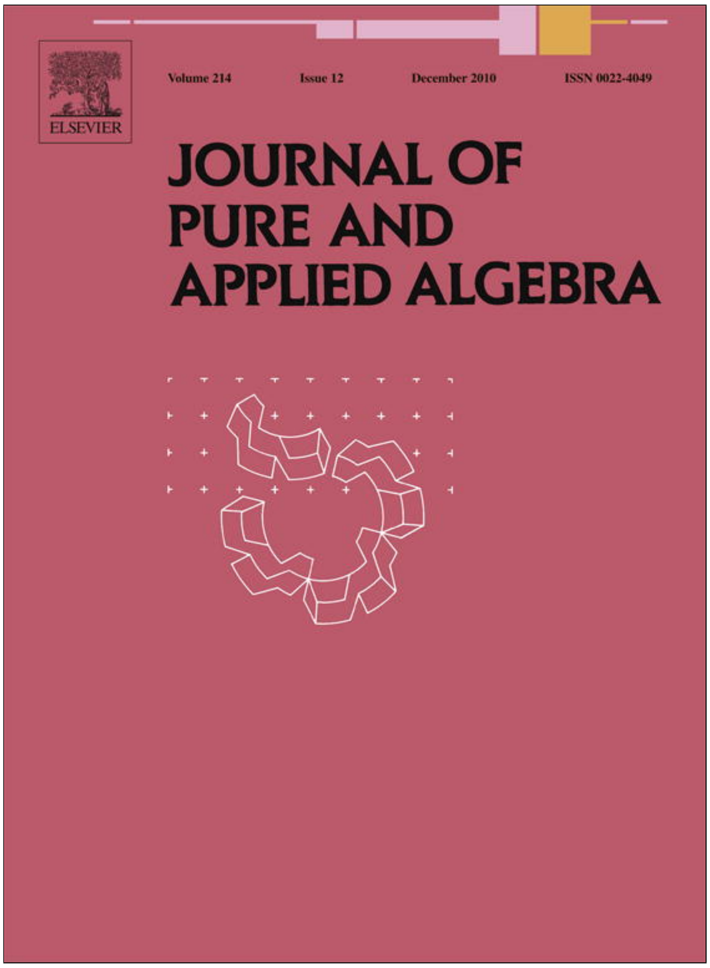
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On the arithmetic of tame monoids with applications to Krull monoids and Mori domains☆

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ABSTRACT

Let H be an atomic monoid (e.g., the multiplicative monoid of a noetherian domain). For an element $b \in H$, let $\omega(H, b)$ be the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ having the following property: if $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in H$ are such that b divides $a_1 \cdot \ldots \cdot a_n$, then b already divides a subproduct of $a_1 \cdot \ldots \cdot a_n$ consisting of at most N factors. The monoid H is called tame if $\sup\{\omega(H, u) \mid u \text{ is an atom of } H\} < \infty$. This is a well-studied property in factorization theory, and for various classes of domains there are explicit criteria for being tame. In the present paper, we show that, for a large class of Krull monoids (including all Krull domains), the monoid is tame if and only if the associated Davenport constant is finite. Furthermore, we show that tame monoids satisfy the Structure Theorem for Sets of Lengths. That is, we prove that in a tame monoid there is a constant M such that the set of lengths of any element is an almost arithmetical multiprogression with bound M.

1. Introduction

By an atomic monoid we mean a commutative cancellative semigroup with unit element such that every non-unit may be written as a finite product of atoms (irreducible elements). The main examples we have in mind are the multiplicative monoids consisting of the nonzero elements from a noetherian domain. Let H be an atomic monoid and $b \in H$. We denote by $\omega(H, b)$ the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ having the following property: if $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in H$ are such that b divides $a_1 \cdot \ldots \cdot a_n$, then b already divides a subproduct of $a_1 \cdot \ldots \cdot a_n$ consisting of at most N factors. Thus, by definition, b is a prime element of H if and only if $\omega(H, b) = 1$. The $\omega(H, \cdot)$-invariants have been studied in factorization theory for many years, but only recently was it shown that in a v-noetherian monoid we have $\omega(H, a) < \infty$ for all $a \in H$ (see [24]).

The monoid H is said to be tame if the invariant $\omega(H) = \sup\{\omega(H, u) \mid u \text{ is an atom of } H\}$ is finite. Indeed, this is not the original definition but a new characterization achieved in the present paper (see Proposition 3.5). Tameness implies a variety of further arithmetical finiteness properties (such as the finiteness of the catenary degree and of the elasticity), and local tameness is a central finiteness property in factorization theory (we refer to the monograph [22] and some recent publications [9,8,27]). Finitely generated monoids and Krull monoids with finite class group are simple examples of tame monoids. A non-principal order $\mathfrak{o}$ in an algebraic number field is locally tame, and it is tame if and only if for every prime ideal $\mathfrak{p}$ containing the conductor there is precisely one prime ideal $\overline{\mathfrak{p}}$ in the principal order $\overline{\mathfrak{o}}$ such that $\overline{\mathfrak{p}} \cap \mathfrak{o} = \mathfrak{p}$. More examples (including various classes of Mori domains) are discussed in Sections 3 and 4 (see Example 3.2 and Theorem 4.2).

Krull monoids and Krull domains have been in the center of interest of factorization theory since its very beginning. Their arithmetic is completely determined by the class group and the distribution of prime divisors in the classes. If the class group is finite, then the main invariants of factorization theory are finite too (this is relatively simple to show, but to obtain precise

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values for the invariants is open in most cases; see [18]). Suppose the class group is infinite. If every class contains a prime divisor, then the main invariants of factorization theory are infinite, and in particular, every finite nonempty set $L \subset \mathbb{N}$ occurs as a set of lengths (see [32,22] and [24, Theorem 4.4]). If there are classes without prime divisors, then the knowledge on the arithmetic is still very limited. Chapman et al. studied the arithmetic of a Krull monoid with infinite cyclic class group (see [6,7]; for more on the arithmetic in the case of infinite class groups see [28,19,21]). In Theorem 4.2 of the present paper, we prove that, for a large class of Krull monoids (including all Krull monoids with torsion class group and all Krull domains) the monoid is tame if and only if the associated Davenport constant is finite.

If an element $a \in H$ has a factorization of the form $a = u_1 \cdot \ldots \cdot u_k$, where $k \in \mathbb{N}$ and $u_1, \ldots, u_k \in H$ are atoms, then k is called the length of the factorization, and the set $\mathsf{L}(a)$ of all possible lengths is called the set of lengths of a. Sets of lengths (and all invariants derived from them, as the elasticity or the set of distances) are among the most investigated invariants in factorization theory. If H is v-noetherian, then all sets of lengths are finite, and it is easy to observe that either all sets of lengths are singletons or that for every $N \in \mathbb{N}$ there is an element $a \in H$ such that $|\mathsf{L}(a)| \ge N$. The Structure Theorem for Sets of Lengths states that all sets of lengths in a given monoid are almost arithmetical multiprogressions with universal bounds for all parameters (roughly speaking, these are finite unions of arithmetical progressions having the same difference). This Structure Theorem holds true for a great variety of monoids (among them are tame and non-tame monoids) which satisfy suitable finiteness conditions; see Remark 5.2. Recently, Schmid established a realization theorem showing that this structural description of sets of lengths is sharp (see [37]). In Theorem 5.1 of the present paper, we show that every tame monoid satisfies the Structure Theorem. The proof uses the general machinery (as presented in [22, Section 4.3]) and crucial new ideas introduced in [20].

2. Preliminaries

Our notation and terminology are consistent with [22]. We briefly gather some key notions. Let $\mathbb{N}$ denote the set of positive integers, and put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For integers $a, b \in \mathbb{Z}$ we set $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$. For a real number $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x, and $\lceil x \rceil$ denotes the smallest integer that is greater than or equal to x. By a *monoid* we mean a commutative semigroup with unit element which satisfies the cancellation laws.

Let G be an additive abelian group and $G_0 \subset G$ a subset. Then $[G_0] \subset G$ denotes the submonoid generated by G_0 and $\langle G_0 \rangle \subset G$ denotes the subgroup generated by G_0. A family $(e_i)_{i \in I}$ of elements of G is said to be *independent* if $e_i \ne 0$ for all $i \in I$ and, for every family $(m_i)_{i \in I} \in \mathbb{Z}^{(I)}$,

$$\sum_{i \in I} m_i e_i = 0 \quad \text{implies} \quad m_i e_i = 0 \quad \text{for all } i \in I.$$

The subset $G_0 \subset G$ is called independent if the family $(g)_{g \in G_0}$ is independent, and it is called a *basis* if it is independent and $\langle G_0 \rangle = G$. The *total rank* of G is the maximum of the cardinalities of maximal independent subsets, and the *torsion free rank* of G is the cardinality of a maximal independent subset consisting of elements of infinite order.

Let $A, B \subset G$ be nonempty subsets. Then $A + B = \{a + b \mid a \in A, b \in B\}$ denotes their *sumset* and, for $k \in \mathbb{N}$, $kA = A + \cdots + A$ denotes the *k-fold sumset* of A. Now suppose that $A \subset \mathbb{Z}$. We denote by $\Delta(A)$ the *set of distances* of A, that is the set of all $d \in \mathbb{N}$ for which there exists $l \in A$ such that $A \cap [l, l+d] = \{l, l+d\}$. Two distinct elements $k, l \in A$ are called *adjacent* if either $A \cap [k, l] = \{k, l\}$ or $A \cap [l, k] = \{k, l\}$. In particular, $\Delta(\emptyset) = \emptyset$, and if $A = \{a_1, \ldots, a_t\}$ is finite, with $t \in \mathbb{N}$ and $a_1 < \cdots < a_t$, then $\Delta(A) = \{a_{\nu+1} - a_\nu \mid \nu \in [1, t-1]\}$). Clearly, $\Delta(A) \subset \{d\}$ if and only if A is an arithmetical progression with difference d. If $A \subset \mathbb{N}$, we call

$$\rho(A) = \sup\left\{ \frac{m}{n} \;\middle|\; m, n \in A \right\} = \frac{\sup A}{\min A} \in \mathbb{Q}_{\ge 1} \cup \{\infty\}$$

the *elasticity* of A, and we set $\rho(\{0\}) = 1$.

Throughout this paper, let H be a monoid.

We denote by $\mathcal{A}(H)$ the set of atoms (irreducible elements) of H, by $\mathsf{q}(H)$ a quotient group of H, by $H^\times$ the group of invertible elements and by $H_{\mathrm{red}} = \{aH^\times \mid a \in H\}$ the associated reduced monoid of H. We say that H is reduced if $H^\times = \{1\}$.

For a set P we denote by $\mathcal{F}(P)$ the *free (abelian) monoid* with basis P. Then every $a \in \mathcal{F}(P)$ has a unique representation in the form

$$a = \prod_{p \in P} p^{\mathsf{v}_p(a)} \quad \text{with } \mathsf{v}_p(a) \in \mathbb{N}_0 \text{ and } \mathsf{v}_p(a) = 0 \text{ for almost all } p \in P,$$

and we call $\mathrm{supp}_P(a) = \mathrm{supp}(a) = \{p \in P \mid \mathsf{v}_p(a) > 0\} \subset P$ the *support* of a. For a subset $P_0 \subset P$, we set

$$\mathsf{v}_{P_0}(a) = \sum_{p \in P_0} \mathsf{v}_p(a), \quad \text{and we call } |a|_F = |a| = \mathsf{v}_P(a) \text{ the } \textit{length} \text{ of } a.$$

The free monoid $\mathsf{Z}(H) = \mathcal{F}\big(\mathcal{A}(H_{\mathrm{red}})\big)$ is called the *factorization monoid* of H, and the unique homomorphism

$$\pi : \mathsf{Z}(H) \to H_{\mathrm{red}} \quad \text{satisfying } \pi(u) = u \text{ for all } u \in \mathcal{A}(H_{\mathrm{red}})$$

is called the *factorization homomorphism* of H. For $a \in H$, the set

$$\mathsf{Z}(a) = \pi^{-1}(aH^\times) \subset \mathsf{Z}(H) \quad \text{is the } \textit{set of factorizations} \text{ of } a,$$

$$\mathsf{L}(a) = \big\{|z| \ \big| \ z \in \mathsf{Z}(a)\big\} \subset \mathbb{N}_0 \quad \text{is the } \textit{set of lengths} \text{ of } a \quad \text{and}$$

$$\mathcal{L}(H) = \big\{\mathsf{L}(a) \ \big| \ a \in H\big\} \quad \text{denotes the } \textit{system of sets of lengths} \text{ of } H.$$

By definition, we have $\mathsf{Z}(a) = \{1\}$ for all $a \in H^\times$. The monoid H is called

- *atomic* if $\mathsf{Z}(a) \ne \emptyset$ for all $a \in H$;
- *factorial* if $|\mathsf{Z}(a)| = 1$ for all $a \in H$ (equivalently, H is atomic and every atom is a prime);
- *half-factorial* if $|\mathsf{L}(a)| = 1$ for all $a \in H$;
- a BF-*monoid* (a bounded factorization monoid) if $\mathsf{L}(a)$ is finite and nonempty for all $a \in H$.

Let $z,\ z' \in \mathsf{Z}(H)$. Then we can write

$$z = u_1 \cdot \ldots \cdot u_l v_1 \cdot \ldots \cdot v_m \quad \text{and} \quad z' = u_1 \cdot \ldots \cdot u_l w_1 \cdot \ldots \cdot w_n,$$

where $l,\ m,\ n \in \mathbb{N}_0, u_1, \ldots, u_l,\ v_1, \ldots, v_m,\ w_1, \ldots, w_n \in \mathcal{A}(H_{\text{red}})$ are such that

$$\{v_1, \ldots, v_m\} \cap \{w_1, \ldots, w_n\} = \emptyset.$$

We call

$$\mathsf{d}(z, z') = \max\{m,\ n\} = \max\{|z \gcd(z, z')^{-1}|, |z' \gcd(z, z')^{-1}|\} \in \mathbb{N}_0$$

the *distance* of z and z'. If $\pi(z) = \pi(z')$ and $z \ne z'$, then

$$2 + \big||z| - |z'|\big| \le \mathsf{d}(z, z') \tag{2.1}$$

by [22, 1.6.2]. For subsets $X, Y \subset \mathsf{Z}(H)$, we set

$$\mathsf{d}(X, Y) = \min\{\mathsf{d}(x, y) \mid x \in X,\ y \in Y\},$$

and thus $X \cap Y \ne \emptyset$ if and only if $\mathsf{d}(X, Y) = 0$.

3. Tame monoids: examples and first properties

Definition 3.1. Suppose that H is atomic.

1. For $b \in H$, let $\omega(H, b)$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:
   For all $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in H$, if $b \mid a_1 \cdot \ldots \cdot a_n$, then there exists a subset $\Omega \subset [1, n]$ such that $|\Omega| \le N$ and
   $$b \ \Big| \ \prod_{\nu \in \Omega} a_\nu .$$
   Furthermore, we set
   $$\omega(H) = \sup\{\omega(H, u) \mid u \in \mathcal{A}(H)\} \in \mathbb{N}_0 \cup \{\infty\}.$$
2. For $a \in H$ and $x \in \mathsf{Z}(H)$, let $\mathsf{t}(a, x) \in \mathbb{N}_0 \cup \{\infty\}$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:
   If $\mathsf{Z}(a) \cap x\mathsf{Z}(H) \ne \emptyset$ and $z \in \mathsf{Z}(a)$, then there exists $z' \in \mathsf{Z}(a) \cap x\mathsf{Z}(H)$ such that $\mathsf{d}(z, z') \le N$.
   For subsets $H' \subset H$ and $X \subset \mathsf{Z}(H)$, we define
   $$\mathsf{t}(H', X) = \sup\big\{\mathsf{t}(a, x) \ \big| \ a \in H', x \in X\big\} \in \mathbb{N}_0 \cup \{\infty\}.$$
   H is called *locally tame* if $\mathsf{t}(H, u) < \infty$ for all $u \in \mathcal{A}(H_{\text{red}})$, and
   $$\mathsf{t}(H) = \mathsf{t}\big(H, \mathcal{A}(H_{\text{red}})\big) = \sup\{\mathsf{t}(H, u) \mid u \in \mathcal{A}(H_{\text{red}})\} \in \mathbb{N}_0 \cup \{\infty\}$$
   denotes the *tame degree* of H. The monoid H is said to be *tame* if $\mathsf{t}(H) < \infty$.

