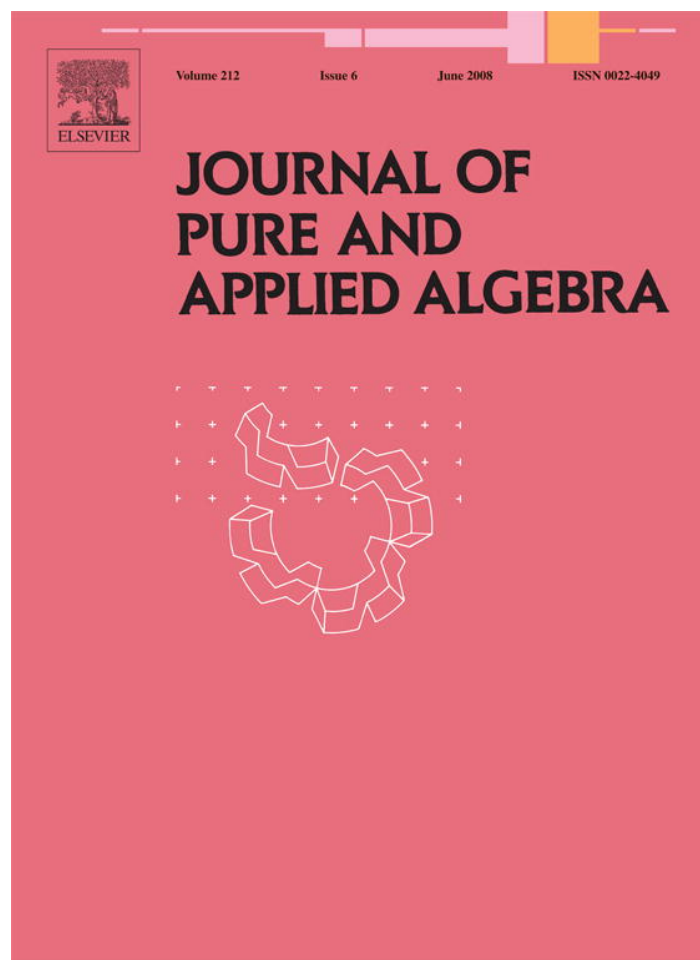


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# Local tameness of $v$ -noetherian monoids

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## Abstract

Let  $H$  be a  $v$ -noetherian monoid, e.g., the multiplicative monoid  $R \setminus \{0\}$  of a noetherian domain  $R$ . We show that, for every  $b \in H$ , there exists a constant  $\omega(H, b) \in \mathbb{N}_0$  having the following property: If  $n \in \mathbb{N}$  and  $a_1, \dots, a_n \in H$  such that  $b$  divides the product  $a_1 \cdot \dots \cdot a_n$ , then  $b$  already divides a subproduct of  $a_1 \cdot \dots \cdot a_n$  consisting of at most  $\omega(H, b)$  factors. Using the  $\omega(H, \cdot)$ -quantities we derive a new characterization of local tameness – a crucial finiteness property in the theory of non-unique factorizations.

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## 1. Introduction

In this paper the term “monoid” always means a commutative cancellative semigroup with unit element. Let  $H$  be a monoid. Recall that  $H$  is said to be  $v$ -noetherian if it satisfies the ascending chain condition on  $v$ -ideals. Krull monoids and the multiplicative monoids of noetherian domains are  $v$ -noetherian. Further examples are discussed in Section 2. Let  $b \in H$ . We denote by  $\omega(H, b)$  the smallest  $N \in \mathbb{N}_0 \cup \{\infty\}$  having the following property: If  $n \in \mathbb{N}$  and  $a_1, \dots, a_n \in H$  such that  $b$  divides  $a_1 \cdot \dots \cdot a_n$ , then  $b$  already divides a subproduct of  $a_1 \cdot \dots \cdot a_n$  consisting of at most  $N$  factors. Thus, by definition,  $b$  is a prime element of  $H$  if and only if  $\omega(H, b) = 1$ . The  $\omega(H, \cdot)$ -invariants, introduced in [12], are well-established invariants in the theory of non-unique factorizations, and they appear also in the context of direct-sum decompositions of modules [6, Remark 1.6].

