) = |B^+|$. If H is a Krull monoid, $\varphi\colon H \to \mathcal{F}(P)$ a cofinal divisor homomorphism and $a \in H$, then we define

$$\Upsilon(a) = \{z \in \mathsf{Z}(a) \mid \overline{\boldsymbol{\beta}}(z) \in \Upsilon(\boldsymbol{\beta}(a))\}.$$

For a pair of monoids $H \subset D$, we recall the definition of the *relative Davenport constant*, originally introduced in [24] and denoted $\mathsf{D}(H, D)$, which is the minimum $N \in \mathbb{N} \cup \{\infty\}$ such that if $z \in \mathsf{Z}(D) = \mathcal{F}(\mathcal{A}(D))$ with $\pi(z) \in H$, then there exists $z'|z$ with $\pi(z') \in H$ and $|z'| \le N$.

Next, we introduce two new monoids associated to $\mathcal{F}(G_0)$. We assume that $\emptyset \ne G_0 \subset \mathbb{Z} \setminus \{0\}$, yet here we do not assume that G_0 is condensed. Consider the free monoid $\mathcal{F}(G_0) \times \mathcal{F}(G_0)$ and let

$$\mathcal{E}(G_0) = \{(S_1, S_2) \in \mathcal{F}(G_0) \times \mathcal{F}(G_0) \mid \sigma(S_1) = \sigma(S_2)\} \subset \mathcal{F}(G_0) \times \mathcal{F}(G_0)$$

the subset of pairs of sequences with equal sum and

$$\mathcal{S}(G_0) = \{(S_1, S_2) \in \mathcal{F}(G_0) \times \mathcal{F}(G_0) \mid S_1 = S_2\} \subset \mathcal{E}(G_0) \subset \mathcal{F}(G_0) \times \mathcal{F}(G_0)$$

the subset of symmetric pairs. Note both $\mathcal{E}(G_0)$ and $\mathcal{S}(G_0)$ are monoids; furthermore, $\mathcal{S}(G_0)$ is saturated and cofinal in $\mathcal{E}(G_0)$, and $\mathcal{E}(G_0)$ is saturated and cofinal in $\mathcal{F}(G_0) \times \mathcal{F}(G_0)$. Thus, if we let G' denote the class group of the inclusion $\mathcal{S}(G_0) \hookrightarrow \mathcal{E}(G_0)$ and let

$$G'_0 = \{[u] \in G' \mid u \in \mathcal{A}(\mathcal{E}(G_0))\} \subset G',$$

then [24, Lemma 4.4] shows that (recall that, due to the cofinality, the definition of the class group in that paper is equivalent to the present one)

$$\mathsf{D}(\mathcal{S}(G_0), \mathcal{E}(G_0)) = \mathsf{D}(G'_0). \tag{7.14}$$

Note that, if $(S_1, S_2) \in \mathcal{A}(\mathcal{E}(G_0))$, then $S_1(-S_2) \in \mathcal{A}(G_0 \cup -G_0)$, whence $|S_1| + |S_2| \le \mathsf{D}(G_0 \cup -G_0)$; by [28, Theorem 3.4.2.1], we know that, for a finite subset P of an abelian group, we have both $\mathsf{D}(P)$ and $\mathcal{A}(P)$ finite. Consequently, if G_0 is finite, then $\mathsf{D}(G_0 \cup -G_0)$ is finite, whence $\mathcal{A}(\mathcal{E}(G_0))$ is finite, which in turn implies G'_0, and hence also $\mathsf{D}(G'_0)$, is finite. Therefore, in view of (7.14), we conclude that

$$\mathsf{D}(\mathcal{S}(G_0), \mathcal{E}(G_0)) < \infty \tag{7.15}$$

for G_0 finite.