Let H be atomic, and for simplicity of notation, suppose that it is reduced. Pick an atom $u \in \mathcal{A}(H)$. Then u is a prime if and only if $\omega(H, u) = 1$. Thus H is factorial if and only if $\omega(H) = 1$. Let $a \in H$. If $u \nmid a$, then $\mathsf{t}(a, u) = 0$ by definition. Suppose that $u \mid a$. Then $\mathsf{t}(a, u)$ is the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property: if $z = a_1 \cdot \ldots \cdot a_n$ is any factorization of a where $a_1, \ldots, a_n$ are atoms, then there exist a subset $\Omega \subset [1, n]$, say $\Omega = [1, k]$, and a factorization $z' = uu_2 \cdot \ldots u_l a_{k+1} \cdot \ldots \cdot a_n \in \mathsf{Z}(a)$, with atoms $u_2, \ldots, u_l$, such that $\max\{k, l\} \le N$. Thus $\mathsf{t}(a, u)$ measures how far away from any given factorization z of a there is a factorization z' of a which contains u. Suppose that u is a prime. Then every factorization of a contains u; we can choose $z' = z$ in the above definition, obtain that $\mathsf{d}(z, z') = \mathsf{d}(z, z) = 0$ and hence $\mathsf{t}(H, u) = 0$. Thus H is factorial if and only if $\mathsf{t}(H) = 0$. If u is not a prime, then $\omega(H, u) \le \mathsf{t}(H, u)$, and hence if H is not factorial, then $\omega(H) \le \mathsf{t}(H)$.

The $\omega(H, \cdot)$-invariants (introduced in [17]) and the tame degrees are well-established invariants in the theory of non-unique factorizations which found much interest in recent literature (for example, see [4] for investigations in the context of integral domains, or [5] for investigations in numerical monoids). Whereas in v-noetherian monoids (these are monoids satisfying the ascending chain condition for v-ideals) we have $\omega(H, u) < \infty$ for all atoms $u \in \mathcal{A}(H)$, this does not hold for the $\mathsf{t}(H, u)$ values (see [25, Corollary 3.6], [24, Theorems 4.2 and 4.4], [23, Theorems 5.3 and 6.7]).

We continue with a list of examples, where tameness is characterized in various classes of monoids and domains. Krull monoids will receive special attention and will be discussed in Section 4 (see in particular Theorem 4.2).

Examples 3.2. Let R be an integral domain. Then $R^\bullet = R \setminus \{0\}$ denotes its multiplicative monoid, $\mathfrak{X}(R)$ the set of minimal nonzero prime ideals, $\widehat{R}$ its complete integral closure and $\mathcal{C}_v(R)$ its v-class group. Clearly, R is a Mori domain if and only if $R^\bullet$ is v-noetherian.

1. *Finitely generated monoids*. If H_{red} is finitely generated, then H is tame (see [22, Theorem 3.1.4]). The domain R is called a (*generalized*) *Cohen–Kaplansky domain* if R has only finitely many nonassociated atoms (if almost all atoms are prime) (see [2,1,3]). Thus generalized Cohen–Kaplansky domains are tame.

2. *Finitely primary monoids*. Let H be finitely primary of rank $s \in \mathbb{N}$. Then H is tame if and only if $s = 1$. Thus a one-dimensional local Mori domain R with $(R : \widehat{R}) \neq \{0\}$ is tame if and only if $|\mathfrak{X}(R)| = 1$ (see [22, Proposition 2.10.7 and Theorem 3.1.5]).

3. *Weakly Krull domains*. Let R be a v-noetherian weakly Krull domain with nonzero conductor $\mathfrak{f} = (R : \widehat{R})$ and finite v-class group $\mathcal{C}_v(R)$. Note that, in particular, orders in algebraic number fields fulfill all these properties.

Then R is tame if and only if for every nonzero minimal prime ideal $\mathfrak{p} \in \mathfrak{X}(R)$ with $\mathfrak{p} \supset \mathfrak{f}$ there is precisely one $\widehat{\mathfrak{p}} \in \mathfrak{X}(\widehat{R})$ such that $\widehat{\mathfrak{p}} \cap R = \mathfrak{p}$ (see [22, Theorem 3.7.1]).

4. *Mori domains*. Let R be a Mori domain with nonzero conductor $\mathfrak{f} = (R : \widehat{R})$ and let

$$S = \operatorname{Reg}(R^\bullet) = \{a \in R^\bullet \mid \text{if } z \in \widehat{R^\bullet} \text{ and } za \in R^\bullet, \text{ then } z \in R^\bullet\}$$

denote the monoid of regular elements of $R^\bullet$ (see [22, Section 2.3]). Suppose that R satisfies the following three finiteness conditions:

- The v-class groups $\mathcal{C}_v(R)$ and $\mathcal{C}_v(\widehat{R})$ are both finite.
- $S^{-1}\widehat{R}$ is semilocal and the Jacobson radical of $S^{-1}\widehat{R}/S^{-1}\mathfrak{f}$ is nilpotent.
- The Jacobson radical of $S^{-1}R/S^{-1}\mathfrak{f}$ is nilpotent.

Then R is tame if and only if the natural map $\operatorname{spec}(\widehat{R}) \to \operatorname{spec}(R)$ is one to one (see [31]). We point out two special cases where the above three finiteness conditions are satisfied. First, if R is weakly Krull and $\mathcal{C}_v(R)$ is finite, then all three finiteness conditions are satisfied (so weakly Krull domains are a special case of the situation discussed here). Second, if the factor ring $R/\mathfrak{f}$ and the class group $\mathcal{C}_v(\widehat{R})$ are both finite, then $R^\bullet$ is a C-monoid (see [22, Theorem 2.11.9]) and the above three finiteness conditions are satisfied. Higher-dimensional finitely generated algebras over $\mathbb{Z}$, whose multiplicative monoids are C-monoids, are discussed in [29,33] and [22, Section 2.11].

We start with two technical lemmas. The first one gathers some simple observations (a proof can be found in [24, Lemma 3.3]).

Lemma 3.3. *Let H be atomic.*

1. *If $b_1, b_2 \in H$, then $\omega(H, b_1) \le \omega(H, b_1b_2) \le \omega(H, b_1) + \omega(H, b_2)$.*
2. *For all $b \in H$, we have $\sup \mathsf{L}(b) \le \omega(H, b)$. In particular, if $\omega(H, u) < \infty$ for all $u \in \mathcal{A}(H)$, then H is a* BF*-monoid.*

Lemma 3.4. *Let H be atomic and reduced.*

1. *If $b \in H$, $z \in \mathsf{Z}(H)$ such that $b \mid \pi(z)$, then there exists $z' \in \mathsf{Z}(H)$ such that $b \mid \pi(z')$, $z' \mid z$ and $|z'| \le \omega(H) \min \mathsf{L}(b)$.*
2. *If $u \in \mathsf{Z}(H)$ and $v \in \mathcal{A}(H)$ such that $v \mid \pi(u)$, then there exists $u' \in \mathsf{Z}(\pi(u))$ such that $v \mid u'$ and $||u| - |u'|| \le \max\{0, \mathsf{t}(H) - 2\}$.*
3. *If $a, b \in H$ with $b \mid a$, then $\min \mathsf{L}(ab^{-1}) \le \min \mathsf{L}(a) + (2\mathsf{t}(H) - 1) \min \mathsf{L}(b)$.*

Proof. 1. Let $b = u_1 \cdot \ldots \cdot u_k$ with $k = \min \mathsf{L}(b)$ and $u_1, \ldots, u_k \in \mathcal{A}(H)$. Then Lemma 3.3.1 shows that $\omega(H, b) \le \omega(H) \min \mathsf{L}(b)$, which implies the assertion.

2. If $v \mid u$, then we set $u' = u$. Suppose that $v \nmid u$. This implies that H is not factorial and $\mathsf{t}(H) \ge 2$. Further, there exists some $u' \in \mathsf{Z}(\pi(u))$ such that $v \mid u'$ and $\mathsf{d}(u, u') \le \mathsf{t}(\pi(u), v) \le \mathsf{t}(H)$. Since $v \nmid u$, we have $u \ne u'$ and by Eq. (2.1) we obtain $||u| - |u'|| \le \mathsf{d}(u, u') - 2 \le \mathsf{t}(H) - 2$.

3. We note first

$$\min \mathsf{L}(a) \ge \max \mathsf{L}(b) + \min \mathsf{L}(ab^{-1}) - \mathsf{t}(a, \mathsf{Z}(b))$$

by [22, 4.3.4.1]. Moreover, by [22, 1.6.5.7] we have $\mathsf{t}(a, \mathsf{Z}(b)) \le 2 \min \mathsf{L}(b)\mathsf{t}(H)$. Hence we obtain

$$\begin{aligned}\min \mathsf{L}(ab^{-1}) &\le \min \mathsf{L}(a) - \max \mathsf{L}(b) + \mathsf{t}(a, \mathsf{Z}(b)) \le \min \mathsf{L}(a) - \min \mathsf{L}(b) + 2\mathsf{t}(H) \min \mathsf{L}(b) \\ &= \min \mathsf{L}(a) + (2\mathsf{t}(H) - 1) \min \mathsf{L}(b). \quad \square\end{aligned}$$

We continue with a characterization of tameness, which is based on a precise recent description of local tameness achieved in [24]. In Example 4.12, we present a monoid H, for which the bound $\mathsf{t}(H) \le \omega(H)^2$ is almost sharp. On the other hand, suppose that H is atomic but not factorial. Then by definition we have $\omega(H) \le \mathsf{t}(H)$. If H is half-factorial, then it can be checked from the definitions that equality holds. In [34, Theorem 3.10], there is a class of non-half-factorial numerical monoids for which $\omega(H) = \mathsf{t}(H)$ holds.

Proposition 3.5. *Let H be atomic. Then $\mathsf{t}(H) \le \omega(H)^2$. In particular, H is tame if and only if $\omega(H) < \infty$.*

Proof. We may assume that H is reduced, and we have to show that $\mathsf{t}(H, u) \le \omega(H)^2$ for all $u \in \mathcal{A}(H)$. Let $u \in \mathcal{A}(H)$ be given. If u is prime, then $\mathsf{t}(H, u) = 0$, and the assertion is clear. Suppose that u is not prime. Then [24, Theorem 3.6] implies that

$$\mathsf{t}(H, u) = \max\{\omega(H, u), 1 + \tau(H, u)\},$$

where

$$\tau(H, u) = \sup\Big\{ \min \mathsf{L}(u^{-1}a) \,\Big|\, a = u_1 \cdot \ldots \cdot u_j \in uH \text{ with } j \in \mathbb{N},\ u_1, \ldots, u_j \in \mathcal{A}(H), \text{ and } u \nmid u_i^{-1}a \text{ for all } i \in [1, j]\Big\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Clearly, $\omega(H, u) \le \omega(H)^2$, and thus we may assume that $\mathsf{t}(H, u) = 1 + \tau(H, u)$. Let $\theta \in \mathbb{N}$ with $\theta \le \tau(H, u)$. By definition, there exist $a = v_1 \cdot \ldots \cdot v_s \in uH$, where $s \in \mathbb{N}$, $v_1, \ldots, v_s \in \mathcal{A}(H)$ and $u \nmid v_i^{-1}a$ for all $i \in [1, s]$, such that $\min \mathsf{L}(u^{-1}a) \ge \theta$. Thus $s \le \omega(H, u)$. We set $u^{-1}a = u_2 \cdot \ldots \cdot u_t$ with $t \in \mathbb{N}$ and $u_2, \ldots, u_t \in \mathcal{A}(H)$ such that $t - 1 = \min \mathsf{L}(u^{-1}a) \ge \theta$. Then $a = v_1 \cdot \ldots \cdot v_s = uu_2 \cdot \ldots \cdot u_t$, and clearly a divides $uu_2 \cdot \ldots \cdot u_t$ but does not divide any proper subproduct. Thus

$$\theta + 1 \le t \le \omega(H, a) = \omega(H, v_1 \cdot \ldots \cdot v_s) \le \omega(H, v_1) + \cdots + \omega(H, v_s) \le s\omega(H) \le \omega(H)^2,$$

and hence

$$\mathsf{t}(H, u) = 1 + \tau(H, u) \le \omega(H)^2. \quad \square$$

It is well known that tame monoids are BF-monoids with finite elasticity and finite catenary degree. Here we provide new proofs showing that the elasticity and the catenary degree are not only bounded by the tame degree $\mathsf{t}(H)$, but they are in fact bounded by $\omega(H)$. This new upper bound is sharp for large classes of Krull monoids (see Example 4.12). We recall the definition of the elasticity and of the catenary degree.

Let H be atomic and $a \in H$. Then $\rho(a) = \rho\big(\mathsf{L}(a)\big)$ is called the *elasticity* of a, and the *elasticity* of H is defined as

$$\rho(H) = \sup\{\rho(L) \mid L \in \mathcal{L}(H)\} \in \mathbb{R}_{\ge 1} \cup \{\infty\}.$$

For $k \in \mathbb{N}$, we set $\rho_k(H) = k$ if $H = H^\times$, and

$$\rho_k(H) = \sup\{\sup L \mid L \in \mathcal{L}(H),\ k \in L\} \in \mathbb{N} \cup \{\infty\}, \quad \text{if } H \ne H^\times.$$

The *catenary degree* $\mathsf{c}(a)$ is the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ such that, for any two factorizations z, z' of a, there exists a finite sequence $z = z_0, z_1, \ldots, z_k = z'$ of factorizations of a satisfying that $\mathsf{d}(z_{i-1}, z_i) \le N$ for all $i \in [1, k]$. Globally, we define

$$\mathsf{c}(H) = \sup\{\mathsf{c}(a) \mid a \in H\} \in \mathbb{N}_0 \cup \{\infty\},$$

and we call $\mathsf{c}(H)$ the *catenary degree* of H. By Proposition 3.6.3, every tame monoid has finite catenary degree. But there are monoids with finite catenary degree which are not tame (see [22, Section 3.7]). In Remark 5.2, we discuss the first example of a Krull monoid with this property.

Proposition 3.6. *Let H be atomic.*

1. *Then $\rho(a) \le \min\{\omega(H, a),\ \omega(H)\}$ for all $a \in H$, and thus $\rho(H) \le \omega(H)$.*
2. *For all $k \ge 2$, we have $\rho_k(H) - \rho_{k-1}(H) \le \max\{1, \omega(H) - 1\}$.*
3. *$\mathsf{c}(H) \le \omega(H)$.*

Proof. Without restriction we may suppose that H is reduced.