Suppose that  $H$  is  $v$ -noetherian, and let  $\widehat{H}$  denote its complete integral closure. One of the main results in this paper is that  $\omega(H, b) < \infty$  for all  $b \in H$  (Theorem 4.2). Furthermore, if the conductor  $(H : \widehat{H})$  of  $H$  is non-empty, then we give an explicit upper bound for  $\omega(H, b)$  (Corollary 4.3). At the end of Section 4 we provide an example of a monoid  $H$  such that  $\omega(H, b) = \infty$  for all non-units  $b \in H$  ( $H$  is constructed as a primary submonoid of  $(\mathbb{N}_0^2, +)$ , see Example 4.7).

The investigation of the  $\omega(H, \cdot)$ -invariants is part of a larger study. Local tameness (see Definition 3.1) is a basic finiteness property in the theory of non-unique factorizations, and in many situations where the finiteness of

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an arithmetical invariant such as the catenary degree or the set of distances is studied, local tameness has to be proved first. In Section 3 we introduce a new arithmetical invariant, denoted  $\tau(H, \cdot)$ , and we show that  $H$  is locally tame if and only if  $\omega(H, u) < \infty$  and  $\tau(H, u) < \infty$  for all atoms  $u \in H$  (see Theorem 3.6). Although this characterization is not hard to prove, it is, together with the finiteness of the  $\omega(H, \cdot)$ -invariants for  $v$ -noetherian monoids, of high conceptual value. Suppose, for instance, that  $H$  is a Krull monoid having the property that every element of the class group contains a prime. Then the  $\tau(H, \cdot)$ -invariants are finite if and only if the class group of  $H$  is finite, and the tame degree of  $H$  depends only on the  $\tau(H, \cdot)$ -invariants and on the Davenport constant of the class group (see Theorem 4.4 and Remark 4.5). We note that the  $\omega(H, \cdot)$  and  $\tau(H, \cdot)$ -invariants (and in particular Theorems 3.6 and 4.2) are fundamental for a detailed arithmetical analysis of a large class of  $v$ -noetherian monoids in [17].

## 2. Preliminaries

Our notation and terminology is consistent with [15]. We briefly gather some key notions and fix the notation for monoids. Let  $\mathbb{N}$  denote the set of positive integers, and let  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . For integers  $a, b \in \mathbb{Z}$  we set  $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$ . By convention, the supremum of the empty set is zero. By a *monoid* we mean a commutative cancellative semigroup with unit element. Apart from Example 4.7 we use multiplicative notation.

*Throughout this paper  $H$  denotes a monoid.*

Let  $H^\times$  denote the set of invertible elements of  $H$ ,  $H_{\text{red}} = \{aH^\times \mid a \in H\}$  the associated reduced monoid,  $\mathfrak{q}(H)$  the quotient group of  $H$ , and

$$\widehat{H} = \{x \in \mathfrak{q}(H) \mid \text{there exists } c \in H \text{ such that } cx^n \in H \text{ for all } n \in \mathbb{N}\}$$

the *complete integral closure* of  $H$ . We say that  $H$  is *completely integrally closed* if  $H = \widehat{H}$ . For a prime element  $p \in H$  we denote by  $v_p: \mathfrak{q}(H) \rightarrow \mathbb{Z}$  the  *$p$ -adic valuation*.

Let  $S \subset H$  be a submonoid. Then  $S \subset H$  is called *saturated* if  $S = \mathfrak{q}(S) \cap H$ , and it is called *divisor-closed* if, for all  $a \in S$  and all  $b \in H$ ,  $b \mid_H a$  implies that  $b \in S$ . For a subset  $T \subset H$  we denote by  $\llbracket T \rrbracket$  the smallest divisor-closed submonoid of  $H$  containing  $T$  (that is,  $\llbracket T \rrbracket$  denotes the set of all  $a \in H$  dividing some product of elements in  $T$ ). For  $a \in H$  we set  $\llbracket a \rrbracket = \llbracket \{a\} \rrbracket$ .

A subset  $X \subset H$  is called an  *$s$ -ideal* of  $H$  if  $XH = X$ . By definition,  $\emptyset$  and  $H$  are  $s$ -ideals of  $H$ . An  $s$ -ideal  $X \subset H$  is called *prime* if  $H \setminus X$  is a submonoid of  $H$ . We denote by  $s\text{-spec}(H)$  the set of all prime  $s$ -ideals of  $H$ . For subsets  $X, Y \subset \mathfrak{q}(H)$  we set

$$(Y : X) = \{a \in \mathfrak{q}(H) \mid aX \subset Y\}, \quad X^{-1} = (H : X), \quad \text{and} \quad X_v = (X^{-1})^{-1}.$$

