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Arithmetic of Mori domains and monoids

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Abstract

In this paper we introduce weakly C-monoids as a new class of v-noetherian monoids. Weakly C-monoids generalize C-monoids and make it possible to study multiplicative properties of a wide class of Mori domains, e.g., rings of generalized power series with coefficients in a field and exponents in a finitely generated monoid. The main goal of the paper is to study the question when a weakly C-monoid is locally tame. After having proved a classification theorem for local tameness, we use it to show that every locally tame weakly C-monoid whose complete integral closure has finite class group has finite catenary degree and finite set of distances.

Keywords: Mori domain; v-Noetherian monoid; C-monoid; Non-unique factorizations; Local tameness; Catenary degree

1. Introduction

Let R be an atomic integral domain. By definition, R is factorial if and only if, given irreducible elements $u_1, \ldots, u_n, v_1, \ldots, v_m$ in R with

$$u_1 \cdot \ldots \cdot u_n = v_1 \cdot \ldots \cdot v_m,$$

then $n = m$, and there exists a permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ and units ε_i such that $u_{\pi(i)} = \varepsilon_i v_i$ for all i, $1 \leqslant i \leqslant n$. The main objective of factorization theory is to investigate non-uniqueness of factorizations in atomic domains and monoids that fail to be factorial. The theory

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has its origin in algebraic number theory, where non-unique factorizations were first observed in rings of algebraic integers. Later, the investigations were extended to larger classes of domains (see the conference proceedings [1,9,11,16] and the monograph [23]), and in the present paper we focus on the study of multiplicative properties of Mori domains and monoids. Mori domains were introduced in the 1970s [38,39] and attracted a lot of attention since that time (see the works of V. Barucci, S. Gabelli, E. Houston, T.G. Lucas, M. Roitman and others [4–6,37,42]).

The main approach to investigate the multiplicative arithmetic of domains is to pass—in a first step—to suitable auxiliary monoids. Then the arithmetic of these monoids is studied in detail, and finally the results are shifted back to the domains of original interest by means of transfer principles. Apart from methodical advantages, this approach opened the door to unexpected applications of the theory to problems in additive number theory (see [21], [23, Chapter 5]) as well as to quantitative investigations of non-unique direct-sum decompositions of finitely generated modules over one-dimensional local rings (see [15,30]). In the early stage of the development of the theory, the investigations were restricted to Krull domains and monoids. Later, also non-integrally closed domains such as orders in algebraic number fields were tackled. Only recently C-monoids were introduced in literature (see [17–19,22,28,29]). This class of monoids includes Krull monoids as well as congruence monoids in Krull domains satisfying natural finiteness conditions. For instance, if R is a Mori domain with complete integral closure $\widehat{R}$ such that the conductor $\mathfrak{f} = (R : \widehat{R})$ is non-zero and the ring $R/\mathfrak{f}$ and the v-class group of $\widehat{R}$ are both finite, then the multiplicative monoid $R \setminus \{0\}$ is a C-monoid (see [23, Theorem 2.11.9]). We note that these finiteness assumptions are satisfied for all orders in algebraic number fields and for a large number of higher-dimensional finitely generated algebras over $\mathbb{Z}$ (see [29,33] for details).

In this paper we generalize the notion of C-monoids by introducing weakly C-monoids as v-noetherian monoids satisfying several natural finiteness conditions (Definition 4.1). The significance of this new class of monoids is twofold. First, as the name already indicates, weakly C-monoids generalize C-monoids (see Proposition 4.8). This generalization makes it possible to study Mori domains which could not be treated before (all our applications of the theory to multiplicative monoids of domains and the monoid of v-invertible v-ideals of domains are gathered in the second part of Section 6; see in particular Theorem 6.7 and Corollary 6.8). Second, weakly C-monoids comprise various types of auxiliary monoids which had to be treated separately before (see Propositions 4.7, 4.8, 6.4 and 6.5). An additional benefit of looking at weakly C-monoids is that this class of monoids is—in contrast to C-monoids—closed under taking finite products (cf. Proposition 4.6).

The ostensible goal of this paper is to study when a weakly C-monoid is locally tame. Local tameness (cf. Definition 5.1) is an important finiteness property in the theory of non-unique factorizations (see, e.g., [23] for a detailed description of the significance of this invariant). Our main result, formulated in Theorem 5.3, provides an explicit algebraic characterization of local tameness. Its proof occupies the whole section, and it is performed in such a way that we obtain explicit, homogeneous bounds for the local tame degrees. This allows us to give a very simple proof of the finiteness of the catenary degree and the set of distances (see Theorem 6.3).

2. Preliminaries

Our notation and terminology is consistent with [23]. By a semigroup we mean a commutative semigroup with neutral element, and by a monoid we mean a semigroup satisfying the cancellation law. All (semigroup and monoid) homomorphisms are assumed to respect the neutral element.

We denote by $\mathbb{N}$ the set of positive integers, and we put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For integers $a, b \in \mathbb{Z}$ we set $[a, b] = \{x \in \mathbb{Z} \mid a \leqslant x \leqslant b\}$, and we define $\sup \emptyset = \max \emptyset = \min \emptyset = 0$. In the following subsections we briefly recall some key notions needed in the sequel.

Throughout this paper H denotes a monoid.

Basic notions on monoids

We denote by $H^\times$ the set of invertible elements of H, by $H_{\mathrm{red}} = H/H^\times = \{aH^\times \mid a \in H\}$ the associated reduced monoid, and by $\mathsf{q}(H)$ the quotient group of H. Furthermore,

$$\widetilde{H} = \left\{x \in \mathsf{q}(H) \,\middle|\, x^n \in H \text{ for some } n \in \mathbb{N}\right\}$$

denotes the *root closure* of H, and

$$\widehat{H} = \left\{x \in \mathsf{q}(H) \,\middle|\, \text{there exists } c \in H \text{ such that } cx^n \in H \text{ for all } n \in \mathbb{N}\right\}$$

denotes the *complete integral closure* of H. Both $\widetilde{H}$ and $\widehat{H}$ are monoids, and we have $H \subset \widetilde{H} \subset \widehat{H} \subset \mathsf{q}(H)$.

For $a, b \in H$ with $b \in aH$ we write as usual $a \mid_H b$ or, when there is no danger of confusion, we simply write $a \mid b$. An element $u \in H$ is called an *atom* of H if $u \notin H^\times$ and, for all $a, b \in H$, $u = ab$ implies $a \in H^\times$ or $b \in H^\times$. We denote by $\mathcal{A}(H)$ the set of all atoms of H, and we call the monoid H *atomic* if every $a \in H \setminus H^\times$ is a product of atoms. An element $p \in H$ is called a *prime element* of H if $H \setminus pH$ is a submonoid of H, and H is called *factorial* if every $a \in H \setminus H^\times$ decomposes into a product of primes. Every prime element is an atom, and a monoid is factorial if and only if it is atomic, and every atom is a prime. If H is atomic and $p \in H$ is a prime, then every $a \in \mathsf{q}(H)$ has a representation $a = p^n bc^{-1}$, where $b, c \in H$, $p \nmid bc$, and $n \in \mathbb{Z}$. The exponent n is uniquely determined, and we call $\mathsf{v}_p(a) = n$ the *p-adic value* of a. The map $\mathsf{v}_p : \mathsf{q}(H) \to \mathbb{Z}$ is a surjective group homomorphism, called the *p-adic valuation* of $\mathsf{q}(H)$.

For a set P we denote by $\mathcal{F}(P)$ the *free* (*abelian*) *monoid* generated by P. Let $a, b \in \mathcal{F}(P)$. Then a has a unique representation of the form

$$a = \prod_{p \in P} p^{\nu_p}, \quad \text{where } \nu_p \in \mathbb{N}_0 \text{ and } \nu_p = 0 \text{ for all but finitely many } p \in P,$$

and we have $\nu_p = \mathsf{v}_p(a)$ for all $p \in P$. We call

$$|a| = |a|_{\mathcal{F}(P)} = \sum_{p \in P} \mathsf{v}_p(a) \in \mathbb{N}_0$$

the *length* of a, and, with $c = \gcd(a, b)$, we call

$$\mathsf{d}(a, b) = \mathsf{d}_{\mathcal{F}(P)}(a, b) = \max\left\{\left|c^{-1}a\right|, \left|c^{-1}b\right|\right\} \in \mathbb{N}_0$$

the *distance* of a and b.

Let F be a factorial monoid and P a maximal set of pairwise non-associated prime elements of F. Then $F = F^\times \times \mathcal{F}(P)$. For $x \in F$ we call $\mathrm{supp}(x) = \mathrm{supp}_P(x) = \{p \in P \mid \mathsf{v}_p(x) \neq 0\}$ the *support* of x (with respect to P). For a subset $S \subset F$ we set

$$\mathrm{supp}(S) = \mathrm{supp}_P(S) = \bigl\{\mathrm{supp}(x) \bigm| x \in S\bigr\},$$

and, if $Q \subset P$, we define

$$\mathsf{v}_Q(x) = \sum_{p \in Q} \mathsf{v}_p(x).$$

Ideal theory of monoids

In this subsection we recall some basic ideal-theoretic terminology for monoids (see [23] or [27] for more details). A subset $X \subset H$ is called an *s-ideal* of H if $XH = X$. By definition, $\emptyset$ and H are s-ideals of H. An s-ideal $X \subset H$ is called *prime* if $H \setminus X$ is a submonoid of H. We denote by $s\text{-spec}(H)$ the set of all prime s-ideals of H, and we denote by $\mathfrak{X}(H)$ the set of all minimal non-empty prime s-ideals of H. The monoid H is called a G-*monoid* if the intersection of all non-empty prime s-ideals of H is non-empty. Thus, if $s\text{-spec}(H)$ is finite, then H is a G-monoid. For subsets $X, Y \subset \mathsf{q}(H)$ we set

$$(Y : X) = \bigl\{a \in \mathsf{q}(H) \bigm| aX \subset Y\bigr\}, \quad X^{-1} = (H : X), \quad \text{and} \quad X_v = \bigl(X^{-1}\bigr)^{-1}.$$

We say that X is a *v-ideal* of H if $X \subset H$ and $X_v = X$. We denote by $v\text{-spec}(H)$ the set of all v-ideals of H that are prime. The monoid H is called *v-noetherian* if it satisfies the ascending chain condition on v-ideals. We denote by $\mathcal{I}_v^*(H)$ the monoid of v-invertible v-ideals with v-multiplication.

The set $(H : \widehat{H}) = \{x \in \mathsf{q}(H) \mid x\widehat{H} \subset H\}$ is called the *conductor* of H. The monoid H is said to be a *Krull monoid* if H is v-noetherian and $H = \widehat{H}$. If R is an integral domain, then $R^\bullet = R \setminus \{0\}$ is a monoid, and a subset $X \subset R^\bullet$ is a v-ideal of $R^\bullet$ if and only if $X \cup \{0\}$ is a divisorial ideal of R. In particular, $R^\bullet$ is v-noetherian if and only if R is a Mori domain. It can be shown [25, Section 2] that this extends to Mori rings having non-trivial zerodivisors. Rings of that kind were introduced in [35].

The monoid H is said to be *primary* if $H \neq H^\times$ and $s\text{-spec}(H) = \{\emptyset, H \setminus H^\times\}$ (equivalently, $H \neq H^\times$, and, for all $a, b \in H \setminus H^\times$, there exists $n \in \mathbb{N}$ such that $a \mid b^n$). The monoid H is called a *discrete valuation monoid* if $H_{\mathrm{red}} \cong (\mathbb{N}_0, +)$ (equivalently, H is a primary Krull monoid, see [23, Theorem 2.3.8] and [26, Lemma 3.1.1]).

Let $T \subset H$ be a subset. We denote by $[T] = [T]_H$ the smallest submonoid of H containing T. We say that T is a *divisor-closed* subset of H if $b \mid_H a$ implies $b \in T$ for all $a \in T$, $b \in H$. Thus T is a divisor-closed subset of H if and only if $H \setminus T$ is an s-ideal of H, and T is a divisor-closed submonoid of H if and only if $H \setminus T$ is a prime s-ideal of H. We denote by $[\![T]\!] = [\![T]\!]_H$ the smallest divisor-closed submonoid of H containing T (that is, $[\![T]\!]$ denotes the set of all $a \in H$ dividing some $c \in [T]$). For $a \in H$ we set $[\![a]\!] = [\![\{a\}]\!]$. The monoid H is a G-monoid if and only if there exists $a \in H$ such that $H = [\![a]\!]$, cf. [23, Definition 2.7.6 and Lemma 2.7.7].

Let $S \subset H$ be a submonoid. We say that $S \subset H$ is *cofinal* if, for every $a \in H$, there exists $b \in S$ such that $a \mid_H b$. Furthermore, $S \subset H$ is called *saturated* if $\mathsf{q}(S) \cap H \subset S$ (and then $\mathsf{q}(S) \cap H = S$). If S is a divisor-closed submonoid of H, then $S \subset H$ is saturated.

Class groups, class semigroups, and C-monoids

Let D be a monoid, and suppose that $H \subset D$ is a submonoid. In this paper we call

$$\frac{\mathsf{q}(D)}{\mathsf{q}(H)D^\times} \tag{1}$$

the *class group* of $H \subset D$. Consider the map

$$\varphi : H \to D/D^\times,$$

defined by $\varphi(a) = aD^\times$ for all $a \in H$. The group $\mathsf{q}(D)/D^\times$ is the quotient group of $D/D^\times$, and $\mathsf{q}(H)D^\times/D^\times$ is the quotient group of $\varphi(H) = D^\times H/D^\times$. Since we have a natural isomorphism

$$\big(\mathsf{q}(D)/D^\times\big)\big/\big(\mathsf{q}(H)D^\times/D^\times\big) \;\cong\; \mathsf{q}(D)/\mathsf{q}(H)D^\times,$$

the class group of $H \subset D$ defined in (1) is nothing else but the class group of φ in the sense of [23, Section 2.4].

Let D be a monoid and $H \subset D$ a submonoid. Two elements $y, y' \in D$ are called (H, D)-*equivalent* if $y^{-1}H \cap D = y'^{-1}H \cap D$ (that is, $ya \in H$ if and only if $y'a \in H$ for all $a \in D$). (H, D)-equivalence is a congruence relation on D, i.e., (H, D)-equivalence is compatible with the semigroup operation on D. For $y \in D$ we denote by $[y]_H^D$ the (H, D)-equivalence class of y. The semigroup

$$\mathcal{C}(H, D) = \big\{[y]_H^D \;\big|\; y \in D\big\}$$

is called the *class semigroup* of H in D, and

$$\mathcal{C}^*(H, D) = \big\{[y]_H^D \;\big|\; y \in \big(D \setminus D^\times\big) \cup \{1\}\big\}$$

is called the *reduced class semigroup* of H in D.

The monoid H is called a C-*monoid* if it is a submonoid of a factorial monoid F such that $F^\times \cap H = H^\times$ and $\mathcal{C}^*(H, F)$ is finite. In this case we say that H is a C-monoid defined in F. As already pointed out in the introduction, if R is a Mori domain with complete integral closure $\widehat{R}$ such that the conductor $\mathfrak{f} = (R : \widehat{R})$ is non-zero and the ring $R/\mathfrak{f}$ and the v-class group of $\widehat{R}$ are both finite, then the multiplicative monoid $R \setminus \{0\}$ is a C-monoid (see [23, Theorem 2.11.9]).

Next we introduce a generalized Davenport constant and recall the concept of block monoids. Suppose that H is a submonoid of a factorial monoid $F = F^\times \times \mathcal{F}(P)$ such that $F^\times \cap H = H^\times$. For a subset $E \subset P$ we define

$$\mathsf{D}_E(H) = \sup\big\{\mathsf{v}_E(u) \;\big|\; u \in \mathcal{A}(H)\big\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Then $\mathsf{v}_E(a) \leqslant \min \mathsf{L}(a)\mathsf{D}_E(H)$ for every $a \in H$.

Let G be an additively written abelian group and $G_0 \subset G$ a subset (in our applications G will be the class group of a containment $H \subset D$ of monoids, and G_0 will be the set of classes containing prime elements of D). Following the tradition of combinatorial number theory, the elements of $\mathcal{F}(G_0)$ are called *sequences* over G_0. Let $S = g_1 \cdot \ldots \cdot g_l$ be a sequence over G_0. Then $|S| = l \in \mathbb{N}_0$ is called the *length* of S, the quantity $\sigma(S) = g_1 + \cdots + g_l \in G$ is called the *sum* of S, and $\mathrm{supp}_{G_0}(S) = \mathrm{supp}(S) = \{g_1, \ldots, g_l\} \subset G$ is called the *support* of S. S is called a *zero-sum sequence* if $\sigma(S) = 0$. The set of all zero-sum sequences, denoted by $\mathcal{B}(G_0)$, and also called the *block monoid* over G_0, is a saturated submonoid of $\mathcal{F}(G_0)$. Thus $\mathcal{B}(G_0)$ is a Krull monoid (cf. Lemma 2.2.1), and

$$\mathsf{D}(G_0) = \mathsf{D}_{G_0}\big(\mathcal{B}(G_0)\big) = \sup\big\{|S| \,\big|\, S \text{ is an atom of } \mathcal{B}(G_0)\big\} \in \mathbb{N}_0 \cup \{\infty\}$$

is called the *Davenport constant* of G_0 (see [21] and [23, Section 5.1]).

In the following lemma we put together some easy facts about the relationship between class groups and class semigroups. For the definition of transfer homomorphisms see, e.g., [23, Section 3.2].

Lemma 2.1. *Let D be a monoid, and suppose that $H \subset D$ is a cofinal submonoid. Consider the maps*

$$\theta : \mathcal{C}(H, D) \to \mathsf{q}(D)/\mathsf{q}(H) \quad \text{and} \quad \theta^* : \mathcal{C}^*(H, D) \to \mathsf{q}(D)/\mathsf{q}(H)D^\times,$$

given by $\theta([y]_H^D) = y\mathsf{q}(H)$ for all $y \in D$, and $\theta^*([y]_H^D) = y\mathsf{q}(H)D^\times$ for all $y \in (D \setminus D^\times) \cup \{1\}$.

1. *θ and θ^* are epimorphisms, and $H \subset D$ is saturated if and only if θ is an isomorphism.*
2. *If $H \subset D$ is saturated, then $\theta' : \mathcal{C}(HD^\times, D) \to \mathsf{q}(D)/\mathsf{q}(H)D^\times$, given by $\theta'([a]_{HD^\times}^D) = a\mathsf{q}(H)D^\times$ for all $a \in D$, is an isomorphism.*
3. *Suppose that $D = D^\times \times \mathcal{F}(P)$ is factorial and $H \subset D$ is saturated. Put $G_P = \{[p]_{HD^\times}^D \mid p \in P\} \subset \mathcal{C}(HD^\times, D)$. Then the map $\boldsymbol{\beta} : H \to \mathcal{B}(G_P)$, given by*

$$\boldsymbol{\beta}(a) = \prod_{i=1}^{l} [p_i]_{HD^\times}^D,$$
$$\text{where } a = \varepsilon p_1 \cdot \ldots \cdot p_l \in H \text{ with } \varepsilon \in D^\times,\ l \in \mathbb{N} \text{ and } p_1, \ldots, p_l \in P,$$

is a transfer homomorphism, and

$$\mathsf{D}(G_P) = \sup\big\{\mathsf{v}_P(u) \,\big|\, u \in \mathcal{A}(H)\big\}.$$

Proof. The proof of 1. is straightforward and can be found in [23, Proposition 2.8.7]. If $H \subset D$ is saturated, then $HD^\times \subset D$ is saturated and cofinal, whence 2. follows from 1. It remains to show 3. Put $G'_P = \{p\mathsf{q}(H)D^\times \mid p \in P\} \subset \mathsf{q}(D)/\mathsf{q}(H)D^\times$, and let $\boldsymbol{\beta}' : HD^\times/D^\times \to \mathcal{B}(G'_P)$ be defined by

$$\boldsymbol{\beta}'\big(aD^\times\big) = \prod_{i=1}^{l} p_i\mathsf{q}(H)D^\times,$$
$$\text{where } a = \varepsilon p_1 \cdot \ldots \cdot p_l \in H \text{ with } l \in \mathbb{N},\ \varepsilon \in D^\times \text{ and } p_1, \ldots, p_l \in P.$$

Since $HD^\times/D^\times \hookrightarrow D/D^\times$ is cofinal and saturated, $\boldsymbol{\beta}'$ is a transfer homomorphism [23, Proposition 3.4.8.2], and

$$\mathsf{D}\big(G'_P\big) = \sup\big\{\mathsf{v}_P(u) \,\big|\, u \in \mathcal{A}\big(HD^\times/D^\times\big)\big\}$$

[23, Proposition 3.4.5.3]. Since the natural map $H \to HD^\times/D^\times$ is a transfer homomorphism and θ' in 2. induces an isomorphism $\mathcal{B}(G'_P) \to \mathcal{B}(G_P)$, the assertion follows. $\square$

In the following lemma we gather some properties of Krull monoids which will be used tacitly throughout the paper.

Lemma 2.2.

1. *The following statements are equivalent*:
   (a) *H is a Krull monoid.*
   (b) *H is a saturated submonoid of a factorial monoid.*
   (c) *H splits in the form $H = H^\times \times H_0$, where $H_0 \cong H_{\mathrm{red}}$ and H_0 is a saturated submonoid of a free monoid $F_0 = \mathcal{F}(P)$ such that every $a \in F_0$ can be written as $a = \gcd_{F_0}(E)$ for some subset $E \subset H_0$.*
2. *Let $H = H^\times \times H_0$ and $H_0 \subset F_0 = \mathcal{F}(P)$ be as in* 1.(c). *Then the map*

$$\varphi : F_0 \to \mathcal{I}_v^*(H), \quad \textit{defined by } \varphi(a) = (aF_0 \cap H)_v,$$

is an isomorphism satisfying $\varphi(P) = \mathfrak{X}(H)$ and $\varphi(a) = aH$ for all $a \in H$. Moreover, φ induces an isomorphism

$$\overline{\varphi} : \mathsf{q}(F_0)/\mathsf{q}(H_0) \to \mathsf{q}\big(\mathcal{I}_v^*(H)\big)/\mathsf{q}\big(\varphi(H)\big) = \mathcal{C}_v(H)$$

of the class group of $H_0 \subset F_0$ onto the v-class group $\mathcal{C}_v(H)$ of H, and $\overline{\varphi}$ maps the set $G_P = \{p\mathsf{q}(H_0) \mid p \in P\} \subset \mathsf{q}(F_0)/\mathsf{q}(H_0)$ of all classes containing primes onto the set of all v-ideal classes in $\mathcal{C}_v(H)$ containing prime v-ideals.

If H is a Krull monoid, then $\mathcal{C}(H) = \mathcal{C}_v(H)$ is briefly called *the* class group of H.

Next we summarize some basic algebraic properties of C-monoids (for a proof see [23, Section 2.9]).

Theorem 2.3. *Let $F = F^\times \times \mathcal{F}(P)$ be a factorial monoid, and suppose that $H \subset F$ is a submonoid with $H \cap F^\times = H^\times$.*

1. *Suppose that H is a* C*-monoid defined in F. Then H is v-noetherian, $(H : \widehat{H}) \neq \emptyset$, and $\mathcal{C}(\widehat{H})$ is finite. Moreover, every divisor-closed submonoid of H is a* C*-monoid.*
2. *Suppose that P is finite. Then the following statements are equivalent*:
   (a) *H is a* C*-monoid defined in F.*
   (b) *There exist $\alpha \in \mathbb{N}$ and a subgroup $V \subset F^\times$ such that $(F^\times : V) \mid \alpha$, $V \cdot (H \setminus H^\times) \subset H$, and $a \in H$ if and only if $p^\alpha a \in H$ for all $p \in P$ and all $a \in p^\alpha F$.*

   If the equivalent statements (a) *and* (b) *hold, then H is a* G*-monoid.*

Basic notions from factorization theory

Suppose that H is atomic, and put $A = \mathcal{A}(H_{\mathrm{red}})$. The monoid $\mathsf{Z}(H) = \mathcal{F}(A)$ is called the *factorization monoid* of H. Let $\pi = \pi_H : \mathsf{Z}(H) \to H_{\mathrm{red}}$ be the canonical homomorphism. For

$x, y \in \mathsf{Z}(H)$ the integer $\mathsf{d}(x, y) = \mathsf{d}_{\mathsf{Z}(H)}(x, y)$ is called the *distance* of x and y, and we call $|x| = |x|_{\mathsf{Z}(H)}$ the *length* of x. For $a \in H$ the set

$$\mathsf{Z}(a) = \mathsf{Z}_H(a) = \pi^{-1}\big(aH^\times\big) \subset \mathsf{Z}(H)$$

is called the *set of factorizations* of a, and

$$\mathsf{L}(a) = \mathsf{L}_H(a) = \big\{|z| \ \big|\ z \in \mathsf{Z}(a)\big\} \subset \mathbb{N}_0$$

is called the *set of lengths* of a. H is called a BF-*monoid* if $\mathsf{L}(a)$ is a finite set for all $a \in H$. All v-noetherian monoids are BF-monoids. For more refined arithmetical notions such as local tameness and the catenary degree see Definitions 5.1 and 6.1.

3. v-Noetherian monoids: algebraic properties

Suppose that D is a monoid and $H \subset D$ is a submonoid. In this section we study whether certain ideal-theoretic properties of D are preserved when passing to H. For instance, if $H \subset D$ is saturated and D is v-noetherian or has non-empty conductor, does H inherit these properties? We start with a key result regarding the structure of the complete integral closure of a v-noetherian monoid with non-empty conductor (see [23, Theorems 2.6.5 and 2.7.9]).