1. By definition, we have $\rho(1) = 1$. Pick an element $a \in H \setminus \{1\}$. If $a = u_1 \cdot \ldots \cdot u_k = v_1 \cdot \ldots \cdot v_l$ with $u_1, \ldots, u_k, v_1, \ldots, v_l \in \mathcal{A}(H)$, then, by [24, Lemma 3.3], we get

$$k \le \omega(H, u_1 \cdot \ldots \cdot u_k) = \omega(H, v_1 \cdot \ldots \cdot v_l) \le \omega(H, v_1) + \cdots + \omega(H, v_l) \le l \min\{\omega(H), \omega(H, a)\},$$

and hence

$$\rho(a) = \sup\left\{ \frac{r}{s} \,\middle|\, r, s \in \mathsf{L}(a) \right\} \le \min\{\omega(H, a),\ \omega(H)\}.$$

2. If $\omega(H) = \infty$, then nothing has to be done. Suppose that $\omega(H) < \infty$. Then H is a BF-monoid by Lemma 3.3. Let $k \in \mathbb{N}_{\ge 2}$. If $\rho_k(H) = k$, then $\rho_{k-1}(H) = k - 1$, and the assertion follows. Suppose that $\rho_k(H) > k$. Then H is not factorial, $\omega(H) > 1$, and we pick an $a \in H$ with $k \in \mathsf{L}(a)$ and $\ell = \max \mathsf{L}(a) > k$. We have to show that $\ell \le \omega(H) - 1 + \rho_{k-1}(H)$. Let

$$a = u_1 \cdot \ldots \cdot u_k = v_1 \cdot \ldots \cdot v_\ell,$$

where $u_1, \ldots, u_k, v_1, \ldots, v_\ell \in \mathcal{A}(H)$. There is a subset $\Omega \subset [1, \ell]$, say $\Omega = [1, j]$, such that $j \le \omega(H)$ and $u_1 \mid v_1 \cdot \ldots \cdot v_j$. Since $\omega(H) > 1$ and $l > k \ge 1$, we may suppose that $j \ge 2$. Then $u_2 \cdot \ldots \cdot u_k = (u_1^{-1} v_1 \cdot \ldots \cdot v_j) v_{j+1} \cdot \ldots \cdot v_\ell$,

$$1 + (\ell - j) \le \max \mathsf{L}(u_1^{-1} v_1 \cdot \ldots \cdot v_j) + \max \mathsf{L}(v_{j+1} \cdot \ldots \cdot v_\ell) \le \max \mathsf{L}(u_2 \cdot \ldots \cdot u_k) \le \rho_{k-1}(H),$$

and thus $\ell \le \omega(H) - 1 + \rho_{k-1}(H)$.

3. As in 2, we may suppose that $\omega(H) < \infty$, and we have to show that $\mathsf c(a) \le \omega(H)$ for all $a \in H$. To do so, we proceed by induction on $\max \mathsf L(a)$. If $\max \mathsf L(a) \le 1$, then $a \in \mathcal A(H) \cup \{1\}$ and $\mathsf c(a) = 0$. Suppose that $\max \mathsf L(a) > 1$ and that $\mathsf c(b) \le \omega(H)$ for all $b \in H$ with $\max \mathsf L(b) < \max \mathsf L(a)$. Let $z \in \mathsf Z(a)$. It is sufficient to find an $\omega(H)$-chain of factorizations from z to every $z' \in \mathsf Z(a)$ with $|z'| = \max \mathsf L(a)$. Let $z' = v_1 \cdot \ldots \cdot v_l \in \mathsf Z(a)$ with $v_1, \ldots, v_l \in \mathcal A(H)$ and $|z'| = l = \max \mathsf L(a)$. Pick a $u \in \mathcal A(H)$ such that $z \in u\mathsf Z(H)$. After renumbering if necessary, there is a $k \in [1, l]$ with $k \le \omega(H, u) \le \omega(H)$ such that $u \mid v_1 \cdot \ldots \cdot v_k$. Consider a factorization $v_1 \cdot \ldots \cdot v_k = uu_2 \cdot \ldots \cdot u_m$ with $u_2, \ldots, u_m \in \mathcal A(H)$. Since $|z'| = \max \mathsf L(a)$, it follows that $m \le k$, and thus

$$\mathsf d(z', uu_2 \cdot \ldots \cdot u_m v_{k+1} \cdot \ldots \cdot v_l) = \mathsf d(v_1 \cdot \ldots \cdot v_k, uu_2 \cdot \ldots \cdot u_m) \le \max\{k, m\} \le \omega(H).$$

Since $\max \mathsf L(u^{-1}a) < \max \mathsf L(a)$, there exists an $\omega(H)$-chain from $u^{-1}z \in \mathsf Z(u^{-1}a)$ to $u_2 \cdot \ldots \cdot u_m v_{k+1} \cdot \ldots \cdot v_l \in \mathsf Z(u^{-1}a)$, and thus there is an $\omega(H)$-chain from z to $uu_2 \cdot \ldots \cdot u_m v_{k+1} \cdot \ldots \cdot v_l$ and to z'. □

4. Tame Krull monoids

The main aim of this section is to derive a characterization for being tame, which is valid for a large class of Krull monoids (see Theorem 4.2). Krull monoids can be characterized by ideal theoretic or by divisor theoretic tools. We briefly gather the necessary terminology. For details we refer to one of the monographs [26,22].

Let D be a monoid. A homomorphism $\varphi \colon H \to D$ is called

- *cofinal* if for every $a \in D$ there exists some $u \in H$ such that $a \mid \varphi(u)$;
- a *divisor homomorphism* if $\varphi(u) \mid \varphi(v)$ implies $u \mid v$ for all $u, v \in H$;
- a *divisor theory* (for H) if $D = \mathcal F(P)$ for some set P, φ is a divisor homomorphism and, for every $p \in P$ (equivalently for every $a \in F$), there exists a finite subset $\emptyset \ne X \subset H$ satisfying $p = \gcd\big(\varphi(X)\big)$.

Note that, by definition, every divisor theory is cofinal. Now suppose that $H \subset D$ and $\mathsf q(H) \subset \mathsf q(D)$. Then $H \subset D$ is called *saturated* (resp. *cofinal*) if the inclusion $H \hookrightarrow D$ is a divisor homomorphism (resp. cofinal). For $a \in \mathsf q(D)$, we denote by $[a] = [a]_{D/H} = a\,\mathsf q(H) \in \mathsf q(D)/\mathsf q(H)$ the class containing a, and we set $D/H = \{[a] \mid a \in D\} \subset \mathsf q(D)/\mathsf q(H)$. The quotient group $\mathsf q(D)/\mathsf q(H)$ is called the *class group* of D modulo H, and $H \subset D$ is cofinal if and only if $D/H = \mathsf q(D)/\mathsf q(H)$ (see [22, Corollary 2.4.3]). Class groups will be written additively whence $[1]$ is the zero element of D/H. If $H \subset D$ is saturated and $a, b \in D$ with $[a] = [b]$, then $a \in H$ if and only if $b \in H$. The monoid H is called a *Krull monoid* if it satisfies one of the following equivalent conditions (see [22, Theorem 2.4.8]):

- H is v-noetherian and completely integrally closed.
- H has a divisor theory.
- H_{red} is a saturated submonoid of a free monoid.

In particular, H is a Krull monoid if and only if H_{red} is a Krull monoid. Let H be a Krull monoid and $F = \mathcal F(P)$ a free monoid. Then F is called a *monoid of divisors* and P a *set of prime divisors* for H if $H_{\mathrm{red}} \subset F$ is a submonoid, and the inclusion $H_{\mathrm{red}} \hookrightarrow F$ is a divisor theory. The monoid of divisors and the set of prime divisors are uniquely determined (up to canonical isomorphism). Hence the class group of $H_{\mathrm{red}} \subset F$,

$$\mathcal C(H) = F/H_{\mathrm{red}} \quad \text{and} \quad \text{the subset } G_P = \{[p] \in \mathcal C(H) \mid p \in P\}$$

of all classes containing prime divisors, are uniquely determined by H. Clearly, we have $[G_P] = \mathcal C(H)$, and conversely, there is the following realization result (see [22, Theorems 2.5.4 and 3.7.8]).

Lemma 4.1. *Let G be an abelian group, $(m_g)_{g \in G}$ a family of cardinal numbers, $G_0 = \{g \in G \mid m_g \ne 0\}$ and $G_1 = \{g \in G \mid m_g = 1\}$. Then the following statements are equivalent:*

(a) *There exists a Krull monoid H and a group isomorphism $\Phi \colon G \to \mathcal C(H)$ such that $\mathrm{card}(P \cap \Phi(g)) = m_g$ for every $g \in G$.*
(b) *$G = [G_0]$, and $G = [G_0 \setminus \{g\}]$ for every $g \in G_1$.*

Let H be a Krull monoid and $F = \mathcal F(P)$ a monoid of divisors for H. Then we say that H has the *approximation property* if it satisfies the following condition (see [22, Proposition 2.5.2]):

- For all $n \in \mathbb N$, distinct $p_1, \ldots, p_n \in P$ and $e_1, \ldots, e_n \in \mathbb N_0$, there exists some $a \in H$ such that $\mathsf v_{p_i}(a) = e_i$ for all $i \in [1, n]$.

Let R be an integral domain. Then R is a Krull domain if and only if its multiplicative monoid $R^\bullet = R \setminus \{0\}$ is a Krull monoid, and if this holds, then $R^\bullet$ has the approximation property.

Next we discuss a Krull monoid of crucial importance, the monoid of zero-sum sequences over a subset of an abelian group. Let G be an additive abelian group and $G_0 \subset G$ a subset. According to the tradition of combinatorial number theory, the elements of $\mathcal F(G_0)$ are called *sequences over* G_0. Thus a sequence $S \in \mathcal F(G_0)$ will be written in the form

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G_0} g^{\mathsf v_g(S)},$$

and we use all notions (such as length and support) as in general free monoids. Furthermore, we denote by $\sigma(S) = g_1 + \cdots + g_l$ the *sum* of S and by

$$\Sigma(S) = \Big\{ \sum_{i\in I} g_i \mid \emptyset \ne I \subset [1,l] \Big\} \subset G \quad \text{the set of subsums of } S.$$

The monoid

$$\mathcal{B}(G_0) = \{ S \in \mathcal{F}(G_0) \mid \sigma(S) = 0 \}$$

is called the *monoid of zero-sum sequences* over G_0.

Clearly, $\mathcal{B}(G_0)$ is reduced, $\mathcal{B}(G_0) \subset \mathcal{F}(G_0)$ is saturated, and hence $\mathcal{B}(G_0)$ is a reduced Krull monoid. Moreover, $\mathcal{F}(G_0)$ is a monoid of divisors for $\mathcal{B}(G_0)$ if and only if $\langle G_0 \rangle = [G_0 \setminus \{g\}]$ for every $g \in G_0$ (see [22, Proposition 2.5.6]).

For every arithmetical invariant $*(H)$ defined for a monoid H, we write $*(G_0)$ instead of $*\big(\mathcal{B}(G_0)\big)$. In particular, we set $\mathcal{A}(G_0) = \mathcal{A}\big(\mathcal{B}(G_0)\big)$, $\omega(G_0) = \omega\big(\mathcal{B}(G_0)\big)$, $\mathsf{t}(G_0) = \mathsf{t}\big(\mathcal{B}(G_0)\big)$, and so on. We define the *Davenport constant* of G_0 by

$$\mathsf{D}(G_0) = \sup\big\{ |U| \bigm| U \in \mathcal{A}(G_0) \big\} \in \mathbb{N}_0 \cup \{\infty\},$$

which is a central invariant in zero-sum theory (see [15,18] for its relevance in factorization theory). We will use without further mention that for a finite set G_0 we have $\mathsf{D}(G_0) < \infty$ (see [22, Theorem 3.4.2]).

Let H be a reduced Krull monoid, $H \subset F = \mathcal{F}(P)$ a divisor theory and $G_P = \{[p] \mid p \in P\} \subset F/H_{\mathrm{red}}$ the set of classes containing prime divisors. The homomorphism $\widetilde{\boldsymbol{\beta}}\colon \mathcal{F}(P) \to \mathcal{F}(G_P)$, mapping an element $p \in P$ onto its class $[p] \in G_P$, induces a transfer homomorphism $\boldsymbol{\beta} = \widetilde{\boldsymbol{\beta}} \mid H\colon H \to \mathcal{B}(G_P)$. In particular, $\boldsymbol{\beta}\big(\mathcal{A}(H)\big) = \mathcal{A}(G_P)$ and

$$\mathsf{D}(H) = \sup\{ |u| \mid u \in \mathcal{A}(H) \} = \mathsf{D}(G_P)$$

(for details see [22, Section 3.4 and Theorem 5.1.5]).

Recall that, for a constant $m \in \mathbb{N}$, mG_0 denotes the m-fold sumset. Clearly, the condition $-G_0 \subset m(G_0 \cup \{0\})$ is equivalent to the condition that for every $g \in G_0$ there exists a zero-sum sequence $U_g \in \mathcal{B}(G_0)$ such that $g \mid U_g$ and $|U_g| \le m+1$.

Now we can formulate the main result of this section.

Theorem 4.2. *Let H be a Krull monoid, $F = \mathcal{F}(P)$ a monoid of divisors and $G_P = \{[p] \mid p \in P\} \subset F/H_{\mathrm{red}} = G$ the set of classes containing prime divisors. Suppose that one of the following conditions hold:*

(a) *H has the approximation property.*
(b) *Every $g \in G_P$ contains at least two prime divisors.*
(c) *There is an $m \in \mathbb{N}$ such that $-G_P \subset m(G_P \cup \{0\})$.*
(d) *The torsion free rank of G is finite.*

Then H is tame if and only if $\mathsf{D}(G_P) < \infty$. Moreover, we have:

1. *If* (a) *or* (b) *or* $\big($(c) *with* $m = 1\big)$ *holds, then* $\omega(H) = \mathsf{D}(G_P)$.
2. *Suppose that either the total rank of G is finite or that there is an $m \in \mathbb{N}$ such that $G = m(G_P \cup \{0\})$. Then H is tame if and only if G_P is finite.*

We will see that the finiteness of the Davenport constant implies (almost trivially) that the monoid is tame. But the converse needs some additional assumption. Indeed, in Example 4.13 we will point out a tame Krull monoid with $\mathsf{D}(G_P) = \infty$. But before that we start with the proof of Theorem 4.2, which will be done in a series of lemmas. We fix our notations for the rest of this section.

Let H be a reduced Krull monoid, $H \subset F = \mathcal{F}(P)$ a monoid of divisors and $G_P = \{[p] \mid p \in P\} \subset F/H = G$ the set of classes containing prime divisors. We define subsets P_s, P_m, P_t and P_f of P by

$$P_s = \{p \in P \mid P \cap [p] = \{p\}\}, \quad P_m = P \setminus P_s, \quad P_t = \{p \in P \mid \mathrm{ord}([p]) < \infty\} \quad \text{and} \quad P_f = P \setminus P_t.$$

For $a \in \{s, m\}$, $b \in \{f, t\}$, we set $P_{a,b} = P_a \cap P_b$. In order to get lower bounds on $\omega(H, \cdot)$, we make the following definition. Let $P_0, P_1 \subset P$ be finite subsets. We call (P_0, P_1) *independent of order* $(\alpha_p)_{p\in P_0} \in \mathbb{N}^{P_0}$ if the following two conditions hold:

1. For any $p \in P_0$ there exists $a_p \in H$ such that $\mathsf{v}_p(a_p) = \alpha_p$ and $\mathsf{v}_q(a_p) = 0$ for all $q \in P_0 \setminus \{p\}$.
2. There exists some $b \in H$ such that $P_1 \subset \mathrm{supp}(b) \subset P \setminus P_0$.

The first lemma is well known. Since it is the starting point of our investigations, we present its short and simple proof.

Lemma 4.3.

1. *For every $a \in H$, we have $\omega(H, a) \le |a|$.*
2. *$\omega(H) \le \mathsf{D}(G_P)$. In particular, if $\mathsf{D}(G_P) < \infty$, then H is tame.*

Proof. 1. Let $a \in H$ and $a_1, \ldots, a_n \in H$ such that $a \mid a_1 \cdot \ldots \cdot a_n$. Suppose that $a = p_1 \cdot \ldots \cdot p_l$ with $|a| = l \in \mathbb{N}$ and $p_1, \ldots, p_l \in P$. Clearly, there is a subset $\Omega \subset [1, n]$ such that $|\Omega| \le l$ and

$$a = p_1 \cdot \ldots \cdot p_l \quad \text{divides} \quad \prod_{\nu \in \Omega} a_\nu \text{ in } F, \text{ and hence in } H.$$

2. Obviously, 1 implies that

$$\omega(H) = \sup\{\omega(H, u) \mid u \in \mathcal{A}(H)\} \le \sup\{|u| \mid u \in \mathcal{A}(H)\} = \mathsf{D}(G_P).$$

Thus, if $\mathsf{D}(G_P) < \infty$, then $\omega(H) < \infty$, and hence H is tame by Proposition 3.5. □

Lemma 4.4.