Theorem 7.3. *Let H be a Krull monoid and $\varphi\colon H \to \mathcal{F}(P)$ a cofinal divisor homomorphism into a free monoid such that the class group $G = \mathcal{C}(\varphi)$ is an infinite cyclic group that we identify with $\mathbb{Z}$. Let $G_P \subset G$ denote the set of classes containing prime divisors, and suppose that G_P^- is finite. Let $a \in H$ and $M = |\min(\operatorname{supp}(\boldsymbol{\beta}(a)))|$.*

1. *For any factorization $z \in \mathsf{Z}(a)$, there exists a factorization $y \in \Upsilon(a)$ and a chain of factorizations $z = z_0, \ldots, z_r = y$ of a such that*

$$|z| = |z_0| \le \cdots \le |z_r| = |y| \quad \text{and} \quad \mathsf{d}(z_i, z_{i+1}) \le \max\{M \cdot \mathsf{D}(\mathcal{S}(G_P^-), \mathcal{E}(G_P^-)), 2\} < \infty$$

   for all $i \in [0, r-1]$; in fact $\overline{\boldsymbol{\beta}}(z_0)^+ \le \overline{\boldsymbol{\beta}}(z_1)^+ \le \ldots \le \overline{\boldsymbol{\beta}}(z_r)^+$, where $\le$ is the partial order from the definition of $\Upsilon(\boldsymbol{\beta}(a))$.
2. *For any two factorizations $z, y \in \Upsilon(a)$ with $\overline{\boldsymbol{\beta}}(z)^+ = \overline{\boldsymbol{\beta}}(y)^+$, there exists a chain of factorizations $z = z_0, \ldots, z_r = y$ of a such that*

$$\overline{\boldsymbol{\beta}}(z)^+ = \overline{\boldsymbol{\beta}}(z_i)^+ = \overline{\boldsymbol{\beta}}(y)^+ \quad \text{and} \quad \mathsf{d}(z_i, z_{i+1}) \le \max\{\mathsf{D}(\mathcal{S}(G_P^-), \mathcal{E}(G_P^-)), 2\} < \infty$$

   for all $i \in [0, r-1]$; in particular, $|z| = |z_i| = |y|$ for all $i \in [0, r]$.

Proof. We set $B = \boldsymbol{\beta}(a)$. By Lemma 3.3, it suffices to prove the assertion for $\mathcal{B}(G_P)$ and B. As 0 is a prime divisor of $\mathcal{B}(G_P)$, we may w.l.o.g. assume $0 \notin \text{supp}(B)$.

Note $\mathsf{D}(\mathcal{S}(G_P^-), \mathcal{E}(G_P^-)) < \infty$ follows from (7.15). Also, for $z_i,\ z_{i+1} \in \mathsf{Z}(S)$, we have $|z_i| \le |z_{i+1}|$ whenever $z_i^+ \le z_{i+1}^+$, and $|z_i| = |z_{i+1}|$ whenever $z_i^+ = z_{i+1}^+$ (where $\le$ is the partial order from the definition of $\Upsilon(B)$). Let $z \in \mathsf{Z}(B)$ and let $y \in \Upsilon(B)$ with $z^+ \le y^+$. We will construct a chain of factorizations $z = z_0, \ldots, z_r$ of B such that $z_i^+ \le z_{i+1}^+$, either $z_r = y$ or $z^+ < z_r^+$, and

$$\mathsf{d}(z_i, z_{i+1}) \le M \cdot \mathsf{D}(\mathcal{S}(G_P^-), \mathcal{E}(G_P^-)) < \infty \quad (\text{when } z_i^+ < z_{i+1}^+) \tag{7.16}$$

$$\mathsf{d}(z_i, z_{i+1}) \le \mathsf{D}(\mathcal{S}(G_P^-), \mathcal{E}(G_P^-)) < \infty \quad (\text{when } z_i^+ = z_{i+1}^+), \tag{7.17}$$

for $i \in [0, r-1]$. Since both parts of the proposition follow by repeated application of this statement, the proof will be complete once we show the existences of such a chain of factorizations $z = z_0, \ldots, z_r = y$.