Theorem 3.1. *Suppose that H is v-noetherian and $(H : \widehat{H}) \neq \emptyset$. Then $\widehat{H}$ is a Krull monoid. If, moreover, H is a* G*-monoid, then H is finitary, s-*spec(H) *and s-*spec$(\widehat{H})$ *are both finite, and $\widehat{H}_{\mathrm{red}}$ is finitely generated.*

Lemma 3.2. *Suppose that D is a monoid such that $H \subset D$ is a saturated submonoid. Let $T \subset H$ be a divisor-closed submonoid.*

1. *$T = H \cap [\![T]\!]_D$, and $T \subset [\![T]\!]_D$ is saturated and cofinal.*
2. *The homomorphism $\iota : \mathsf{q}([\![T]\!]_D)/\mathsf{q}(T)D^\times \to \mathsf{q}(D)/\mathsf{q}(H)D^\times$, defined by $\alpha\mathsf{q}(T)D^\times \mapsto \alpha\mathsf{q}(H)D^\times$, is injective.*
3. *The class group of $\widehat{H} \subset \widehat{D}$ is a homomorphic image of the class group of $H \subset D$.*

Proof. 1. Suppose $a \in H \cap [\![T]\!]_D$. Then there exists $b \in T$ such that $a \mid_D b$. Since $H \subset D$ is saturated it follows that $a \mid_H b$, and since $T \subset H$ is divisor-closed, we obtain $a \in T$. Thus $T = H \cap [\![T]\!]_D$. We have

$$T \subset [\![T]\!]_D \cap \mathsf{q}(T) \subset D \cap \mathsf{q}(H) \cap \mathsf{q}(T) = H \cap \mathsf{q}(T) = T$$

from which it follows that $T \subset [\![T]\!]_D$ is saturated. By the very definition of $[\![T]\!]_D$ we see that $T \subset [\![T]\!]_D$ is cofinal.

2. We show that $\mathsf{q}(H)D^\times \cap \mathsf{q}([\![T]\!]_D) = \mathsf{q}(T)D^\times$. For, let $x = ab^{-1} \in \mathsf{q}(H)D^\times \cap \mathsf{q}([\![T]\!]_D)$, with $a, b \in [\![T]\!]_D$. Without loss of generality we may suppose that $b \in T$. Then $xb = a \in \mathsf{q}(H)D^\times \cap [\![T]\!]_D$, and there exists $\varepsilon \in D^\times$ such that

$$\varepsilon xb \in \mathsf{q}(H) \cap [\![T]\!]_D = \mathsf{q}(H) \cap D \cap [\![T]\!]_D = H \cap [\![T]\!]_D = T.$$

It follows that $x \in \mathsf{q}(T)D^\times$.

3. Clearly, the natural homomorphism

$$\mathsf{q}(D)/\mathsf{q}(H)D^\times \to \mathsf{q}(D)/\mathsf{q}(H)\widehat{D}^\times = \mathsf{q}(\widehat{D})/\mathsf{q}(\widehat{H})\widehat{D}^\times$$

is surjective. □

Lemma 3.3. *Suppose that $S \subset H$ is a saturated and cofinal submonoid.*

1. *$\widehat{S} \subset \widehat{H}$ is saturated.*
2. *If $(H : \widehat{H}) \neq \emptyset$, then $(S : \widehat{S}) \neq \emptyset$.*

Proof. 1. We shall prove that $\widehat{H} \cap \mathsf{q}(S) \subset \widehat{S}$. Let $x \in \widehat{H} \cap \mathsf{q}(S)$. Then there exists $d \in H$ such that $dx^n \in H$ for all $n \in \mathbb{N}$. If $c \in H$ such that $cd \in S$, then $(cd)x^n \in H \cap \mathsf{q}(S) = S$ for all $n \in \mathbb{N}$, and thus $x \in \widehat{S}$.

2. Let $f \in (H : \widehat{H})$ and $c \in H$ such that $cf \in S$. Then $cf \in (H : \widehat{H})$ and $cf\widehat{S} \subset cf\widehat{H} \cap \mathsf{q}(S) \subset H \cap \mathsf{q}(S) = S$. It follows that $cf \in (S : \widehat{S})$. □

In 3. of the following lemma we prove that lying-over holds for arbitrary prime s-ideals in almost integral extensions of v-noetherian monoids (cf. [7, Proposition 1.1 and Theorem 1.4]).

Lemma 3.4. *Suppose that H is v-noetherian.*

1. *Let $x = c^{-1}b \in \widehat{H}$, with $b, c \in H$. Then there exists $m \in \mathbb{N}$ such that $c^m x^n \in H$ for all $n \in \mathbb{N}$. In particular, $\widehat{H}^\times \cap H = H^\times$.*
2. *Let $S \subset H$ be a submonoid. Then $S^{-1}H$ is v-noetherian and $\widehat{S^{-1}H} = S^{-1}\widehat{H}$. If $(H : \widehat{H}) \neq \emptyset$, then $(S^{-1}H : \widehat{S^{-1}H}) = S^{-1}(H : \widehat{H})$.*
3. *For every $\mathfrak{p} \in s\text{-spec}(H)$ there exists $\widehat{\mathfrak{p}} \in s\text{-spec}(\widehat{H})$ such that $\widehat{\mathfrak{p}} \cap H = \mathfrak{p}$. Moreover, if $\mathfrak{p} \in \mathfrak{X}(H)$ and $\widehat{H}$ is v-noetherian, then there exists $\widehat{\mathfrak{p}} \in v\text{-spec}(\widehat{H})$ such that $\widehat{\mathfrak{p}} \cap H = \mathfrak{p}$.*

Proof. 1. Since $x \in \widehat{H}$ there exists $d \in H$ such that $dx^n \in H$ for all $n \in \mathbb{N}$. Thus the set $\{x^n \mid n \in \mathbb{N}_0\}$ is H-fractional, and by [23, Proposition 2.1.10] there exists $m \in \mathbb{N}$ such that

$$\{x^n \mid n \in \mathbb{N}_0\}_v = \{x^n \mid n \in [0, m]\}_v.$$

Since $c^m x^n \in H$ for all $n \in [0, m]$ it follows that

$$\{c^m x^n \mid n \in \mathbb{N}_0\} \subset c^m\{x^n \mid n \in \mathbb{N}_0\}_v = \{c^m x^n \mid n \in [0, m]\}_v \subset H.$$

Clearly, we have $H^\times \subset \widehat{H}^\times \cap H$. Conversely, pick $a \in \widehat{H}^\times \cap H$. Then there exists $m \in \mathbb{N}$ such that $a^m(a^{-1})^n \in H$ for all $n \in \mathbb{N}$, and hence $a^{-1} = a^m(a^{-1})^{m+1} \in H$.

2. $S^{-1}H$ is v-noetherian by [23, Proposition 2.2.8.4], and [23, Theorem 2.3.5] implies that $\widehat{S^{-1}H} = S^{-1}\widehat{H}$. Using [23, Theorem 2.2.8.1], we infer that

$$\left(S^{-1}H : \widehat{S^{-1}H}\right) = \left(S^{-1}H : S^{-1}\widehat{H}\right) = S^{-1}(H : \widehat{H}).$$

3. Without loss of generality we may suppose that $\mathfrak{p} \in s\text{-spec}(H) \setminus \{\emptyset\}$. By 1. and 2. it follows that $H_\mathfrak{p}$ is v-noetherian and $\widehat{H_\mathfrak{p}}^\times \cap H_\mathfrak{p} = {H_\mathfrak{p}}^\times$. Since $H_\mathfrak{p} \ne H_\mathfrak{p}^\times$, we get $\widehat{H_\mathfrak{p}} \ne \widehat{H_\mathfrak{p}}^\times$. Hence $\widehat{H_\mathfrak{p}} \setminus \widehat{H_\mathfrak{p}}^\times \in s\text{-spec}(\widehat{H_\mathfrak{p}}) \setminus \{\emptyset\}$,

$$\widehat{\mathfrak{p}} = \big(\widehat{H_\mathfrak{p}} \setminus \widehat{H_\mathfrak{p}}^\times\big) \cap \widehat{H} \in s\text{-spec}(\widehat{H}),$$

and

$$\widehat{\mathfrak{p}} \cap H = \big(\widehat{H_\mathfrak{p}} \setminus \widehat{H_\mathfrak{p}}^\times\big) \cap H_\mathfrak{p} \cap H = \big(H_\mathfrak{p} \setminus H_\mathfrak{p}^\times\big) \cap H = \mathfrak{p}H_\mathfrak{p} \cap H = \mathfrak{p}.$$

Suppose that $\mathfrak{p} \in \mathfrak{X}(H)$ and that $\widehat{H}$ is v-noetherian. Then $\widehat{H_\mathfrak{p}} = \widehat{(H \setminus \mathfrak{p})^{-1}H} = (H \setminus \mathfrak{p})^{-1}\widehat{H}$ is v-noetherian by 2., and since $\widehat{H_\mathfrak{p}} \ne \widehat{H_\mathfrak{p}}^\times$, there exists $\mathfrak{q} \in v\text{-spec}(\widehat{H_\mathfrak{p}}) \setminus \{\emptyset\}$. Then $\mathfrak{q} \cap H_\mathfrak{p} \in s\text{-spec}(H_\mathfrak{p}) = \{\emptyset, \mathfrak{p}H_\mathfrak{p}\}$, and by [23, Proposition 2.3.4.2] we have $\mathfrak{q} \cap H_\mathfrak{p} \ne \emptyset$. Since $\mathfrak{q} \in \widehat{H_\mathfrak{p}} = (H \setminus \mathfrak{p})^{-1}\widehat{H}$, [23, Proposition 2.2.8.3] implies that $\widehat{\mathfrak{p}} = \mathfrak{q} \cap \widehat{H}$ is a v-ideal of $\widehat{H}$. Thus it follows that $\widehat{\mathfrak{p}} \in v\text{-spec}(\widehat{H})$, and clearly

$$\widehat{\mathfrak{p}} \cap H = \mathfrak{q} \cap H_\mathfrak{p} \cap H = \mathfrak{p}H_\mathfrak{p} \cap H = \mathfrak{p}. \qquad \square$$

Lemma 3.5. *Suppose that H is v-noetherian and $S \subset H$ is a saturated submonoid.*

1. *S is v-noetherian, and $\widehat{S} \subset \widehat{H}$ is saturated.*
2. *If $(H : \widehat{H}) \ne \emptyset$, then $(S : \widehat{S}) \ne \emptyset$.*
3. *Assume that $\widehat{H}$ is a Krull monoid. Then $\widehat{S}$ is a Krull monoid. If, further, $\mathcal{C}(\widehat{H})$ and the class group of $S \subset H$ are both finite, then $\mathcal{C}(\widehat{S})$ is finite.*

Proof. 1. S is v-noetherian by [23, Proposition 2.4.4.2], and it remains to verify that $\mathsf{q}(S) \cap \widehat{H} \subset \widehat{S}$. Let $x = c^{-1}b \in \mathsf{q}(S) \cap \widehat{H}$, with $b, c \in S$. Then Lemma 3.4.1 implies that there exists $m \in \mathbb{N}$ such that $c^m x^n \in H \cap \mathsf{q}(S) = S$ for all $n \in \mathbb{N}$. It follows that $x \in \widehat{S}$.

2. We have $\widehat{S} \subset S^{-1}H \cap \widehat{H} \subset \widehat{H}$, and if $(H : \widehat{H}) \ne \emptyset$, then these three sets are H-fractional. By [23, Proposition 2.1.10] there exists a finite set $E \subset S^{-1}H \cap \widehat{H}$ such that $E_v = (S^{-1}H \cap \widehat{H})_v$. Then

$$E_v \supset S^{-1}H \cap \widehat{H} \supset \widehat{S}.$$

If $t \in S$ with $tE \subset H$, then $t\widehat{S} \subset tE_v \cap \mathsf{q}(S) = (tE)_v \cap \mathsf{q}(S) \subset H \cap \mathsf{q}(S) = S$, and we see that $t \in (S : \widehat{S})$.

3. Let F be a factorial monoid such that $\widehat{H} \subset F$ is saturated and $\mathsf{q}(F)/\mathsf{q}(H)F^\times = \mathcal{C}(\widehat{H})$. By 1. $\widehat{S} \subset \widehat{H}$ is saturated, and hence $\widehat{S} \subset F$ is saturated. Since the class group of $S \subset H$ is finite, the class group of $\widehat{S} \subset \widehat{H}$ is finite by Lemma 3.2.3, and thus the class group of $\widehat{S} \subset F$ is finite. Now [23, Theorem 2.4.7] implies that $\mathcal{C}(\widehat{S})$ is finite. $\square$

Suppose that H is v-noetherian and $S \subset H$ is a divisor-closed submonoid. Then $\widehat{S} \subset \widehat{H}$ is saturated but need not be divisor-closed any more, as the following example shows. Suppose that

$$H = \big(\mathbb{N}_0^2 \setminus \big\{(2k+1, 0) \bigm| k \in \mathbb{N}_0\big\}, +\big) \subset \big(\mathbb{N}_0^2, +\big) = F$$

and

$$S = \{(2k, 0) \mid k \in \mathbb{N}_0\} \subset H.$$

Then, by Theorem 2.3, H is a C-monoid defined in F, $\widehat{H} = F$, and $(1,1) + F \subset H$. Moreover, $\widehat{S} = S \subset H$ is divisor-closed but $\widehat{S} \subset \widehat{H}$ is not divisor-closed.

The following lemma generalizes [32, Lemmas 2 and 3].

Lemma 3.6. *Suppose that H is v-noetherian and $T \subset H$ is a divisor-closed submonoid.*

1. $\widehat{T} = (T^{-1}H)^\times \cap \widehat{H}$ *and* $\mathsf{q}(T) = (T^{-1}H)^\times$.
2. $T = [\![T]\!]_{\widehat{H}} \cap H$ *and* $\mathsf{q}([\![T]\!]_{\widehat{H}}) = (T^{-1}\widehat{H})^\times$.
3. $\widehat{T} \subset [\![T]\!]_{\widehat{H}}$ *is saturated. Further, if F is a monoid such that $\widehat{H} \subset F$ is saturated, then $[\![T]\!]_{\widehat{H}} \subset [\![T]\!]_F$ is saturated.*

Proof. 1. Since $T \subset H$ is saturated it follows that $T = H \cap \mathsf{q}(T)$. An easy calculation shows that $\mathsf{q}(T) = T^{-1}T = (T^{-1}H)^\times$. Obviously we have $\widehat{T} \subset \mathsf{q}(T) \cap \widehat{H} = (T^{-1}H)^\times \cap \widehat{H}$. Conversely, let $x = t^{-1}s \in (T^{-1}H)^\times \cap \widehat{H}$, with $s, t \in T$. By Lemma 3.4.1 there exists $m \in \mathbb{N}$ such that $t^m x^n \in H$ for all $n \in \mathbb{N}$. It follows that $t^m x^n \in H \cap (T^{-1}H)^\times = T$ for all $n \in \mathbb{N}$, and therefore $x \in \widehat{T}$.

2. Clearly, we have $T \subset [\![T]\!]_{\widehat{H}} \cap H$. Conversely, let $x \in [\![T]\!]_{\widehat{H}} \cap H$. Then there exists $z \in \widehat{H}$ such that $xz = t \in T$. Since $z = x^{-1}t$, Lemma 3.4.1 implies that there exists $m \in \mathbb{N}$ such that $x^m z^n \in H$ for all $n \in \mathbb{N}$. From this we see that

$$x^m z^{m+1} = x^m \frac{t^{m+1}}{x^{m+1}} = \frac{t^{m+1}}{x} \in H,$$

and hence we obtain $x \in [\![t]\!]_H \subset T$. To prove that $\mathsf{q}([\![T]\!]_{\widehat{H}}) = (T^{-1}\widehat{H})^\times$ put $M = [\![T]\!]_{\widehat{H}}$. Clearly, M is a divisor-closed submonoid of $\widehat{H}$. Therefore the same argument as in the proof of 1. shows that $\mathsf{q}(M) = M^{-1}M = (M^{-1}\widehat{H})^\times$. The assertion now follows since $M^{-1}\widehat{H} = T^{-1}\widehat{H}$.

3. By Lemma 3.5.1 the inclusion $\widehat{T} \subset \widehat{H}$ is saturated. Clearly, $\widehat{T}$ is contained in $[\![T]\!]_{\widehat{H}}$. Thus we obtain

$$\widehat{T} \subset \mathsf{q}(T) \cap [\![T]\!]_{\widehat{H}} \subset \mathsf{q}(T) \cap \widehat{H} = \widehat{T},$$

and it follows that $\widehat{T} \subset [\![T]\!]_{\widehat{H}}$ is saturated. Lemma 3.2.1 (applied to the monoids $[\![T]\!]_{\widehat{H}} \subset \widehat{H} \subset F$) implies that $[\![T]\!]_{\widehat{H}} \subset [\![T]\!]_F$ is saturated. □

Lemma 3.7. *Suppose that H is v-noetherian and $F = F^\times \times \mathcal{F}(P)$ is a factorial monoid such that $\widehat{H} \subset F$ is saturated.*

1. *Let $a, b \in H$. Then $b \in [\![a]\!]_H$ if and only if* $\operatorname{supp}(b) \subset \operatorname{supp}(a)$.
2. *Let $T \subset H$ be a divisor-closed submonoid and $Q = \{p \in P \mid pF \cap T \neq \emptyset\}$.*
   (a) $T = \{a \in H \mid \operatorname{supp}(a) \subset Q\}$ *and* $[\![T]\!]_F = F^\times \times \mathcal{F}(Q)$.
   (b) *If $T' \subset H$ is a divisor-closed submonoid and $Q' = \{p \in P \mid pF \cap T' \neq \emptyset\}$, then $T' \subset T$ if and only if $Q' \subset Q$.*
   (c) $\widehat{T} \subset F^\times \times \mathcal{F}(Q)$ *is saturated and cofinal.*
   (d) $\widehat{T}$ *is a discrete valuation monoid if and only if* $|\operatorname{supp}(\widehat{T} \setminus \widehat{T}^\times)| = 1$.

Proof. 1. If $b \in [\![a]\!]_H$, then there exists $n \in \mathbb{N}$ such that $b \mid a^n$ in H. This implies that $b \mid a^n$ in F, whence $\mathrm{supp}(b) \subset \mathrm{supp}(a^n) = \mathrm{supp}(a)$. Conversely, suppose that $\mathrm{supp}(b) \subset \mathrm{supp}(a)$. Then there exists $n \in \mathbb{N}$ such that $b \mid a^n$ in F. Since $\widehat{H} \subset F$ is saturated, we have $b \mid a^n$ in $\widehat{H}$, and thus $b^{-1}a^n \in \widehat{H}$. Lemma 3.4.1 implies that there exists $m \in \mathbb{N}$ such that $b^m(b^{-1}a^n)^k \in H$ for all $k \in \mathbb{N}$. For $k = m+1$ it follows that $b^{-1}a^{n(m+1)} \in H$. Therefore $b \mid a^{n(m+1)}$ in H, and it follows that $b \in [\![a]\!]_H$.

2.(a) If $a \in T$ and $p \in \mathrm{supp}(a)$, then $pF \cap T \neq \emptyset$. Thus $\mathrm{supp}(a) \subset Q$. Conversely, let $a \in H$ with $\mathrm{supp}(a) \subset Q$. Then, for every $p \in \mathrm{supp}(a)$, there is $b_p \in T$ with $p \mid b_p$, and thus there exists $b \in T$ with $\mathrm{supp}(a) \subset \mathrm{supp}(b)$. Therefore 1. implies that $a \in [\![b]\!]_H \subset T$.

Since $T \subset F^\times \times \mathcal{F}(Q)$ and $F^\times \times \mathcal{F}(Q)$ is a divisor-closed submonoid of F it follows that $[\![T]\!]_F \subset F^\times \times \mathcal{F}(Q)$. Conversely, since $F^\times \subset [\![T]\!]_F$ and $Q \subset [\![T]\!]_F$, we obtain $F^\times \times \mathcal{F}(Q) \subset [\![T]\!]_F$.

2.(b) This follows immediately from 2.(a).

2.(c) Since $\widehat{T} \subset \widehat{H}$ is saturated by Lemma 3.5.1, we infer that

$$\widehat{T} = \mathsf{q}(T) \cap \widehat{H} = \mathsf{q}(T) \cap \big(\mathsf{q}(H) \cap F\big) = \mathsf{q}(T) \cap F.$$

If $F_0 = F^\times \times \mathcal{F}(Q)$, then $F_0 = \mathsf{q}(F_0) \cap F$, and we get

$$\mathsf{q}(\widehat{T}) \cap F_0 = \mathsf{q}(T) \cap \mathsf{q}(F_0) \cap F = \mathsf{q}(T) \cap F = \widehat{T},$$

whence $\widehat{T} \subset F_0$ is saturated. Since $T \subset [\![T]\!]_F = F_0$ is cofinal, $\widehat{T} \subset F_0$ is cofinal.

2.(d) By 2.(c) $\widehat{T}$ is a Krull monoid, and hence it is a discrete valuation monoid if and only if it is primary. Suppose that $|\mathrm{supp}(\widehat{T} \setminus \widehat{T}^\times)| = 1$. Then $\mathrm{supp}(a) = Q$ for all $a \in \widehat{T} \setminus \widehat{T}^\times$. Thus, for all $a, b \in \widehat{T} \setminus \widehat{T}^\times$, there exists $n \in \mathbb{N}$ such that $a \mid b^n$ in F_0. This implies that $a \mid b^n$ in $\widehat{T}$, and therefore $\widehat{T}$ is primary. Conversely, suppose that $\widehat{T}$ is primary, and let $a, b \in \widehat{T} \setminus \widehat{T}^\times$. Then there are $m, n \in \mathbb{N}$ such that $a \mid b^m$, $b \mid a^n$, and hence $\mathrm{supp}(a) = \mathrm{supp}(b)$ by 1. □

Lemma 3.8. *Suppose that H is a submonoid of a factorial monoid $F = F^\times \times \mathcal{F}(P)$, where P is a finite set, and assume that $\widehat{H} \subset F$ is saturated. Then the following statements are equivalent*:

(a) *$\widetilde{H}$ is a* C*-monoid defined in F.*
(b) *The class group of $\widehat{H} \subset F$ is finite, and $\widetilde{H} = \{a \in \widehat{H} \mid \emptyset \neq \mathrm{supp}(a) \in \mathrm{supp}(H)\} \cup \widetilde{H}^\times$.*

Proof. Since $\widehat{H} \subset F$ is saturated $\widehat{H}$ is a Krull monoid, and thus $\widehat{H} \subset \widehat{\widetilde{H}} \subset \widehat{\widehat{H}} = \widehat{H}$. Let G denote that class group of $\widehat{H} \subset F$.

(a) ⇒ (b) Suppose that $\widetilde{H}$ is a C-monoid defined in F, where the parameter $\alpha \in \mathbb{N}$ and the subgroup $V \subset F^\times$ satisfy the conditions in Theorem 2.3.2.(b). Since $\widetilde{H}$ is a C-monoid defined in F, the reduced class semigroup $\mathcal{C}^*(\widetilde{H}, F)$ is finite. The natural homomorphism $\mathcal{C}^*(\widetilde{H}, F) \to \mathcal{C}(\widehat{\widetilde{H}} F^\times, F) = \mathcal{C}(\widehat{H} F^\times, F)$ is surjective (apply Lemma 4.2.1.(a) with $D = \widehat{H} F^\times$). By Lemma 2.1.2 it now follows that G is finite.

Clearly, we have $\widetilde{H} \subset \{a \in \widehat{H} \mid \emptyset \neq \mathrm{supp}(a) \in \mathrm{supp}(H)\}$. Conversely, let

$$a = \varepsilon \prod_{p \in Q} p^{k_p} \in \widehat{H}, \quad \text{where } \varepsilon \in F^\times,\ Q \subset P \text{ and } k_p \in \mathbb{N},$$

and suppose that $\emptyset \neq Q \in \operatorname{supp}(H)$. Then it follows by [23, Proposition 2.9.6.2.(b)] that $\varepsilon^\alpha \in V$ and $a^\alpha = \varepsilon^\alpha \prod_{p \in P} p^{\alpha k_p} \in \widetilde{H}$. Since $\widetilde{H}$ is root-closed, we see that $a \in \widetilde{H}$.