1. *Suppose there exists a constant $m \in \mathbb{N}$ such that $-G_P \subset m(G_P \cup \{0\})$. Then $\omega(H, a) \ge |a|/(m + 1)$ for all $a \in H$, and if $m = 1$ and $a \in \mathcal{A}(H)$, then $\omega(H, a) \ge |a|$. Thus H is tame if and only if $\mathsf{D}(G_P) < \infty$.*
2. *Suppose there exists a constant $m \in \mathbb{N}$ such that $G = m(G_P \cup \{0\})$. Then H is tame if and only G_P is finite.*

Proof. 1. Let $a = p_1 \cdot \ldots \cdot p_l \in H$ with $l \in \mathbb{N}$ and $p_1, \ldots, p_l \in P$. For every $i \in [1, l]$, there is an $a_i \in H$ with $p_i \mid a_i$ and $|a_i| \le m + 1$. Then $a \mid a_1 \cdot \ldots \cdot a_l$, and there is a subset $\Omega \subset [1, l]$, say $\Omega = [1, \lambda]$, such that a divides $a_1 \cdot \ldots \cdot a_\lambda$ but no proper subproduct. Then we obtain

$$|a| \le |a_1 \cdot \ldots \cdot a_\lambda| \le \lambda(m + 1) \quad \text{and hence} \quad \omega(H, a) \ge \lambda \ge \frac{|a|}{m + 1}.$$

Now suppose that $m = 1$ and $a \in \mathcal{A}(H)$. For $l = 1$, the assertion is clear. Suppose that $l = 2$. Since $H \subset F$ is a divisor theory, there are $a_1, a_2 \in H$ such that $p_1 \mid a_1, p_2 \nmid a_1, p_2 \mid a_2$ and $p_1 \nmid a_2$. Thus $a \mid a_1 a_2$, but $a \nmid a_1, a \nmid a_2$, and hence $\omega(H, a) \ge 2$. Suppose that $l \ge 3$. For $i \in [1, l]$, let $q_i \in -[p_i] \cap P$ and $a_i = p_i q_i \in H$. Clearly, $a \mid a_1 \cdot \ldots \cdot a_l$, and we assert that a does not divide a proper subproduct, which implies that $\omega(H, a) \ge |a|$. Assume to the contrary that there is an $\Omega \subsetneq [1, l]$ such that a divides $\prod_{\mu \in \Omega} a_\mu$. Then $\prod_{\lambda \in [1, l] \setminus \Omega} p_\lambda$ divides $\prod_{\mu \in \Omega} q_\mu$ in F, and hence there is a $\lambda \in [1, l] \setminus \Omega$ and a $\mu \in \Omega$ such that $p_\lambda = q_\mu$. But this implies that $[p_\lambda p_\mu] = [p_\lambda] - [q_\mu] = 0$, and hence $p_\lambda p_\mu \in H$, a contradiction to $l \ge 3$ and $a \in \mathcal{A}(H)$.

Using Lemma 4.3, we infer that $\omega(H) < \infty$ if and only if $\mathsf{D}(G_P) < \infty$, and thus Proposition 3.6 shows that H is tame if and only if $\mathsf{D}(G_P) < \infty$.

2. If G_P is infinite, then G is infinite, $\Delta(H)$ is infinite by [28], and hence H is not tame by Proposition 3.6.3. If G_P is finite, then $\mathsf{D}(G_P) < \infty$, and H is tame by Lemma 4.3. □

Remark 4.5.

1. By definition, a group is called bounded if there is an $N \in \mathbb{N}$ such that $Ng = 0$ for all $g \in G$. Note that a bounded group is a direct sum of finite cyclic groups with a bound on their orders; see e.g. [22, Corollary A.4]. Now suppose that G is a bounded group, and let $G_0 \subset G$ be a subset with $0 \in G_0$. Then, for every $g \in G_0$, there is an $n \in [1, N]$ such that $-g = (n - 1)g$, and thus $-G_0 \subset NG_0$.

2. Let R be a Krull domain with class group G and $G_P \subset G$ the set of classes containing prime divisors. If R is either a finitely generated $\mathbb{Z}$-algebra or a finitely generated k-algebra over some infinite perfect field k, then there is an $m \in \mathbb{N}$ such that $G = m(G_P \cup \{0\})$ (see [33, Propositions 4.2 and 4.4]).

Lemma 4.6. *Let $a \in H$, $P_0, P_1 \subset P$ be such that $\mathrm{supp}(a) = P_0 \cup P_1$ and (P_0, P_1) is independent of order $(\alpha_p)_{p \in P_0} \in \mathbb{N}^{P_0}$. Then*

$$\omega(H, a) \ge \sum_{p \in P_0} \left\lceil \frac{\mathsf{v}_p(a)}{\alpha_p} \right\rceil.$$

Proof. For all $p \in P_0$, let $a_p \in H$ and let $b \in H$ be as in the definition of independence. We set, for all $p \in P_0$, $e_p = \lceil \mathsf{v}_p(a)/\alpha_p \rceil$, $M = \max\{\mathsf{v}_p(a) \mid p \in P_1\}$ and

$$u = \prod_{p \in P_0} a_p^{e_p} b^M.$$

Then $u \in H$, $\mathsf{v}_p(a) \le \mathsf{v}_p(u)$ for all $p \in P$, and hence $a \mid u$. Thus there exist $(\beta_p)_{p \in P_0} \in \mathbb{N}_0^{P_0}$ and $M' \in [0, M]$ such that $\beta_p \le e_p$ for all $p \in P_0$, $\sum_{p \in P_0} \beta_p + M' \le \omega(H, a)$ and

$$a \quad \text{divides} \quad u' = \prod_{p \in P_0} a_p^{\beta_p} b^{M'}.$$

Then, for any $p \in P_0$, we have $\mathsf{v}_p(a) \le \mathsf{v}_p(u') = \alpha_p \beta_p$. Hence $\beta_p \ge \lceil \mathsf{v}_p(a)/\alpha_p \rceil = e_p$, and therefore $\beta_p = e_p$ for all $p \in P_0$. This implies that

$$\omega(H, a) \ge \sum_{p \in P_0} \beta_p + M' \ge \sum_{p \in P_0} \left\lceil \frac{\mathsf{v}_p(a)}{\alpha_p} \right\rceil. \quad \square$$

Lemma 4.7. *Let $P_0, P_0', P_1 \subset P$ be finite subsets and $p \in P \setminus P_1$.*

1. *If H has the approximation property, then $(P_0, \emptyset)$ is independent of order $(1)_{p\in P_0}$.*
2. *$(\{p\}, P_1)$ is independent of order 1.*
3. *Suppose that P_0, P_0', P_1 have the following properties:*
   (a) $P_0 \subset P_m$.
   (b) $P_0' \subset P_{s,t}$.
   (c) $P_1 \subset P \setminus (P_0 \cup P_0')$.
   (d) $[p] \neq [q]$, *if $p, q \in P_0$ are different or if $p \in P_0$ and $q \in P_0'$.*
   For $p \in P_0 \cup P_1$, set
$$\alpha_p = \begin{cases} 1 & p \in P_0 \\ \operatorname{ord}([p]) & p \in P_0'. \end{cases}$$
   Then $(P_0 \cup P_0', P_1)$ is independent of order $(\alpha_p)_{p\in P_0\cup P_0'}$.

Proof. 1. This follows immediately from the definition of independence and the approximation property (for b in the definition of independence take $b = 1$).

2. Since $H \subset F$ is a divisor theory, there are $a, b \in H$ such that $\mathsf v_p(a) = 1$, $\prod_{p\in P_1} p \mid b$ and $p \nmid b$. Then a and b satisfy all conditions for independence of order 1 of $(\{p\}, P_1)$.

3. First note that $P_0 \cap P_0' = P_0 \cap P_1 = P_0' \cap P_1 = \emptyset$. Since $P_0 \subset P_m$, we can choose for any $p \in P_0$ some $p' \in P \setminus \{p\}$ such that $[p'] = [p]$. Then, by construction and condition (3d), we have $\{p' \mid p \in P_0\} \cap (P_0 \cup P_0') = \emptyset$. We continue with the following assertion.

A. If $u \in H$ and $Q \subset \operatorname{supp}(u) \cap P_t$, then there exists $u' \in H$ such that $\operatorname{supp}(u') = \operatorname{supp}(u) \setminus Q$.

Proof of A. Let u and Q be as above, and set $u = u_1u_2$, where $u_1, u_2 \in \mathcal F(P)$ are uniquely determined by $\operatorname{supp}(u_1) = \operatorname{supp}(u) \setminus Q$ and $\operatorname{supp}(u_2) = Q$. Set $N = \operatorname{lcm}\{\operatorname{ord}([p]) \mid p \in Q\}$. Then $u_2^N \in H$, and from $u^N = u_1^N u_2^N$ we obtain $u_1^N \in H$. Since $\operatorname{supp}(u_1^N) = \operatorname{supp}(u_1) = \operatorname{supp}(u) \setminus Q$, the claim follows. □

In order to show that $(P_0 \cup P_0', P_1)$ is independent of order $(\alpha_p)_{p\in P_0\cup P_0'}$, we verify the two conditions in the definition of independence.

First, we show that there exists $b \in H$ such that $P_1 \subset \operatorname{supp}(b) \subset P \setminus (P_0 \cup P_0')$. Since $H \subset F$ is a divisor theory, we may choose $b_1 \in H$ such that $P_1 \subset \operatorname{supp}(b_1)$. Using **A** with $Q = \operatorname{supp}(b_1) \cap P_0'$ and using $P_0' \cap P_1 = \emptyset$, we see that there exists $b_2 \in H$ such that $P_1 \subset \operatorname{supp}(b_2) \subset P \setminus P_0'$. Set $S = \operatorname{supp}(b_2)$. Then $b_2 = c_1c_2$, where

$$c_1 = \prod_{p\in S\setminus P_0} p^{\mathsf v_p(b_1)}, \qquad c_2 = \prod_{p\in S\cap P_0} p^{\mathsf v_p(b_2)}.$$

Now set

$$c_2' = \prod_{p\in S\cap P_0} p'^{\mathsf v_p(c_2)}$$

and $b = c_1c_2'$. From $[c_2'] = [c_2]$, it follows that $b \in H$. By construction, we have $P_1 \subset \operatorname{supp}(b) \subset P \setminus (P_0 \cup P_0')$.

Second, we pick $p \in P_0 \cup P_0'$ and show that there exists some $a \in H$ such that $\mathsf v_p(a) = \alpha_p$ and $\mathsf v_q(a) = 0$ for all $q \in (P_0 \cup P_0') \setminus \{p\}$.

If $p \in P_0'$, then $\alpha_p[p] = 0$, and hence $a = p^{\alpha_p} \in H$ has the required property. Suppose that $p \in P_0$. Choose $a_1 \in H$ with $P_0 \subset \operatorname{supp}(a_1)$. Then, using $P_0 \cap P_0' = \emptyset$ and **A** with $Q = \operatorname{supp}(a_1) \cap P_0'$, we obtain an element $a_2 \in H$ such that $P_0 \subset \operatorname{supp}(a_2) \subset P \setminus P_0'$. Now set

$$a = pp'^{\mathsf v_p(a_2)-1} \cdot \prod_{q\in(\operatorname{supp}(a_2)\setminus\{p\})\cap P_0} q'^{\mathsf v_p(a_2)} \cdot \prod_{q\in\operatorname{supp}(a_2)\setminus P_0} q^{\mathsf v_q(a_2)}.$$

Since $[a_2] = [a]$, it follows that $a \in H$. From $\{q' \mid q \in P_0\} \cap (P_0 \cup P_0') = \emptyset$ we obtain $\mathsf v_p(a) = 1 = \alpha_p$ and $\mathsf v_q(a) = 0$ for all $q \in (P_0 \cup P_0') \setminus \{p\}$. □

Corollary 4.8. *Let $a \in H$.*

1. *Suppose that $[p] \neq [q]$ for all distinct $p, q \in \operatorname{supp}(a) \cap P_m$. Then*
$$\omega(H, a) \ge \mathsf v_{P_m}(a) + \sum_{p\in P_{s,t}} \left\lceil \frac{\mathsf v_p(a)}{\operatorname{ord}([p])} \right\rceil.$$
2. *If H has the approximation property, then $\omega(H, a) = |a|$, and if $P = P_m$, then there is an $a' \in \boldsymbol\beta^{-1}\big(\boldsymbol\beta(a)\big)$ such that $\omega(H, a') = |a'| = |a|$.*
3. *$\omega(H, a) \ge \max\{\mathsf v_p(a) \mid p \in P\}$.*

Proof. 1. We set $P_0 = \mathrm{supp}(a) \cap P_m$, $P_0' = \mathrm{supp}(a) \cap P_{s,t}$ and $P_1 = \mathrm{supp}(a) \cap P_{s,f}$. Then $\mathrm{supp}(a) = (P_0 \cup P_0') \cup P_1$. For $p \in P_0 \cup P_0'$, set

$$\alpha_p = \begin{cases} 1 & p \in P_0 \\ \mathrm{ord}([p]) & p \in P_0'. \end{cases}$$

By Lemma 4.7.3, $(P_0 \cup P_0', P_1)$ is independent of order $(\alpha_p)_{p \in P_0 \cup P_0'}$. By Lemma 4.6, we obtain

$$\omega(H, a) \ge \sum_{p \in P_0 \cup P_0'} \left\lceil \frac{\mathsf v_p(a)}{\alpha_p} \right\rceil = \mathsf v_{P_m}(a) + \sum_{p \in P_{s,t}} \left\lceil \frac{\mathsf v_p(a)}{\mathrm{ord}([p])} \right\rceil.$$

2. By Lemma 4.3.1, it suffices to show that $\omega(H, a) \ge |a|$ and $\omega(H, a') \ge |a'| = |a|$, respectively.

Suppose that H has the approximation property and set $P_0 = \mathrm{supp}(a)$, $P_0' = P_1 = \emptyset$ and $\alpha = (1)_{p \in P_0}$. Then, by Lemma 4.7.1, (P_0, P_1) is independent of order α, and thus Lemma 4.6 implies that $\omega(H, a) \ge |a|$.

Suppose that $P = P_m$ and set $\boldsymbol\beta(a) = g_1^{k_1} \cdot \ldots \cdot g_s^{k_s}$ with $k_1, \ldots, k_s \in \mathbb N$ and $g_1, \ldots, g_s \in G$ pairwise distinct. For $i \in [1, s]$, we pick $p_i \in P \cap g_i$ and define $a' = p_1^{k_1} \cdot \ldots \cdot p_s^{k_s}$. Now we set $P_0 = \mathrm{supp}(a')$, $P_0' = P_1 = \emptyset$ and $\alpha = (1)_{p \in P_0}$. Then, by Lemma 4.7.3, (P_0, P_1) is independent of order α, and thus Lemma 4.6 implies that $\omega(H, a') \ge |a'| = |a|$.