Since $z^+ \le y^+$, we have

$$z = A_1 \cdot \ldots \cdot A_n$$
$$y = (B_{1,1} \cdot \ldots \cdot B_{1,k_1}) \cdot (B_{2,1} \cdot \ldots \cdot B_{2,k_2}) \cdot \ldots \cdot (B_{n,1} \cdot \ldots \cdot B_{n,k_n})$$

with $A_j,\ B_{j,i} \in \mathcal{A}(G_0)$ and $A_j^+ = B_{j,1}^+ \cdot \ldots \cdot B_{j,k_j}^+$, for $j \in [1, n]$ and $i \in [1, k_j]$. Then $A_j^+ = B_{j,1}^+ \cdot \ldots \cdot B_{j,k_j}^+$ and $\sigma(A_j) = \sigma(B_{j,i}) = 0$, for all j and i. Thus, for $j \in [1, n]$, let

$$T_j = (A_j^-, (B_{j,1}^- \cdot \ldots \cdot B_{j,k_j}^-)) \in \mathcal{E}(G_P^-).$$

For each $j \in [1, n]$, let

$$T_{j,1} \cdot \ldots \cdot T_{j,l_j} \in \mathsf{Z}(\mathcal{E}(G_P^-))$$

be a factorization of T_j with each $T_{j,i} \in \mathcal{A}(\mathcal{E}(G_P^-))$. Now let

$$T = \prod_{j=1}^{n} \prod_{i=1}^{l_j} T_{j,i} \in \mathsf{Z}(\mathcal{E}(G_P^-)). \tag{7.18}$$

However, since $z,\ y \in \mathsf{Z}(B)$ both factor the same element B, we in fact have

$$\pi(T) \in \mathcal{S}(G_P^-).$$

Let $T = T'T''$ where $T'|T$ is the maximal length sub-factorization with all atoms dividing T' from $\mathcal{S}(G_P^-)$.

If $T'' = 1$, then $A_j = \prod_{i=1}^{k_j} B_{j,i}$ for every $j \in [1, n]$. In view of $A_j, B_{j,i} \in \mathcal{A}(G_P)$, we get $k_j = 1$ for every $j \in [1, n]$, that is $z = y$, and so there is nothing to show. Therefore we may assume T'' is nontrivial and proceed by induction on $|z|$ and then $|T''|$, assuming (7.16) and (7.17) hold for $z' \in \mathsf{Z}(B)$ when $z^+ < z'^+$ or when $z^+ = z'^+$ and $|R''| < |T''|$, where R'' is defined for z' as T'' was for z.

Let $W = \prod_{j \in J} \prod_{i \in I_j} T_{j,i}$ be a nontrivial subsequence of T'', where $J \subset [1, n]$ and $I_j \subset [1, l_j]$ for $j \in J$, such that $\pi(W) \in \mathcal{S}(G_P^-)$. Note, since $\pi(T') \in \mathcal{S}(G_P^-)$ (by definition) and since $\pi(T) \in \mathcal{S}(G_P^-)$ (by (7.18)), we have $\pi(T'') \in \mathcal{S}(G_P^-)$, whence we may w.l.o.g. assume $|W| \le \mathsf{D}(\mathcal{S}(G_P^-), \mathcal{E}(G_P^-))$ (in view of the definition of the relative Davenport constant). Write $W = \prod_{j \in J} W_j$ with each $W_j = \prod_{i \in I_j} T_{j,i} \in \mathsf{Z}(\mathcal{E}(G_P^-))$. Moreover, for $j \in J$, let $\pi(W_j) = (X_j, Y_j) \in \mathcal{E}(G_P^-)$.