(b) $\Rightarrow$ (a) It follows directly from the assumptions that $F^\times \cap \widetilde{H} = \widetilde{H}^\times$. Next we verify condition 2.(b) of Theorem 2.3, where we put $V = F^\times$ and $\alpha = \exp(G)$. If $a \in \widehat{H} \setminus \widehat{H}^\times$ and $\varepsilon \in \widehat{H}^\times$, then $\operatorname{supp}(a) = \operatorname{supp}(\varepsilon a)$. Therefore we have $V \cdot (\widetilde{H} \setminus \widetilde{H}^\times) \subset \widetilde{H}$. Let $p \in P$, and suppose that $a \in p^\alpha F$. Since $\widehat{H} \subset F$ is saturated and $\alpha = \exp(G)$ it follows that $a \in \widehat{H} F^\times$ if and only if $p^\alpha a \in \widehat{H} F^\times$. Furthermore, $\operatorname{supp}(a) = \operatorname{supp}(p^\alpha a)$. Hence we see that $a \in \widetilde{H}$ if and only if $p^\alpha a \in \widetilde{H}$. $\square$

Lemma 3.9. *Suppose that H is v-noetherian, $(H : \widehat{H}) \neq \emptyset$, and assume that $F = F^\times \times \mathcal{F}(P)$ is a factorial monoid such that $\widehat{H} \subset F$ is saturated.*

1. *If $p \in P \cap \widetilde{H}$, then there exists $\alpha \in \mathbb{N}$ such that, for all $a \in p^\alpha F$, we have $a \in H$ if and only if $p^\alpha a \in H$.*
2. *If $\mathsf{q}([\![a]\!]_{\widehat{H}})/\mathsf{q}([\![a]\!]_H)$ is a torsion group for every $a \in H$, then*

$$\widetilde{H} = \bigl\{a \in \widehat{H} \bigm| \operatorname{supp}(a) \in \operatorname{supp}(H)\bigr\}.$$

Proof. 1. Pick $f \in (H : \widehat{H})$, and let $k \in \mathbb{N}$ such that $b = p^k \in H$. For all $m \in \mathbb{N}$ we have

$$\bigl(H : b^m\bigr) = \bigl\{x \in \mathsf{q}(H) \bigm| b^m x \in H\bigr\} \in \mathcal{F}_v(H),$$

and since $(H : \widehat{H}) \neq \emptyset$, [23, Theorem 2.3.5] implies that $\mathfrak{a}_m = (H : b^m) \cap \widehat{H} \in \mathcal{F}_v(H)$. Then

$$f\mathfrak{a}_1 \subset f\mathfrak{a}_2 \subset \cdots$$

is an ascending chain of v-ideals in H, and hence there exists $n \in \mathbb{N}$ such that $f\mathfrak{a}_n = f\mathfrak{a}_{n+m}$ for all $m \in \mathbb{N}_0$. We set $\alpha = kn$ and let $a = p^\alpha x \in p^\alpha F$, with $x \in F$. Suppose that $a \in H$. Then $p^\alpha a \in H$ since $p^\alpha \in H$. Conversely, suppose that $p^\alpha a = p^{2\alpha} x = b^{2n} x \in H$. Then $x \in F \cap \mathsf{q}(H) = \widehat{H}$ and $x \in (H : b^{2n}) = (H : b^n)$. This implies that $a = p^\alpha x = b^n x \in H$.

2. If $a \in \widetilde{H}$, then there exists $n \in \mathbb{N}$ such that $a^n \in H$. Thus $\operatorname{supp}(a) = \operatorname{supp}(a^n) \in \operatorname{supp}(H)$. Conversely, let $a \in \widehat{H}$, and suppose that $a' \in H$ such that $\operatorname{supp}(a') = \operatorname{supp}(a)$. By Lemma 3.7.1 it follows that $[\![a]\!]_{\widehat{H}} = [\![a']\!]_{\widehat{H}}$, and by Lemma 3.5.2 there exists

$$f \in \bigl([\![a']\!]_H : \widehat{[\![a']\!]_H}\bigr) \subset \widehat{[\![a']\!]_H} \subset [\![a']\!]_{\widehat{H}} = [\![a]\!]_{\widehat{H}}.$$

There exists $n \in \mathbb{N}$ such that $f \mid a^n$ in $\widehat{H}$, and since $\mathsf{q}([\![a']\!]_{\widehat{H}})/\mathsf{q}([\![a']\!]_H)$ is a torsion group, we may choose n in such a way that

$$a^n \in \mathsf{q}\bigl([\![a']\!]_H\bigr) \cap \widehat{H} = \widehat{[\![a']\!]_H}.$$

The last equal sign holds since $\widehat{[\![a']\!]_H} \subset \widehat{H}$ is saturated (Lemma 3.5.1). It follows that f divides a^n in $\widehat{[\![a']\!]_H}$, and then

$$a^n = f\bigl(f^{-1}a^n\bigr) \in f\widehat{[\![a']\!]_H} \subset [\![a']\!]_H \subset H,$$

showing that $a \in \widetilde{H}$. $\square$

4. Weakly C-monoids: definition and algebraic properties

In this section we introduce weakly C-monoids. We first give their definition, and then we discuss and justify conditions **(C1)** and **(C2)**.

Definition 4.1. The monoid H is called a *weakly* C*-monoid* if it is a submonoid of a factorial monoid $F = F^\times \times \mathcal{F}(P)$ such that the following two conditions are satisfied:

(C1) H is v-noetherian, $(H : \widehat{H}) \neq \emptyset$ and $\widehat{H} \subset F$ is saturated and cofinal.
(C2) There exist an equivalence relation $\sim$ on P and a constant $\lambda \in \mathbb{N}$ such that
- $P/\sim$ is finite, and
- for all $p_1, p_1', \ldots, p_\lambda, p_\lambda' \in P$ with $p_1 \sim p_1' \sim \cdots \sim p_\lambda \sim p_\lambda'$ there exists $\varepsilon \in F^\times$ such that $[p_1 \cdot \ldots \cdot p_\lambda]_H^F = [\varepsilon p_1' \cdot \ldots \cdot p_\lambda']_H^F$.

We refer to these properties by saying that *H is a weakly* C*-monoid defined in F with equivalence relation $\sim$ and parameter λ*.

In this paper we are interested in arithmetical finiteness properties of v-noetherian monoids. To make conditions (**C1**) and (**C2**) of Definition 4.1 plausible, we consider two extreme cases of v-noetherian monoids that are intended to be weakly C-monoids. First, suppose that H is v-noetherian and completely integrally closed, that is, H is a Krull monoid. If its class group $\mathcal{C}(H)$ is infinite, then it is well known that H may fail to be locally tame. More precisely, if every class of $\mathcal{C}(H)$ contains a prime, then H is locally tame if and only if $\mathcal{C}(H)$ is finite [25, Theorem 4.4]. If H is a weakly C-monoid, then—roughly speaking—the finiteness of $P/\sim$ implies that the class group $\mathcal{C}(\widehat{H})$ of $\widehat{H}$ is finitely generated (see Proposition 4.4.2). To obtain our arithmetical main result (Theorem 5.3) we need to assume, in addition, that $\mathcal{C}(\widehat{H})$ is even finite. A second extreme case of v-noetherian monoids we want to describe are primary monoids, that is, monoids which have only a single non-empty prime s-ideal. If the conductor $(H : \widehat{H})$ of such a monoid H is empty, then H may fail to be locally tame (examples of such monoids can be found in [26]). Therefore it is natural to claim, in Definition 4.1, that H has non-empty conductor.

Suppose now that H is v-noetherian and $(H : \widehat{H}) \neq \emptyset$. Then $\widehat{H}$ is a Krull monoid, and hence there exists a factorial monoid F such that $\widehat{H} \subset F$ is saturated and cofinal. Thus **(C1)** holds.

Condition **(C2)** is necessary to enforce a "uniform" behavior of the prime elements of F in the case there are infinitely many, e.g., if H is a C-monoid or if H is the multiplicative monoid of a higher-dimensional domain described in Theorem 6.7. Note that, if H is a G-monoid, and F is a factorial monoid such that **(C1)** is satisfied, then F has only finitely many pairwise non-associated primes. In this case condition **(C2)** is necessarily satisfied.

The main examples of weakly C-monoids we have in mind are

- v-noetherian G-monoids with $(H : \widehat{H}) \neq \emptyset$ (cf. Proposition 4.7); the equivalence relation $\sim$ is equality,
- C-monoids (cf. Proposition 4.8); the equivalence relation $\sim$ is (H, F)-equivalence,
- the monoid of non-zero elements and the monoid of v-invertible v-ideals of Mori domains satisfying certain finiteness conditions (cf. Theorem 6.7 and Corollary 6.8).

After having proved several technical results gathered in the following lemmas and propositions, we show that there is a canonical choice for the monoid F occurring in Definition 4.1 in the case $\mathcal{C}(\widehat{H})$ is finite (Proposition 4.5). Then we show that weakly C-monoids are stable under algebraic standard operations such as taking finite products or passing to divisor-closed submonoids (Proposition 4.6). After that we study v-noetherian G-monoids, and we show that all finitely generated monoids are weakly C-monoids. Further, we characterize all Krull monoids that are weakly C-monoids (Propositions 4.7 and 4.8).

Lemma 4.2. *Assume that H is a submonoid of a monoid F.*

1. *Let $D \subset F$ be a saturated submonoid.*
   (a) *Suppose that $H \subset D \subset \mathsf{q}(H)$ and $(H : D) \neq \emptyset$. If $x, y \in F$ and $[x]_H^F = [y]_H^F$, then $[x]_D^F = [y]_D^F$.*
   (b) *If $x, y \in D$, then $[x]_H^F = [y]_H^F$ if and only if $[x]_H^D = [y]_H^D$.*
2. *If $S \subset H$ is a saturated submonoid, $f \in (S : \widehat{S})$, and $x \in S$ with $[x]_H^F = [f]_H^F$, then $x \in (S : \widehat{S})$.*
3. *If $x, y \in F$ with $[x]_H^F = [y]_H^F$, then $[x]_{\widetilde{H}}^F = [y]_{\widetilde{H}}^F$.*

Proof. 1.(a) Let $f \in (H : D)$, and suppose that $x, y \in F$ are (H, F)-equivalent. If $z \in F$ with $xz \in D$, then $xzf \in H$, and thus $yzf \in H$. This implies that $yz \in \mathsf{q}(H) \cap F = D$, and hence x, y are (D, F)-equivalent.

1.(b) Let $x, y \in D$. If x, y are (H, F)-equivalent, then they are (H, D)-equivalent. Conversely, suppose that $[x]_H^D = [y]_H^D$. If $z \in F$ with $zx \in H$, then $z = x^{-1}(zx) \in \mathsf{q}(D) \cap F = D$. Thus $[x]_H^D = [y]_H^D$ implies that $zy \in H$, and it follows that $[x]_H^F = [y]_H^F$.

2. If $z \in \widehat{S}$, then $fz \in S \subset H$, and hence $xz \in \mathsf{q}(S) \cap H = S$. This implies that $x\widehat{S} \subset S$, and it follows that $x \in (S : \widehat{S})$.

3. Let $x, y \in F$ such that $[x]_H^F = [y]_H^F$, and suppose that $z \in F$ with $xz \in \widetilde{H}$. Then there exists $n \in \mathbb{N}$ such that $x^n z^n \in H$. Since $[x]_H^F = [y]_H^F$ it follows that $x^{n-\nu} y^\nu z^n \in H$ for all $\nu \in [0, n]$. Thus we have $y^n z^n \in H$, and therefore $yz \in \widetilde{H}$. □

Lemma 4.3. *Suppose that H is a weakly* C*-monoid defined in $F = F^\times \times \mathcal{F}(P)$ with equivalence relation $\sim$ and parameter λ.*

1. *Let $k \in \mathbb{N}$ with $k \geqslant \lambda$ and $p_1, p_1', \ldots, p_k, p_k' \in P$ with $p_1 \sim p_1' \sim \cdots \sim p_k \sim p_k'$. Then there exists $\varepsilon \in F^\times$ such that $[p_1 \cdot \ldots \cdot p_k]_H^F = [\varepsilon p_1' \cdot \ldots \cdot p_k']_H^F$.*
2. *Let $a, b \in F$ with $\mathsf{v}_\tau(a) = \mathsf{v}_\tau(b) \in \mathbb{N}_{\geqslant \lambda} \cup \{0\}$ for every $\tau \in P/\sim$. Then there exists $\varepsilon \in F^\times$ such that $[a]_H^F = [\varepsilon b]_H^F$. Moreover, if $a, b \in \widehat{H}$, then $\varepsilon \in \widehat{H}^\times$.*
3. *The equivalence relation $\sim$ can be extended to a congruence relation on F (again denoted by $\sim$) such that $F/\sim \cong \mathcal{F}(P/\sim)$ and such that, for all $a, b \in F$, the following conditions are satisfied:*
   (a) *$a \sim b$ if and only if $\mathsf{v}_\tau(a) = \mathsf{v}_\tau(b)$ for all $\tau \in P/\sim$.*
   (b) *If $a \sim b$, then $[a]_{\widehat{H}F^\times}^F = [b]_{\widehat{H}F^\times}^F$ and $a\mathsf{q}(H)F^\times = b\mathsf{q}(H)F^\times$.*
4. *Let $m \in \mathbb{N}_0$ and $a, b_1, \ldots, b_m \in F$ such that $\mathsf{v}_\tau(b_1 \cdot \ldots \cdot b_m) \leqslant \mathsf{v}_\tau(a)$ and $\mathsf{v}_\tau(b_i) \in \lambda\mathbb{N}_0$ for all $i \in [1, m]$ and all $\tau \in P/\sim$. Then there exist $b_1', \ldots, b_m' \in F$ such that $b_1' \cdot \ldots \cdot b_m' \mid_F a$, $[b_i']_H^F = [b_i]_H^F$, and $\mathsf{v}_\tau(b_i') = \mathsf{v}_\tau(b_i)$ for all $\tau \in P/\sim$ and all $i \in [1, m]$.*

Proof. 1. We proceed by induction on k. If $k=\lambda$, then the assertion follows directly from **(C2)**. Suppose that $k>\lambda$, and assume the assertion holds for all $l \in [\lambda, k-1]$. Then there exist $\varepsilon_1, \varepsilon_2 \in F^\times$ such that

$$\begin{aligned}[p_1 \cdot \ldots \cdot p_k]_H^F &= [p_1 \cdot \ldots \cdot p_{k-1}]_H^F + [p_k]_H^F \\ &= \big[\varepsilon_1 p_1' \cdot \ldots \cdot p_{k-1}'\big]_H^F + [p_k]_H^F = \big[\varepsilon_1 p_1' \cdot \ldots \cdot p_{k-1}' p_k\big]_H^F \\ &= \big[\varepsilon_1 p_1'\big]_H^F + \big[p_2' \cdot \ldots \cdot p_{k-1}' p_k\big]_H^F \\ &= \big[\varepsilon_1 p_1'\big]_H^F + \big[\varepsilon_2 p_2' \cdot \ldots \cdot p_{k-1}' p_k'\big]_H^F \\ &= \big[\varepsilon_1 \varepsilon_2 p_1' \cdot \ldots \cdot p_k'\big]_H^F.\end{aligned}$$

2. By 1. there exists some $\varepsilon \in F^\times$ such that $[a]_H^F = [\varepsilon b]_H^F$. Suppose that $a, b \in \widehat{H}$ and pick $f \in (H : \widehat{H})$. Then $fa \in H$, $fb \in H$, $f\varepsilon b \in H$ and $\varepsilon = (fb)^{-1} f\varepsilon b \in \mathsf{q}(H) \cap F^\times \in \widehat{H}^\times$.

3. Let $\mu : F \to \mathcal{F}(P/\sim)$ be the unique homomorphism satisfying $\mu(\varepsilon) = 1$ for all $\varepsilon \in F^\times$ and $\mu(p) = [p]_\sim \in P/\sim$ for all $p \in P$ and define, for all $a, b \in F$,

$$a \sim b \quad \text{if and only if} \quad \mu(a) = \mu(b).$$

Then $\sim$ is a congruence relation on F, $F/\sim \,\cong \mathcal{F}(P/\sim)$, and for $a, b \in F$ we have $a \sim b$ if and only if $\mathsf{v}_\tau(a) = \mathsf{v}_\tau(b)$ for all $\tau \in P/\sim$.

Let $a, b \in F$ with $a \sim b$. Clearly, there is an element $c \in F$ such that $\mathsf{v}_\tau(ac) = \mathsf{v}_\tau(bc) \in \mathbb{N}_{\geqslant \lambda} \cup \{0\}$ for all $\tau \in P/\sim$. By 2. there exists $\varepsilon \in F^\times$ such that $[ac]_H^F = [\varepsilon bc]_H^F$. Then Lemma 4.2.1.(a) implies that $[ac]_{\widehat{H}}^F = [\varepsilon bc]_{\widehat{H}}^F$, and hence $[ac]_{\widehat{H}F^\times}^F = [bc]_{\widehat{H}F^\times}^F$.

Since $\widehat{H} \subset F$ is saturated and cofinal, $\widehat{H}F^\times \subset F$ is saturated and cofinal. By Lemma 2.1.2 the map $\mathcal{C}(\widehat{H}F^\times, F) \to \mathsf{q}(F)/\mathsf{q}(H)F^\times$, defined by $[z]_{\widehat{H}F^\times}^F \mapsto z\mathsf{q}(H)F^\times$ for all $z \in F$, is an isomorphism. Thus $\mathcal{C}(\widehat{H}F^\times, F)$ is a group, and therefore $[ac]_{\widehat{H}F^\times}^F = [bc]_{\widehat{H}F^\times}^F$ implies that $[a]_{\widehat{H}F^\times}^F = [b]_{\widehat{H}F^\times}^F$ and $a\mathsf{q}(H)F^\times = b\mathsf{q}(H)F^\times$.

4. We proceed by induction on m. If $m = 0$ there is nothing to do. Suppose that $m \geqslant 1$ and $b_1, \ldots, b_m \in F$ with $\mathsf{v}_\tau(b_1 \cdot \ldots \cdot b_m) \leqslant \mathsf{v}_\tau(a)$. Then $\mathsf{v}_\tau(b_1 \cdot \ldots \cdot b_{m-1}) \leqslant \mathsf{v}_\tau(a)$, and by the induction hypothesis there exist $b_1', \ldots, b_{m-1}' \in F$ such that $b_1' \cdot \ldots \cdot b_{m-1}' \mid_F a$, $[b_i']_H^F = [b_i]_H^F$, and $\mathsf{v}_\tau(b_i') = \mathsf{v}_\tau(b_i)$ for all $\tau \in P/\sim$ and all $i \in [1, m-1]$. We set $a' = (b_1' \cdot \ldots \cdot b_{m-1}')^{-1} a$. Then there exists $b_m'' \in F$ with $b_m'' \mid_F a'$ and $\mathsf{v}_\tau(b_m'') = \mathsf{v}_\tau(b_m)$ for all $\tau \in P/\sim$. By 3. there exists $\varepsilon \in F^\times$ such that $[b_m]_H^F = [\varepsilon b_m'']_H^F$. Therefore the assertion follows if we set $b_m' = \varepsilon b_m''$. $\square$

Proposition 4.4. *Suppose that H is a weakly* C*-monoid defined in $F = F^\times \times \mathcal{F}(P)$ with equivalence relation $\sim$ and parameter λ.*

1. *H is a* BF*-monoid and $H \cap F^\times = H^\times$.*
2. *The set $G_P = \{p\mathsf{q}(H)F^\times \mid p \in P\} \subset \mathsf{q}(F)/\mathsf{q}(H)F^\times$ is finite, $\mathsf{D}(G_P) = \sup\{\mathsf{v}_P(u) \mid u \in \mathcal{A}(\widehat{H})\} < \infty$, and in particular the group $\mathsf{q}(F)/\mathsf{q}(H)F^\times$ is finitely generated.*
3. *$\widehat{H}$ is a Krull monoid and the set of classes $g \in \mathcal{C}(\widehat{H})$ containing primes $\mathfrak{p} \in \mathfrak{X}(\widehat{H})$ is finite. In particular, $\mathcal{C}(\widehat{H})$ is finitely generated.*

Proof. 1. Since $\widehat{H} \subset F$ is saturated, it follows that $\widehat{H} \cap F^\times = \widehat{H}^\times$. Since H is v-noetherian and $(H : \widehat{H}) \neq \emptyset$, [23, Theorem 2.3.5] implies that $H \cap \widehat{H}^\times = H^\times$. Thus we obtain

$$H^\times = H \cap \widehat{H}^\times = H \cap F^\times.$$

Since H is v-noetherian it is a BF-monoid [23, Theorem 2.2.9].

2. G_P is finite by Lemma 4.3.3.(b), and clearly G_P generates $\mathsf{q}(F)/\mathsf{q}(H)F^\times$. Since the Davenport constant of finite subsets of abelian groups is finite [23, Theorem 3.4.2], it follows that $\mathsf{D}(G_P) < \infty$. It follows from Lemma 2.1.3 (with $H = \widehat{H}$ and $D = F$) that $\mathsf{D}(G_P) = \sup\{\mathsf{v}_P(u) \mid u \in \mathcal{A}(\widehat{H})\}$.

3. $\widehat{H}$ is a Krull monoid by Theorem 3.1. Since $\widehat{H} \subset F$ is saturated and cofinal, there is a cofinal divisor homomorphism $\varphi : \widehat{H} \to \mathcal{F}(P)$. For $a \in \mathcal{F}(P)$ we set $[a]_\varphi = a\mathsf{q}(\varphi(\widehat{H}))$. By [23, Theorem 2.4.7]

$$F_0 = \bigl\{\gcd\bigl(\varphi(X)\bigr) \bigm| \emptyset \neq X \subset \widehat{H}\bigr\} \subset \mathcal{F}(P) \quad \text{and} \quad \mathcal{C}_0 = \bigl\{[a]_\varphi \bigm| a \in F_0\bigr\} \subset \mathsf{q}\bigl(\mathcal{F}(P)\bigr)/\mathsf{q}\bigl(\varphi(\widehat{H})\bigr)$$

are submonoids. Furthermore, there are epimorphisms $\varphi^* : F_0 \to \mathcal{I}_v^*(\widehat{H})$ and $\overline{\varphi} : \mathcal{C}_0 \to \mathcal{C}_v(\widehat{H})$, given by

$$\varphi^*(a) = \varphi^{-1}(aF)_v \quad \text{and} \quad \overline{\varphi}\bigl([a]_\varphi\bigr) = \bigl[\varphi^{-1}(aF)_v\bigr] \quad \text{for all } a \in F_0,$$

and for $x \in \widehat{H}$ we have $\varphi^* \circ \varphi(x) = x\widehat{H}$. Clearly, there is an isomorphism

$$\psi : \mathsf{q}(F)/\mathsf{q}(H)F^\times \to \mathsf{q}\bigl(\mathcal{F}(P)\bigr)/\mathsf{q}\bigl(\varphi(\widehat{H})\bigr),$$

and for $G_0 = \{[p]_\varphi \mid p \in P\} \subset \mathcal{C}_0$ we have $\psi(G_P) = G_0$. In particular, G_0 is finite and $\mathsf{D}(G_0) = \mathsf{D}(G_P) < \infty$.

If $\mathfrak{p} \in \mathfrak{X}(\widehat{H})$, then there exists $u \in F_0$ such that $\mathfrak{p} = \varphi^*(u)$. Since φ^* maps non-units to non-units and $\mathfrak{p} \in \mathcal{I}_v^*(\widehat{H})$ is a prime, u must be an atom of F_0. We shall prove that the set $\{[u]_\varphi \mid u \in \mathcal{A}(F_0)\}$ is finite.

Let $c \in \widehat{H}$ and $u \in F_0$ such that $\varphi(c) \mid u$ in $\mathcal{F}(P)$. We assert that $\varphi(c) \mid u$ in F_0. Let $\emptyset \neq X \subset \widehat{H}$ such that $u = \gcd(\varphi(X))$. Since $\varphi(c)^{-1}\gcd(\varphi(X)) \in \mathcal{F}(P)$, it follows that $\varphi(c)^{-1}\varphi(X) \subset \mathcal{F}(P)$, and we see that $c^{-1}X \subset \widehat{H}$. Thus $\gcd(\varphi(c^{-1}X)) \in F_0$ and $u = \gcd(\varphi(X)) = \varphi(c)\gcd(\varphi(c^{-1}X))$.

Let $u \in \mathcal{A}(F_0)$, $k \in \mathbb{N}$ and $p_1, \ldots, p_k \in P$ such that $u = p_1 \cdot \ldots \cdot p_k$. We assert that $k \leqslant \mathsf{D}(G_0)$. Assume to the contrary that $k > \mathsf{D}(G_0)$. By [23, Theorem 5.1.5] there exists $I \subset [1, k]$, say $I = [1, l]$, such that $l \leqslant \mathsf{D}(G_0)$ and $c = p_1 \cdot \ldots \cdot p_l \in \varphi(\widehat{H})$. Then $c \mid u$ in $\mathcal{F}(P)$ and hence in F_0, and since $c^{-1}u = p_{l+1} \cdot \ldots \cdot p_k \neq 1$, we get a contradiction to $u \in \mathcal{A}(F_0)$. Therefore we obtain

$$\mathcal{A}(F_0) \subset \bigl\{p_1 \cdot \ldots \cdot p_k \bigm| p_1, \ldots, p_k \in P,\ k \leqslant \mathsf{D}(G_0)\bigr\}$$

and

$$\bigl\{[u]_\varphi \bigm| u \in \mathcal{A}(F_0)\bigr\} \subset \bigl\{g_1 + \cdots + g_k \bigm| g_1, \ldots, g_k \in G_0,\ k \leqslant \mathsf{D}(G_0)\bigr\}. \qquad \square$$

Next we focus on the role of the monoid F in Definition 4.1. First we show that the imposition that $\widehat{H} \subset F$ be cofinal can be dropped without loss of generality (in Definition 4.1 we claim cofinality of $\widehat{H} \subset F$ to simplify technical arguments). Suppose that H is a weakly C-monoid.

Then H is a weakly C-monoid defined in a factorial monoid F such that $F^\times = \widehat{H}^\times$ (see Proposition 4.6.2). If $\mathcal{C}(\widehat{H})$ is finite, then H is a weakly C-monoid defined in a factorial monoid F such that $F \cong \widehat{H}^\times \times \mathcal{I}_v^*(\widehat{H})$ (see Proposition 4.5.2).