3. Let $p \in \mathrm{supp}(a)$. By Lemma 4.7.2 $(\{p\}, \mathrm{supp}(a) \setminus \{p\})$ is independent of order 1. Hence $\omega(H, a) \ge \mathsf v_p(a)$ by 4.6. □

Corollary 4.9. *Let $a \in H$ and suppose that $[p] \ne [q]$ for all distinct $p, q \in \mathrm{supp}(a) \cap P_m$. Then*

$$|\mathrm{supp}(a)| \le \omega(H, a)\, \omega(H) + |\mathrm{supp}(a) \cap P_{s,f}|.$$

Proof. By 4.8.1, we have

$$\omega(H, a) \ge \mathsf v_{P_m}(a) + \sum_{p \in P_{s,t}} \left\lceil \frac{\mathsf v_p(a)}{\mathrm{ord}([p])} \right\rceil \ge |\mathrm{supp}(a) \cap P_m| + \sum_{p \in \mathrm{supp}(a) \cap P_{s,t}} \frac{1}{\mathrm{ord}([p])}.$$

Set $A = \max\{\mathrm{ord}([p]) \mid p \in \mathrm{supp}(a) \cap P_{s,t}\}$. Then we obtain

$$\omega(H, a) \ge |\mathrm{supp}(a) \cap P_m| + \frac{1}{A} |\mathrm{supp}(a) \cap P_{s,t}| \ge \frac{1}{A}(|\mathrm{supp}(a)| - |\mathrm{supp}(a) \cap P_{s,f}|).$$

It remains to show that $A \le \omega(H)$. So let $p \in P_{s,t} \cap \mathrm{supp}(a)$. Then $u = p^{\mathrm{ord}([p])}$ is an atom of H and by 4.8.3 we obtain $\mathrm{ord}([p]) = \mathsf v_p(u) \le \omega(H, u) \le \omega(H)$. □

Lemma 4.10. *Let A be an abelian group, $G_0 \subset A$ be a subset such that $[G_0] = A$ and $M = \sup\{\mathsf v_g(U) \mid g \in G_0,\ U \in \mathcal A(G_0)\} < \infty$, and set $G_{0,f} = \{g \in G_0 \mid \mathrm{ord}(g) = \infty\}$. If the torsion free rank of A is finite, then the set $\{M!g \mid g \in G_{0,f}\}$ is finite. If the total rank of A is finite, then G_0 is finite.*

Proof. First suppose that the torsion free rank of A is finite. Let $E \subset G_{0,f}$ be a maximal independent set. Then E is finite by assumption. Since $[G_0] = A$, there exists, for each $h \in E$, some $S_h \in \mathcal F(G_0)$ such that $hS_h \in \mathcal B(G_0)$. Then the set

$$E_1 = E \cup \bigcup_{h \in E} \mathrm{supp}(S_h)$$

is finite, and we claim that for any $g \in G_{0,f}$ there exists some $C \in \mathcal A(\{g\} \cup E_1)$ such that $g \mid C$. Let $g \in G_{0,f}$. Clearly, it is sufficient to find some $B \in \mathcal B(\{g\} \cup E_1)$ such that $g \mid B$. If $g \in E$, then gS_g does the job. So suppose that $g \notin E$. Since $E \cup \{g\}$ is not independent, there are $\alpha \in \mathbb Z$ and, for every $h \in E$, an element $\beta_h \in \mathbb Z$ such that $\alpha g + \sum_{h \in E} \beta_h h = 0$ and $\alpha g \ne 0$ or $\beta_h h \ne 0$ for some $h \in E$. Since E is independent, we get $\alpha \ne 0$, and suppose without restriction that $\alpha > 0$. Then, clearly we have

$$B = g^{\alpha} \prod_{\substack{h \in E \\ \beta_h \ge 0}} h^{\beta_h} \prod_{\substack{h \in E \\ \beta_h < 0}} S_h^{-\beta_h} \in \mathcal B(\{g\} \cup E_1) \quad \text{and} \quad g \mid B.$$

Since the set

$$E_2 = \Big\{ -\sum_{g \in E_1} \beta_g g \mid \beta_g \in [0, M \cdot M!] \text{ for all } g \in E_1 \Big\}$$

is finite, it suffices to show that for every $g \in G_{0,f}$ we have $M!g \in E_2$. Pick $g \in G_{0,f}$ and a $C \in \mathcal A(\{g\} \cup E_1)$ such that $g \mid C$, say $C = g^{\alpha} \prod_{g \in E_1} g^{\beta_g}$. Then $\alpha \in [1, M]$ and $\beta_g \in [0, M]$ for all $g \in E_1$. Since $\alpha g + \sum_{g \in E_1} \beta_g g = 0$, it follows that $M!g \in E_2$.

Now suppose that the total rank of A is finite, and for a prime $p \in \mathbb P$ let

$$\mathbb Z(p^{\infty}) = \left\{ \frac{m}{p^k} + \mathbb Z \mid m \in \mathbb Z, k \in \mathbb N \right\} \subset \mathbb Q/\mathbb Z$$

denote the Prüfer group of type p^∞. Then, by [14, Sections 23 and 24],

$$A \subset \mathbb{Q}^s \oplus \bigoplus_{i=1}^{t} \mathbb{Z}(p_i^\infty), \tag{$*$}$$

where $s,\ t \in \mathbb{N}_0,\ s+t$ is the total rank of A and $p_1, \ldots, p_t \in \mathbb{P}$. Obviously, $(*)$ implies that for every $n \in \mathbb{N}$ the subgroup $\{g \in A \mid Ng = 0\}$ is finite. This and the finiteness of $\{M!g \mid g \in G_{0,f}\}$ imply that $G_{0,f}$ is finite. Let $G_{0,t} = G_0 \setminus G_{0,f}$. If $g \in G_{0,t}$, then $g^{\mathrm{ord}(g)} \in \mathcal{A}(G_0)$, and hence $\mathrm{ord}(g) \le M$. As before, this implies that $G_{0,t}$ is finite, and hence G_0 is finite. □

Proposition 4.11. *Suppose that the torsion free rank of G is finite and that $\omega(H) < \infty$. Then $\mathsf{D}(G_P) < \infty$.*

Proof. Since $\omega(H) < \infty$, Corollary 4.8.3 implies that $M = \sup\{\mathsf{v}_p(u) \mid u \in \mathcal{A}(H),\ p \in P\} < \infty$. Thus by Lemma 4.10 the set $\{M![p] = [p^{M!}] \mid p \in P_{s,f}\}$ is finite, and hence there exists a finite subset $P' \subset P_{s,f}$ such that $\{[p^{M!}] \mid p \in P_{s,f}\} = \{[p^{M!}] \mid p \in P'\}$. We pick an $u \in \mathcal{A}(H)$ and claim that $|u| \le \omega(H)^3 + |P'|\omega(H)$. We start with the following assertion.

A. There exists $a \in H$ such that:
- $\min \mathsf{L}(a) \le M!$;
- $|u^{M!}| = |a|$;
- $[p] \ne [q]$ for distinct $p, q \in \mathrm{supp}(a) \cap P_m$;
- $\mathrm{supp}(a) \cap P_{s,f} \subset P'$.

Proof of A. Let $\boldsymbol{\beta}\colon H \to \mathcal{B}(G_P)$ be the block homomorphism and let $\boldsymbol{\beta}(u) = g_1^{k_1} \cdot \ldots \cdot g_r^{k_r}$ with pairwise distinct $g_1, \ldots, g_r \in G_P$. For each $i \in [1, r]$, let $p_i \in P$ be such that $[p_i] = g_i$. Then $u' = p_1^{k_1} \cdot \ldots \cdot p_r^{k_r} \in \mathcal{A}(H)$ and $|u| = |u'|$. Thus, after replacing u by u' if necessary, we can suppose that $[p] \ne [q]$ for distinct $p, q \in \mathrm{supp}(u)$. By definition of P' there is a map $\theta : P_{s,f} \to P'$ such that $[p^{M!}] = [\theta(p)^{M!}]$ for all $p \in P_{s,f}$. We now set

$$a = \prod_{p \in \mathrm{supp}(u) \cap (P_m \cup P_{s,t})} p^{M!\mathsf{v}_p(u)} \prod_{p \in P_{s,f}} \theta(p)^{M!\mathsf{v}_p(u)}.$$

Then $[a] = [u^{M!}]$, and hence $a \in H$. From $\boldsymbol{\beta}(a) = \boldsymbol{\beta}(u^{M!})$, we obtain $\min \mathsf{L}(a) = \min \mathsf{L}(\boldsymbol{\beta}(a)) \le M!$ and $|a| = |u^{M!}|$. Thus a fulfills all our requirements. □

Now let $a = v_1 \cdot \ldots \cdot v_n$ be a factorization of a such that $n = \min \mathsf{L}(a) \le M!$. Choose some $i \in [1, n]$. Then clearly $\mathrm{supp}(v_i) \cap P_{s,f} \subset P'$ and $[p] \ne [q]$ for distinct $p, q \in \mathrm{supp}(v_i) \cap P_m$. By Corollary 4.9, we obtain

$$|\mathrm{supp}(v_i)| \le \omega(H, v_i)\omega(H) + |\mathrm{supp}(v_i) \cap P_{s,f}| \le \omega(H)^2 + |P'|.$$

Using 4.8.3, we obtain

$$|v_i| = \sum_{p \in \mathrm{supp}(v_i)} \mathsf{v}_p(v_i) \le \sum_{p \in \mathrm{supp}(v_i)} \omega(H, v_i) \le \omega(H)^3 + |P'|\omega(H),$$

and hence

$$|u| = \frac{1}{M!}|a| = \frac{1}{M!}\sum_{i=1}^{n} |v_i| \le \frac{n}{M!}(\omega(H)^3 + |P'|\omega(H)) \le \omega(H)^3 + |P'|\omega(H). \quad \square$$

Proof of Theorem 4.2. If $\mathsf{D}(G_P) < \infty$, then H is tame by Lemma 4.3. Conversely, suppose that H is tame. If Condition (a) or (b) of Theorem 4.2 holds, then all assertions follow from Corollary 4.8.2. If Condition (c) holds, then Lemma 4.4.1 does the job. If the torsion free rank of G is finite, then $\mathsf{D}(G_P) < \infty$ by Proposition 4.11. Suppose that the total rank of G is finite. Since H is tame, the invariant M occurring in Lemma 4.10 is finite, and clearly we have $[G_P] = G$. Thus Lemma 4.10 implies that G_P is finite. If $G = m(G_P \cup \{0\})$ for some $m \in \mathbb{N}$, then the assertion follows from Lemma 4.4. □

We end this section with two examples. The first example shows that several bounds obtained in Section 3 are sharp for Krull monoids. Example 4.13 reveals a tame Krull monoid H with $\mathsf{D}(G_P) = \infty$.

Example 4.12. Let H be a Krull monoid with finite class group G such that every class contains a prime divisor. In order to avoid trivial cases we suppose that $|G| \ge 3$. Then $\omega(H) = \mathsf{D}(G)$ by Theorem 4.2.(c).

1. Suppose that G is isomorphic to C_2^r for some even $r \in \mathbb{N}$. Then, by [22, Corollary 6.5.6], we have

$$\mathsf{D}(C_2^r) = r + 1 = \omega(H) \quad \text{and} \quad \mathsf{t}(H) = 1 + \frac{r^2}{2}.$$

   This shows that the bound given in Proposition 3.5 has the right order of magnitude.
2. Suppose that G is either cyclic or an elementary 2-group. Then $\mathsf{c}(H) = \mathsf{D}(G)$ by [22, Theorem 6.4.7], and thus equality holds in Proposition 3.6.3.

3. Let $k \in \mathbb{N}$. Then $\rho_k(H) - \rho_{k-1}(H) \le \mathsf{D}(G)/2$, and equality holds for a variety of groups (see [22, Section 6.3]. If G is cyclic, then $\rho_{2k}(H) - \rho_{2k-1}(H) = \omega(H) - 1$ by [18, Corollary 5.3.2], and thus equality holds in Proposition 3.6.2.

Example 4.13. We construct a Krull monoid H with $\mathsf{t}(H) = 2$ and $\mathsf{D}(H) = \infty$. To do so we proceed in two steps.

1. Let $n \in \mathbb{N}_{\ge 2}$, $F_n = \oplus_{i=0}^n \langle e_i \rangle \cong \mathbb{Z}^{n+1}$ be an additive free abelian group of rank $n+1$ with basis $\{e_0, \ldots, e_n\}$ and P_n the power set of $[1, n]$. For $I \in P_n$, let $\varphi_I : F_n \to \mathbb{Z}$ be defined by

$$\varphi_I\left(\sum_{i=0}^n x_i e_i\right) = x_0 + \sum_{i \in I} x_i,$$

and hence for $i \in [1, n]$ we have

$$\varphi_I(e_i) = \begin{cases} 1 & i \in I \\ 0 & i \notin I, \end{cases} \qquad \varphi_I(e_0 - e_i) = \begin{cases} 0 & i \in I \\ 1 & i \notin I. \end{cases} \tag{4.1}$$

We define

$$H_n = \{x \in F_n \mid \varphi_I(x) \ge 0 \text{ for all } I \in P_n\}.$$

Then $\boldsymbol{\varphi} = \big(\varphi_I \mid H_n : H_n \to \mathbb{N}_0\big)_{I \in P_n}$ is a defining family for H_n; hence

$$\varphi_n : H_n \to \mathbb{N}_0^{P_n}, \qquad x \mapsto \big(\varphi_I(x)\big)_{I \in P_n}$$

is a cofinal divisor homomorphism (see [22, Proposition 2.6.2]), and therefore H is a Krull monoid.

1(a) We assert that

$$H_n = [e_1, \ldots, e_n, e_0 - e_1, \ldots, e_0 - e_n].$$

The inclusion $\supset$ follows from Eqs. (4.1). Conversely, let $\sum_{i=0}^n x_i e_i \in H_n$. Then, for every $I \in P_n$, we have

$$x_0 + \sum_{i \in I} x_i \ge 0. \tag{4.2}$$

For $i \in [1, n-1]$, set $b_i = \max\{0, -x_i\} \in \mathbb{N}_0$. Taking $I_1 = \{i \in [1, n-1] \mid x_i < 0\}$ in (4.2), we obtain

$$x_0 - \sum_{i=1}^{n-1} b_i = x_0 + \sum_{i \in I_1} x_i \ge 0.$$

We define $b_n = x_0 - \sum_{i=1}^{n-1} b_i \in \mathbb{N}_0$. Taking $I_2 = I_1 \cup \{n\}$ in (4.2), we obtain

$$b_n + x_n = x_0 + \sum_{i \in I_2} x_i \ge 0.$$

It follows that $a_i = x_i + b_i \in \mathbb{N}_0$ for all $i \in [1, n]$. Now we have

$$\sum_{i=0}^n x_i e_i = \left(\sum_{i=1}^n b_i\right) e_0 + \sum_{i=1}^n (a_i - b_i) e_i = \sum_{i=1}^n a_i e_i + \sum_{i=1}^n b_i (e_0 - e_i) \in [e_1, \ldots, e_n, e_0 - e_1, \ldots, e_0 - e_n].$$

1(b) We assert that φ_n is a divisor theory. For $I, J \in P_n$, we set $H_{n,I} = \{e_i \mid i \in I\} \cup \{e_0 - e_i \mid i \in [1, n] \setminus I\} \subset H_n$, and claim that

$$\min \varphi_J(H_{n,I}) = \begin{cases} 1 & J = I \\ 0 & J \ne I. \end{cases}$$

Then every element of $\mathbb{N}_0^{P_n}$ is a greatest common divisor of a finite subset of elements of $\varphi_n(H_n)$. If $J = I$, then $\varphi_I(a) = 1$ for all $a \in H_{n,I}$, by (4.1). Now let $J \ne I$. If $I \not\subset J$, take $i \in I \setminus J$. Then $e_i \in H_{n,I}$ and $\varphi_J(e_i) = 0$. If $I \subset J$, then $J \setminus I \ne \emptyset$, and we take $i \in J \setminus I$. Then $e_0 - e_i \in H_{n,I}$ and $\varphi_J(e_0 - e_i) = 0$.

1(c) Since the family $(e_0, \ldots, e_n)$ is independent, it follows immediately that

$$\mathcal{A}(H_n) = \{e_1, \ldots, e_n, e_0 - e_1, \ldots, e_0 - e_n\},$$

and hence

$$\mathsf{D}(H_n) = \max\left\{\sum_{I \in P_n} \varphi_I(u) \mid u \in \mathcal{A}(H_n)\right\} = 2^{n-1}.$$

1(d) Finally, we show that $\mathsf{t}(H_n) = 2$. Pick some $u \in \mathcal{A}(H_n)$ and consider the automorphism $f : F_n \to F_n$ defined by $f(e_0) = e_0$ and $f(e_i) = e_0 - e_i$ for every $i \in [1, n]$. Then $f(H_n) = H_n$, and hence we may assume that $u = e_j$ for some $j \in [1, n]$.