Define a new factorization $z_1 = z_1' \cdot \ldots \cdot z_n' \in \mathsf{Z}(G_P^-)$ by letting $z_j' = A_j$ for $j \notin J$ and letting $z_j' \in \mathsf{Z}(A_j X_j^{-1} Y_j)$ for $j \in J$—by construction X_j is a subsequence of A_j, and since $(X_j, Y_j) \in \mathcal{E}(G_P^-)$, we have $\sigma(X_j) = \sigma(Y_j)$, and thus $\sigma(A_j X_j^{-1} Y_j) = \sigma(A_j) = 0$ for all $j \in J$, so z_1 is well defined. Also, since $\pi(W) = \pi(\prod_{j \in J} W_j) \in \mathcal{S}(G_P^-)$, it follows (by definition of $\mathcal{S}(G_P^-)$) that

$$\prod_{j \in J} X_j = \prod_{j \in J} Y_j,$$

and thus $z_1 \in \mathsf{Z}(B)$. Moreover, by construction, we have $z^+ \le z_1^+$, and by Lemma 4.3, we have $|B_j| \le M$ for all j. Thus

$$\mathsf{d}(z, z_1) \le M|J| \le M|W| \le M \cdot \mathsf{D}(\mathcal{S}(G_P^-), \mathcal{E}(G_P^-)). \tag{7.19}$$

Additionally, if $z \in \Upsilon(B)$, then $z^+ \le z_1^+$ implies that $z^+ = z_1^+ = y^+$, whence $|z| = |z_1|$ and $|z_j'| = 1$ for all j, in which case the estimate (7.19) improves to

$$\mathsf{d}(z, z_1) \le |J| \le |W| \le \mathsf{D}(\mathcal{S}(G_P^-), \mathcal{E}(G_P^-)).$$

Finally, if $z^+ = z_1^+$, then, by construction, the sequence $R = R'R''$—whose role for z_1 is analogous to the role of $T = T'T''$ for z—can be defined so that $R'' = T''W^{-1}$, in which case $|R''| < |T''|$. Consequently, applying the induction hypothesis to z_1 completes the proof. $\square$

Corollary 7.4. *Let H be a Krull monoid and $\varphi\colon H \to \mathcal{F}(P)$ a cofinal divisor homomorphism into a free monoid such that the class group $G = \mathcal{C}(\varphi)$ is an infinite cyclic group that we identify with $\mathbb{Z}$. Let $G_P \subset G$ denote the set of classes containing prime divisors, and suppose that G_P^- is finite.*

Let $a \in H$ with $\max \mathsf{L}(a) = |\boldsymbol{\beta}(a)^+| + \mathsf{v}_0\big(\boldsymbol{\beta}(a)\big)$ and let $M = |\min(\operatorname{supp}(\boldsymbol{\beta}(a)))|$. Then, for any factorization $z \in \mathsf{Z}(a)$ and any factorization $y \in \mathsf{Z}(a)$ with $|y| = |\max \mathsf{L}(a)|$, there exists a chain of factorizations $z = z_0, \ldots, z_r = y$ of a such that $|z| = |z_0| \le \cdots \le |z_r| = |y|$ and

$$\mathsf{d}(z_i, z_{i+1}) \le \max\{M \cdot \mathsf{D}(\mathcal{S}(G_P^-), \mathcal{E}(G_P^-)), 2\} \le \max\{|\min G_P| \cdot \mathsf{D}(\mathcal{S}(G_P^-), \mathcal{E}(G_P^-)), 2\} < \infty$$

for all $i \in [0, r-1]$.

Proof. This follows directly from Theorem 7.3 in view of (7.13). □

We end this section with a result showing that the assumption $\max \mathsf{L}(a) = |\boldsymbol{\beta}(a)^+| + \mathsf{v}_0\big(\boldsymbol{\beta}(a)\big)$ holds for a large class of $a \in H$. We formulate the result in the setting of zero-sum sequences. Since $\mathcal{B}(G_P)$ is factorial when $M = |\min G_P| \le 1$, the assumption $M \ge 2$ below is purely for avoiding distracting technical points in the statement and proof.