Proposition 4.5. *Suppose that H is v-noetherian with $(H : \widehat{H}) \neq \emptyset$, and assume that $F = F^\times \times \mathcal{F}(P)$ is a factorial monoid such that $\widehat{H} \subset F$ is saturated.*

1. *Suppose there exist an equivalence relation $\sim$ on P and $\lambda \in \mathbb{N}$ such that* **(C2)** *is fulfilled. Then there exists $P' \subset P$ such that H is a weakly* C*-monoid defined in $F' = F^\times \times \mathcal{F}(P')$ with equivalence relation $\sim'$ and parameter λ', where $\sim'$ is the restriction of $\sim$ to P' and $\lambda' = \lambda$.*
2. *Suppose that H is a weakly* C*-monoid and that the class group of $\widehat{H} \subset F$ is finite. Then H is a weakly* C*-monoid defined in F. In particular, if $\mathcal{C}(\widehat{H})$ is finite, then H is a weakly* C*-monoid defined in a factorial monoid isomorphic to $\widehat{H}^\times \times \mathcal{I}_v^*(\widehat{H})$.*

Proof. 1. Lemma 3.7.2 (applied with $T = H$) implies that there exists $P' \subset P$ such that $\widehat{H} \subset F' = F^\times \times \mathcal{F}(P')$ is saturated and cofinal. Thus $H \subset F'$ satisfies **(C1)**, and to verify **(C2)** it suffices to show that

$$[x]_H^F = [y]_H^F \quad \text{if and only if} \quad [x]_H^{F'} = [y]_H^{F'}$$

for all $x, y \in F'$. But this follows from Lemma 4.2.1.(b).

2. First we deduce the "In particular…" statement from the main statement. Let $\widehat{H} = \widehat{H}^\times \times H_0 \subset F_0 = \widehat{H}^\times \times \mathcal{F}(P_0)$ such that $H_0 \cong \widehat{H}_{\mathrm{red}}$ and $H_0 \hookrightarrow \mathcal{F}(P_0)$ is a divisor theory (see Lemma 2.2.2). Then $\mathcal{F}(P_0) \cong \mathcal{I}_v^*(\widehat{H})$, and the class group of $\widehat{H} \subset F_0$ is isomorphic to $\mathcal{C}(\widehat{H})$.

Now we prove the main statement. Suppose that H is a weakly C-monoid defined in $F' = F'^\times \times \mathcal{F}(P')$ with equivalence relation $\sim'$ and parameter λ'. By Lemma 4.3.3 we can extend $\sim'$ to a congruence relation on F' (again denoted by $\sim'$) such that $F'/{\sim'} \cong \mathcal{F}(P'/{\sim'})$. We let e denote the exponent of the class group of $\widehat{H} \subset F$, and we put $G_P = \{\pi \mathsf{q}(H)F^\times \mid \pi \in P\} \subset \mathsf{q}(F)/\mathsf{q}(H)F^\times$. We define an equivalence relation $\sim$ on F by setting $x \sim y$ if and only if

- $x\mathsf{q}(H)F^\times = y\mathsf{q}(H)F^\times$,
- $dx\,\varepsilon \sim' dy\,\eta$ for all $d \in F$ and all $\varepsilon, \eta \in F^\times$ such that $dx\,\varepsilon, dy\,\eta \in \widehat{H}$.

Clearly, $\sim$ is an equivalence relation on F, and we denote its restriction to P again by $\sim$. We assert that $\sim$ is even a congruence relation on F. Indeed, if $x, y, z \in F$ and $x \sim y$, then $x\mathsf{q}(H)F^\times = y\mathsf{q}(H)F^\times$ implies $xz\mathsf{q}(H)F^\times = yz\mathsf{q}(H)F^\times$. If $d \in F$ and $\varepsilon, \eta \in F^\times$ such that $dx\,z\varepsilon, dy\,z\eta \in \widehat{H}$, then $(dz)x\varepsilon \sim' (dz)y\eta$, and therefore $xz \sim yz$.

We continue by proving the following three assertions.

A1. For all $x \in F^\times$ we have $x \sim 1$.
A2. For all $x, y \in \widehat{H}$ we have $x \sim y$ if and only if $x \sim' y$.
A3. $P/{\sim}$ is finite.

Proof of A1. Let $x \in F^\times$. Then $x\mathsf{q}(H)F^\times = \mathsf{q}(H)F^\times$. Let $d \in F$ and $\varepsilon, \eta \in F^\times$ such that $dx\,\varepsilon, d\eta \in \widehat{H}$. Then $x^{-1}\varepsilon^{-1}\eta = (dx\,\varepsilon)^{-1}(d\eta) \in \mathsf{q}(H) \cap F^\times = \widehat{H}^\times$, $x^{-1}\varepsilon^{-1}\eta \sim' 1$ and $dx\,\varepsilon \sim' dx\,\varepsilon(x^{-1}\varepsilon^{-1}\eta) = d\eta$. $\square$

Proof of A2. Let $x, y \in \widehat{H}$. Then $x\mathsf{q}(H)F^\times = y\mathsf{q}(H)F^\times = \mathsf{q}(H)F^\times$, and thus $x \sim y$ if and only if $dx\,\varepsilon \sim' dy\,\eta$ for all $d \in F$, $\varepsilon, \eta \in F^\times$ for which $dx\,\varepsilon, dy\,\eta \in \widehat{H}$. Let $d \in F$, $\varepsilon, \eta \in F^\times$ such that $dx\,\varepsilon, dy\,\eta \in \widehat{H}$. Since $x, y \in \widehat{H}$, we get $d\varepsilon, d\eta \in \mathsf{q}(H) \cap F = \widehat{H}$ and $\varepsilon^{-1}\eta \in \widehat{H}^\times$. Since $\sim'$ is a cancellative congruence relation, it follows that $(d\varepsilon)x \sim' dy\,\eta = (d\varepsilon)(\varepsilon^{-1}\eta)y$ if and only if $x \sim' y$. Thus the assertion follows. $\square$

Proof of A3. Since G_P is finite, it suffices to prove that, for every $\pi \in P$, the set $\{[p]_\sim \mid p \in \pi\mathsf{q}(H)F^\times \cap P\}$ is finite. Thus let $\pi \in P$, and fix some $f \in \pi^{-1}\mathsf{q}(H)F^\times \cap F$. If $p \in \pi\mathsf{q}(H)F^\times \cap P$, then there exists $\varepsilon \in F^\times$ such that $pf\varepsilon \in \widehat{H}$, and we obtain $\max \mathsf{L}_{\widehat{H}}(pf\varepsilon) \leqslant \mathsf{v}_P(pf\varepsilon) = 1 + \mathsf{v}_P(f)$. Next we decompose $pf\varepsilon$ in F'. By Proposition 4.4.2

$$G' = \left\{p'\mathsf{q}(H)F'^\times \mid p' \in P'\right\} \subset \mathsf{q}(F')/\mathsf{q}(H)F'^\times$$

is finite, and therefore we obtain

$$\mathsf{v}_{P'}(pf\varepsilon) \leqslant \max \mathsf{L}_{\widehat{H}}(pf\varepsilon)\mathsf{D}(G') \leqslant \big(1 + \mathsf{v}_P(f)\big)\mathsf{D}(G'),$$

whence

$$\left|\left\{[pf\varepsilon]_{\sim'} \mid p \in \pi\mathsf{q}(H)F^\times \cap P\right\}\right| \leqslant |P'/{\sim'}|^{(1+\mathsf{v}_P(f))\mathsf{D}(G')}.$$

Thus, in order to complete the proof of **A3**, it suffices to show that the relation $p_1f\varepsilon \sim' p_2f\eta$ implies $p_1 \sim p_2$ for all $p_1, p_2 \in \pi\mathsf{q}(H)F^\times \cap P$ and $\varepsilon, \eta \in F^\times$ for which $p_1f\varepsilon, p_2f\eta \in \widehat{H}$.

Let $p_1, p_2 \in \pi\mathsf{q}(H)F^\times \cap P$ and $\varepsilon, \eta \in F^\times$ such that $p_1f\varepsilon, p_2f\eta \in \widehat{H}$ and $p_1f\varepsilon \sim' p_2f\eta$. Since $p_1\mathsf{q}(H)F^\times = p_2\mathsf{q}(H)F^\times$, we must prove that $dp_1\,\alpha \sim' dp_2\,\beta$ for all $d \in F$ and all $\alpha, \beta \in F^\times$ for which $dp_1\alpha, dp_2\beta \in \widehat{H}$. If $dp_1\alpha, dp_2\beta \in \widehat{H}$, then

$$(p_1f\varepsilon)(dp_1\,\alpha) \sim' (p_2f\eta)(dp_1\,\alpha) = (p_1f\varepsilon)\big(\varepsilon^{-1}\beta^{-1}\eta\alpha\big)(dp_2\,\beta).$$

Since

$$\varepsilon^{-1}\beta^{-1}\eta\alpha = (dp_1\,\alpha)(p_2f\eta)(p_1f\varepsilon)^{-1}(dp_2\,\beta)^{-1} \in F^\times \cap \mathsf{q}(H) = \widehat{H}^\times,$$

and since $F'/{\sim'}$ is cancellative, it follows that $dp_1\alpha \sim' dp_2\beta$. $\square$

Now we prove that H is a weakly C-monoid defined in F with equivalence relation $\sim$ and parameter $\lambda = \lambda' e\mathsf{D}(G_P)$.

Let $p_1, \ldots, p_\lambda, p'_1, \ldots, p'_\lambda \in P$ such that $p_1 \sim \cdots \sim p_\lambda \sim p'_1 \sim \cdots \sim p'_\lambda$. Then $e \mid \lambda$ implies that there exist $\varepsilon, \eta \in F^\times$ such that $\varepsilon p_1 \cdot \ldots \cdot p_\lambda$, $\eta p'_1 \cdot \ldots \cdot p'_\lambda \in \widehat{H}$. Since $\sim$ is a congruence relation, it follows that $\varepsilon p_1 \cdot \ldots \cdot p_\lambda \sim \eta p'_1 \cdot \ldots \cdot p'_\lambda$. Thus, by **A2**, we get $\varepsilon p_1 \cdot \ldots \cdot p_\lambda \sim' \eta p'_1 \cdot \ldots \cdot p'_\lambda$, and Lemma 4.3.3.(a) implies that $\mathsf{v}_\tau(\varepsilon p_1 \cdot \ldots \cdot p_\lambda) = \mathsf{v}_\tau(\eta p'_1 \cdot \ldots \cdot p'_\lambda)$ for all $\tau \in P'/{\sim'}$. We assert that $\mathsf{v}_\tau(\varepsilon p_1 \cdot \ldots \cdot p_\lambda) \in \mathbb{N}_{\geqslant \lambda'} \cup \{0\}$ for all $\tau \in P'/{\sim'}$. Suppose that $\varepsilon p_1 \cdot \ldots \cdot p_\lambda = u_1 \cdot \ldots \cdot u_t$, with $t \in \mathbb{N}$ and $u_1, \ldots, u_t \in \mathcal{A}(\widehat{H})$. Then $t \geqslant \mathsf{D}(G_P)^{-1}\lambda \geqslant \lambda'$, $u_1 \sim \cdots \sim u_t$, and hence $u_1 \sim' \cdots \sim' u_t$. Therefore $[u_1]_{\sim'} = \cdots = [u_t]_{\sim'} \in F'/{\sim'} \cong \mathcal{F}(P'/{\sim'})$, and for every $\tau \in P'/{\sim'}$ it follows that

$$\mathsf{v}_\tau(\varepsilon p_1 \cdot \ldots \cdot p_\lambda) = \mathsf{v}_\tau\big([u_1]^t_{\sim'}\big) \in \mathbb{N}_{\geqslant t} \cup \{0\}.$$

Hence, by Lemma 4.3.2, there exists $\gamma \in \widehat{H}^\times$ such that $[\varepsilon p_1 \cdot \ldots \cdot p_\lambda]_H^{F'} = [\gamma\eta p_1' \cdot \ldots \cdot p_\lambda']_H^{F'}$, and we assert that $[\varepsilon p_1 \cdot \ldots \cdot p_\lambda]_H^F = [\gamma\eta p_1' \cdot \ldots \cdot p_\lambda']_H^F$. If $f \in F$ and $f\varepsilon p_1 \cdot \ldots \cdot p_\lambda \in H$, then $f = (\varepsilon p_1 \cdot \ldots \cdot p_\lambda)^{-1} f\varepsilon p_1 \cdot \ldots \cdot p_\lambda \in \mathsf{q}(H) \cap F = \widehat{H} \subset F'$, and hence $f\gamma\eta p_1' \cdot \ldots \cdot p_\lambda' \in H$. Conversely, if $f \in F$ and $f\gamma\eta p_1' \cdot \ldots \cdot p_\lambda' \in H$, then $f = (\gamma\eta p_1' \cdot \ldots \cdot p_\lambda')^{-1} f\gamma\eta p_1' \cdot \ldots \cdot p_\lambda' \in \mathsf{q}(H) \cap F = \widehat{H} \subset F'$, and thus $f\varepsilon p_1 \cdot \ldots \cdot p_\lambda \in H$. □

Proposition 4.6.

1. *If H is a weakly* C*-monoid and $T \subset H$ is a divisor-closed submonoid, then T is a weakly* C*-monoid. More precisely, if H is defined in $F = F^\times \times \mathcal{F}(P)$ with equivalence relation $\sim$, then T is a weakly* C*-monoid defined in $F^\times \times \mathcal{F}(Q)$, where $Q = \{p \in P \mid pF \cap T \neq \emptyset\}$, and the equivalence relation on Q is the restriction of $\sim$ to Q.*
2. *H is a weakly* C*-monoid if and only if H_{red} is a weakly* C*-monoid. Moreover, if H is a weakly* C*-monoid, then H is a weakly* C*-monoid defined in a factorial monoid F such that $F^\times = \widehat{H}^\times$.*
3. *Let H_1 and H_2 be submonoids of H such that $H = H_1 \times H_2$. Then H is a weakly* C*-monoid if and only if H_1 and H_2 are weakly* C*-monoids.*
4. *If H is a weakly* C*-monoid and $\widetilde{H}$ is v-noetherian, then $\widetilde{H}$ is a weakly* C*-monoid.*

Proof. 1. Suppose that H is defined in $F = F^\times \times \mathcal{F}(P)$ with equivalence relation $\sim$ and parameter λ. If $T \subset H$ is a divisor-closed submonoid, then T is v-noetherian, $(T : \widehat{T}) \neq \emptyset$, and $\widehat{T} \subset [\![T]\!]_F$ is saturated by Lemmas 3.5 and 3.6. By Lemma 3.7.2 we have $[\![T]\!]_F = F^\times \times \mathcal{F}(Q)$, with $Q = \{p \in P \mid pF \cap T \neq \emptyset\}$. Clearly, $Q/{\sim} \subset P/{\sim}$ is finite, and in order to prove the assertion it is enough to verify that

$$[x]_H^F = [y]_H^F \quad \text{implies} \quad [x]_T^{[\![T]\!]_F} = [y]_T^{[\![T]\!]_F} \quad \text{for all } x, y \in [\![T]\!]_F.$$

Let $x, y \in [\![T]\!]_F$ such that $[x]_H^F = [y]_H^F$, and let $z \in [\![T]\!]_F$ with $xz \in T$. Then Lemma 3.6.2 implies that

$$yz \in H \cap [\![T]\!]_F = H \cap \widehat{H} \cap [\![T]\!]_F = H \cap [\![T]\!]_{\widehat{H}} = T,$$

and thus $[x]_T^{[\![T]\!]_F} = [y]_T^{[\![T]\!]_F}$.

2. H is v-noetherian if and only if $H_{\mathrm{red}} = H/H^\times$ is v-noetherian. Further, $\widehat{H_{\mathrm{red}}} = \widehat{H}/H^\times$ [23, Proposition 2.3.4.1]. We have

$$\big(H/H^\times : \widehat{H}/H^\times\big) = \big\{fH^\times \bigm| f \in (H : \widehat{H})\big\},$$

and hence $(H : \widehat{H}) \neq \emptyset$ if and only if $(H_{\mathrm{red}} : \widehat{H_{\mathrm{red}}}) \neq \emptyset$. If H is v-noetherian and $(H : \widehat{H}) \neq \emptyset$, then $\widehat{H}$ and $\widehat{H_{\mathrm{red}}}$ are Krull monoids with $\mathcal{C}(\widehat{H}) = \mathcal{C}_v(\widehat{H}) = \mathcal{C}_v(\widehat{H}/H^\times) = \mathcal{C}(\widehat{H_{\mathrm{red}}})$.

If $F = F^\times \times \mathcal{F}(P)$ is factorial and $\widehat{H} \subset F$ is saturated and cofinal, then $\widehat{H_{\mathrm{red}}} = \widehat{H}/H^\times \subset F/H^\times \times \mathcal{F}(P)$ is saturated and cofinal. For all $x, y \in F$ we have

$$[x]_H^F = [y]_H^F \quad \text{if and only if} \quad \big[xH^\times\big]_{H/H^\times}^{F/H^\times} = \big[yH^\times\big]_{H/H^\times}^{F/H^\times}.$$

Thus, since $H \subset F$ satisfies **(C2)**, it follows that $H/H^\times \subset F/H^\times$ satisfies **(C2)**.

Conversely, suppose that H_{red} is a weakly C-monoid defined in F, say

$$H/H^\times \subset \widehat{H}/H^\times \subset F = F^\times \times \mathcal{F}(P).$$

Let

$$\pi : \mathsf{q}(H) = \mathsf{q}(\widehat{H}) \to \mathsf{q}(H)/H^\times = \mathsf{q}\big(H/H^\times\big) = \mathsf{q}\big(\widehat{H}/H^\times\big)$$

denote the canonical epimorphism, and let $\theta : F \to \mathcal{F}(P)$ be defined by $\theta(\varepsilon a) = a$ for all $\varepsilon \in F^\times$ and all $a \in \mathcal{F}(P)$. Since $\widehat{H}$ is a Krull monoid, Lemma 2.1 implies that $\widehat{H} = \widehat{H}^\times \times H_0$. It follows that $\mathsf{q}(H) = \widehat{H}^\times \times \mathsf{q}(H_0)$, and hence there is an epimorphism $\lambda : \mathsf{q}(H) \to \widehat{H}^\times$ which is the identity on $\widehat{H}^\times$. Therefore the map

$$\iota : \widehat{H} \to \overline{F} = \widehat{H}^\times \times \mathcal{F}(P) \quad \text{defined by } \iota(a) = \big(\lambda(a), \theta\big(\pi(a)\big)\big)$$

is an injective divisor homomorphism. Clearly, it is sufficient to show that $\iota(H)$ is a weakly C-monoid defined in $\overline{F}$. For, it remains to verify the following assertion.

A. Let $a, b \in \mathcal{F}(P)$ and $\varepsilon \in F^\times$ such that $[a]^F_{H_{\mathrm{red}}} = [\varepsilon b]^F_{H_{\mathrm{red}}}$. Then there exists $\bar{\varepsilon} \in \widehat{H}^\times$ such that $[(1, a)]^{\overline{F}}_{\iota(H)} = [(\bar{\varepsilon}, b)]^{\overline{F}}_{\iota(H)}$.

Proof of A. Lemma 4.2.1.(a) implies that $[a]^F_{\widehat{H}/H^\times} = [\varepsilon b]^F_{\widehat{H}/H^\times}$, and hence $a\mathsf{q}(H)/H^\times = \varepsilon b \mathsf{q}(H)/H^\times$ by Lemma 2.1. Thus there exists $u \in \mathsf{q}(H)$ such that $a = \varepsilon b \pi(u)$, and we assert that $\bar{\varepsilon} = \lambda(u)^{-1} \in \widehat{H}^\times$ has the required property. We pick $(s, c) \in \overline{F}$ and proceed in two steps.

Suppose that $(1, a)(s, c) \in \iota(H)$, say $(s, ac) = \iota(h) = (\lambda(h), \theta(\pi(h)))$ for some $h \in H$. If $\gamma \in F^\times$ such that $\pi(h) = \gamma\theta(\pi(h))$, then $\gamma ca = \pi(h) \in H/H^\times$. This implies that $\gamma c \varepsilon b \in H/H^\times$, say $\gamma c\varepsilon b = \pi(h')$ for some $h' \in H$, and then $\theta(\pi(h')) = bc$. Since

$$\pi(h) = \gamma ac = \gamma \varepsilon b \pi(u) c = \pi(h')\pi(u) = \pi(h'u),$$

there is $\sigma \in H^\times$ with $h = h'u\sigma$, and hence $\lambda(h) = \lambda(h')\lambda(u)\sigma = \lambda(h')\bar{\varepsilon}^{-1}\sigma$. Thus we obtain that

$$\iota\big(h'\sigma^{-1}\big) = \big(\lambda(h'), \theta\big(\pi(h')\big)\big) = \big(\lambda(h)\bar{\varepsilon}, bc\big) = (\bar{\varepsilon}s, bc) = (\bar{\varepsilon}, b)(s, c).$$

Conversely, suppose that $(\bar{\varepsilon}, b)(s, c) \in \iota(H)$, say $(\bar{\varepsilon}s, bc) = \iota(h) = (\lambda(h), \theta(\pi(h)))$ for some $h \in H$. If $\gamma \in F^\times$ such that $\pi(h) = \gamma\theta(\pi(h))$, then $\pi(h) = \gamma bc = \gamma\varepsilon^{-1}c(\varepsilon b) \in H/H^\times$. This implies that $\gamma\varepsilon^{-1}ca \in H/H^\times$, say $\gamma\varepsilon^{-1}ca = \pi(h')$ for some $h' \in H$, and hence $\theta(\pi(h')) = ac$. Since

$$\pi(h') = \gamma\varepsilon^{-1}ca = \gamma\varepsilon^{-1}c\varepsilon b\pi(u) = \pi(h)\pi(u) = \pi(hu),$$

there is $\sigma \in H^\times$ with $h' = hu\sigma$, and hence $\lambda(h') = \lambda(h)\lambda(u)\sigma = \lambda(h)\bar{\varepsilon}^{-1}\sigma$. Thus we obtain that

$$\iota\big(h'\sigma^{-1}\big) = \big(\lambda(h'), \theta\big(\pi(h')\big)\big) = \big(\lambda(h)\bar{\varepsilon}^{-1}, ac\big) = (s, ac) = (1, a)(s, c). \qquad \square$$

3. By 2. we may suppose that H_1 and H_2 are both reduced. If H is a weakly C-monoid, then, for every $i \in [1,2]$, the submonoid $H_i \subset H$ is divisor-closed, and it follows by 1. that H_i is a weakly C-monoid. Conversely, suppose that, for every $i \in [1,2]$, H_i is a weakly C-monoid defined in $F_i = F_i^\times \times \mathcal{F}(P_i)$ with equivalence relation $\sim_i$ and parameter λ_i. Set $F = F_1 \times F_2$ and $\lambda = \max\{\lambda_1, \lambda_2\}$. Then $F = F^\times \times \mathcal{F}(P)$, where P is the disjoint union of P_1 and P_2. We define $\sim$ on P by $p \sim q$ if and only if

$$(p, q \in P_1 \text{ and } p \sim_1 q) \quad \text{or} \quad (p, q \in P_2 \text{ and } p \sim_2 q).$$

We claim that H is a weakly C-monoid defined in F with equivalence relation $\sim$ and parameter λ. Clearly, H is v-noetherian, $(H : \widehat{H}) \neq \emptyset$, and $\widehat{H} = \widehat{H_1} \times \widehat{H_2} \subset F = F_1 \times F_2$ is saturated and cofinal. Furthermore, it is easy to see that for all $p_1, p_1', \ldots, p_\lambda, p_\lambda' \in P$ with $p_1 \sim p_1' \sim \cdots \sim p_\lambda \sim p_\lambda'$ there exists $\varepsilon \in F^\times$ such that $[p_1 \cdot \ldots \cdot p_\lambda]_H^F = [\varepsilon p_1' \cdot \ldots \cdot p_\lambda']_H^F$.

4. Suppose that H is defined in $F = F^\times \times \mathcal{F}(P)$ with equivalence relation $\sim$ and parameter λ. Since $\widehat{H}$ is a Krull monoid, we infer that $\widehat{\widehat{H}} = \widehat{H}$. If $f \in (H : \widehat{H})$, then $f\widehat{H} \subset H \subset \widetilde{H}$. Hence we see that $(\widetilde{H} : \widehat{\widetilde{H}}) \neq \emptyset$. Now Lemma 4.2.3 implies that $\widetilde{H}$ is a weakly C-monoid defined in F with equivalence relation $\sim$ and parameter λ. □

We do not know whether or not it is necessary to impose, in 4., that $\widetilde{H}$ be v-noetherian. It might be true that the root closure of a weakly C-monoid is always v-noetherian, though, we have not been able to work out a proof of this.

Next we consider v-noetherian G-monoids. This class of monoids represents the prototype of auxiliary monoids in factorization theory (cf. [23, Section 2.7]). In the following proposition we consider some algebraic properties of G-monoids (for results concerning their arithmetic see Propositions 6.4 and 6.5).

Let H be a v-noetherian G-monoid. Then s-spec(H) is finite by Theorem 3.1, H has only finitely many divisor-closed submonoids, and for every divisor-closed submonoid $T \subset H$ there exists $a \in T$ such that $T = [\![a]\!]_H$ [23, Lemma 2.2.1].

Proposition 4.7.