Pick some $i \in [1, n] \setminus \{j\}$. Then the equation $e_i + (e_0 - e_i) = e_0 = e_j + (e_0 - e_j)$ shows that e_j is not prime, and hence $\mathsf{t}(H_n, e_j) \geq 2$. To show the reverse inequality, take an element $h \in e_j + H_n$ and a $z = \sum_{i=1}^{n} a_i e_i + \sum_{i=1}^{n} b_i(e_0 - e_i) \in \mathsf{Z}(h)$. We show that there is some $z' \in \mathsf{Z}(h)$ such that $e_j \mid z'$ (in $\mathsf{Z}(H_n)$) and $\mathsf{d}(z, z') \leq 2$.

To see this, we set $J = \{i \in [1, n] \mid a_i > 0\}$ and $K = \{i \in [1, n] \mid b_i > 0\}$. If $J \cap K \neq \emptyset$, then, for some $k \in J \cap K$, we set

$$z' = \sum_{i \in [1,n] \setminus \{k\}} (a_i e_i + b_i(e_0 - e_i)) + (a_k - 1)e_k + (b_k - 1)(e_0 - e_k) + e_j + (e_0 - e_j).$$

Then $z' \in \mathsf{Z}(h)$, $e_j \mid z'$ and $\mathsf{d}(z, z') \leq 2$. Suppose that $J \cap K = \emptyset$. Assume to the contrary that $j \notin J$. Then, taking

$$x = h - e_j = \left(\sum_{i \in K} b_i \right) e_0 + \sum_{i \in J} a_i e_i - e_j + \sum_{i \in K} (-b_i) e_i$$

and $I = K \cup \{j\}$ in (4.2), we obtain

$$0 \leq \sum_{i \in K} b_i + (-1) + \sum_{i \in K} (-b_i) = -1,$$

a contradiction. Thus we have $j \in J$ and hence $e_j \mid z$.

2. Let

$$H = \coprod_{n \geq 2} H_n$$

be the coproduct of all H_n. Then H is Krull monoid, the coproduct of the divisor theories φ_n is a divisor theory of H,

$$\mathcal{A}(H) = \bigcup_{n \geq 2} \mathcal{A}(H_n), \quad \text{hence} \quad \mathsf{D}(H) = \sup\{\mathsf{D}(H_n) \mid n \geq 2\} = \infty$$

and, by [22, Proposition 1.6.8],

$$\mathsf{t}(H) = \sup\{\mathsf{t}(H_n) \mid n \geq 2\} = 2.$$

5. Sets of lengths in tame monoids

Let H be a tame monoid. By Lemma 3.3, all sets of lengths in H are finite. Suppose there is an $a \in H$ such that $|\mathsf{L}(a)| > 1$, say $a = u_1 \cdot \ldots \cdot u_k = v_1 \cdot \ldots \cdot v_l$ with $k < l$ and $u_1, \ldots, u_k, v_1, \ldots, v_l \in \mathcal{A}(H)$. Then, for every $N \in \mathbb{N}$, we have

$$a^N = (u_1 \cdot \ldots \cdot u_k)^\nu (v_1 \cdot \ldots \cdot v_l)^{N-\nu} \quad \text{for all } \nu \in [0, N],$$

whence $\{\nu k + l(N - \nu) \mid \nu \in [0, N]\} \subset \mathsf{L}(a^N)$ and $|\mathsf{L}(a^N)| \geq N + 1$. This shows that sets of lengths get arbitrarily large, but by the following Theorem 5.1 they have a well-defined structure. Indeed, sets of lengths are AAMPs (almost arithmetical multiprogressions) with universal bounds for all parameters. We recall the definitions involved.

To begin with, let

$$\Delta(H) = \bigcup_{L \in \mathcal{L}(H)} \Delta(L) \subset \mathbb{N}$$

denote the *set of distances* of H. A simple calculation (based on Eq. (2.1)) shows that $2 + \sup \Delta(H) \leq \mathsf{c}(H)$, and hence by Proposition 3.5 it follows that in a tame monoid the set of distances is finite.

Let $d \in \mathbb{N}$, $M \in \mathbb{N}_0$ and $\{0, d\} \subset \mathcal{D} \subset [0, d]$. A subset $L \subset \mathbb{Z}$ is called an *almost arithmetical multiprogression* (AAMP for short) with *difference* d, *period* $\mathcal{D}$, and *bound* M, if

$$L = y + (L' \cup L^* \cup L'') \subset y + \mathcal{D} + d\mathbb{Z},$$

where

- L^* is finite nonempty with $\min L^* = 0$ and $L^* = (\mathcal{D} + d\mathbb{Z}) \cap [0, \max L^*]$;
- $L' \subset [-M, -1]$ and $L'' \subset \max L^* + [1, M]$;
- $y \in \mathbb{Z}$.

If $a \in H$ and $k \in \mathbb{Z}$, then $\mathsf{Z}_k(a) = \{z \in \mathsf{Z}(a) \mid |z| = k\} \subset \mathsf{Z}(a)$ denotes the set of factorizations of a having length k. Now we can formulate the main result of this section.

Theorem 5.1. *Let H be a tame monoid. Then there exists a constant $M \in \mathbb{N}_0$ such that for all $a \in H$ the following two properties hold:*

(a) *The set of lengths $\mathsf{L}(a)$ is an* AAMP *with difference $d \in \Delta(H)$ and bound M.*
(b) *For each two adjacent lengths $k, l \in \mathsf{L}(a) \cap [\min \mathsf{L}(a) + M, \max \mathsf{L}(a) - M]$ we have $\mathsf{d}\big(\mathsf{Z}_k(a), \mathsf{Z}_l(a)\big) \leq M$.*

In the following remark we discuss Statement (a) of Theorem 5.1. Its proof will require the rest of this section. More on Statement (b) can be found in Remark 5.5.

Remark 5.2. We say that *the Structure Theorem for Sets of Lengths holds for the monoid H* if H is atomic and there exist some $M^* \in \mathbb{N}_0$ and a finite nonempty set $\Delta^* \subset \mathbb{N}$ such that every $L \in \mathcal{L}(H)$ is an AAMP with some difference $d \in \Delta^*$ and bound M^*.

1. An overview of further classes of monoids and domains satisfying the Structure Theorem can be found in [22, Section 4.7]. But the structure of sets of lengths was open for general tame monoids, and in particular for the tame Mori domains discussed in Example 3.2.4. Let H be a tame monoid. By [16, Theorems 3.5 and 4.2] and Proposition 3.6.2, all sufficiently large unions of sets of lengths are even AAMPs with period $\{0, \min \Delta(H)\}$.

By a recent Realization Theorem (due to Schmid, [37]) the Structure Theorem is sharp for Krull monoids with finite class group.

2. Suppose that H is a Krull monoid with finite class group G. Then H satisfies the Structure Theorem with the set

$$\Delta^* = \Delta^*(G) = \{\min \Delta(G_0) \mid G_0 \subset G \text{ with } \Delta(G_0) \neq \emptyset\} \subset \Delta(H).$$

The set $\Delta^*(G)$ has been investigated in detail (see [35,10,36]), and in general it is a proper subset of $\Delta(H)$. Theorem 5.1 shows that for tame monoids the Structure Theorem holds with $\Delta^* = \Delta(H)$, and by [22, Example 4.8.10] this cannot be improved in general.

3. There is a Dedekind domain R (in particular, $R^\bullet$ is a Krull monoid with approximation property) with finite catenary degree (hence with a finite set of distances) which does not satisfy the Structure Theorem for Sets of Lengths. In particular, R not tame, and if $G_P \subset \mathcal{C}(R)$ denotes the set of classes containing prime ideals, then $\mathsf{D}(G_P) = \infty$.

Proof. By [22, Theorem 4.8.4], there is a Krull monoid H with finite catenary degree which does not satisfy the Structure Theorem for Sets of Lengths. We may suppose that H is reduced and consider a divisor theory $H \subset F = \mathcal{F}(P)$ with class group $G = F/H$ and $G_0 \subset G$ being the set of classes containing primes. Then $\mathsf{c}(G_0) \le \mathsf{c}(H) < \infty$. By Claborn's Realization Theorem there is a Dedekind domain R and an isomorphism $\psi : G \to \mathcal{C}(R)$ mapping G_0 onto the set of classes $G_P \subset \mathcal{C}(R)$ containing prime ideals. Then $\mathcal{L}(H) = \mathcal{L}(G_0) = \mathcal{L}(G_P) = \mathcal{L}(R)$, and hence R does not satisfy the Structure Theorem for Sets of Lengths. By Theorem 5.1, R is not tame, and by Theorem 4.2.(a) it follows that $\mathsf{D}(G_P) = \infty$. Let $\boldsymbol{\beta} : R \to \mathcal{B}(G_P)$ denote the block homomorphism. Then [22, Theorem 3.4.10] implies that

$$\mathsf{c}(R) \le \max\{\mathsf{c}(G_P), 2\} = \max\{\mathsf{c}(G_0), 2\} < \infty,$$

and hence R has finite catenary degree. $\square$

In order to prove Theorem 5.1, we apply the machinery presented in [22, Section 4.3]. In order to do so, we need one more concept, that of tamely generated pattern ideals.

Definition 5.3. Let H be atomic, $\mathfrak{a} \subset H$ and $A \subset \mathbb{Z}$ be a finite nonempty subset.

1. We say that a subset $L \subset \mathbb{Z}$ *contains the pattern* A if there exists some $y \in \mathbb{Z}$ such that $y + A \subset L$. We denote by $\Phi(A) = \Phi_H(A)$ the set of all $a \in H$ for which $\mathsf{L}(a)$ contains the pattern A.
2. $\mathfrak{a}$ is called a *pattern ideal* if $\mathfrak{a} = \Phi(B)$ for some finite, nonempty subset $B \subset \mathbb{Z}$.
3. A subset $E \subset H$ is called a *tame generating set* of $\mathfrak{a}$ if $E \subset \mathfrak{a}$ and there exists some $N \in \mathbb{N}$ with the following property: for every $a \in \mathfrak{a}$, there exists some $e \in E$ such that

$$e \mid a, \quad \sup \mathsf{L}(e) \le N \quad \text{and} \quad \mathsf{t}(a, \mathsf{Z}(e)) \le N.$$

In this case, we call E a *tame generating set with bound N*, and we say that $\mathfrak{a}$ is called *tamely generated*.

4. If $\mathfrak{a}$ is tamely generated, then we denote by $\varphi(\mathfrak{a})$ the smallest $N \in \mathbb{N}_0$ such that $\mathfrak{a}$ has a tame generating set with bound N. Otherwise, we define $\varphi(\mathfrak{a}) = \infty$, and we set $\varphi(A) = \varphi\big(\Phi(A)\big)$.

The significance of tamely generated pattern ideals stems from the following result.

Proposition 5.4. *Let H be a BF-monoid with finite nonempty set of distances $\Delta(H)$, and suppose that all pattern ideals of H are tamely generated. Then there exists a constant $M \in \mathbb{N}_0$ such that for all $a \in H$ the following properties are satisfied:*

(a) *The set of lengths $\mathsf{L}(a)$ is an AAMP with difference $d \in \Delta(H)$ and bound M.*
(b) *For each two adjacent lengths $k, l \in \mathsf{L}(a) \cap [\min \mathsf{L}(a) + M, \max \mathsf{L}(a) - M]$ we have $\mathsf{d}\big(\mathsf{Z}_k(a), \mathsf{Z}_l(a)\big) \le M$.*

Proof. We use Theorem 4.3.11 of [22]. Then (a) follows immediately.

The proof of (b) uses the same ideas used in the proof of (a) in [22]. For that we will need some further notations. For a finite subset $L \subset \mathbb{Z}$ and $\theta \in \mathbb{N}$, we set

$$\kappa_\theta(L) = \max\big\{|L \cap [y+1, y+\theta]| \mid y \in L\big\} \in [0, \theta].$$

Note that for finite $L_1 \subset L_2 \subset \mathbb{Z}$ and $m \in \mathbb{Z}$ we have $\kappa_\theta(L_1) \le \kappa_\theta(L_2)$ and $\kappa_\theta(m + L_1) = \kappa_\theta(L_1)$. If $y \in L$ is such that $\kappa_\theta(L) = |[y+1, y+\theta] \cap L|$, and if we set $\mathcal{D} = \big([y, y+\theta] \cap L\big) - y$, then $\kappa_\theta(L) = \kappa_\theta(\mathcal{D})$, $0 \in \mathcal{D} \subset [0, \theta]$ and L contains the pattern $\mathcal{D}$.

If $A \subset B$ are subsets of $\mathbb{Z}$, then we say that A is an interval of B, if $\emptyset \ne A = B \cap [a, b]$ for some $a, b \in \mathbb{Z}$. Note if $B \subset C$ and A is an interval of C, then A is an interval of B, too.

We choose $\theta \in \mathbb{N}$ such that $\theta \ge 2 \max \Delta(H) + 1$. We set

$$M = 2 \max \big\{ \varphi(A) \mid A \subset [0, \theta] \big\}.$$

Now, suppose that $a \in H$ such that $\mathsf{L}(a) \not\subset \min \mathsf{L}(a) + [0, M]$. We choose $\mathcal{D} \subset \mathbb{Z}$ such that $0 \in \mathcal{D} \subset [0, \theta]$, $\kappa_\theta(\mathcal{D}) = \kappa_\theta(\mathsf{L}(a))$ and $a \in \Phi(\mathcal{D})$. Since $\Phi(\mathcal{D})$ is tamely generated, there exists $a^* \in \Phi(\mathcal{D})$ such that $a^* \mid a$, $\max \mathsf{L}(a^*) \le \varphi(\mathcal{D}) \le M$ and $\mathsf{t}(a, \mathsf{Z}(a^*)) \le \varphi(\mathcal{D}) \le M/2$. Let $m \in \mathbb{Z}$ be such that $m + \mathcal{D} \subset \mathsf{L}(a^*)$.

Let $b \in H$ be such that $a = a^* b$. Then $\mathsf{L}(a) \not\subset \min \mathsf{L}(a) + [0, 2\mathsf{t}(a, \mathsf{Z}(a^*))]$, and by Proposition 4.3.4.2 of [22] we obtain that $\mathsf{L}(b)$ contains at least two elements. So we can choose $x \in \mathbb{Z}$ and $d \in \Delta(\mathsf{L}(b))$ such that $\{x, x+d\} \subset \mathsf{L}(b)$.

Now we set $L_1 = m + \mathcal{D} \subset \mathsf{L}(a^*)$, $L_2 = \mathsf{L}(b)$, $L^* = L_1 + L_2$, $L = \mathsf{L}(a) \cap [\min L^*, \max L^*]$, $\mathcal{D}' = \mathcal{D} \cap [0, d]$. Then $L^* \subset L$, L is an interval of $\mathsf{L}(a)$ and

$$\kappa_\theta(\mathcal{D}) = \kappa_\theta(L_1) = \kappa_\theta(L_1 + x) \le \kappa_\theta(L^*) \le \kappa_\theta(L) \le \kappa_\theta(\mathsf{L}(a)) = \kappa_\theta(\mathcal{D}).$$

Hence equality holds, and by Theorem 4.2.20 of [22] we obtain $L^* = L$, L_1 is an interval of $\min L_1 + \mathcal{D}' + d\mathbb{Z}$ and L^* is an interval of $\min L^* + \mathcal{D}' + d\mathbb{Z}$ (i.e. L_1 and L^* are AMPs with period $\mathcal{D}'$ and difference d). From $L_1 = m + \mathcal{D}$ and $m = \min(L_1)$, we obtain $\mathcal{D} \subset \mathcal{D}' + d\mathbb{Z}$, and hence $\mathcal{D} + d\mathbb{Z} = \mathcal{D}' + d\mathbb{Z}$.