Proposition 7.5. *Let $G_0 \subset \mathbb{Z} \setminus \{0\}$ be a condensed set with $|G_0| \ge 2$. Let $B \in \mathcal{B}(G_0)$ be such that, for $M = |\min(\operatorname{supp}(B))|$, we have $M \ge 2$ and $\min(\operatorname{supp}(B)^+) \ge M(M^2-1)$. Then, at least one of the following statements holds:*

(a) *There exists a subset $A \subset \operatorname{supp}(B^-)$ and a factorization $z \in \mathsf{Z}(B)$ such that $\langle \operatorname{supp}(B^+)\rangle \not\subset \langle A\rangle$ (in particular, $\langle A\rangle \ne \mathbb{Z}$) and every atom $U|z$ has*

$$\mathsf{v}_x(U) \le 2M-2 \quad \text{for all } x \in \operatorname{supp}(B) \setminus A. \tag{7.20}$$

(b) (i) $\max \mathsf{L}(B) = |B^+|$, *and*
(ii) *for any factorization $z \in \mathsf{Z}(B)$, there exists a chain of factorizations $z = z_0, \ldots, z_r$ of B such that*

$$|z| = |z_0| < \cdots < |z_r| = |B^+| \quad \text{and} \quad \mathsf{d}(z_i, z_{i+1}) \le M^2$$

for all $i \in [0, r-1]$.

Proof. We assume (a) fails and show that (b) follows. Note, by Lemma 4.3, that $\mathsf{v}_x(U) \le M \le 2M-2$ holds for any atom $U \in \mathcal{A}(G_0)$ and $x \ge 0$, whence (7.20) can only fail for some $x \in G_0^-$. To establish (i) and (ii), we need only show that, given an arbitrary factorization $z \in \mathsf{Z}(B)$ with $|z| < |B^+|$, there is another factorization $z' \in \mathsf{Z}(B)$ with $|z| < |z'|$ and $\mathsf{d}(z, z') \le M^2$. We proceed to do so.

Let $z \in \mathsf{Z}(B)$ with $|z| < |B^+|$. Then there must exists some atom $U_0|z$ such that $|U_0^+| \ge 2$. Let $A \subset \operatorname{supp}(B)$ be all those a for which there exists some atom $V|z$ with $\mathsf{v}_a(V) \ge 2M-1$. We must have

$$\langle \operatorname{supp}(B^+)\rangle \subset \langle A\rangle, \tag{7.21}$$

else (a) holds. Let $a_1, \ldots, a_t \in A$ be those elements such that $\mathsf{v}_a(U_0) \le M-2$, let $a_{t+1}, \ldots, a_{|A|}$ be the remaining element of A and, for all $i \in [1, t]$, let $U_i|z$ be an atom with $\mathsf{v}_{a_i}(U_i) \ge 2M-1$. Note that $U_i \ne U_0$ for $i \le t$ since otherwise

$$2M-1 \le \mathsf{v}_{a_i}(U_i) = \mathsf{v}_{a_i}(U_0) \le M-2 \le 2M-2,$$

a contradiction. Also, $t < |A| \le M$ since otherwise

$$2M(M^2-1) \le 2\min(\operatorname{supp}(B^+)) \le \sigma(U_0^+) = -\sigma(U_0^-) \le M(2M-2),$$

a contradiction.

We proceed to describe a procedure to swap only negative integers between the U_i which results in new blocks $U_0', U_1', \ldots, U_t' \in \mathcal{B}(G_0)$ with $U_0'U_1' \cdot \ldots \cdot U_t' = U_0U_1 \cdot \ldots \cdot U_t$, with $U_i'^+ = U_i^+$ for all i, and with U_0' not an atom. Once this is done, then, letting $z_i \in \mathsf{Z}(U_i')$, we can define z' to be

$$z' = z_0z_1 \cdot \ldots \cdot z_tU_0^{-1}U_1^{-1} \cdot \ldots \cdot U_t^{-1}z.$$

Then $|z'| > |z|$ in view of U_0' not being an atom, while, in view of $t \le |A|-1 \le M-1$ and Lemma 4.3, we have

$$\mathsf{d}(z, z') \le \sum_{i=0}^{t} |U_i^+| \le (t+1)M \le M^2.$$

Thus the proof of (i) and (ii) will be complete once we show that such a process exists.