1. *The following statements are equivalent*:
   (a) *H is a v-noetherian* G-*monoid with $(H : \widehat{H}) \neq \emptyset$.*
   (b) *H is a weakly* C-*monoid where the equivalence relation $\sim$ can be chosen to be equality.*
2. *Suppose that H is a v-noetherian* G-*monoid with $(H : \widehat{H}) \neq \emptyset$.*
   (a) *Every saturated submonoid $S \subset H$ is a v-noetherian* G-*monoid with $(S : \widehat{S}) \neq \emptyset$.*
   (b) *Assume that $\widehat{H}^\times / H^\times$ is finite and that, for all $a \in H$, the class group of $\widehat{[\![a]\!]_H} \subset [\![a]\!]_{\widehat{H}}$ is finite. Then $\widetilde{H}$ is a* C-*monoid if and only if $\mathcal{C}(\widehat{H})$ is finite.*

Proof. 1. (a) ⇒ (b) By Theorem 3.1 $\widehat{H}_{\mathrm{red}}$ is a finitely generated Krull monoid, and therefore there exists a factorial monoid $F = \widehat{H}^\times \times \mathcal{F}(P)$, where P is finite, such that $\widehat{H} \subset F$ is saturated and cofinal. For two primes $p, p' \in P$ we define $p \sim p'$ if and only if $p = p'$. Then **(C2)** holds for $\lambda = 1$.

(b) ⇒ (a) Let H be a weakly C-monoid defined in $F = F^\times \times \mathcal{F}(P)$ with equivalence relation being equality. Then **(C1)** implies that H is v-noetherian with $(H : \widehat{H}) \neq \emptyset$, and **(C2)** implies that P is finite. Therefore $\widehat{H}_{\mathrm{red}}$ is finitely generated, and thus $\widehat{H}$ is a G-monoid [23, Theorem 2.7.13]. It follows from [23, Theorem 2.7.9] that s-spec$(\widehat{H})$ is finite. By Lemma 3.4.3 we see that s-spec(H) is finite, and hence H is a G-monoid.

2.(a) Let $S \subset H$ be a saturated submonoid. Then S is a v-noetherian G-monoid by [23, Theorem 2.7.9], and Lemma 3.5.2 implies that $(S : \widehat{S}) \neq \emptyset$.

2.(b) By Theorem 3.1 there exists a finite set P such that $\widehat{H} \subset F = \widehat{H}^\times \times \mathcal{F}(P)$ is saturated. Let $a \in H$. Since $\widehat{H}^\times / H^\times$ and $\mathsf{q}([\![a]\!]_{\widehat{H}})/\widehat{H}^\times \mathsf{q}([\![a]\!]_H)$ are both finite it follows that $\mathsf{q}([\![a]\!]_{\widehat{H}})/\mathsf{q}([\![a]\!]_H)$ is finite. Now the assertion follows from Lemmas 3.8 and 3.9.2. □

The monoid H is said to be *seminormal* if $x^2, x^3 \in H$ implies $x \in H$ for all $x \in \mathsf{q}(H)$ (cf. [3, 10,36]). Thus every root-closed monoid is seminormal. If H is a seminormal G-monoid, then $(H : \widehat{H}) \neq \emptyset$ by [24, Proposition 4.8], and hence every seminormal v-noetherian G-monoid is a weakly C-monoid. We do not know if all assumptions on H in Proposition 4.7.2.(b) are necessary. In the special case of a primary v-noetherian monoid H with $(H : \widehat{H}) \neq \emptyset$, the root closure of H is a weakly C-monoid if and only if $\mathcal{C}(\widehat{H})$ is finite (see [26, Theorem 3.5]). The next proposition sheds light on the relationship between weakly C-monoids on the one hand, and C-monoids, Krull monoids and finitely generated monoids on the other hand.

Proposition 4.8.

1. *Every* C*-monoid is a weakly* C*-monoid.*
2. *Suppose that H is a Krull monoid with class group $\mathcal{C}(H)$.*
   (a) *H is a weakly* C*-monoid if and only if the set of classes in $\mathcal{C}(H)$ containing primes is finite.*
   (b) *H is a* C*-monoid if and only if $\mathcal{C}(H)$ is finite.*
3. *Suppose that H_{red} is finitely generated.*
   (a) *H is a weakly* C*-monoid.*
   (b) *H is a* C*-monoid if and only if $\mathcal{C}(\widehat{H})$ is finite.*

Proof. 1. Assume that H is a C-monoid. By [23, Theorem 2.9.11] there exists a factorial monoid $F = F^\times \times \mathcal{F}(P)$ such that H is a C-monoid defined in F and $\widehat{H} \subset F$ is saturated with finite class group. By Theorem 2.3 **(C1)** is satisfied. We define the equivalence relation $\sim$ on P as the restriction of the (H, F)-equivalence. Then $P/{\sim} \subset \mathcal{C}^*(H, F)$ is finite, and **(C2)** holds with parameter $\lambda = 1$.

2.(a) Suppose that $H = \widehat{H} = H^\times \times H_0 \subset H^\times \times \mathcal{F}(P) = F$ such that $\mathcal{C}(H) = \mathsf{q}(\mathcal{F}(P))/\mathsf{q}(H)$ and $\{p\mathsf{q}(H_0) \mid p \in P\}$ is finite (see Lemma 2.2.2). We define an equivalence relation $\sim$ on P by setting $p \sim p'$ if and only if $p\mathsf{q}(H_0) = p'\mathsf{q}(H_0)$. Since $\mathcal{C}(H) = \mathsf{q}(F)/\mathsf{q}(H)$, Lemma 2.1.1 implies that, for all $p, p' \in P$, we have $p \sim p'$ if and only if $[p]_H^F = [p']_H^F$. Thus H is a weakly C-monoid defined in F with equivalence relation $\sim$ and parameter $\lambda = 1$.

Conversely, if H is a weakly C-monoid, then the set of classes of $\mathcal{C}(\widehat{H}) = \mathcal{C}(H)$ containing primes is finite by Proposition 4.4.3.

2.(b) This follows from [23, Theorem 2.9.12] and Theorem 2.3.

3.(a) H is a v-noetherian G-monoid with $(H : \widehat{H}) \neq \emptyset$ by [23, Theorem 2.7.13], and hence it is a weakly C-monoid by Proposition 4.7.1.

3.(b) If H is a C-monoid, then $\mathcal{C}(\widehat{H})$ is finite by Theorem 2.3. Conversely, suppose that $\mathcal{C}(\widehat{H})$ is finite. By [23, Proposition 2.7.11 and Theorem 2.7.13] H is a v-noetherian G-monoid, $\widehat{H} = \widetilde{H}$, $(H : \widehat{H}) \neq \emptyset$, and $\widehat{H}/H^\times$ is finitely generated. Moreover, $P = \mathfrak{X}(\widehat{H})$ is finite, and $\widehat{H} \subset F = \widehat{H}^\times \times \mathcal{F}(P)$ is saturated with class group isomorphic to $\mathcal{C}(\widehat{H})$. We verify the condition in Theorem 2.3.2.(b). Since $\widehat{H}/H^\times$ is finitely generated it follows that $\widetilde{H}^\times/H^\times$ is a finitely generated torsion group. Hence $\widetilde{H}^\times/H^\times$ is finite. Since P is finite there exists, by Lemma 3.9.1, an integer $\alpha \in \mathbb{N}$ such that, for all $p \in P$ and $a \in p^\alpha F$, we have $a \in H$ if and only if $p^\alpha a \in H$. Without loss of generality we may choose α to be a multiple of $(\widetilde{H}^\times : H^\times)$. Now the condition in Theorem 2.3.2.(b) is satisfied with α and $V = H^\times$. $\square$

5. Weakly C-monoids: arithmetic properties

We open this section with the definition of the local tame degrees $\mathsf{t}(H, \cdot)$ and the associated invariants $\omega(H, \cdot)$ and $\tau(H, \cdot)$. For general information on local tameness and its relevance in factorization theory we refer to [23]. Recent results can be found in [2,8,12,13,25,26]. The main result of this paper is Theorem 5.3 which states that a weakly C-monoid H for which the class group $\mathcal{C}(\widehat{H})$ is finite is locally tame if and only if the algebraic condition we call **(C3)** is fulfilled (a simple example where **(C3)** fails is given in Example 6.11). The proof of Theorem 5.3 is based on the following two recent results [25, Theorems 3.6 and 4.2]:

- An atomic monoid H is locally tame if and only if $\omega(H, u) < \infty$ and $\tau(H, u) < \infty$ for all $u \in \mathcal{A}(H)$.
- If H is v-noetherian, then $\omega(H, b) < \infty$ for all $b \in H$.

Therefore it suffices to show that the $\tau(H, \cdot)$ invariants are finite if and only if **(C3)** holds. The proof occupies the whole section, and the crucial steps are Propositions 5.8 and 5.9. The explicit upper bounds given in Theorem 5.3 are essential for proving the finiteness of the catenary degree (see Theorem 6.3).

Definition 5.1. Suppose that H is atomic.

1. For $b \in H$ let $\omega(H, b)$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ having the following property:
   For all $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in H$, if $b \mid a_1 \cdot \ldots \cdot a_n$, then there exists a subset $\Omega \subset [1, n]$ such that $|\Omega| \leqslant N$ and
$$b \,\Big|\, \prod_{\nu \in \Omega} a_\nu.$$
2. For $b \in H$ we define
$$\tau(H, b) = \sup\Big\{\min \mathsf{L}\big(b^{-1}a\big) \,\Big|\, a = u_1 \cdot \ldots \cdot u_k \in bH \text{ with } k \in \mathbb{N},\ u_1, \ldots, u_k \in \mathcal{A}(H), \text{ and } b \nmid u_i^{-1}a \text{ for all } i \in [1, k]\Big\}.$$
3. For $a \in H$ and $x \in \mathsf{Z}(H)$ let $\mathsf{t}(a, x) \in \mathbb{N}_0 \cup \{\infty\}$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:
   If $\mathsf{Z}(a) \cap x\mathsf{Z}(H) \neq \emptyset$ and $z \in \mathsf{Z}(a)$, then there exists $z' \in \mathsf{Z}(a) \cap x\mathsf{Z}(H)$ such that $\mathsf{d}(z, z') \leqslant N$.
   For subsets $H' \subset H$ and $X \subset \mathsf{Z}(H)$ we define

$$\mathsf{t}(H', X) = \sup\{\mathsf{t}(a, x) \mid a \in H',\ x \in X\} \in \mathbb{N}_0 \cup \{\infty\}.$$

H is called *locally tame* if $\mathsf{t}(H, u) < \infty$ for all $u \in \mathcal{A}(H_{\mathrm{red}})$.

Definition 5.2. Suppose that H is a weakly C-monoid defined in $F = F^\times \times \mathcal{F}(P)$ with equivalence relation $\sim$ and parameter λ.

1. For $x \in F$ we set

$$\mathrm{supp}_\sim(x) = \{[p]_\sim \mid p \in P \text{ with } \mathsf{v}_p(x) > 0\} \subset P/\sim,$$

and for a subset $S \subset F$ we set

$$\mathrm{supp}_\sim(S) = \{\mathrm{supp}_\sim(x) \mid x \in S\}.$$

2. An element $p \in P$ is called *H-essential* if there exists $a \in H$ with $\mathrm{supp}_\sim(p) = \mathrm{supp}_\sim(a)$. We denote by $\mathcal{E}(H)$ the set of all H-essential primes.
3. A submonoid $T \subset H$ is called *support-closed* if $\mathrm{supp}_\sim(b) \subset \mathrm{supp}_\sim(a)$ implies $b \in T$ for all $a \in T$ and $b \in H$ (obviously, a support-closed submonoid is divisor-closed).

In Theorem 5.3 we need the generalized Davenport constant $\mathsf{D}_E(H)$ introduced prior to Lemma 2.1. Note that, by Proposition 4.5.2, if H is a weakly C-monoid such that $\mathcal{C}(\widehat{H})$ is finite, then H is a weakly C-monoid defined in a factorial monoid F such that the class group of $\widehat{H} \subset F$ is finite.

Theorem 5.3 *(Main Theorem). Suppose that H is a weakly* C*-monoid defined in $F = F^\times \times \mathcal{F}(P)$ with equivalence relation $\sim$ and parameter λ. Assume that the class group of $\widehat{H} \subset F$ is finite. Then H is locally tame if and only if the following condition is fulfilled*:

(C3) *For every support-closed submonoid $T \subset H$ with $|\mathrm{supp}_\sim(\widehat{T} \setminus \widehat{T}^\times)| = 1$ the class group of $\widehat{T} \subset [\![T]\!]_{\widehat{H}}$ is finite.*

More precisely, if **(C3)** *is fulfilled and $\mathcal{E} = \mathcal{E}(H) \subset P$ is the set of H-essential primes, then there exist $K_1 \in \mathbb{N}$ (K_1 is the same constant as in Proposition* 5.8) *such that, for all $u \in \mathcal{A}(H)$ and all $k \in \mathbb{N}$,*

$$\tau(H, u) \leqslant \omega(H, u)\mathsf{D}_{\mathcal{E}}(H) + K_1,$$

and, for every $f \in (H : \widehat{H})$,

$$\mathsf{t}(H, uH^\times) \leqslant (\mathsf{v}_P(u) + \omega(H, f)) \max\{\mathsf{D}_{\mathcal{E}}(H), 1\} + K_1 + 1.$$

In the following lemma we show, among other things, that for weakly C-monoids our definition of an H-essential prime is consistent with [23, Definition 2.9.5].

Lemma 5.4. *Suppose that H is a submonoid of a factorial monoid $F = F^\times \times \mathcal{F}(P)$, and put $\mathcal{E} = \{p \in P \mid$ there exists $n \in \mathbb{N}$ such that $p^n \in HF^\times\}$.*

1. *Suppose that $H \cap F^\times = H^\times$, $(H : \widehat{H}) \neq \emptyset$ and $\widehat{H} \subset F$ is saturated.*
   (a) *If $f \in (H : \widehat{H})$, $p \in \mathcal{E}$ and $c \in H$ with $\mathrm{supp}(c) = \{p\}$, then $\mathsf{v}_p(u) \leqslant \mathsf{v}_p(c)(\omega(H, f) + 1)$ for every $u \in \mathcal{A}(H)$.*
   (b) *If $\mathcal{E}$ is finite, then $\mathsf{D}_{\mathcal{E}}(H) < \infty$.*
2. *Suppose that H is a weakly C-monoid defined in F with equivalence relation $\sim$ and parameter λ.*
   (a) *$\mathcal{E}$ is the set of H-essential primes.*
   (b) *$\mathsf{D}_{\mathcal{E}}(H) < \infty$.*

Proof. 1.(a) Let $u \in \mathcal{A}(H)$ and assume to the contrary that $\mathsf{v}_p(u) > \mathsf{v}_p(c)(\omega(H, f) + 1)$. Then $c^{\omega(H,f)+1} \mid u$ (in F and hence in $\widehat{H}$), $b = c^{-\omega(H,f)-1}u \in \widehat{H}$ and $c^{\omega(H,f)+1}b \in H$. Clearly, $\mathsf{v}_p(b) > 0$ implies that $b \notin H^\times$. By [25, Lemma 3.4.2.(a)] there exists $k \in [0, \omega(H, f)]$ such that $c^k b \in H$. Thus $u = (c^k b)c^{\omega(H,f)+1-k}$ is a product of two non-units of H, a contradiction.

1.(b) For $p \in \mathcal{E}$ let $c_p \in H$ with $\mathrm{supp}(c_p) = \{p\}$, and let $M = \max\{\mathsf{v}_p(c_p) \mid p \in \mathcal{E}\}$. If $f \in (H : \widehat{H})$ and $u \in \mathcal{A}(H)$, then 1. implies that

$$\mathsf{v}_{\mathcal{E}}(u) = \sum_{p \in \mathcal{E}} \mathsf{v}_p(u) \leqslant |\mathcal{E}| M\big(\omega(H, f) + 1\big).$$

2.(a) If $p \in \mathcal{E}$, then there are $n \in \mathbb{N}$ and $\varepsilon \in F^\times$ such that $p^n\varepsilon \in H$. Since $\mathrm{supp}_\sim(p) = \{[p]_\sim\} = \mathrm{supp}_\sim(p^n\varepsilon)$, the prime p is H-essential. Conversely, suppose that $p \in P$ is H-essential. Then there exists $a = \varepsilon p_1 \cdot \ldots \cdot p_n \in H$, where $\varepsilon \in F^\times$, $n \in \mathbb{N}$ and $p_1, \ldots, p_n \in P$, such that $\mathrm{supp}_\sim(p) = \mathrm{supp}_\sim(a) = \mathrm{supp}_\sim(p_i)$ for all $i \in [1, n]$. It follows that $p \sim p_1 \sim \cdots \sim p_n$, and by Lemma 4.3.2 there is an $\eta \in F^\times$ such that $[a^\lambda]_H^F = [\eta p^{\lambda n}]_H^F$. Since $a^\lambda \in H$ it follows that $\eta p^{\lambda n} \in H$, and hence $p \in \mathcal{E}$.

2.(b) Let $f \in (H : \widehat{H})$ and put $m = \omega(H, f) + 1$. For every $\sigma \in \mathcal{E}/\sim$ we fix an element $a_\sigma \in H$ such that $\mathrm{supp}_\sim(a_\sigma) = \{\sigma\}$ and $\mathsf{v}_\sigma(a_\sigma) \in \lambda\mathbb{N}$. Let $u \in \mathcal{A}(H)$. We assert that $\mathsf{v}_\sigma(u) \leqslant m\mathsf{v}_\sigma(a_\sigma)$ for all $\sigma \in \mathcal{E}/\sim$. If this is proved, then we obtain

$$\mathsf{v}_{\mathcal{E}}(u) = \sum_{\sigma \in \mathcal{E}/\sim} \mathsf{v}_\sigma(u) \leqslant m \sum_{\sigma \in \mathcal{E}/\sim} \mathsf{v}_\sigma(a_\sigma).$$

Assume to the contrary that there exists $\sigma \in \mathcal{E}/\sim$ such that $\mathsf{v}_\sigma(u) > m\mathsf{v}_\sigma(a_\sigma)$. Then $\mathsf{v}_\tau(a_\sigma^m) \leqslant \mathsf{v}_\tau(u)$ and $\mathsf{v}_\tau(a_\sigma) \in \lambda\mathbb{N}_0$ for all $\tau \in P/\sim$. By Lemma 4.3.4 there exist $b_1, \ldots, b_m \in F$ such that

(a) $\mathsf{v}_\tau(b_1) = \cdots = \mathsf{v}_\tau(b_m) = \mathsf{v}_\tau(a_\sigma)$ for all $\tau \in P/\sim$,
(b) $[b_1]_H^F = \cdots = [b_m]_H^F = [a_\sigma]_H^F$, and
(c) $b_1 \cdot \ldots \cdot b_m \mid u$ (in F and hence in $\widehat{H}$).

Therefore $u = b_1 \cdot \ldots \cdot b_m u'$, where $b_1, \ldots, b_m \in [a_\sigma]_H^F \subset H$ and $u' = (b_1 \cdot \ldots \cdot b_m)^{-1}u \in \widehat{H}$. Moreover, $\mathsf{v}_\sigma(b_1) > 0, \ldots, \mathsf{v}_\sigma(b_m) > 0, \mathsf{v}_\sigma(u') > 0$ implies that $\{b_1, \ldots, b_m, u'\} \cap H^\times = \emptyset$. By [25, Lemma 3.4.2.(a)] there exists a proper subset $\Omega \subsetneq [1, m]$ such that

$$\Big(\prod_{j \in \Omega} b_j\Big)u' \in H.$$

Thus

$$u = \Bigl(\prod_{j\in[1,m]\setminus\Omega} b_j\Bigr)\Bigl(u'\prod_{j\in\Omega} b_j\Bigr)$$

is a product of two non-units of H, a contradiction. □

Definition 5.5. Suppose that H is a weakly C-monoid defined in $F = F^\times \times \mathcal{F}(P)$ with equivalence relation $\sim$ and parameter λ.

1. For a support-closed submonoid $S \subset H$ we denote by $\psi_\sim(S)$ the number of non-trivial support-closed submonoids. (Observe that $\psi_\sim(H^\times) = 0$ and that $\psi_\sim(H) < \infty$ because $P/\sim$ is finite.)
2. For a subset $U \subset H$ we denote by $[\![U]\!]_\sim$ the set of all $a \in H$ with $\operatorname{supp}_\sim(a) \subset \operatorname{supp}_\sim(c)$ for some $c \in [U]$. Then $[\![U]\!]_\sim$ is the smallest support-closed submonoid of H containing U. For $a \in H$ we set $[\![a]\!]_\sim = [\![\{a\}]\!]_\sim$.

The following technical lemma is invoked in the proof of Proposition 5.7.

Lemma 5.6. *Let G be an abelian group, $G_0 \subset G$ a finite subset, and $A, C \in \mathcal{B}(G_0)$ with $\emptyset \neq \operatorname{supp}(A) \subsetneq \operatorname{supp}(C) = G_0$. Then there exists $B \in \mathcal{B}(G_0)$ with $\operatorname{supp}(A) \setminus \operatorname{supp}(B) \neq \emptyset$, $\operatorname{supp}(B) \setminus \operatorname{supp}(A) \neq \emptyset$, and $\operatorname{supp}(AB) = G_0$.*

Proof. We set $G_0 = \{g_1, \ldots, g_t\}$, $C = g_1^{m_1} \cdot \ldots \cdot g_t^{m_t}$ and $A = g_1^{l_1} \cdot \ldots \cdot g_s^{l_s}$, where $t, l_1, \ldots, l_s, m_1, \ldots, m_t \in \mathbb{N}$ and $s \in [1, t-1]$. Put $l = \operatorname{lcm}(l_1, \ldots, l_s)$, and let $k_i \in \mathbb{N}$ such that $l_i k_i = l m_i$ for all $i \in [1, s]$. After a suitable renumbering of the indices, we may suppose that $k_1 = \min\{k_1, \ldots, k_s\}$. We assert that

$$B = A^{-k_1} C^l$$

has the required properties. For every $i \in [1, s]$ we have

$$\mathsf{v}_{g_i}(B) = -k_1 l_i + l m_i \geqslant -k_i l_i + l m_i = 0,$$

whence $B \in \mathcal{B}(G_0)$. Clearly, $\mathsf{v}_{g_1}(B) = 0$, and thus $\{g_{s+1}, \ldots, g_t\} \subset \operatorname{supp}(B) \subset \{g_2, \ldots, g_t\}$. It follows that $g_1 \in \operatorname{supp}(A) \setminus \operatorname{supp}(B)$, $g_t \in \operatorname{supp}(B) \setminus \operatorname{supp}(A)$, and $\operatorname{supp}(AB) = \{g_1, \ldots, g_t\} = G_0$. □

The following proposition will serve as the induction basis in the proof of Proposition 5.8. Proposition 5.8 is the crucial ingredient for proving that **(C3)** implies local tameness.

Proposition 5.7. *Suppose that H is a weakly* C*-monoid defined in $F = F^\times \times \mathcal{F}(P)$ with equivalence relation $\sim$ and parameter λ, and assume that H contains no proper support-closed submonoid. Put $G_P = \{p\mathsf{q}(H)F^\times \mid p \in P\} \subset \mathsf{q}(F)/\mathsf{q}(H)F^\times$, and let $\mathsf{D}(G_P)$ be the Davenport constant of this set.*

1. *If* $|\mathrm{supp}_\sim(\widehat{H} \setminus \widehat{H}^\times)| = 1$, *then*

$$\max \mathsf{L}(a) \leqslant \mathsf{v}_\tau(a) \leqslant \mathsf{v}_P\big(f^2\big)\mathsf{D}(G_P) \min \mathsf{L}(a)$$

for all $a \in H$, $\tau \in P/\sim$, *and all* $f \in (H : \widehat{H})$ *satisfying* $\mathsf{v}_\sigma(f) \in \lambda\mathbb{N}$ *for each* $\sigma \in P/\sim$.

2. *If* $|\mathrm{supp}_\sim(\widehat{H} \setminus \widehat{H}^\times)| > 1$, *then*

$$\sup\big\{\min \mathsf{L}(a) \bigm| a \in H\big\} < \infty.$$

Proof. Since $\widehat{H} \subset F$ is cofinal, for every $p \in P$ there exists $a \in \widehat{H}$ with $\mathsf{v}_p(a) > 0$. If $a, b \in \widehat{H}$, then $\mathrm{supp}_\sim(ab) = \mathrm{supp}_\sim(a) \cup \mathrm{supp}_\sim(b)$. Thus there exists $a \in \widehat{H}$ with $\mathrm{supp}_\sim(a) = P/\sim$. If $f \in (H : \widehat{H})$, then $fa \in H$ and $\mathrm{supp}_\sim(fa) = P/\sim$. Since H contains no proper support-closed submonoid, it follows that $\mathrm{supp}_\sim(b) = P/\sim$ for all $b \in H \setminus H^\times$. This implies that $\max \mathsf{L}(b) \leqslant \mathsf{v}_\tau(b)$ for each $\tau \in P/\sim$ and all $b \in H \setminus H^\times$.

1. By assumption we have $\mathrm{supp}_\sim(v) = P/\sim$ for all $v \in \widehat{H} \setminus \widehat{H}^\times$. Let $u \in \mathcal{A}(H)$, and let $f \in (H : \widehat{H})$ with $\mathsf{v}_\tau(f) \in \lambda\mathbb{N}$ for all $\tau \in P/\sim$. Suppose that $u = v_1 \cdot \ldots \cdot v_t$, with $t \in \mathbb{N}$ and $v_1, \ldots, v_t \in \mathcal{A}(\widehat{H})$. Then it follows that

$$t \leqslant \sum_{i=1}^{t} \mathsf{v}_\tau(v_i) = \mathsf{v}_\tau(u) \quad \text{for all } \tau \in P/\sim. \tag{2}$$

We continue with the following assertion.

A1. $t \leqslant \min\{\mathsf{v}_\tau(u) \mid \tau \in P/\sim\} < 2\max\{\mathsf{v}_\tau(f) \mid \tau \in P/\sim\} \leqslant \mathsf{v}_P(f^2)$.