Next we apply Proposition 4.2.19.1 and assertion **A.** of the proof of Proposition 4.2.19.3 of [22] to $L_1 - \min L_1$ and L_2. We obtain $\Delta(L_2) \subset L_1 - \min L_1 = \mathcal{D}$ and that for any $y \in L_2$ the set $y + L_1$ is an interval of $\min L_2 + L_1 + d\mathbb{Z}$. Hence $\max \Delta(\mathsf{L}(b)) = \max \Delta(L_2) \le \max \mathcal{D}$, and we claim that for any $y \in L_2$ the set $y + L_1$ is an interval of L^*. To see this it is enough to show that $L^* \subset \min L_2 + L_1 + d\mathbb{Z}$. This follows from

$$L^* \subset \min L^* + \mathcal{D} + d\mathbb{Z} = \min L_1 + \min L_2 + \mathcal{D} + d\mathbb{Z} = \min L_2 + L_1 + d\mathbb{Z}.$$

Using Proposition 4.3.4.1 of [22], we see that

$$\max \mathsf{L}(a) \le \max L^* - \mathsf{t}(a, \mathsf{Z}(a^*)), \qquad \min \mathsf{L}(a) \ge \min L^* - \mathsf{t}(a, \mathsf{Z}(a^*)),$$

and hence

$$\mathsf{L}(\mathsf{a}) \cap [\min \mathsf{L}(a) + M, \max \mathsf{L}(a) - M] \subset L^*.$$

Now we set $\mathcal{D} = \{0 = \delta_0, \ldots, \delta_\mu\}$ and $\mathsf{L}(b) = \{\epsilon_1, \ldots, \epsilon_s\}$ with $0 = \delta_0 < \delta_1 < \cdots < \delta_\mu$ and $\epsilon_1 < \cdots < \epsilon_s$. We show first that, if $x \in L^*$ with $x < \max L^*$, then $x = m + \delta_i + \epsilon_j$ with $i \in [0, \mu - 1]$ and $j \in [1, s]$. Indeed, since $L^* = L_1 + \mathsf{L}(b)$, there are $i \in [0, \mu]$ and $j \in [1, s]$ such that $x = m + \delta_i + \epsilon_j$. Suppose that $i = \mu$. Since $x < \max L^*$, it follows that $j < s$. Since

$$\epsilon_{j+1} - \epsilon_j \le \max \Delta(\mathsf{L}(b)) \le \max \mathcal{D} = \delta_\mu - \delta_0,$$

we infer that

$$m + \delta_0 + \epsilon_{j+1} \le m + \delta_\mu + \epsilon_j < m + \delta_\mu + \epsilon_{j+1}.$$

Since $\epsilon_{j+1} + L_1$ is an interval of L^*, it follows that $x = m + \delta_\mu + \epsilon_j \in \epsilon_{j+1} + m + \mathcal{D}$. From $\epsilon_j < \epsilon_{j+1}$, we obtain $x = m + \delta_{i_0} + \epsilon_{j+1}$ for some $i_0 \in [0, \mu - 1]$. Now let $k, l \in \mathsf{L}(a) \cap [\min \mathsf{L}(a) + M,\ \max \mathsf{L}(a) - M] \subset L^*$ be two adjacent lengths with $k < l$. Then $k < \max L^*$ and by the above there are $i \in [0, \mu - 1]$ and $j \in [1, s]$ such that $k = m + \delta_i + \epsilon_j$. Since $\epsilon_j + L_1$ is an interval of L^*, it follows that $l = m + \delta_{i+1} + \epsilon_j$. Now choose factorizations $x_i, x_{i+1} \in \mathsf{Z}(a^*)$ with $|x_i| = m + \delta_i$, $|x_{i+1}| = m + \delta_{i+1}$ and $y \in \mathsf{Z}(b)$ with $|y| = \epsilon_j$. Then $z = x_i y$, $z' = x_{i+1} y \in \mathsf{Z}(a)$ with $|z| = k$, $|z'| = l$ and $\mathsf{d}(z, z') = \mathsf{d}(x_i, x_{i+1}) \le \max \mathsf{L}(a^*) \le \varphi(\mathcal{D}) \le M$. $\square$

Remark 5.5. In [13, Theorem 3.1], it is proved that C-monoids satisfy Property (b) of Theorem 5.4. Note that there are finitely primary monoids – they have finite catenary degree and their pattern ideals are tamely generated – where Property (b) does not hold for all adjacent lengths $k, l \in \mathsf{L}(a)$. We do not know whether in tame monoids there is an $M \in \mathbb{N}$ such that for all $a \in H$ and all adjacent lengths $k, l \in \mathsf{L}(a)$ we have $\mathsf{d}\big(\mathsf{Z}_k(a), \mathsf{Z}_l(a)\big) \le M$.

Several conditions, stronger than Property (b) above, have been studied in the literature. The interested reader is referred to [11–13,30]. We only want to recall that also in tame monoids the successive distance $\delta(H)$ and the monotone catenary degree might be infinite (see [11, Example 4.5]). We do not know if this might happen in tame Krull monoids.

In tame monoids there is a simple characterization for tamely generated ideals. We formulate a variant which is suitable for our purposes.

Lemma 5.6. *Let H be a tame monoid and $\mathfrak{a} \subset H$ an s-ideal. Then the following statements are equivalent:*

(a) *$\mathfrak{a}$ is tamely generated.*
(b) *There is a constant $\psi \in \mathbb{N}$ such that $\{a' \in \mathfrak{a} \mid \min \mathsf{L}(a') \le \psi\}H = \mathfrak{a}$.*

If (b) holds, then $\varphi(\mathfrak{a}) \le 2\psi\,\mathsf{t}(H)$.

Proof. The implication (a) $\Rightarrow$ (b) follows from the definition. Conversely, suppose that (b) holds. We set $E = \{e \in \mathfrak{a} \mid \min \mathsf{L}(e) \le \psi\}$ and have to verify that E is a tame generating set of $\mathfrak{a}$. Let $a \in \mathfrak{a}$. Then there is an $e \in E$ such that $e \mid a$ and $\min \mathsf{L}(e) \le \psi$. By Proposition 3.6.1, we have $\max \mathsf{L}(e) \le \rho(H) \min \mathsf{L}(e) \le \psi \mathsf{t}(H)$. By [22, Lemma 1.6.5.7], it follows that

$$\mathsf{t}(a, \mathsf{Z}(e)) \le 2 \min \mathsf{L}(e) \mathsf{t}(H) \le 2\psi \mathsf{t}(H).$$

Thus E is a tame generating set of $\mathfrak{a}$ with bound $2\psi \mathsf{t}(H)$. $\square$

Now we start with the more specific preparations for the proof of Theorem 5.1. The constructions follow the ideas developed in the setting of Krull monoids (see [20]).

Let H be a tame monoid with $\mathsf{t}(H) \ge 3$, $N = \mathsf{t}(H) - 2$ and $a \in H$. We denote by $\Delta_{\text{cat}}(a)$ the set of all integers $d \in [-N, N]$ for which there exists a divisor b of a and $z, z' \in \mathsf{Z}(b)$ such that $\min \mathsf{L}(b) \le \omega(H)$ and $d = |z| - |z'|$. Then, by definition, we have $\Delta_{\text{cat}}(a) = -\Delta_{\text{cat}}(a)$ and $\Delta_{\text{cat}}(a) \subset \Delta_{\text{cat}}(a')$ for every $a' \in aH$.

Let $b \in H$ with $b \mid a$ and $w, w' \in \mathsf{Z}(a)$. We call a triple $((w_j)_{j \in [0,\ell]}, T, t)$ adapted to (a, b, w, w') if the following properties hold:

P1: $(w_j)_{j\in[0,\ell]}$ is a finite sequence in $\mathsf{Z}(a)$ with $w_0 = w$ and $w_\ell = w'$, $T \in \mathsf{Z}(H)$ is a divisor of w and $t \in [0, \ell]$.
P2: $t \le \omega(H) \min \mathsf{L}(b)$.
P3: For every $j \in [1, \ell]$ we have $\big||w_{j-1}| - |w_j|\big| \le N$, and for every $j \in [t+1, \ell]$ we have $|w_{j-1}| - |w_j| \in \Delta_{\text{cat}}(ab^{-1})$.
P4: wT^{-1} divides w_t.
P5: $|T| \le 2\omega(H)^2 \min \mathsf{L}(b)$ and $b \mid \pi(T)$.

We show the existence of adapted triples in three steps.

Lemma 5.7. *Let $a, b \in H$ with $b \mid a$ and $w, w' \in \mathsf{Z}(a)$. Then there exists a finite sequence $(w_j)_{j\in[0,\ell]}$ in $\mathsf{Z}(a)$ and some $t \in [0, \ell]$ such that the following properties hold:*

1. *$w_0 = w$ and $w' = w_\ell$.*
2. *$t \le \omega(H) \min \mathsf{L}(b)$.*
3. *For every $j \in [1, \ell]$ we have $|w_{j-1} \gcd(w_{j-1}, w_j)^{-1}| \le \omega(H)$ and $\big||w_{j-1}| - |w_j|\big| \le N$.*
4. *$b \mid \pi(\gcd(w_t, w'))$.*
5. *For every $j \in [1, \ell]$, $\gcd(w_{j-1}, w')$ divides $\gcd(w_j, w')$.*

Proof. Since $b \mid a = \pi(w')$, there is, by Lemma 3.4.1, a divisor $\widetilde{w}$ of w' such that $|\widetilde{w}| \le \omega(H) \min \mathsf{L}(b)$ and $b \mid \pi(\widetilde{w})$.

Let $z \in \mathsf{Z}(a) \setminus \{w'\}$. Call a factorization $\bar{z} \in \mathsf{Z}(a)$ an elementary transformation of z if it can be constructed in the following way. Let

$$v \in \mathcal{A}(H) \quad \text{be such that} \begin{cases} v \mid \widetilde{w} \gcd(z, \widetilde{w})^{-1} & \text{if } \widetilde{w} \nmid z \\ v \mid w' \gcd(z, w')^{-1} & \text{if } \widetilde{w} \mid z. \end{cases}$$

Note that $z \ne w'$ implies that $z \nmid w'$ because $\pi(z) = \pi(w')$. Since $\widetilde{w} \gcd(z, \widetilde{w})^{-1} \mid w' \gcd(z, w')^{-1}$, it follows that $v \mid w' \gcd(z, w')^{-1}$, and hence $v \mid \pi(w' \gcd(z, w')^{-1}) = \pi(z \gcd(z, w')^{-1})$. Again, by Lemma 3.4.1, there is a $u \in \mathsf{Z}(H)$ such that $u \mid z \gcd(z, w')^{-1}$, $v \mid \pi(u)$ and $|u| \le \omega(H)$. By Lemma 3.4.2, there is a $u' \in \mathsf{Z}(\pi(u))$ such that $v \mid u'$ and $\big||u| - |u'|\big| \le N$.

Now set $\bar{z} = zu^{-1}u'$, and observe that

$$z \gcd(z, \bar{z})^{-1} = u \gcd(u, u')^{-1} \quad \text{and} \quad \bar{z} \gcd(z, \bar{z})^{-1} = u' \gcd(u, u')^{-1}.$$

Now let $\bar{z} \in \mathsf{Z}(a)$ be an elementary transformation of $z \in \mathsf{Z}(a) \setminus \{w'\}$, and let v, u, u' be as in the above construction. Then we have

(a) Since $v \mid w' \gcd(z, w')^{-1}$, it follows that $v \gcd(z, w') \mid w'$, and by construction we have

$$\gcd(z, w') v \,\Big|\, \frac{z}{u} v \,\Big|\, \frac{z}{u} u'.$$

Thus $\gcd(z, w')$ is a proper divisor of $\gcd(\bar{z}, w')$.

(b) $|z \gcd(z, \bar{z})^{-1}| \le \omega(H)$.

(c) $\big||z| - |\bar{z}|\big| = \big||u| - |u'|\big| \le N$.

Now choose for any $z \in \mathsf{Z}(a) \setminus \{w'\}$ an elementary transformation $\bar{z}$ of z, and set $\overline{w'} = w'$. Define the sequence $(w_j)_{j\ge0}$ inductively by $w_0 = w$ and $w_j = \bar{w}_{j-1}$ for $j \ge 1$. Whenever $w_j \ne w'$, we have by (a) that $\gcd(w_j, w')$ is a proper divisor of $\gcd(w_{j+1}, w')$. Hence there is some $\ell \le |w'|$ such that $w_\ell = w'$. Similarly, there is some $t \le |\widetilde{w}| \le \omega(H) \min \mathsf{L}(b)$ such that $\widetilde{w} \mid w_t$, and hence $b \mid \pi(\widetilde{w}) \mid \pi(\gcd(w_t, w'))$. Hence the sequence $(w_j)_{j\in[0,\ell]}$ and t fulfill all our properties 1–5. $\square$

Lemma 5.8. *Let $a, b \in H$ with $b \mid a$, $w, w' \in \mathsf{Z}(a)$, $(w_j)_{j\in[0,\ell]}$ be a sequence in $\mathsf{Z}(a)$ and $t \in [0, \ell]$ such that Properties 1–5 of Lemma 5.7 are satisfied. Set*

$$T' = w \gcd(w, w_t)^{-1} \in \mathsf{Z}(H).$$

Then we have

1. $|T'| \le \omega(H)^2 \min \mathsf{L}(b)$.
2. $b \mid \pi(T')\pi(\gcd(w, w'))$.
3. $T' \mid w \gcd(w, w')^{-1}$.
4. $w \gcd(w, w')^{-1} T'^{-1} \mid w_t \gcd(w, w')^{-1}$ *(note by 5.7.5 that* $\gcd(w, w')$ *divides* w_t*).*

Proof. We introduce some abbreviations:

$$u_j = w_{j-1} \gcd(w_{j-1}, w_j)^{-1} \quad \text{for all } j \in [1, t]$$
$$x_0 = \gcd(w, w')$$
$$x_t = \gcd(w_t, w')$$
$$\bar{x}_t = \gcd(w, w_t).$$

By definition, we have $w_{j-1} \mid w_j u_j$ for every $j \in [1, t]$. Hence we obtain $w = w_0 \mid w_t u_1 \cdot \ldots \cdot u_t$. We obtain $T' \mid u_1 \cdot \ldots \cdot u_t$, and by Lemma 5.8 $|T'| \le |u_1| + \cdots + |u_t| \le t\omega(H) \le \omega(H)^2 \min \mathsf{L}(b)$, so that 1 holds.

3. follows from $\gcd(w, w') \mid \gcd(w, w_t)$ (Lemma 5.7.5) and 4 follows from the definition of T'.

From 4, we obtain $w \mid w_t T'$, and hence $w \bar{x}_t^{-1} \mid T'$. To prove 2, it is therefore enough to show that $b \mid \pi(w \bar{x}_t^{-1} x_0) = \pi(w_t \bar{x}_t^{-1} x_0)$.

Since $x_0 \mid w_t$, we have $x_0 = \gcd(w, w') = \gcd(w, w_t, w') = \gcd(\gcd(w, w_t), \gcd(w_t, w')) = \gcd(\bar{x}_t, x_t)$. Hence $x_t \bar{x}_t x_0^{-1} = \operatorname{lcm}(\bar{x}_t, x_t) \mid w_t$ and therefore $x_t \mid w_t \bar{x}_t^{-1} x_0$. Since, by 5.7.4, $b \mid \pi(x_t)$, we obtain $b \mid \pi(w_t \bar{x}_t^{-1} x_0)$. □

Lemma 5.9. *Let* $a, b \in H$ *with* $b \mid a$ *and* $w, w' \in \mathsf{Z}(a)$*. Then there exists a triple adapted to* (a, b, w, w')*.*

Proof. Let $(w_j)_{j \in [0, \ell]}$ and $t \in [0, \ell]$ be as in Lemma 5.7 and define T' as in Lemma 5.8. Then we have $b \mid \pi(T')\pi(\gcd(w, w'))$. Using Lemma 3.4.1, we obtain a divisor u of $\gcd(w, w')$ such that $b \mid \pi(T'u)$ and $|u| \le \omega(H) \min \mathsf{L}(b)$. We set $T = T'u$ and show that $((w_j)_{j \in [0, \ell]}, T, t)$ is an adapted triple for (a, b, w, w').

P1: It remains to show that $T \mid w$. Lemma 5.8.3 shows that $T' \gcd(w, w') \mid w$. Since $u \mid \gcd(w, w')$, we obtain that $T = T'u$ divides w.

P2: This is Lemma 5.7.2.