Observe, for $i \in [1, t]$, that we can exchange $a_i^{c_{i,j}}|U_i$ for $c_{i,j}^{a_i}|U_0$ provided there is some term $c_{i,j} \in \operatorname{supp}(U_0^-)$ with $\mathsf{v}_{c_{i,j}}(U_0) \ge a_i$ and $\mathsf{v}_{a_i}(U_i) \ge c_{i,j}$, and this will result in two new zero-sum subsequences obtained by only exchanging negative terms. The idea in general is to repeatedly and simultaneously perform such swaps for the a_i using disjoint sequences

$$\prod_{i=1}^{t}\big(c_{i,1}^{a_i}c_{i,2}^{a_i} \cdot \ldots \cdot c_{i,r_i}^{a_i}\big) \;\Big|\; U_0a_{t+1}^{-M+1} \cdot \ldots \cdot a_{|A|}^{-M+1} \tag{7.22}$$

with

$$\sum_{j=1}^{r_i-1}|c_{i,j}|<M-1 \quad \text{but} \quad \sum_{j=1}^{r_i}|c_{i,j}|\geq M-1 \tag{7.23}$$

for all $i\in[1,t]$, and let $U_0',U_1',\ldots,U_t'$ be the resulting zero-sum sequences. Then $\mathsf{v}_{a_i}(U_0')\geq M-1$ for $i\geq t+1$ by construction, and $\mathsf{v}_{a_i}(U_0')\geq\sum_{j=1}^{r_i}|c_{i,j}|\geq M-1$ for $i\leq t$; consequently, in view of $\min(\operatorname{supp}(B^+))\geq M(M^2-1)\geq(M-1)^2$ and $|U_0'^+|=|U_0^+|\geq 2$, we see that we can apply Lemma 7.2 to U_0', whence (7.21) and $\mathsf{v}_a(U_0')\geq M-1$ for $a\in A$ imply that U_0' cannot be an atom, and hence the U_i' have the desired properties. Thus it remains to show that a sequence satisfying (7.22) and (7.23) exists and that each a_i, for all $i\in[1,t]$, has sufficient multiplicity in U_i.

Note that (7.23) and the definition of $a_i\in A$ imply

$$\sum_{j=1}^{r_i}|c_{i,j}|\leq\sum_{j=1}^{r_i-1}|c_{i,j}|+|c_{i,r_i}|\leq M-2+M\leq\mathsf{v}_{a_i}(U_i)$$

for all $i\in[1,t]$. Thus the multiplicity of each a_i in U_i is large enough to perform such simultaneous swaps. Also,

$$\left|\sigma\left(\prod_{i=1}^{t}(c_{i,1}^{a_i}c_{i,2}^{a_i}\cdot\ldots\cdot c_{i,r_i}^{a_i})\right)\right|\leq\sum_{i=1}^{t}(2M-2)|a_i|. \tag{7.24}$$

We turn our attention now to showing (7.22) and (7.23) hold.

We can continue to remove subsequences $c_{i,j}^{a_i}|U_0a_{t+1}^{-M+1}\cdot\ldots\cdot a_{|A|}^{-M+1}$ until the multiplicity of every term is less than M. But this means a sequence satisfying (7.22) and (7.23) can be found, in view of the estimate (7.24), provided

$$|\sigma(U_0^-)|-(M-1)\sum_{i=t+1}^{|A|}|a_i|-M(M-1)|\operatorname{supp}(B^-)|\geq\sum_{i=1}^{t}(2M-2)|a_i|.$$