Proof of A1. The first inequality follows from (2), and the last inequality is obvious. To prove the intermediate one, assume to the contrary that $\min\{\mathsf{v}_\tau(u) \mid \tau \in P/\sim\} \geqslant 2\max\{\mathsf{v}_\tau(f) \mid \tau \in P/\sim\}$. This implies that

$$\mathsf{v}_\tau(u) \geqslant \mathsf{v}_\tau\big(f^2\big) \quad \text{for all } \tau \in P/\sim.$$

By Lemma 4.3.4 there exist $f', f'' \in F$ with $f'f'' \mid_F u$, $[f']_H^F = [f]_H^F = [f'']_H^F$ and $\mathsf{v}_\tau(f') = \mathsf{v}_\tau(f) = \mathsf{v}_\tau(f'')$ for all $\tau \in P/\sim$. It follows that $f', f'' \in (H : \widehat{H})$ (apply Lemma 4.2.2 with $S = H$), $b = (f'f'')^{-1}u \in \widehat{H}$ and $f'b \in H$, and therefore $u = f''(f'b)$ is a non-trivial decomposition of u. This is a contradiction to $u \in \mathcal{A}(H)$. $\square$

By **A1** and Lemma 2.1.3 we obtain

$$\begin{aligned}\max\big\{\mathsf{v}_\tau(u) \bigm| \tau \in P/\sim\big\} &\leqslant t \max\big\{\mathsf{v}_\tau(v) \bigm| \tau \in P/\sim,\ v \in \mathcal{A}(\widehat{H})\big\} \\ &< \mathsf{v}_P\big(f^2\big)\mathsf{D}(G_P).\end{aligned}$$

Now let $a \in H$, $a = u_1 \cdot \ldots \cdot u_l$ with $u_1, \ldots, u_l \in \mathcal{A}(H)$, and $\tau \in P/\sim$. Then

$$l \leqslant \mathsf{v}_\tau(a) = \sum_{i=1}^{l} \mathsf{v}_\tau(u_i) \leqslant l\mathsf{v}_P\big(f^2\big)\mathsf{D}(G_P),$$

and hence $\max \mathsf{L}(a) \leqslant \mathsf{v}_\tau(a) \leqslant \mathsf{v}_P(f^2)\mathsf{D}(G_P) \min \mathsf{L}(a)$.

2. We start with two assertions.

A2. There exist elements $a, b \in \widehat{H}$ such that $\operatorname{supp}_\sim(a) \cup \operatorname{supp}_\sim(b) = P/\sim$, $\operatorname{supp}_\sim(a) \setminus \operatorname{supp}_\sim(b) \neq \emptyset$, and $\operatorname{supp}_\sim(b) \setminus \operatorname{supp}_\sim(a) \neq \emptyset$.

A3. Let $a \in \widehat{H}$ with $\operatorname{supp}_\sim(a) = P/\sim$. Then there exists $n \in \mathbb{N}$ such that $a^n \in (H : \widehat{H})$.

Proof of A2. We distinguish two cases.

Case 1. The congruence relation $\sim$ and the $(\widehat{H}F^\times, F)$-equivalence do not coincide on P.

Then Lemma 4.3.3.(b) implies that there exist $p_1, p_2 \in P$ with $[p_1]_\sim \neq [p_2]_\sim$ and $[p_1]^F_{\widehat{H}F^\times} = [p_2]^F_{\widehat{H}F^\times}$. Put $\tau_i = [p_i]_\sim$ for $i \in [1, 2]$, and let $c \in \widehat{H}$ with $\operatorname{supp}_\sim(c) = P/\sim$. Let a' denote the element arising from c after replacing (in the factorization of c in F) all primes $q \in \tau_2$ by p_1. Since $[q]^F_{\widehat{H}F^\times} = [p_2]^F_{\widehat{H}F^\times} = [p_1]^F_{\widehat{H}F^\times}$, we have $a' \in \widehat{H}F^\times$, and by construction it follows that $\mathsf{v}_{\tau_2}(a') = 0$ and $\mathsf{v}_\tau(a') \geqslant \mathsf{v}_\tau(c)$ for all $\tau \in P/\sim \setminus \{\tau_2\}$. Similarly, let b' denote the element arising from c after replacing (in the factorization of c in F) all primes $q \in \tau_1$ by p_2. By construction, there exist units $\varepsilon, \eta \in F^\times$ such that $a = \varepsilon a' \in \widehat{H}$ and $b = \eta b' \in \widehat{H}$. Then a and b have the required properties.

Case 2. The congruence relation $\sim$ and the $(\widehat{H}F^\times, F)$-equivalence coincide on P.

Let $\boldsymbol{\beta} : \widehat{H} \to \mathcal{B}(G_P)$ denote the block homomorphism of $\widehat{H}$ defined in Lemma 2.1.3. Then $\operatorname{supp}_\sim(h) = \operatorname{supp}(\boldsymbol{\beta}(h))$ for every $h \in \widehat{H}$, and thus the assertion follows from Lemma 5.6. □

Proof of A3. Let $f \in (H : \widehat{H})$ with $\mathsf{v}_\tau(f) \in \lambda\mathbb{N}_0$ for all $\tau \in P/\sim$. Since $\operatorname{supp}_\sim(a) = P/\sim$, there exists $n \in \mathbb{N}$ such that

$$\mathsf{v}_\tau\big(f^2\big) \leqslant \mathsf{v}_\tau\big(a^n\big) \quad \text{for all } \tau \in P/\sim.$$

By Lemma 4.3.4 there exist $f', f'' \in F$ with $[f']^F_H = [f]^F_H = [f'']^F_H$, $\mathsf{v}_\tau(f') = \mathsf{v}_\tau(f) = \mathsf{v}_\tau(f'')$ for all $\tau \in P/\sim$, and $f'f'' \mid a^n$. Since $f \in (H : \widehat{H})$ it follows that $f', f'' \in (H : \widehat{H}) \subset H$, and we obtain $(f'f'')^{-1}a^n \in \widehat{H}$,

$$f''\big((f'f'')^{-1}a^n\big) \in H, \quad \text{and} \quad a^n = f'f''\big((f'f'')^{-1}a^n\big) \in f'H \subset (H : \widehat{H}). \qquad \square$$

Let $a, b \in \widehat{H}$ be as in **A2**, and suppose that $\sigma \in \operatorname{supp}_\sim(a) \setminus \operatorname{supp}_\sim(b)$ and $\varrho \in \operatorname{supp}_\sim(b) \setminus \operatorname{supp}_\sim(a)$. Replacing a and b by a suitable power if necessary, **A3** shows that we may assume that $f = ab \in (H : \widehat{H})$ and that $\mathsf{v}_\tau(a), \mathsf{v}_\tau(b) \in \mathbb{N}_{\geqslant\lambda} \cup \{0\}$ for all $\tau \in P/\sim$.

Let $c \in H$, and let $l \in \mathbb{N}_0$ be maximal such that

$$\mathsf{v}_\tau\big(f^l\big) \leqslant \mathsf{v}_\tau(c) \quad \text{for all } \tau \in P/\sim.$$

Then there exists $\tau \in P/\sim$ such that $\mathsf{v}_\tau(f^{l+1}) > \mathsf{v}_\tau(c)$. If $l \leqslant 3$, then

$$\min \mathsf{L}(c) \leqslant \max \mathsf{L}(c) \leqslant \mathsf{v}_\tau(c) < \mathsf{v}_\tau\big(f^4\big) \leqslant \mathsf{v}_P\big(f^4\big).$$

Suppose that $l \geqslant 4$. By Lemma 4.3.4 there exist $a_1, \ldots, a_l, b_1, \ldots, b_l \in F$ such that $a_1 \cdot \ldots \cdot a_l b_1 \cdot \ldots \cdot b_l \,|_F\, c$, $[a_i]_H^F = [a]_H^F$, $[b_i]_H^F = [b]_H^F$, $\mathsf{v}_\tau(a_i) = \mathsf{v}_\tau(a)$ and $\mathsf{v}_\tau(b_i) = \mathsf{v}_\tau(b)$ for all $i \in [1, l]$ and all $\tau \in P/{\sim}$. Lemma 4.2.1.(a) implies that $a_i \in [a_i]_{\widehat{H}}^F = [a]_{\widehat{H}}^F \subset \widehat{H}$, $b_i \in [b_i]_{\widehat{H}}^F = [b]_{\widehat{H}}^F \subset \widehat{H}$ and that $f_i = a_i b_i \in (H : \widehat{H})$ for all $i \in [1, l]$. Then $c' = (f_1 \cdot \ldots \cdot f_l)^{-1} c \in \widehat{H}$, $f_l c' \in H$, and we consider the product decomposition

$$c = f_1 \cdot \ldots \cdot f_{l-1}(f_l c').$$

If $\tau \in P/{\sim}$ with $\mathsf{v}_\tau(f^{l+1}) > \mathsf{v}_\tau(c)$, then

$$\mathsf{v}_P\big(f^2\big) \geqslant \mathsf{v}_\tau\big(f^2\big) > \mathsf{v}_\tau(f_l c') \geqslant \max \mathsf{L}(f_l c').$$

We have $f_1 \cdot \ldots \cdot f_{l-1} = g' g''$, where

$$g' = f_{l-2} a_1 \cdot \ldots \cdot a_{l-4} b_{l-3} \in H \quad \text{and} \quad g'' = f_{l-1} b_1 \cdot \ldots \cdot b_{l-4} a_{l-3} \in H.$$

By construction we have

$$\max \mathsf{L}(g') \leqslant \mathsf{v}_\varrho(g') = \mathsf{v}_\varrho(f_{l-2} b_{l-3}) \leqslant \mathsf{v}_\varrho\big(f^2\big),$$
$$\max \mathsf{L}(g'') \leqslant \mathsf{v}_\sigma(g'') = \mathsf{v}_\sigma(f_{l-1} a_{l-3}) \leqslant \mathsf{v}_\sigma\big(f^2\big),$$

and hence

$$\min \mathsf{L}(f_1 \cdot \ldots \cdot f_{l-1}) \leqslant \mathsf{v}_P\big(f^2\big).$$

Putting all together we obtain

$$\min \mathsf{L}(c) \leqslant \min \mathsf{L}(f_1 \cdot \ldots \cdot f_{l-1}) + \max \mathsf{L}(f_l c') \leqslant \mathsf{v}_P\big(f^2\big) + \mathsf{v}_P\big(f^2\big) = \mathsf{v}_P\big(f^4\big). \qquad \square$$

Proposition 5.8. *Suppose that H is a weakly* C*-monoid defined in $F = F^\times \times \mathcal{F}(P)$ with equivalence relation $\sim$ and parameter λ. Assume that the class group of $\widehat{H} \subset F$ is finite and that condition* **(C3)** *in Theorem* 5.3 *is fulfilled. Let $\mathcal{E} \subset P$ denote the set of H-essential primes. Then there exists $K_1 \in \mathbb{N}$ such that*

$$\min \mathsf{L}(a) \leqslant \mathsf{v}_{\mathcal{E}}(a) + K_1$$

for all $a \in H$.

Proof. Suppose that $S \subset H$ is a support-closed submonoid. We put $F_S = [\![S]\!]_F$, and we denote by $\sim_S$ the restriction of $\sim$ to $P_S = F_S \cap P$. By Proposition 4.6.1 S is a weakly C-monoid defined in F_S with equivalence relation $\sim_S$ and parameter λ. Note that we have $\mathrm{supp}_\sim(E) = \mathrm{supp}_{\sim_S}(E)$ for every subset $E \subset F_S$.

Suppose now that $S \subset H$ is support-closed, $\psi_\sim(S) = 1$, and $|\,\mathrm{supp}_\sim(\widehat{S} \setminus \widehat{S}^\times)| > 1$. Then

$$K_S = \sup\big\{\min \mathsf{L}_S(a) \,\big|\, a \in S\big\} = \sup\big\{\min \mathsf{L}_H(a) \,\big|\, a \in S\big\}$$

is finite by Proposition 5.7. Define

$$K_0 = \max\{K_S \mid S \subset H \text{ is support-closed, } \psi_\sim(S) = 1, \text{ and } |\operatorname{supp}_\sim(\widehat{S} \setminus \widehat{S}^\times)| > 1\} \in \mathbb{N}_0$$

(recall that we defined $\max \emptyset = 0$). For every non-trivial support-closed submonoid $S \subset H$ we fix an element $f_S \in (S : \widehat{S})$ such that the following conditions are satisfied:

- f_S is not a unit of S.
- $\mathsf{v}_\tau(f_S) \in \lambda\mathbb{N}_0$ for all $\tau \in P/\sim$.
- If $S' \subset H$ is support-closed and $S' \subset S$, then $\mathsf{v}_\tau(f_{S'}) \leqslant \mathsf{v}_\tau(f_S)$ for all $\tau \in P/\sim$.

Let $\psi \in \mathbb{N}$. We show by induction on ψ that

$$\min \mathsf{L}(a) \leqslant \mathsf{v}_{\mathcal{E}}(a) + \psi!\left(K_0 + \mathsf{v}_P\left(f_H^2\right)\sum_{j=1}^{\psi}\frac{1}{j!}\right) \tag{3}$$

for all $a \in H$ with $\psi_\sim([\![a]\!]_\sim) \leqslant \psi$. Let $a \in H \setminus H^\times$ with $\psi_\sim([\![a]\!]_\sim) \leqslant \psi$. We set $T = [\![a]\!]_\sim$ throughout the rest of the proof. Suppose that $\psi = 1$. Then $\psi_\sim(T) = 1$. If $|\operatorname{supp}_\sim(\widehat{T} \setminus \widehat{T}^\times)| > 1$, then

$$\min \mathsf{L}_H(a) = \min \mathsf{L}_T(a) \leqslant K_0.$$

Suppose that $|\operatorname{supp}_\sim(\widehat{T} \setminus \widehat{T}^\times)| = 1$. Since H satisfies **(C3)**, the class group of $\widehat{T} \subset [\![T]\!]_{\widehat{H}}$ is finite. Further, since the class group of $[\![T]\!]_{\widehat{H}} \subset F_T$ is a subgroup of the class group of $\widehat{H} \subset F$ (Lemma 3.2), it follows that the class group of $\widehat{T} \subset F_T$ is finite. Therefore $|\operatorname{supp}_\sim(F_T \setminus (F_T)^\times)| = |\operatorname{supp}_\sim(\widehat{T} \setminus \widehat{T}^\times)| = 1$. Since F_T is factorial and $|\operatorname{supp}_\sim(F_T \setminus (F_T)^\times)| = 1$, it follows that $\operatorname{supp}_\sim(F_T \setminus {F_T}^\times) = \{\{\tau\}\}$, where $\tau \in P_T/\sim_T$. Hence $\tau = P_T \cap \mathcal{E}$. Proposition 5.7.1 yields

$$\min \mathsf{L}(a) \leqslant \max \mathsf{L}(a) \leqslant \mathsf{v}_\tau(a) = \mathsf{v}_{\mathcal{E}}(a).$$

Thus we have shown that $\min \mathsf{L}(a) \leqslant \mathsf{v}_{\mathcal{E}}(a) + K_0$ for all $a \in H$ with $\psi_\sim([\![a]\!]_\sim) \leqslant 1$.

Suppose now that $\psi > 1$, and assume that (3) is true for all $b \in H$ with $\psi_\sim([\![b]\!]_\sim) < \psi$. We suppose that $\psi_\sim(T) = \psi$, and distinguish two cases.

Case 1. Every $c \in H \setminus H^\times$ with $c \mid_H a$ satisfies $[\![c]\!]_\sim = T$.

We assert that

$$\min \mathsf{L}(a) < \mathsf{v}_P\left(f_T^2\right).$$

Write $a = u_1 \cdot \ldots \cdot u_k$, with $k = \min \mathsf{L}(a)$ and $u_1, \ldots, u_k \in \mathcal{A}(H)$. Then, since $[\![a]\!]_\sim = [\![u_i]\!]_\sim$ for all $i \in [1, k]$, we have $\operatorname{supp}_\sim(u_1) = \cdots = \operatorname{supp}_\sim(u_k) = \operatorname{supp}_\sim(a)$ (cf. Lemma 3.7.2.(b)). Thus $\mathsf{v}_\tau(a) \geqslant k$ for all $\tau \in \operatorname{supp}_\sim(a)$ and $\mathsf{v}_\tau(a) = 0$ for all $\tau \in (P/\sim) \setminus \operatorname{supp}_\sim(a)$. Since $\psi_\sim(T) > 1$ there exists a non-trivial support-closed submonoid S properly contained in T, and we have $\mathsf{v}_\tau(f_T) \geqslant \mathsf{v}_\tau(f_S)$ for all $\tau \in P/\sim$. Assume, by way of contradiction, that $k \geqslant \mathsf{v}_P(f_T^2)$. This implies that

$$\mathsf{v}_\tau(a) \geqslant \mathsf{v}_P\left(f_T^2\right) \geqslant \mathsf{v}_\tau(f_T f_S) \quad \text{for all } \tau \in P/\sim.$$

By Lemma 4.3.4 there exist $f'_T, f'_S \in F$ such that $f'_T f'_S \mid_F a$, $[f'_T]^F_H = [f_T]^F_H$, $[f'_S]^F_H = [f_S]^F_H$, and $\mathsf{v}_\tau(f'_T) = \mathsf{v}_\tau(f_T)$ and $\mathsf{v}_\tau(f'_S) = \mathsf{v}_\tau(f_S)$ for all $\tau \in P/{\sim}$. Since the obvious map $\{[x]^F_H \mid x \in F_T\} \to \mathcal{C}(T, F_T)$ is well-defined (see the proof of Proposition 4.6.1), it follows that $[f'_T]^{F_T}_T = [f_T]^{F_T}_T$. Since $f_S \in (S : \widehat{S})$ and $[f_S]^F_H = [f'_S]^F_H$, Lemma 4.2.2 implies that $f'_S \in (S : \widehat{S})$. The same argument shows that $f'_T \in (T : \widehat{T})$. We infer that

$$\left(f'_T f'_S\right)^{-1} a \in \mathsf{q}(T) \cap F = \widehat{T}, \qquad f'_T\left(f'_T f'_S\right)^{-1} a \in T,$$

and hence a has a decomposition

$$a = f'_S\left(f'_T\left(f'_T f'_S\right)^{-1} a\right),$$

where f'_S is a divisor with $H^\times \subsetneq [\![f'_S]\!]_\sim \subset S \subsetneq T$. This is a contradiction.

Case 2. There exists $c \in H \setminus H^\times$ with $c \mid_H a$ and $[\![c]\!]_\sim \subsetneq T$.

Proposition 4.4.1 implies that H is a BF-monoid. Therefore there exists $l \in \mathbb{N}_0$ being maximal such that a has a product decomposition

$$a = c_1 \cdot \ldots \cdot c_l a',$$

where $a' \in H$ and $c_1, \ldots, c_l \in H \setminus H^\times$ with $[\![c_i]\!]_\sim \subsetneq T$ for all $i \in [1, l]$. By assumption $l \geqslant 1$, and the maximality of l implies that $[\![c]\!]_\sim = T$ for every divisor $c \in H \setminus H^\times$ of a'. By gathering the c_i with the same $\sim$-support, we obtain a product decomposition

$$a = b_1 \cdot \ldots \cdot b_k a',$$

where $[\![b_1]\!]_\sim, \ldots, [\![b_k]\!]_\sim$ are pairwise distinct non-trivial and proper support-closed submonoids of T. Then $k < \psi_\sim(T)$, and

$$\min \mathsf{L}(a) \leqslant \sum_{i=1}^{k} \min \mathsf{L}(b_i) + \min \mathsf{L}(a').$$

Since $\psi_\sim([\![b_i]\!]_\sim) < \psi_\sim(T) = \psi$ for each $i \in [1, k]$, the induction hypothesis and Case 1 imply that

$$\begin{aligned}
\min \mathsf{L}(a) &\leqslant \sum_{i=1}^{k} \min \mathsf{L}(b_i) + \min \mathsf{L}(a') \\
&\leqslant \sum_{i=1}^{k} \left(\mathsf{v}_{\mathcal{E}}(b_i) + (\psi - 1)! \left(K_0 + \mathsf{v}_P\left(f_H^2\right) \sum_{j=1}^{\psi-1} \frac{1}{j!} \right) \right) + \mathsf{v}_P\left(f_H^2\right) \\
&\leqslant \mathsf{v}_{\mathcal{E}}(a) + \psi! \left(K_0 + \mathsf{v}_P\left(f_H^2\right) \sum_{j=1}^{\psi-1} \frac{1}{j!} \right) + \mathsf{v}_P\left(f_H^2\right) \\
&= \mathsf{v}_{\mathcal{E}}(a) + \psi! \left(K_0 + \mathsf{v}_P\left(f_H^2\right) \sum_{j=1}^{\psi} \frac{1}{j!} \right). \qquad \square
\end{aligned}$$

The following proposition yields the "only if" part in Theorem 5.3.

Proposition 5.9. *Suppose that H is a weakly* C*-monoid defined in $F = F^\times \times \mathcal{F}(P)$ with equivalence relation $\sim$ and parameter λ, and assume that the class group of $\widehat{H} \subset F$ is finite. Then, if* **(C3)** *does not hold, H fails to be locally tame.*

Proof. Let $T \subset H$ be a support-closed submonoid such that $|\operatorname{supp}_\sim(\widehat{T} \setminus \widehat{T}^\times)| = 1$ and such that the class group of $\widehat{T} \subset [\![T]\!]_{\widehat{H}}$ is infinite. We put $F_T = [\![T]\!]_F$, $P_T = P \cap F_T$, and define the equivalence relation $\sim_T$ on P_T to be the restriction of $\sim$ to P_T. Since T is a weakly C-monoid defined in F_T (Proposition 4.6.1), Proposition 4.4.2 implies that the class group of $\widehat{T} \subset F_T$ is finitely generated. Since the class group of $\widehat{T} \subset [\![T]\!]_{\widehat{H}}$ injects into $G = \mathsf{q}(F_T)/\mathsf{q}(T)F^\times = \mathcal{C}(\widehat{T}F^\times, F_T)$, the group G is infinite. Therefore G contains elements of infinite order, and so does any set of generators of G. In particular, $\{[p]^{F_T}_{\widehat{T}F^\times} \mid p \in P_T\}$ contains an element $[p]^{F_T}_{\widehat{T}F^\times}$ of infinite order. Choose $w \in F_T$ with $[w]^{F_T}_{\widehat{T}F^\times} = -[p]^{F_T}_{\widehat{T}F^\times}$. Then w can be written as $w = p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$, where $p_i \in P_T$, $p_i \neq p_j$ if $i \neq j$, and $\alpha_i \in \mathbb{N}$. Consider the set

$$S = \left\{ I \subset [1,r] \;\middle|\; \text{there exist } n \in \mathbb{N} \text{ and } (\beta_i)_{i \in I} \in \mathbb{N}_0^I \text{ such that } \big[p^n\big]^{F_T}_{\widehat{T}F^\times} = -\sum_{i \in I} \big[p_i^{\beta_i}\big]^{F_T}_{\widehat{T}F^\times} \right\}.$$

Let $I_0 \in S$ be minimal with respect to inclusion, and let $n_0 \in \mathbb{N}$ and $(\beta_i)_{i \in I_0} \in \mathbb{N}^{I_0}$ such that

$$n_0 [p]^{F_T}_{\widehat{T}F^\times} = -\sum_{i \in I_0} \beta_i [p_i]^{F_T}_{\widehat{T}F^\times}. \tag{4}$$

Then $I_0 \neq \emptyset$ since $[p]^{F_T}_{\widehat{T}F^\times}$ has infinite order and $n_0 \geqslant 1$, and by the minimal choice of I_0 it follows that all β_i are non-zero. Define

$$M = \left\{ \big(\gamma_0, (\gamma_i)_{i \in I_0}\big) \in \mathbb{N}_0^{1+|I_0|} \;\middle|\; \gamma_0 [p]^{F_T}_{\widehat{T}F^\times} = -\sum_{i \in I_0} \gamma_i [p_i]^{F_T}_{\widehat{T}F^\times} \right\} \subset \big(\mathbb{N}_0^s, +\big).$$

We continue with two assertions.

A1. Let $\boldsymbol{\gamma} = (\gamma_0, (\gamma_i)_{i \in I_0}) \in M \setminus \{\mathbf{0}\}$. Then $\gamma_0 > 0$ and $\gamma_i > 0$ for all $i \in I_0$.
A2. M is a discrete valuation monoid.

Proof of A1. Suppose we have proved that $\gamma_0 > 0$. Then it follows by the minimality of I_0 that $\gamma_i > 0$ for all $i \in I_0$. Hence it suffices to show that $\gamma_0 > 0$. Assume to the contrary that $\gamma_0 = 0$. Since $\boldsymbol{\gamma} \neq \mathbf{0}$ there exist non-zero elements contained in $Q = \{\frac{\gamma_i}{\beta_i} \mid i \in I_0\}$. Let $i_0 \in I_0$ such that $\frac{\gamma_{i_0}}{\beta_{i_0}} = \max Q$, and put $\widetilde{n}_0 = n_0\gamma_{i_0} - \gamma_0\beta_{i_0} = n_0\gamma_{i_0} > 0$ and $\widetilde{\gamma}_i = \beta_i\gamma_{i_0} - \gamma_i\beta_{i_0} \geqslant 0$ for all $i \in I_0$. Then it follows from (4) and $\gamma_0[p]^{F_T}_{\widehat{T}F^\times} = -\sum_{i \in I_0} \gamma_i [p_i]^{F_T}_{\widehat{T}F^\times}$ that

$$\widetilde{n}_0 [p]^{F_T}_{\widehat{T}F^\times} = -\sum_{i \in I_0} \widetilde{\gamma}_i [p_i]^{F_T}_{\widehat{T}F^\times}. \tag{5}$$

Since $\widetilde{\gamma}_{i_0} = 0$ and $\widetilde{n}_0 \neq 0$ Eq. (5) contradicts the minimal choice of I_0. □

Proof of A2. A simple calculation shows that $M \subset (\mathbb{N}_0^s, +)$ is a saturated submonoid, and therefore M is a Krull monoid. Thus it remains to show that M is primary. Let $\boldsymbol{\gamma} = (\gamma_0, (\gamma_i)_{i\in I_0})$, $\boldsymbol{\gamma}' = (\gamma_0', (\gamma_i')_{i\in I_0}) \in M \setminus \{\mathbf{0}\}$. It suffices to show that either $\boldsymbol{\gamma} \leqslant \boldsymbol{\gamma}'$ or $\boldsymbol{\gamma}' \leqslant \boldsymbol{\gamma}$.