P3: Let $j \in [1, \ell]$. By Lemma 5.7.3, we have $\big||w_{j-1}| - |w_j|\big| \le N$. Now suppose that $j \ge t + 1$. Set $y = w_{j-1} \gcd(w_{j-1}, w_j)^{-1}$, $y' = w_j \gcd(w_{j-1}, w_j)^{-1}$ and $c = \pi(y) = \pi(y')$. Then $\big||w_{j-1}| - |w_j|\big| = \big||y| - |y'|\big|$, and from Lemma 5.7.3 we obtain $\min \mathsf{L}(c) \le |y| \le \omega(H)$. Hence we only have to show that $c \mid ab^{-1}$ or equivalently $b \mid ac^{-1} = \pi(\gcd(w_{j-1}, w_j))$. But, by 5.7.4, we have $b \mid \pi(\gcd(w_t, w'))$. Since $t < j$, Lemma 5.7.5 implies that

$$\gcd(w_t, w') \mid \gcd(w_{t+1}, w') \mid \ldots \mid \gcd(w_{j-1}, w') \mid \gcd(w_j, w'),$$

and hence $\gcd(w_t, w') \mid \gcd(w_{j-1}, w') \mid \gcd(w_{j-1}, w_j)$.

P4: This follows from $u \mid \gcd(w, w')$ and Lemma 5.8.4.

P5: By construction, we have $b \mid \pi(T)$, and by Lemma 5.8.1, we get $|T| = |T'| + |u| \le \omega(H)^2 \min \mathsf{L}(b) + \omega(H) \min \mathsf{L}(b) \le 2\omega(H)^2 \min \mathsf{L}(b)$. □

Lemma 5.10. *Let* $B \subset \mathbb{Z} \setminus \{0\}$ *by a finite nonempty subset with* $-B = B$*,* $d = \gcd(B)$*,* $N' = \max(B)/d$ *and* $M' \in \mathbb{N}$*. Then there exists an* $S \in \mathcal{F}(B)$ *such that* $|S| \le 2M' + 3N' - 3$ *and* $d\mathbb{Z} \cap [-dM', dM'] \subset \Sigma(S)$*.*

Proof. Since $2\lfloor (M' + 1)/N' \rfloor + 3N' - 5 \le 2M' + 3N' - 3$, this follows from [20, Lemma 5.1]. □

Proof of Theorem 5.1. Let H be tame. By Lemma 3.3 and Proposition 3.6.3, H is a BF-monoid with finite catenary degree and with finite set of distances $\Delta(H)$. If $\Delta(H) = \emptyset$, then all sets of lengths are singletons, and the assertion is clear. Suppose that $\Delta(H)$ is nonempty. Then, by Proposition 5.4, it suffices to show that every pattern ideal is tamely generated. Let $A = \{d_0, \ldots, d_s\} \subset \mathbb{Z}$ be a finite nonempty subset. If $|A| = 1$, then $\Phi(A) = H$, and $\{1\}$ is a tame generating set of H. Suppose that $|A| \ge 2$. By Lemma 5.6, we have to show that there is a constant $\psi \in \mathbb{N}$ such that for every $a \in \Phi(A)$ there is an $a' \in \Phi(A)$ with $a' \mid a$ and $\min \mathsf{L}(a') \le \psi$.

We need one more definition. For $c \in H$ and $\theta \in \mathbb{N}$, we say that $d \in [-N, N]$ is θ-deficient in c if there exists some divisor c' of c such that $\min \mathsf{L}(c') \le \theta$ and $d \notin \Delta_{\mathrm{cat}}(cc'^{-1})$. Since $-\Delta_{\mathrm{cat}} = \Delta_{\mathrm{cat}}$, d is θ-deficient in c if and only if $-d$ is θ-deficient in c. If $\theta \le \theta'$ and d is θ-deficient in c then d is θ'-deficient in c, too.

Now set $M = \max(A) - \min(A)$ and $N = \mathsf{t}(H) - 2$. Then $M \ge 1$,

$$\mathsf{t}(H) \ge \omega(H) \ge \mathsf{c}(H) \ge 2 + \max \Delta(H) \ge 3,$$

and hence $N = \mathsf{t}(H) - 2 \ge 1$. We define a sequence $(s_j)_{j \in [-1, N]}$ of nonnegative integers by

$$s_{-1} = 0,$$
$$s_k = (2s\omega(H)^2 + 2\omega(H)^2 N + 2N + 4)s_{k-1} + (2M + 3N - 4)\omega(H) \quad \text{for } k \in [0, N-1] \quad \text{and}$$
$$s_N = 2s\omega(H)^2 s_{N-1}.$$

We assert that $\psi = s_N$ has the required property, and we pick an $a \in \Phi(A)$.

Clearly, $(s_j)_{j\in[-1,N]}$ is increasing and $s_{N-1} \geq s_0 = (2M+3N-4)\omega(H) \geq 1$. Using this and $s \geq 1$, $\omega(H) \geq 1$, we obtain for all $k \in [0, N-1]$ that

$$\psi = 2s\omega(H)^2 s_{N-1} \geq \omega(H)^2 s_{N-1} + \omega(H)^2 s_{N-1} \geq s_{N-1} + \omega(H) \geq s_k + \omega(H). \tag{5.1}$$

Setting $k = 0$, $B' = \emptyset$, $b = 1$, we see that there exist $k \in [0, N]$, $B' \subset [1, N]$ and a divisor b of a such that

$$|B'| = k, \quad \min \mathsf{L}(b) \leq s_{k-1} \quad \text{and} \quad B' \cap \Delta_{\text{cat}}(ab^{-1}) = \emptyset. \tag{5.2}$$

We take k to be maximal in $[1, N]$ such that there exist $B' \subset [1, N]$ and a divisor b of a such that (5.2) holds and we choose such B' and b. We claim that

$$\text{no } d \in [1, N] \setminus B' \text{ is } (s_k - s_{k-1})\text{-deficient in } b^{-1}a. \tag{5.3}$$

Indeed, suppose that $d \in [1, N] \setminus B'$ is $(s_k - s_{k-1})$-deficient in $b^{-1}c$. Then $B' \neq [1, N]$, and hence $k \leq N-1$. We choose a divisor c of $b^{-1}a$ such that $\min \mathsf{L}(c) \leq s_k - s_{k-1}$ and $d \notin \Delta_{\text{cat}}(ab^{-1}c^{-1})$ and set $B'' = B' \cup \{d\}$. Then $|B''| = k+1$, $\min \mathsf{L}(bc) \leq \min \mathsf{L}(b) + \min \mathsf{L}(c) \leq s_{k-1} + s_k - s_{k-1} = s_k$ and $B'' \cap \Delta_{\text{cat}}(ab^{-1}c^{-1}) = B' \cap \Delta_{\text{cat}}(ab^{-1}c^{-1}) \subset B' \cap \Delta_{\text{cat}}(ab^{-1}) = \emptyset$. But this contradicts the maximality of k.

For every $i \in [0, s]$, we pick a factorization $w_i \in \mathsf{Z}(a)$ with length $|w_i| = d_i$. By Lemma 5.9, there exists for every $i \in [1, s]$ a triple $((w_{i,j})_{j\in[0,\ell_i]}, t_i, T_i)$ adapted to (a, b, w_0, w_i). Define $T = \operatorname{lcm}(T_1, \ldots, T_s)$ and $a_0 = \pi(T)$. Then, from the definitions, we obtain

$$\min \mathsf{L}(a_0) \leq |T| \leq |T_1| + \cdots + |T_s| \leq 2s\omega(H)^2 \min \mathsf{L}(b) \leq 2s\omega(H)^2 s_{k-1}, \tag{5.4}$$

$$T \mid w_0 \text{ and hence } a_0 \mid a, \tag{5.5}$$

$$b \mid \pi(T) = a_0 \quad \text{and} \tag{5.6}$$

$$w_0 T^{-1} \mid w_{i,t_i} \quad \text{for every } i \in [1, s]. \tag{5.7}$$

Finally, we set $B = -\big([1, N] \setminus B'\big) \cup \big([1, N] \setminus B'\big)$. Since $\Delta_{\text{cat}}(ab^{-1}) \cap B' = \emptyset$, we obtain $\Delta_{\text{cat}}(ab^{-1}) \subset B \cup \{0\}$. Applying **P3**, we obtain

$$|w_{i,j}| - |w_{i,j-1}| \in B \cup \{0\} \quad \text{for all } j \in [t_i + 1, \ell_i] \text{ and all } i \in [1, s]. \tag{5.8}$$

For the construction of the required element $a' \in \Phi(A)$ we distinguish two cases.

Case 1: $k \leq N-1$.

Then $\emptyset \neq B \subset [-N, N] \setminus \{0\}$ and $-B = B$. We set $d = \gcd(B)$, $N' = \max(B)/d \leq N$ and $M' = M + \omega(H)s_{k-1}N$. Applying Lemma 5.10, we obtain a sequence $S = r_1 \cdot \ldots \cdot r_m \in \mathcal{F}(B)$ such that

$$m \leq 2M' + 3N' - 3 \leq 2M + 2\omega(H)s_{k-1}N + 3N - 3 \quad \text{and} \quad d\mathbb{Z} \cap [-dM', dM'] \subset \Sigma(r_1 \cdot \ldots \cdot r_m). \tag{5.9}$$

We choose $m' \in [0, m]$ maximal, $a_1, \ldots, a_{m'} \in H$ and, for all $\nu \in [1, m']$, $z_\nu, z'_\nu \in \mathsf{Z}(a_\nu)$ such that

$$\min \mathsf{L}(a_\nu) \leq \omega(H), \quad r_\nu = |z'_\nu| - |z_\nu| \quad \text{and} \quad a_1 \cdot \ldots \cdot a_{m'} \mid aa_0^{-1}. \tag{5.10}$$

We claim that $m' = m$. Indeed, suppose that $m' \leq m-1$. Then $r_{m'+1} \notin \Delta_{\text{cat}}(aa_0^{-1}a_1^{-1} \cdot \ldots \cdot a_{m'}^{-1}) = \Delta_{\text{cat}}\big((ab^{-1})(a_0b^{-1}a_1 \cdot \ldots \cdot a_{m'})^{-1}\big)$ (note (5.6)). Hence $r_{m'+1}$ is $\min \mathsf{L}(a_0 \cdot \ldots \cdot a_{m'}b^{-1})$-deficient in ab^{-1}. By (5.3), we obtain $\min \mathsf{L}(a_0 \cdot \ldots \cdot a_{m'}b^{-1}) > s_k - s_{k-1}$. But we have

$$\begin{aligned}
\min \mathsf{L}(a_0 \cdot \ldots \cdot a_{m'}b^{-1}) &\leq \text{(Lemma 3.4.3)} \\
&\leq \min \mathsf{L}(a_0) + \min \mathsf{L}(a_1) + \cdots + \min \mathsf{L}(a_{m'}) + (2N+3)\min \mathsf{L}(b) \leq \text{((5.4), (5.10), (5.2))} \\
&\leq 2s\omega(H)^2 s_{k-1} + (m-1)\omega(H) + (2N+3)s_{k-1} \leq \text{(5.9)} \\
&\leq (2s\omega(H)^2 + 2N + 3)s_{k-1} + (2M + 2\omega(H)s_{k-1}N + 3N - 4)\omega(H) \\
&= (2s\omega(H)^2 + 2\omega(H)^2 N + 2N + 3)s_{k-1} + (2M + 3N - 4)\omega(H) \\
&= (2s\omega(H)^2 + 2\omega(H)^2 N + 2N + 4)s_{k-1} + (2M + 3N - 4)\omega(H) - s_{k-1} \\
&= s_k - s_{k-1},
\end{aligned}$$

a contradiction.

We set $a' = a_0 \cdot \ldots \cdot a_m$. Then, by construction, $a' \mid a$ and

$$\begin{aligned}
\min \mathsf{L}(a') &\leq \min \mathsf{L}(a_0) + \min \mathsf{L}(a_1) + \ldots + \min \mathsf{L}(a_m) \leq \text{((5.4), (5.10))} \\
&\leq 2s\omega(H)^2 s_{k-1} + m\omega(H) \leq \text{(5.9)} \\
&\leq 2s\omega(H)^2 s_{k-1} + (2M + 2\omega(H)s_{k-1}N + 3N - 3)\omega(H) \\
&= (2s\omega(H)^2 + 2\omega(H)^2 N)s_{k-1} + (2M + 3N - 3)\omega(H) \\
&\leq (2s\omega(H)^2 + 2\omega(H)^2 N + 2N + 4)s_{k-1} + (2M + 3N - 4)\omega(H) + \omega(H) \\
&= s_k + \omega(H) \leq \text{(5.1)}\ \psi.
\end{aligned}$$

It remains to show that $a' \in \Phi(A)$. For that we set $w' = Tz_1 \cdot \ldots \cdot z_m \in \mathsf{Z}(a')$ and, for $I \subset [1, m]$ and $i \in [1, s]$, we define

$$w'_{I,i} = T^{-1}w' \left(\prod_{\nu \in I} z_\nu\right)^{-1} \prod_{\nu \in I} z'_\nu \,(T w_0^{-1} w_{i,t_i}) \in \mathsf{Z}(a') \quad \text{(note (5.7))}.$$

We have

$$\begin{aligned} |w'_{I,i}| &= |w'| + |w_{i,t_i}| - |w_0| + \sum_{\nu \in I} (|z'_\nu| - |z_\nu|) \\ &= |w'| + |w_i| - |w_0| - (|w_i| - |w_{i,t_i}|) + \sum_{\nu \in I} r_\nu \\ &= |w'| + d_i - d_0 - (|w_i| - |w_{i,t_i}|) + \sum_{\nu \in I} r_\nu. \end{aligned} \tag{5.11}$$

We claim that $|w_i| - |w_{i,t_i}| \in \Sigma(r_1 \cdot \ldots \cdot r_m)$. Indeed, in view of (5.9), we have to show that $|w_i| - |w_{i,t_i}| \in d\mathbb{Z}$ and $\big||w_i| - |w_{i,t_i}|\big| \le M'$. The first assertion follows from $d\mathbb{Z} = \langle B \cup \{0\} \rangle$, (5.8) and

$$|w_i| - |w_{i,t_i}| = \sum_{j=t_i+1}^{\ell_i} (|w_{i,j}| - |w_{i,j-1}|).$$

To obtain the inequality, we use **P2**, **P3** and (5.2) to get

$$\begin{aligned} \big||w_i| - |w_{i,t_i}|\big| &= \big||w_i| - |w_0| + |w_0| - |w_{i,t_i}|\big| \\ &\le |d_i - d_0| + \sum_{j=1}^{t_i} \big||w_{i,j-1}| - |w_{i,j}|\big| \\ &\le M + t_i N \le M + \omega(H) \min \mathsf{L}(b) N \le M + \omega(H) s_{k-1} N = M'. \end{aligned}$$

We can now choose for every $i \in [1, s]$ a subset $I_i \subset [1, m]$ such that $|w_i| - |w_{i,t_i}| = \sum_{\nu \in I_i} r_\nu$. Then, by (5.11), $|w'_{I_i,i}| = |w'| + d_i - d_0$. Hence $|w'| - d_0 + A \subset \mathsf{L}(a')$ and $a' \in \Phi(A)$.

Case 2: $k = N$.

Then $B' = [1, N]$ and $B = \emptyset$. From (5.8), we obtain, for all $i \in [1, s]$,

$$|w_{i,t_i}| = |w_i|. \tag{5.12}$$

We set $a' = a_0$. Then $a' \mid a$, and by (5.4) we get $\min \mathsf{L}(a') \le 2s\omega(H)^2 s_{N-1} = s_N = \psi$. To show that $a' \in \Phi(A)$, we consider the following factorizations of a' (again note (5.7)):

$$T,\ T w_0^{-1} w_{1,t_1}, \ldots,\ T w_0^{-1} w_{s,t_s}.$$

For every $i \in [1, s]$, we have, using (5.12),

$$|T w_0^{-1} w_{i,t_i}| = |T| + |w_{i,t_i}| - |w_0| = |T| + |w_i| - |w_0| = |T| + d_i - d_0.$$

Hence $|T| - d_0 + A \subset \mathsf{L}(a')$ and $a' \in \Phi(A)$. □

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