However, if this fails, then we have (since $|U_0^+|\geq 2$)

$$\begin{aligned}2M(M^2-1) &\leq 2\min(\operatorname{supp}(B^+))\leq\sigma(U_0^+)=-\sigma(U_0^-)=|\sigma(U_0^-)|\\ &<\sum_{i=1}^{t}(2M-2)|a_i|+(M-1)\sum_{i=t+1}^{|A|}|a_i|+M(M-1)|\operatorname{supp}(B^-)|\\ &<(2M-2)\sum_{i=1}^{|A|}|a_i|+M(M^2-1)\leq(2M-2)\sum_{i=1}^{M}i+M(M^2-1)\\ &=2M(M^2-1),\end{aligned}$$

a contradiction, completing the proof. □

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References

[1] D.F. Anderson, S.T. Chapman, W.W. Smith, Some factorization properties of Krull domains with infinite cyclic divisor class group, J. Pure Appl. Algebra 96 (1994) 97–112.
[2] P. Baginski, S.T. Chapman, R. Rodriguez, G.J. Schaeffer, Y. She, On the delta set and catenary degree of Krull monoids with infinite cyclic divisor class group, J. Pure Appl. Algebra 214 (2010) 1334–1339.
[3] L. Carlitz, A characterization of algebraic number fields with class number two, Proc. Amer. Math. Soc. 11 (1960) 391–392.
[4] S.T. Chapman, P.A. García-Sánchez, D. Llena, The catenary and tame degree of numerical monoids, Forum Math. 21 (2009) 117–129.
[5] S.T. Chapman, P.A. García-Sánchez, D. Llena, V. Ponomarenko, J.C. Rosales, The catenary and tame degree in finitely generated commutative cancellative monoids, Manuscr. Math. 120 (2006) 253–264.
[6] S.T. Chapman, N. Kaplan, J. Daigle, R. Hoyer, Delta sets of numerical monoids using non-minimal sets of generators, Commun. Algebra (in press).
[7] S.T. Chapman, U. Krause, E. Oeljeklaus, Monoids determined by a homogenous linear Diophantine equation and the half-factorial property, J. Pure Appl. Algebra 151 (2000) 107–133.
[8] S.T. Chapman, U. Krause, E. Oeljeklaus, On Diophantine monoids and their class groups, Pacific J. Math. 207 (2002) 125–147.
[9] S.T. Chapman, G.J. Schaeffer, A constructive proof of Kainrath's theorem and length-sequence complete subsets of $\mathbb{Z}$, manuscript.
[10] A. Facchini, Direct sum decomposition of modules, semilocal endomorphism rings, and Krull monoids, J. Algebra 256 (2002) 280–307.
[11] A. Facchini, Krull monoids and their application in module theory, in: A. Facchini, K. Fuller, C. M. Ringel, C. Santa-Clara (Eds.), Algebras, Rings and their Representations, World Scientific, 2006, pp. 53–71.
[12] A. Facchini, W. Hassler, L. Klingler, R. Wiegand, Direct-sum decompositions over one-dimensional Cohen-Macaulay local rings, in: J.W. Brewer, S. Glaz, W. Heinzer, B. Olberding (Eds.), Multiplicative Ideal Theory in Commutative Algebra, Springer, 2006, pp. 153–168.