We define $\zeta_i = \gamma_i/\gamma_i'$ for all $i \in I_0 \cup \{0\}$. Let $j_0 \in I_0 \cup \{0\}$ be an index such that $\zeta_{j_0} = \max\{\zeta_j \mid j \in I_0 \cup \{0\}\}$. Put $\widehat{\gamma_i} = \gamma_i'\gamma_{j_0} - \gamma_i\gamma_{j_0}'$ for each $i \in I_0 \cup \{0\}$. Then it follows from $\gamma_0[p]^{F_T}_{\widehat{T}F^\times} = -\sum_{i\in I_0}\gamma_i[p_i]^{F_T}_{\widehat{T}F^\times}$ and $\gamma_0'[p]^{F_T}_{\widehat{T}F^\times} = -\sum_{i\in I_0}\gamma_i'[p_i]^{F_T}_{\widehat{T}F^\times}$ that

$$\widehat{\gamma_0}[p]^{F_T}_{\widehat{T}F^\times} = -\sum_{i\in I_0}\widehat{\gamma_i}[p_i]^{F_T}_{\widehat{T}F^\times}.$$

Since $\widehat{\gamma_i} \geqslant 0$ for all $i \in I_0 \cup \{0\}$, the element $c = (\widehat{\gamma_0}, (\widehat{\gamma_i})_{i\in I_0})$ is contained in M. By the consideration above it follows that all $\widehat{\gamma_i}$ must be non-zero if c is non-zero. But $\widehat{\gamma}_{j_0} = 0$. Therefore it follows that $c = 0$. Hence we obtain $\gamma_i' = \gamma_i(\gamma_{j_0}'/\gamma_{j_0})$ for all $i \in I_0 \cup \{0\}$. But this implies that either $\boldsymbol{\gamma} \leqslant \boldsymbol{\gamma}'$ or $\boldsymbol{\gamma}' \leqslant \boldsymbol{\gamma}$. □

Note that, by Lemma 3.2, the class group of $[\![T]\!]_{\widehat{H}} \subset F_T$ is finite. We set $e = \exp(\mathcal{C}([\![T]\!]_{\widehat{H}}F^\times, F_T))$. Since $|\operatorname{supp}_\sim(\widehat{T} \setminus \widehat{T}^\times)| = 1$ and $(T : \widehat{T}) \neq \emptyset$, there exists $N \in \mathbb{N}$ such that $(\widehat{T} \setminus \widehat{T}^\times)^N \subset (T : \widehat{T})$. We define $v_0 = \varepsilon p^{n_0 e}$ and $w_0 = \eta\prod_{i\in I_0} p_i^{\beta_i e}$, where the units ε and η are chosen in such a way that v_0 and w_0 are contained in $[\![T]\!]_{\widehat{H}}$. By construction we have $v_0w_0 \in \mathsf{q}(T)F^\times \cap [\![T]\!]_{\widehat{H}} = \widehat{H}^\times\widehat{T}$. Let $\chi \in \widehat{H}^\times$ such that $\chi v_0 w_0 \in \widehat{T}$, and put

$$v = v_0^N \quad \text{and} \quad w = (\chi w_0)^N.$$

Then $vw = \chi^N v_0^N w_0^N \in (T : \widehat{T})$.

Let $f \in (H : \widehat{H})$. Our goal is to show that $\mathsf{t}(H, \mathsf{Z}(f^2)) = \infty$. Then H is not locally tame by [23, Theorem 1.6.7]. We claim that

$$\max \mathsf{L}_H\big(fw^n\big) \leqslant \mathsf{v}_P(f) \quad \text{and} \quad \max \mathsf{L}_H\big(fv^n\big) \leqslant \mathsf{v}_P(f) \quad \text{for all } n \in \mathbb{N}. \tag{6}$$

Assume to the contrary that $\max \mathsf{L}_H(fw^n) > \mathsf{v}_P(f)$, and let $fw^n = u_1 \cdot \ldots \cdot u_k$, with $k > \mathsf{v}_P(f)$, be a factorization into atoms u_i of H. Then there exists $j \in [1,k]$ such that $u_j \mid_F w^n$. Write $u_j = \varepsilon_j\prod_{i\in I_0} p_i^{\xi_i}$, where $\varepsilon_j \in F^\times$ and $\xi_i \in \mathbb{N}_0$ for all $i \in I_0$. Since $u_j \in H \cap F_T = T$ it follows that

$$\sum_{i\in I_0}\xi_i[p_i]^{F_T}_{\widehat{T}F^\times} = 0.$$

But by the minimal choice of I_0 it follows that $\xi_i = 0$ for all $i \in I_0$, contradiction. Therefore it follows that $\max \mathsf{L}_H(fw^n) \leqslant \mathsf{v}_P(f)$ for all $n \geqslant 1$, and the same arguments imply that $\max \mathsf{L}_H(fv^n) \leqslant \mathsf{v}_P(f)$ for all $n \geqslant 1$.

We now consider the sequence $b_n = f^2v^nw^n$ in H. By (6) we have $\min \mathsf{L}_H(b_n) \leqslant 2\mathsf{v}_P(f)$. We claim that the length of the shortest factorization of $f^{-2}b_n = v^nw^n$ into atoms of H is at least $n/2$. Indeed, by the minimal choice of I_0, every $h \in \widehat{T}$ with $h \mid_F v^nw^n$ is of the form $h = \varepsilon v^q w^q$, where $\varepsilon \in F^\times$ and q is a positive rational number. Since $\widehat{T} \subset F$ is saturated and $vw \in \widehat{T}$, the element h can only be an atom of $\widehat{T}$ if $q \leqslant 1$. Since $(\widehat{T} \setminus \widehat{T}^\times)^N \subset T$, it follows that if $h = \varepsilon v^q w^q \in \widehat{T}$ is an irreducible element of T, then $q < 2$. Therefore $\min \mathsf{L}_T(v^n w^n) > n/2$, and we conclude that $\mathsf{t}(H, \mathsf{Z}(f^2)) = \infty$. □

Proof of Theorem 5.3. If **(C3)** does not hold, then Proposition 5.9 implies that H is not locally tame. To prove the converse, suppose that **(C3)** holds.

Let $u \in \mathcal{A}(H)$. Suppose that $a \in uH$ and $u_1, \ldots, u_k \in \mathcal{A}(H)$ with $a = u_1 \cdot \ldots \cdot u_k$ and $u \nmid u_i^{-1} a$ for all $i \in [1, k]$. Then $k \leqslant \omega(H, u)$. By Proposition 5.8 it follows that

$$\min \mathsf{L}\big(u^{-1}a\big) \leqslant \mathsf{v}_{\mathcal{E}}\big(u^{-1}a\big) + K_1,$$

where K_1 is a constant that only depends on H. Lemma 5.4.2.(b) implies that $\mathsf{D}_{\mathcal{E}}(H) < \infty$, and we have

$$\mathsf{v}_{\mathcal{E}}\big(u^{-1}a\big) \leqslant \mathsf{v}_{\mathcal{E}}(a) \leqslant \mathsf{v}_{\mathcal{E}}(u_1 \cdot \ldots \cdot u_k) = \sum_{i=1}^{k} \mathsf{v}_{\mathcal{E}}(u_i) \leqslant k\mathsf{D}_{\mathcal{E}}(H).$$

Hence

$$\min \mathsf{L}\big(u^{-1}a\big) \leqslant \mathsf{v}_{\mathcal{E}}\big(u^{-1}a\big) + K_1 \leqslant k\mathsf{D}_{\mathcal{E}}(H) + K_1 \leqslant \omega(H, u)\mathsf{D}_{\mathcal{E}}(H) + K_1.$$

From this it follows that $\tau(H, u) \leqslant \omega(H, u)\mathsf{D}_{\mathcal{E}}(H) + K_1$. If $f \in (H : \widehat{H})$, then [25, Corollary 4.3.2] implies that

$$\omega(H, u) \leqslant \mathsf{v}_P(u) + \omega(H, f).$$

If u is a prime, then $\mathsf{t}(H, uH^\times) = 0$, and the assertion follows. If u is not a prime, then [25, Theorem 3.6] implies that

$$\begin{aligned} \mathsf{t}\big(H, uH^\times\big) &= \max\big\{\omega(H, u), 1 + \tau(H, u)\big\} \\ &\leqslant \big(\mathsf{v}_P(u) + \omega(H, f)\big) \max\big\{\mathsf{D}_{\mathcal{E}}(H), 1\big\} + K_1 + 1. \qquad \square \end{aligned}$$

6. Weakly C-monoids: further arithmetic properties and examples

In this section we prove that a locally tame weakly C-monoid H with finite class group $\mathcal{C}(\widehat{H})$ has finite catenary degree (Theorem 6.3). After that we show how our abstract results apply to Mori domains (see Theorem 6.7 and Corollary 6.8).

For a subset $L \subset \mathbb{Z}$ we denote by $\Delta(L)$ the set of all $d \in \mathbb{N}$ for which there exists $m \in L$ with $[m, m + d] \cap L = \{m, m + d\}$. The set $\Delta(L)$ is called the *set of* (*successive*) *distances* of L.

Definition 6.1. Suppose that H is atomic.

1. Let $a \in H$, $z, z' \in \mathsf{Z}(a)$ and $N \in \mathbb{N}_0 \cup \{\infty\}$. An *$N$-chain of factorizations of a from z to z'* is a finite sequence $(z_i)_{0 \leqslant i \leqslant k}$ of factorizations $z_i \in \mathsf{Z}(a)$ such that $z = z_0$, $z' = z_k$ and $\mathsf{d}(z_{i-1}, z_i) \leqslant N$ for all $i \in [1, k]$.
2. Let $a \in H$. The *catenary degree* $\mathsf{c}(a) \in \mathbb{N}_0 \cup \{\infty\}$ is the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ such that, for any two factorizations $z, z' \in \mathsf{Z}(a)$, there is an N-chain of factorizations of a from z to z'.

3. We call

$$\mathsf{c}(H) = \sup\{\mathsf{c}(a) \mid a \in H\} \in \mathbb{N}_0 \cup \{\infty\}$$

the *catenary degree* of H, and

$$\Delta(H) = \bigcup_{a \in H} \Delta(\mathsf{L}(a))$$

is called the *set of distances* of H.

By definition, if $|\Delta(H)| = 1$, then all sets of lengths are arithmetical progressions of the same difference. The catenary degree is a much more subtle arithmetical invariant than the set of distances (for more information on this invariant we refer to [23, Sections 1.6, 6.4 and 7.6]). Note that, in general, local tameness does not imply the finiteness of the catenary degree (there are even locally tame Krull monoids with infinite catenary degree, see [23, Theorem 4.8.4]).

Recall [24] that a monoid H is called *finitary* if it is a BF-monoid and there exist a finite set $U \subset H \setminus H^\times$ and $M \in \mathbb{N}$ such that $(H \setminus H^\times)^M \subset UH$. By Theorem 3.1 a v-noetherian G-monoid with $(H : \widehat{H}) \neq \emptyset$ is finitary. Neither C-monoids nor weakly C-monoids are finitary in general, but we show that there exists a (not necessarily finite) set $U \subset H$ such that $(H \setminus H^\times)^M \subset UH$ and such that U behaves arithmetically uniform.

Lemma 6.2. *Suppose that H is a* BF*-monoid. Assume $U \subset H \setminus H^\times$ is a subset and $M \in \mathbb{N}$ such that $(H \setminus H^\times)^M \subset UH$ and $\mathsf{t}(H, \mathsf{Z}(u)) \leqslant M$ for all $u \in U$. Then $\mathsf{c}(H) \leqslant M$.*

Proof. We may suppose without loss of generality that H is reduced. We show that $\mathsf{c}(a) \leqslant M$ for all $a \in H$. To this end, we proceed by induction on $\max \mathsf{L}(a)$. If $\max \mathsf{L}(a) < M$, then $\mathsf{c}(a) \leqslant \max \mathsf{L}(a) < M$. Suppose that $\max \mathsf{L}(a) \geqslant M$. Then $a \in (H \setminus H^\times)^M \subset UH$, and therefore $a = ub$ for some $u \in U$ and some $b \in H$. Let $z, z' \in \mathsf{Z}(a)$ and $v \in \mathsf{Z}(u)$. Then there are factorizations $y, y' \in \mathsf{Z}(a) \cap v\mathsf{Z}(H)$ such that $\max\{\mathsf{d}(z, y), \mathsf{d}(z', y')\} \leqslant \mathsf{t}(H, \mathsf{Z}(u)) \leqslant M$. Since $\max \mathsf{L}(b) < \max \mathsf{L}(a)$, the induction hypothesis yields an M-chain $v^{-1}y = x_0, x_1, \ldots, x_k = v^{-1}y'$ in $\mathsf{Z}(b)$ concatenating x_0 and x_k. Then $z, y = vx_0, vx_1, \ldots, vx_k = y', z'$ is an M-chain in $\mathsf{Z}(a)$ concatenating z and z'. □

Theorem 6.3. *Suppose that H is a weakly* C*-monoid defined in $F = F^\times \times \mathcal{F}(P)$.*

1. *There exist $U \subset H \setminus H^\times$ and $N \in \mathbb{N}$ such that $(H \setminus H^\times)^N \subset UH$ and such that $\{\mathsf{v}_P(u) \mid u \in U\}$ is bounded.*
2. *If H is locally tame and has finite class group $\mathcal{C}(\widehat{H})$, then H has finite catenary degree and finite set of distances.*

Proof. 1. Suppose that H is defined in F with equivalence relation $\sim$ and parameter λ. For every support-closed submonoid $T \subset H$ we pick $f_T \in (T : \widehat{T})$ such that $\mathsf{v}_\tau(f_T) \in \lambda\mathbb{N}$ for all $\tau \in \bigcup_{\Sigma \in \mathrm{supp}_\sim(T)} \Sigma$ (this is possible since there exists $t \in T$ such that $\mathrm{supp}_\sim(t)$ is the maximal element of $\mathrm{supp}_\sim(T)$. Pick $f_0 \in (T : \widehat{T})$, and define $f_T = f_0^\lambda t^\lambda$). Define

$$U_T = \{u \in T \mid \mathsf{v}_\tau(u) = \mathsf{v}_\tau(f_T) \text{ for all } \tau \in P/{\sim}\},$$

and put $U = \bigcup_T U_T$, where T ranges over all non-trivial support-closed submonoids of H. It is clear from the definition of U that $\{\mathsf{v}_P(u) \mid u \in U\}$ is bounded. We define $N = M\psi_\sim(H)$, where

$$M = \max\bigl\{2\mathsf{v}_p(f_T) \bigm| T \subset H \text{ is support-closed and } p \in P\bigr\}.$$

We claim that $(H \setminus H^\times)^N \subset UH$. To prove this, let $a_1, \ldots, a_N \in H \setminus H^\times$. Then there exists a divisor-closed submonoid $T \subset H$ and a set $I \subset [1, N]$ such that $|I| \geqslant M$ and $[\![a_i]\!]_\sim = T$ for all $i \in I$. Without loss of generality we may assume that $I = [1, M]$. Then $\mathsf{v}_\tau(f_T^2) \leqslant \mathsf{v}_\tau(a_1 \cdot \ldots \cdot a_M)$ for all $\tau \in P/\!\sim$. By Lemma 4.3.4 there exist $f_T', f_T'' \in F$ such that

(a) $[f_T']_H^F = [f_T'']_H^F = [f_T]_H^F$,
(b) $\mathsf{v}_\tau(f_T') = \mathsf{v}_\tau(f_T'') = \mathsf{v}_\tau(f_T)$ for all $\tau \in P/\!\sim$, and
(c) $f_T' f_T'' \mid a_1 \cdot \ldots \cdot a_M$ (in F and hence in $\widehat{H}$).

By (a) it follows that f_T' and f_T'' are contained in H. By (b) we see that $f_T', f_T'' \in F_T$, where $F_T = [\![T]\!]_F$. Thus $f_T', f_T'' \in H \cap F_T = H \cap \widehat{H} \cap F_T = H \cap [\![T]\!]_{\widehat{H}} = T$ (see Lemma 3.6). By (a) we moreover have $[f_T']_T^{F_T} = [f_T'']_T^{F_T} = [f_T]_T^{F_T}$ (see the proof of Proposition 4.6.1), and therefore it follows that $f_T', f_T'' \in (T : \widehat{T})$. By (c) we have $(f_T' f_T'')^{-1} a_1 \cdot \ldots \cdot a_M \in F_T \cap \mathsf{q}(T) = \widehat{T}$. Thus $(f_T'')^{-1} a_1 \cdot \ldots \cdot a_M \in T$. Since $f_T'' \in U_T$, we infer that $a_1 \cdot \ldots \cdot a_M \in U_T T$. Therefore we have $a_1 \cdot \ldots \cdot a_N \in U_T H \subset UH$.

2. By 1. there exist $M, N \in \mathbb{N}$ and a subset $U \subset H \setminus H^\times$ such that $(H \setminus H^\times)^N \subset UH$ and $\mathsf{v}_P(a) \leqslant M$ for all $a \in U$. We show that there exists $M^* \in \mathbb{N}$ such that $\mathsf{t}(H, \mathsf{Z}(a)) \leqslant M^*$ for all $a \in U$. Then $\mathsf{c}(H) \leqslant \max\{N, M^*\}$ by Lemma 6.2, and it follows by [23, Theorem 1.6.3] that the set of distances of H is finite. Let $f \in (H : \widehat{H})$, $a \in U$ and $x = u_1 \cdot \ldots \cdot u_k \in \mathsf{Z}(a)$ with $k \in \mathbb{N}$ and $u_1, \ldots, u_k \in \mathcal{A}(H_{\text{red}})$. Then $k \leqslant \mathsf{v}_P(a) \leqslant M$, and by [23, Lemma 1.6.5.5] the tame degree $\mathsf{t}(H, \mathsf{Z}(a))$ is bounded from above by

$$2\mathsf{t}(H, u_1) + \cdots + 2\mathsf{t}(H, u_k).$$

Theorem 5.3 provides an upper bound for each $\mathsf{t}(H, u_j)$, and this bound depends only on $\mathsf{v}_P(u_j)$ and some universal parameters. Taking into account that $\mathsf{v}_P(u_j) \leqslant M$, we obtain an upper bound M^* for the local tame degrees. □

Suppose that H is a weakly C-monoid such that $\mathcal{C}(\widehat{H})$ is finite. In the next proposition we show that if H is locally tame and a G-monoid, then every saturated submonoid of H with finite class group is locally tame, too. We do not know if this remains true when we drop the assumption that H be a G-monoid. (Note that, in general, a monoid with finite catenary degree can have a saturated submonoid with finite class group and infinite catenary degree, see [23, Section 3.6].) After having proved Proposition 6.4, we apply the result to a class of v-noetherian G-monoids occurring in the study of one-dimensional domains.

Proposition 6.4. *Suppose that H is a v-noetherian* G*-monoid with $(H : \widehat{H}) \neq \emptyset$ and finite group $\mathcal{C}(\widehat{H})$. Assume that H is locally tame. If $S \subset H$ is a saturated submonoid with finite class group, then S is a locally tame weakly* C*-monoid, $\mathcal{C}(\widehat{S})$ is finite, and S has finite catenary degree and finite set of distances.*

Proof. Let $S \subset H$ be a saturated submonoid with finite class group. By Proposition 4.7.1 and Lemma 3.5.3, S is a weakly C-monoid with finite class group $\mathcal{C}(\widehat{S})$. By Theorem 6.3 it remains to verify that S is locally tame. By Theorem 5.3 it suffices to show that condition **(C3)** holds for S. Let $T_0 \subset S$ be a divisor-closed submonoid such that $|\operatorname{supp}(\widehat{T_0} \setminus \widehat{T_0}^\times)| = 1$. We have to show that the class group

$$\frac{\mathsf{q}([\![T_0]\!]_{\widehat{S}})}{\mathsf{q}(T_0)\widehat{S}^\times}$$

of $\widehat{T_0} \subset [\![T_0]\!]_{\widehat{S}}$ is finite.

We set $T = [\![T_0]\!]_H$ and see that $[\![T]\!]_{\widehat{H}} = [\![T_0]\!]_{\widehat{H}}$. Lemma 3.7.2.(d) implies that $\widehat{T_0}$ is primary. We continue with the following assertion.

A1. $\widehat{T}$ is a discrete valuation monoid.

Proof of A1. Since $\widehat{T}$ is a Krull monoid, it suffices to show that it is primary. Since the class group of $S \subset H$ is finite, Lemma 3.2.2 implies that

$$\frac{\mathsf{q}([\![T_0]\!]_H)}{\mathsf{q}(T_0)H^\times}$$

is finite. By Lemmas 3.2.3 and 3.5.1 it follows that $\widehat{T_0} \subset \widehat{[\![T_0]\!]_H}$ is saturated with finite class group. This implies that

$$M = \widehat{T_0}\widehat{T}^\times \subset \widehat{[\![T_0]\!]_H} = \widehat{T}$$

is saturated, and for every $x \in \widehat{T}$ there exists $n \in \mathbb{N}$ such that $x^n \in M$. Then [14, Proposition 5] implies that there is a bijection from s-spec$(\widehat{T})$ to s-spec(M). Since $\widehat{T_0}$ is primary, M is primary, and hence $\widehat{T}$ is primary. □

Since $\widehat{T}$ is a discrete valuation monoid, Lemma 3.7.2.(d) implies that $|\operatorname{supp}(\widehat{T} \setminus \widehat{T}^\times)| = 1$. Since H is locally tame, Theorem 5.3 implies that the group $\frac{\mathsf{q}([\![T]\!]_{\widehat{H}})}{\mathsf{q}(T)\widehat{H}^\times}$ is finite. From the finiteness of $\frac{\mathsf{q}([\![T_0]\!]_H)}{\mathsf{q}(T_0)H^\times}$ it follows that

$$\frac{\mathsf{q}([\![T_0]\!]_{\widehat{H}})}{\mathsf{q}(T_0)\widehat{H}^\times}$$

is finite. Since the natural homomorphism

$$\frac{\mathsf{q}([\![T_0]\!]_{\widehat{S}})}{\mathsf{q}(T_0)\widehat{S}^\times} \to \frac{\mathsf{q}([\![T_0]\!]_{\widehat{H}})}{\mathsf{q}(T_0)\widehat{H}^\times}$$

is a monomorphism, the first group is finite. □

A monoid H is said to be *strongly primary* if it is finitary and primary (recall that every primary monoid is a G-monoid). The multiplicative monoid of a one-dimensional local Mori domain is v-noetherian and primary (see [23, Proposition 2.10.7.1]), and every v-noetherian primary monoid is finitary [23, Theorem 2.7.9] and thus strongly primary. Suppose that H is strongly primary, $\widehat{H}$ is a Krull monoid, and $(H : \widehat{H}) \neq \emptyset$. Then H is locally tame [26, Theorem 3.5] (note that this result holds without imposing any conditions on the class group of $\widehat{H}$). In the next proposition we study saturated submonoids, with finite class group, of a finite product of strongly primary monoids. Via transfer principles saturated submonoids of products of strongly primary monoids reflect the multiplicative arithmetic of (v-noetherian) weakly Krull domains (see [23, Sections 3.6 and 4.5]). Therefore arithmetical results on these monoids have direct bearings on the structure of factorizations in, e.g., one-dimensional noetherian domains.

Proposition 6.5. *Let $D_1, \ldots, D_n$ be strongly primary monoids, put $D = D_1 \times \cdots \times D_n$, and suppose that $H \subset D$ is a saturated submonoid with finite class group.*

1. *If $T \subset H$ is a divisor-closed submonoid, then there exists $I \subset [1, n]$ such that*

$$T \subset \prod_{i \in I} D_i \times \prod_{i \in [1,n] \setminus I} D_i^{\times}$$

is saturated with finite class group.

2. *$\widehat{H} \subset \widehat{D}$ is saturated. If $(D_i : \widehat{D_i}) \neq \emptyset$ for all $i \in [1, n]$, then $(H : \widehat{H}) \neq \emptyset$. If all $\widehat{D_i}$ are Krull monoids (with finite class group), then $\widehat{H}$ is a Krull monoid (with finite class group).*
3. *If all D_i are v-noetherian with $(D_i : \widehat{D_i}) \neq \emptyset$ and such that $\mathcal{C}(\widehat{D_i})$ is finite, then H is a locally tame weakly* C*-monoid, $\mathcal{C}(\widehat{H})$ is finite, and H has finite catenary degree and finite set of distances.*

Proof. 1. Let $T \subset D$ be a divisor-closed submonoid. Since s-spec(D) is finite, s-spec(H) is finite by [23, Corollary 2.4.3.3]. Thus there exists $a \in H$ such that $T = [\![a]\!]_H$. We set $T_D = [\![T]\!]_D = [\![a]\!]_D$. Then Lemma 3.2 implies that $T \subset T_D$ is saturated and that $\mathsf{q}(T_D)/\mathsf{q}(T)D^{\times}$ is isomorphic to a subgroup of $\mathsf{q}(D)/\mathsf{q}(H)D^{\times}$. In particular, the class group of $T \subset T_D$ is finite. Let $a = a_1 \cdot \ldots \cdot a_n$ with $a_i \in D_i$, and put

$$I = \{ i \in [1, n] \mid a_i \notin D_i^{\times} \}.$$

Then we see that

$$T_D = \prod_{i \in I} D_i \times \prod_{i \in [1,n] \setminus I} D_i^{\times}.$$

2. Since $H \subset D$ has finite class group, $H \subset D$ is cofinal, and hence $\widehat{H} \subset \widehat{D}$ is saturated with finite class group by Lemma 3.3.1. If $(D_i : \widehat{D_i}) \neq \emptyset$ for all $i \in [1, n]$, then $(D : \widehat{D}) \neq \emptyset$, and hence $(H : \widehat{H}) \neq \emptyset$ by Lemma 3.3.2. Suppose that all $\widehat{D_i}$ are Krull monoids. Then $\widehat{D} = \widehat{D_1} \times \cdots \times \widehat{D_n}$ is a Krull monoid, and $\widehat{H}$ is a Krull monoid since it is a saturated submonoid of a Krull monoid. If all $\mathcal{C}(\widehat{D_i})$ are finite, then there exists a factorial monoid F such that $\widehat{D} \subset F$ is saturated with finite class group, and hence $\widehat{H} \subset F$ is saturated with finite class group. Therefore the class group $\mathcal{C}(\widehat{H})$ is finite by [23, Theorem 2.4.7.2].

3. If all D_i are as asserted, then D is a v-noetherian G-monoid with finite class group $\mathcal{C}(\widehat{D})$, and D is locally tame by [26, Theorem 3.5]. Thus H has the asserted properties by Proposition 6.4. □

In the remainder of the paper we show how the theory of weakly C-monoids applies to domains. The main results are Theorem 6.7 and Corollary 6.8. In Remark 6.9 we re-formulate the main assumptions of Theorem 6.7 for noetherian domains. As a special example of noetherian domains we are able to deal with, we look at rings of generalized power series with coefficients in a field K and exponents in a finitely generated monoid (Proposition 6.10). If K is infinite, then the arithmetic of such domains R could not be studied before. In Proposition 6.10 we show that R satisfies the assumptions of Theorem 6.7 for any base field K.

Let R be a Mori domain. We denote by $R^\bullet = R \setminus \{0\}$ the multiplicative monoid of R and by $\widehat{R}$ the complete integral closure of R. It is easy to see that $R^\bullet$ is v-noetherian and that $\widehat{R} = \widehat{R^\bullet} \cup \{0\}$. Suppose that $(R : \widehat{R}) \neq \{0\}$. Then $\widehat{R}$ is a Krull domain, $\widehat{R^\bullet}$ is a Krull monoid, and the divisor class group $\mathcal{C}(\widehat{R})$ of the domain $\widehat{R}$ coincides with the class group of $\widehat{R^\bullet}$. Furthermore, $R^\bullet$ is a G-monoid if and only if R is one-dimensional and semilocal [23, Proposition 2.10.7.2], and hence $\dim(R) > 1$ implies that $R^\bullet$ fails to be a G-monoid. Note that, if v-max (R) is finite, then R is semilocal and $\mathcal{C}_v(R) = 0$ [23, Proposition 2.10.4.1]. We start with a lemma whose proof is due to M. Roitman.

Lemma 6.6. *Let $R \subset S$ be commutative rings and $\mathfrak{f} \lhd R$ an ideal such that $\mathfrak{f}S = \mathfrak{f}$. Then the map*

$$j : \begin{cases} \{\mathfrak{m} \in \max(R) \mid \mathfrak{f} \not\subset \mathfrak{m}\} \to \{\mathfrak{M} \in \max(S) \mid \mathfrak{f} \not\subset \mathfrak{M}\}, \\ \mathfrak{m} \mapsto \mathfrak{m}S \end{cases}$$

is bijective, and for every $\mathfrak{M} \in \max(S)$ with $\mathfrak{f} \not\subset \mathfrak{M}$ we have $j^{-1}(\mathfrak{M}) = \mathfrak{M} \cap R$.

Proof. Let $\mathfrak{m} \in \max(R)$ with $\mathfrak{f} \not\subset \mathfrak{m}$. Since $\mathfrak{m}S = S$ would imply that $\mathfrak{f} = \mathfrak{f}S = \mathfrak{m}\mathfrak{f}S \subset \mathfrak{m}$, it follows that $\mathfrak{m}S \neq S$. Thus $\mathfrak{m}S \cap R = \mathfrak{m}$. Since $S = RS = (\mathfrak{m} + \mathfrak{f})S \subset \mathfrak{m}S + R \subset S$, it follows that $S = \mathfrak{m}S + R$. In particular, $S/\mathfrak{m}S \cong R/\mathfrak{m}$, and hence $\mathfrak{m}S \in \max(S)$. Therefore j is well-defined and injective.

To show that j is surjective, let $\mathfrak{M} \in \max(S)$ with $\mathfrak{f} \not\subset \mathfrak{M}$. Then $\mathfrak{M} + \mathfrak{f} = S$, and if $1 = m + f$ with $m \in \mathfrak{M}$ and $f \in \mathfrak{f}$, then $m = 1 - f \in \mathfrak{M} \cap R$. Therefore we have $(\mathfrak{M} \cap R) + \mathfrak{f} = R$. Since $S/\mathfrak{M} = (R + \mathfrak{M})/\mathfrak{M} \cong R/(\mathfrak{M} \cap R)$, it follows that $\mathfrak{M} \cap R \in \max(R)$. If $x \in \mathfrak{M}$, then $fx \in \mathfrak{M} \cap \mathfrak{f}S \subset \mathfrak{M} \cap R$, $x = mx + fx \in (\mathfrak{M} \cap R)S$, and thus $\mathfrak{M} = (\mathfrak{M} \cap R)S$. Hence j is surjective and $j^{-1}(\mathfrak{M}) = \mathfrak{M} \cap R$. □

Theorem 6.7. *Let R be a semilocal Mori domain such that $\mathfrak{f} = (R : \widehat{R}) \neq \{0\}$, the group $\mathcal{C}(\widehat{R})$ is finite, and $\widehat{R}/\mathfrak{f}$ is semilocal with nilpotent Jacobson radical.*

1. *$R^\bullet$ is a weakly* C*-monoid.*
2. *For every divisor-closed submonoid $T \subset R^\bullet$ the class group of $\widehat{T} \subset [\![T]\!]_{\widehat{R^\bullet}}$ is trivial.*
3. *$R^\bullet$ is locally tame, has finite catenary degree, and finite set of distances.*

Proof. Let $H = \{aR \mid a \in R^\bullet\}$ denote the monoid of non-zero principal ideals of R. Then $(R^\bullet)_{\mathrm{red}} \cong H$, and statements 1., 2. and 3. hold for $R^\bullet$ if and only if they hold for H (for 1. see Proposition 4.6.2, and for 2. this can be checked directly).

1. Since $\widehat{H}$ is a Krull monoid, Lemma 2.2.2 implies that there exists an embedding

$$\widehat{H} \subset F = \widehat{H}^\times \times \mathcal{I}_v^*(\widehat{H}) = \widehat{H}^\times \times \mathcal{F}\big(v\text{-max}\,(\widehat{H})\big)$$

which is cofinal and saturated with class group $\mathcal{C}(\widehat{R})$. We show that H is a weakly C-monoid defined in F. Clearly, condition **(C1)** in Definition 4.1 is fulfilled.

Since $\widehat{R}/\mathfrak{f}$ is semilocal there exist only finitely many maximal ideals of $\widehat{R}$ that contain $\mathfrak{f}$. Therefore it follows by Lemma 6.6 that $\widehat{R}$ is semilocal. We now define an equivalence relation $\sim$ on $P = v\text{-max}(\widehat{H})$. Let $\widehat{\mathfrak{p}}, \widehat{\mathfrak{p}}' \in v\text{-max}(\widehat{H})$. We say that $\widehat{\mathfrak{p}}$ and $\widehat{\mathfrak{p}}'$ are $\sim$-equivalent, $\widehat{\mathfrak{p}} \sim \widehat{\mathfrak{p}}'$, if and only if

$$\big(\widehat{\mathfrak{p}} \subset \mathfrak{M} \Leftrightarrow \widehat{\mathfrak{p}}' \subset \mathfrak{M} \text{ for all } \mathfrak{M} \in \max(\widehat{R})\big) \quad \text{and} \quad \big(\widehat{\mathfrak{p}}\mathsf{q}(\widehat{H}) = \widehat{\mathfrak{p}}'\mathsf{q}(\widehat{H})\big).$$

Put

$$\mathcal{M} = \big\{\mathfrak{M} \in \max(\widehat{R}) \bigm| \mathfrak{f} \subset \mathfrak{M}\big\}.$$

Since the radical of $\widehat{R}/\mathfrak{f}$ is nilpotent, there exists $\alpha \in \mathbb{N}$ such that

$$\mathfrak{a} = \prod_{\mathfrak{M} \in \mathcal{M}} \mathfrak{M}^{\alpha}$$

is contained in $\mathfrak{f}$. By the Chinese Remainder Theorem we have a natural isomorphism

$$\iota : \widehat{R}/\mathfrak{a} \to \prod_{\mathfrak{M} \in \mathcal{M}} \widehat{R}/\mathfrak{M}^{\alpha}.$$

We now define $\lambda = \alpha e$, where $e = \exp(\mathsf{q}(F)/\mathsf{q}(\widehat{H}))$. Let $\mathfrak{p}_1, \mathfrak{p}_1', \ldots, \mathfrak{p}_\lambda, \mathfrak{p}_\lambda' \in P$ such that $\mathfrak{p}_1 \sim \mathfrak{p}_1' \sim \cdots \sim \mathfrak{p}_\lambda \sim \mathfrak{p}_\lambda'$. For $k \in [1, \alpha]$ put $J_k = [(k-1)e + 1, ke]$. Then $d_k = \prod_{j \in J_k} \mathfrak{p}_j$ and $d_k' = \prod_{j \in J_k} \mathfrak{p}_j'$ are contained in $\widehat{H}$ for all $k \in [1, \alpha]$. Furthermore, if d_k is contained in some maximal ideal $\mathfrak{M}$ of $\widehat{R}$, then $d_l, d_l' \in \mathfrak{M}$ for all $l \in [1, \alpha]$. To verify condition **(C2)** in Definition 4.1 it is enough to prove the following statement:

(†) Let $k, k' \geqslant \alpha$, and let $x_1, \ldots, x_k, x_1', \ldots, x_{k'}' \in \widehat{H}$ such that all x_i, x_i' have the same maximal overideals. Then there exist $\eta, \eta' \in \widehat{H}^{\times}$ such that $[\eta x_1 \cdot \ldots \cdot x_k]_H^F = [\eta' x_1' \cdot \ldots \cdot x_k']_H^F$.

To prove (†) put

$$\mathcal{N} = \big\{\mathfrak{M} \in \max(\widehat{R}) \bigm| \text{there exists } \nu \in [1, k] \text{ such that } x_\nu \in \mathfrak{M}\big\}.$$

We then have

$$\iota(x_1 \cdot \ldots \cdot x_k) = \prod_{\mathfrak{M} \in \mathcal{M}} \kappa_{\mathfrak{M}}, \qquad \iota\big(x_1' \cdot \ldots \cdot x_{k'}'\big) = \prod_{\mathfrak{M} \in \mathcal{M}} \kappa_{\mathfrak{M}}',$$

where $\kappa_{\mathfrak{M}} = \kappa_{\mathfrak{M}}' = 0$ if $\mathfrak{M} \in \mathcal{N}$, and $\kappa_{\mathfrak{M}}, \kappa_{\mathfrak{M}}' \in (\widehat{R}/\mathfrak{M}^{\alpha})^{\times}$ if $\mathfrak{M} \notin \mathcal{N}$. Since $\widehat{R}$ is semilocal, the canonical epimorphism $\pi : \widehat{R} \to \widehat{R}/\mathfrak{a}$ induces an epimorphism $\widehat{R}^{\times} \to (\widehat{R}/\mathfrak{a})^{\times}$. Therefore there exist $\eta, \eta' \in \widehat{R}^{\times}$ such that $\iota(\eta)_{\mathfrak{M}} = \iota(\eta')_{\mathfrak{M}} = 1$ if $\mathfrak{M} \in \mathcal{N}$, and $\iota(\eta)_{\mathfrak{M}} = \kappa_{\mathfrak{M}}$, $\iota(\eta')_{\mathfrak{M}} = \kappa_{\mathfrak{M}}'$ if $\mathfrak{M} \notin \mathcal{N}$. If we define $\varepsilon = \eta(\eta')^{-1}$, then

$$f = x_1 \cdot \ldots \cdot x_k - \varepsilon x_1' \cdot \ldots \cdot x_{k'}' \in \mathfrak{a} \subset \mathfrak{f}.$$

This implies that

$$[x_1 \cdot \ldots \cdot x_k]_H^{\widehat{H}} = \left[f + \varepsilon x_1' \cdot \ldots \cdot x_{k'}'\right]_H^{\widehat{H}} = \left[\varepsilon x_1' \cdot \ldots \cdot x_{k'}'\right]_H^{\widehat{H}},$$

and thus $[x_1 \cdot \ldots \cdot x_k]_H^F = [\varepsilon x_1' \cdot \ldots \cdot x_{k'}']_H^F$ by Lemma 4.2.1.(b).

2. Let $T \subset H$ be a divisor-closed submonoid. By Lemma 3.6 we have $\mathsf{q}(T) = (T^{-1}H)^\times$ and $\mathsf{q}([\![T]\!]_{\widehat{H}}) = (T^{-1}\widehat{H})^\times$. Therefore we must prove that the group

$$G = \frac{(T^{-1}\widehat{H})^\times}{(T^{-1}H)^\times \widehat{H}^\times}$$

is trivial. Let $x \in (T^{-1}\widehat{H})^\times$. We show that $x \in (T^{-1}H)^\times \widehat{H}^\times$.

There exist $h \in [\![T]\!]_{\widehat{H}}$ and $t \in T$ such that $x = \frac{h}{t}$. If $u \in T$ such that $\mathrm{supp}_\sim(u)$ is maximal in the set $\{\mathrm{supp}_\sim(t') \mid t' \in T\}$, then, after replacing h with hu and t with tu, we may assume without restriction that $\mathrm{supp}_\sim(h) = \mathrm{supp}_\sim(u) = \mathrm{supp}_\sim(t)$. By statement (†) there exist $\eta, \eta' \in \widehat{H}^\times$ such that

$$\left[\eta h t^{\alpha-1}\right]_H^F = \left[\eta' t^\alpha\right]_H^F.$$

Thus $z = \eta\eta'^{-1} h t^{\alpha-1} \in [t^\alpha]_H^F \subset H$. From Lemma 3.6.2 it follows that $z \in [\![T]\!]_{\widehat{H}} \cap H = T$. Therefore we obtain

$$x = \eta'\eta^{-1}\frac{z}{t^\alpha} \in \widehat{H}^\times \left(T^{-1}H\right)^\times.$$

3. This follows from 1., 2. and from Theorems 5.3 and 6.3. □

Corollary 6.8. *Let R be a Mori domain such that $\mathfrak{f} = (R : \widehat{R}) \neq \{0\}$, $\mathcal{C}(\widehat{R})$ is finite, and $\widehat{R}/\mathfrak{f}$ is semilocal with nilpotent Jacobson radical. Then the monoid $(\mathcal{I}_v^*(R), \cdot_v)$ of v-invertible v-ideals with v-multiplication is a direct product of a free monoid and a weakly* C*-monoid that satisfies* **(C3)**. *In particular, $\mathcal{I}_v^*(R)$ is locally tame and has finite catenary degree and finite set of distances.*

Proof. Let R denote the set of regular elements on $\widehat{R}/R$, that is,

$$\mathsf{R} = \left\{x \in R^\bullet \;\middle|\; xy \in R \text{ implies } y \in R \text{ for all } y \in \widehat{R}\right\}.$$

Put $\mathcal{P} = \{\mathfrak{p} \in v\text{-spec}(R) \mid \mathfrak{p} \cap \mathsf{R} \neq \emptyset\}$. Then $\mathsf{R}^{-1}R$ is a semilocal Mori domain, all localizations $R_\mathfrak{p}$, for $\mathfrak{p} \in \mathcal{P}$, are discrete valuation domains, and there is an isomorphism

$$\delta_0 : \mathcal{I}_v^*(R) \to \coprod_{\mathfrak{p} \in \mathcal{P}} \left(R_\mathfrak{p}^\bullet\right)_{\mathrm{red}} \times \left(\mathsf{R}^{-1}R^\bullet\right)_{\mathrm{red}}$$

[23, Theorem 2.10.9]. Thus it remains to verify that the domain $\mathsf{R}^{-1}R$ satisfies all assumptions of Theorem 6.7. By Lemma 3.4.2 $\mathsf{R}^{-1}R$ has complete integral closure $\mathsf{R}^{-1}\widehat{R}$ and conductor $(\mathsf{R}^{-1}R : \widehat{\mathsf{R}^{-1}R}) = \mathsf{R}^{-1}\mathfrak{f} \neq \{0\}$. Since R is the complement of the union of those maximal ideals of R that contain $\mathfrak{f}$, [31, Lemma 2.3] implies that the natural map $R/\mathfrak{f} \to \mathsf{R}^{-1}R/\mathfrak{f}\mathsf{R}^{-1}R$ is an isomorphism.

By tensoring this with $\widehat{R}$, we see that the natural map $\widehat{R}/\mathfrak{f} \to \mathsf{R}^{-1}\widehat{R}/\mathfrak{f}\mathsf{R}^{-1}\widehat{R}$ is an isomorphism. By Nagata's Theorem [20, Corollary 7.2] there is an epimorphism $\mathcal{C}(\widehat{R}) \to \mathcal{C}(\mathsf{R}^{-1}\widehat{R})$, and thus $\mathcal{C}(\mathsf{R}^{-1}\widehat{R})$ is finite. □

Remark 6.9. Let R be a semilocal noetherian domain with integral closure $\overline{R}$ and conductor $\mathfrak{f}$, and suppose that $R/\mathfrak{f}$ is artinian. Then $\widehat{R} = \overline{R}$, $\mathfrak{f} \neq \{0\}$, and $\overline{R}/\mathfrak{f}$ is artinian. (Thus, like every artinian ring, $\overline{R}/\mathfrak{f}$ is semilocal and zero-dimensional with nilpotent Jacobson radical.)

Proof. For all noetherian domains the integral closure coincides with the complete integral closure. If $\dim R = 0$, then R is a field, and all assertions are trivially true. Hence suppose that $\dim R > 0$. Since $R/\mathfrak{f}$ is zero-dimensional, $\mathfrak{f}$ must be non-zero. Thus $\overline{R}$ is a finitely generated R-module, hence both $\overline{R}$ and $\overline{R}/\mathfrak{f}$ are noetherian. Since $R/\mathfrak{f} \subset \overline{R}/\mathfrak{f}$ is an integral ring extension, $\overline{R}/\mathfrak{f}$ is zero-dimensional. Therefore $\overline{R}/\mathfrak{f}$ is artinian. □

Rings of generalized power series provide a rich source of interesting examples of noetherian rings. In the next proposition we prove that certain generalized power series rings satisfy the assumptions of Theorem 6.7. We first recall the construction of generalized power series from [40]. Suppose R is a commutative ring and $(S, \leqslant)$ is an additive, partially ordered monoid. Then $R[\![S]\!] = [\![R^{S,\leqslant}]\!]$ denotes the set of all mappings $f: S \to R$ such that $\{s \in S \mid f(s) \neq 0\}$ is an artinian and narrow subset of S. Clearly, $R[\![S]\!]$ is an abelian group with pointwise addition. For $s \in S$ and $f, g \in R[\![S]\!]$ put

$$X_s(f,g) = \big\{(t,u) \in S \times S \bigm| s = t+u,\ f(t) \neq 0,\ g(u) \neq 0\big\}.$$

Since S is artinian and narrow, the sets $X_s(f,g)$ are finite, and the multiplication of $R[\![S]\!]$ is defined by convolution:

$$(fg)(s) = \sum_{(t,u)\in X_s(f,g)} f(t)g(u) \quad \text{for all } s \in S.$$

With these operations $R[\![S]\!]$ is a commutative ring, called the *the ring of generalized power series with coefficients in R and exponents in S*.

Proposition 6.10. *Let K be a field, $s \in \mathbb{N}$, and $S \subset (\mathbb{N}_0^s, +)$ a finitely generated submonoid, endowed with the induced product order of $(\mathbb{N}_0^s, \leqslant)$. Assume that $\widehat{S} = \mathbb{N}_0^s$. Then the ring $K[\![S]\!]$ of generalized power series satisfies the assumptions of Theorem* 6.7.

Proof. The ring $R = K[\![S]\!]$ is a local noetherian domain (see [40, 5.8] and [41, 1.20, 2.3]). We have $R \subset K[\![\widehat{S}]\!] = K[\![\mathbb{N}_0^s]\!]$ [34, Theorem 2.5], and $K[\![\mathbb{N}_0^s]\!]$ is isomorphic to $K[\![X_1, \dots, X_s]\!]$, the formal power series ring in s variables over K [40, Example 3], which is a local noetherian factorial domain. Since S is finitely generated and $\widehat{S} = \mathbb{N}_0^s$, it follows from [23, Theorem 2.7.13] that there exists $\boldsymbol{a} = (a_1, \dots, a_s) \in \mathbb{N}_0^s$ such that $\boldsymbol{a} + \mathbb{N}_0^s \subset S$. If $\alpha = \max\{a_1, \dots, a_s\}$, then an easy calculation shows that $X_i^\alpha K[\![\widehat{S}]\!] \subset R$ for all $i \in [1, s]$. This implies that $K[\![\widehat{S}]\!] = \widehat{R}$, and that $X_1^\alpha \widehat{R} + \cdots + X_s^\alpha \widehat{R} \subset (R : \widehat{R})$. Since $\widehat{R}/(X_1, \dots, X_s)$ is zero-dimensional, $\widehat{R}/(R : \widehat{R})$ is zero-dimensional. Therefore we see that $\widehat{R}/(R : \widehat{R})$ is zero-dimensional and noetherian, and hence an artinian ring. By Remark 6.9, all assumptions of Theorem 6.7 are satisfied. □

We end with a simple example of a v-noetherian G-monoid H with $(H:\widehat{H}) \neq \emptyset$ that fails to be locally tame.

Example 6.11. Let $P = \{p_1, p_2, p_3\}$ be a set with three elements. Put $F = \mathcal{F}(P)$, and let

$$H = p_1^2 F \cup \{(p_2 p_3)^k \mid k \in \mathbb{N}_0\} \subset F.$$

1. H is a G-monoid with $\widehat{H} = F$ and $(H:\widehat{H}) \neq \emptyset$.
2. H is isomorphic to the associated reduced monoid of a divisor-closed submonoid of the domain $R = \mathbb{Q}[X^2, X^3] \subset \mathbb{Q}[X]$.
3. H is a weakly C-monoid that fails to be locally tame.
4. R and $\overline{R} = \mathbb{Q}[X]$ are both one-dimensional and noetherian, $(R:\overline{R}) = X^2$, $\mathcal{C}(\overline{R}) = 0$, and $R/(X^2)$ is artinian. Nevertheless, R fails to be locally tame.

Proof. Since $p_1^2 F \subset H$ it follows that $(H:F) \neq \emptyset$. Therefore we see that $\widehat{H} = F$. It is checked easily that $H = [\![p_1 p_2 p_3]\!]_H$, and thus H is a G-monoid [23, Lemma 2.7.7.3]. Put $R^\bullet = R \setminus \{0\}$, $p_1' = X$, $p_2' = X+1$, and $p_3' = X-1$. Then $p_1'^2 p_2' p_3' = X^4 - X^2 \in R^\bullet$, and an easy calculation shows that the homomorphism $H \to ([\![X^4 - X^2]\!]_{R^\bullet})_{\mathrm{red}}$ induced by $p_i \mapsto p_i'$ is an isomorphism.

Since R is noetherian, $R^\bullet$ is v-noetherian. Hence all divisor-closed submonoids of $R^\bullet$ are v-noetherian (Lemma 3.5.1). It follows that H is v-noetherian, and Proposition 4.7.1 implies that H is a weakly C-monoid.

To prove that H fails to be locally tame we show that H does not satisfy **(C3)**. Clearly, $T = \{(p_2 p_3)^k \mid k \in \mathbb{N}_0\} \subset H$ is a divisor closed submonoid with $T = \widehat{T} \cong (\mathbb{N}_0, +)$, and Lemma 3.7.2.(d) implies that $|\operatorname{supp}(\widehat{T} \setminus \widehat{T}^\times)| = 1$. Since $[\![T]\!]_{\widehat{H}} = \mathcal{F}(\{p_2, p_3\})$, the class group of $\widehat{T} \subset [\![T]\!]_{\widehat{H}}$ is isomorphic to $\mathbb{Z}$, and therefore **(C3)** does not hold.

Clearly, $(R:\overline{R}) = X^2$, and $\mathcal{C}(\overline{R}) = 0$. As a factor ring of a one-dimensional noetherian domain $R/(X^2)$ is artinian. Since R has a divisor-closed submonoid that is not locally tame, R fails to be locally tame. □

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