[13] A. Facchini, D. Herbera, Local morphisms and modules with a semilocal endomorphism ring, Algebr. Represent. Theory 9 (2006) 403–422.
[14] A. Foroutan, Monotone chains of factorizations, in: A. Badawi (Ed.), Focus on commutative rings research, Nova Sci. Publ., New York, 2006, pp. 107–130.
[15] A. Foroutan, A. Geroldinger, Monotone chains of factorizations in C-monoids, in: Arithmetical Properties of Commutative Rings and Monoids, in: Lect. Notes Pure Appl. Math., vol. 241, Chapman & Hall, CRC, 2005, pp. 99–113.
[16] A. Foroutan, W. Hassler, Chains of factorizations and factorizations with successive lengths, Commun. Algebra 34 (2006) 939–972.
[17] A. Foroutan, W. Hassler, Factorization of powers in C-monoids, J. Algebra 304 (2006) 755–781.
[18] M. Freeze, A. Geroldinger, Unions of sets of lengths, Funct. Approximatio, Comment. Math. 39 (2008) 149–162.
[19] M. Freeze, W.A. Schmid, Remarks on a generalization of the Davenport constant (submitted for publication).
[20] W. Gao, A. Geroldinger, Zero-sum problems in finite abelian groups: a survey, Expo. Math. 24 (2006) 337–369.
[21] W. Gao, A. Geroldinger, On products of k atoms, Monatsh. Math. 156 (2009) 141–157.
[22] A. Geroldinger, On non-unique factorizations into irreducible elements. II, in: Number Theory, Vol. II Budapest 1987, in: Colloquia Mathematica Societatis Janos Bolyai, vol. 51, North Holland, 1990, pp. 723–757.
[23] A. Geroldinger, The catenary degree and tameness of factorizations in weakly Krull domains, in: Factorization in Integral Domains, in: Lect. Notes Pure Appl. Math., vol. 189, Marcel Dekker, 1997, pp. 113–153.
[24] A. Geroldinger, Chains of factorizations and sets of lengths, J. Algebra 188 (1997) 331–363.
[25] A. Geroldinger, Additive group theory and non-unique factorizations, in: A. Geroldinger, I. Ruzsa (Eds.), Combinatorial Number Theory and Additive Group Theory, in: Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, 2009, pp. 1–86.
[26] A. Geroldinger, D.J. Grynkiewicz, On the arithmetic of Krull monoids with finite Davenport constant, J. Algebra 321 (2009) 1256–1284.
[27] A. Geroldinger, F. Halter-Koch, Realization theorems for semigroups with divisor theory, Semigroup Forum 44 (1992) 229–237.
[28] A. Geroldinger, F. Halter-Koch, Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory, in: Pure and Applied Mathematics, vol. 278, Chapman & Hall/CRC, 2006.
[29] A. Geroldinger, W. Hassler, Arithmetic of Mori domains and monoids, J. Algebra 319 (2008) 3419–3463.
[30] A. Geroldinger, W. Hassler, Local tameness of v-noetherian monoids, J. Pure Appl. Algebra 212 (2008) 1509–1524.
[31] A. Geroldinger, F. Kainrath, On the arithmetic of tame monoids with applications to Krull monoids and Mori domains, J. Pure Appl. Algebra 214 (12) (2010) 2199–2218.
[32] R. Gilmer, Commutative Semigroup Rings, The University of Chicago Press, 1984.
[33] P.A. Grillet, Commutative Semigroups, Kluwer Academic Publishers, 2001.
[34] D.J. Grynkiewicz, On a conjecture of Hamidoune for subsequence sums, Integers 5 (2) (2005) Paper A07, 11p.
[35] F. Halter-Koch, Elasticity of factorizations in atomic monoids and integral domains, J. Théor. Nombres Bordeaux 7 (1995) 367–385.
[36] F. Halter-Koch, Ideal Systems. An Introduction to Multiplicative Ideal Theory, Marcel Dekker, 1998.
[37] W. Hassler, Factorization properties of Krull monoids with infinite class group, Colloq. Math. 92 (2002) 229–242.
[38] F. Kainrath, Arithmetic of Mori domains and monoids: the global case, manuscript.
[39] F. Kainrath, Elasticity of finitely generated domains, Houston J. Math. 31 (2005) 43–64.
[40] H. Kim, The distribution of prime divisors in Krull monoid domains, J. Pure Appl. Algebra 155 (2001) 203–210.
[41] H. Kim, M.O. Kim, Y.S. Park, Some characterizations of Krull monoids, Algebra Colloq. 14 (2007) 469–477.
[42] H. Kim, Y.S. Park, Krull domains of generalized power series, J. Algebra 237 (2001) 292–301.
[43] J.-L. Lambert, Une borne pour les générateurs des solutions entières positives d'une équation diophantienne linéaire, C.R. Acad. Sci., Paris, Ser. I 305 (1987) 39–40.
[44] W.A. Schmid, Higher-order class groups and block monoids of Krull monoids with torsion class group, J. Algebra Appl. (in press).
[45] W.A. Schmid, A realization theorem for sets of lengths, J. Number Theory 129 (2009) 990–999.
[46] R. Wiegand, Direct-sum decompositions over local rings, J. Algebra 240 (2001) 83–97.