We say that  $X \subset H$  is a  *$v$ -ideal* of  $H$  if  $X_v = X$ . We denote by  $\mathcal{I}_v(H)$  the set of all  $v$ -ideals of  $H$ , and by  $v\text{-spec}(H)$  the set of all prime  $v$ -ideals of  $H$ . The monoid  $H$  is called  *$v$ -noetherian* if it satisfies the ascending chain condition on  $v$ -ideals. If  $X \subset H$ , we call

$$\sqrt{X} = \{a \in H \mid a^n \in X \text{ for some } n \in \mathbb{N}\}$$

the *radical* of  $X$ . If  $H$  is  $v$ -noetherian and  $X \subset H$  is a  $v$ -ideal, then  $\sqrt{X}$  is a  $v$ -ideal [15, Theorem 2.2.5]. By a *radical  $v$ -ideal*  $X \subset H$  we mean a  $v$ -ideal  $X$  of  $H$  such that  $\sqrt{X} = X$ .

The monoid  $H$  is called a *Krull monoid* if it is  $v$ -noetherian and completely integrally closed. For all the terminology used in the theory of Krull monoids (such as the notions of class group and divisor theory) we refer the reader to one of the monographs [15,19,20]. By definition, all Krull monoids are  $v$ -noetherian, and we refer the reader to [15, Examples 2.3.2] for an extensive list of Krull monoids, including examples from analytic number theory and from module theory (see also [7,23]).

Clearly, the multiplicative monoid of a Mori domain is  $v$ -noetherian. Congruence monoids and C-monoids (see [15, Sections 2.9 and 2.11] and [8,9,14,17,21,22]) are purely multiplicative examples of  $v$ -noetherian monoids. We now describe two further classes of  $v$ -noetherian monoids which are – to the knowledge of the authors – not mentioned in the literature so far.

**Example 2.1.** 1. *The multiplicative monoid of regular elements of a Mori ring.* A commutative ring is called a *Mori ring* if it satisfies the ascending chain condition on regular divisorial ideals [27]. Recall [25] that a commutative ring

is called a *Marot ring* if each regular ideal of  $R$  is generated by regular elements. Every integral domain and every noetherian ring is a Marot ring [25, Theorem 7.2].

Let  $R$  be a Marot ring. We denote by  $\mathfrak{z}(R)$  the set of zero divisors of  $R$ , and by  $T$  the total quotient ring of  $R$ . For any subset  $I \subset T$  we put  $I^\bullet = I \setminus \mathfrak{z}(T)$ . Further, we set  $G = \mathfrak{q}(R^\bullet)$ . Then we obviously have  $T^\bullet = G$ .

- (a) For every regular fractional ideal  $I \subset T$  we have  $(R :_T I)^\bullet = (R^\bullet :_G I^\bullet)$ .
- (b) For every  $s$ -ideal  $\mathfrak{a} \subset R^\bullet$  we have  $(R :_T \mathfrak{a})^\bullet = (R^\bullet :_G \mathfrak{a})$ , and for every regular ideal  $I \subset R$  we have  $(R :_T I^\bullet) = (R :_T I)$ .
- (c) The assignment  $I \mapsto I^\bullet$  yields an inclusion-preserving bijection from the set of regular divisorial ideals of  $R$  to the set of  $v$ -ideals of  $R^\bullet$ .
- (d)  $R$  is a Mori ring if and only if  $R^\bullet$  is a  $v$ -noetherian monoid.

**Proof.** (a) The inclusion  $(R :_T I)^\bullet \subset (R^\bullet :_G I^\bullet)$  is obvious. Conversely, let  $z \in (R^\bullet :_G I^\bullet)$ . If  $x \in I$ , then  $x = \xi_1 + \dots + \xi_n$ , with  $\xi_i \in I^\bullet$ , and hence  $z\xi_i \in R^\bullet$ . This implies that  $zx \in R$ . Therefore  $z \in (R :_T I)^\bullet$ .

(b) Obvious.

(c) Let  $I \subset R$  be a regular divisorial ideal. Then

$$(R^\bullet :_G (R^\bullet :_G I^\bullet)) \stackrel{(a)}{=} (R^\bullet :_G (R :_T I)^\bullet) \stackrel{(a)}{=} (R :_T (R :_T I))^\bullet = I^\bullet,$$

and hence  $I^\bullet \subset R^\bullet$  is a  $v$ -ideal. Conversely, let  $\mathfrak{a} \subset R^\bullet$  be a  $v$ -ideal. Then  $(R :_T (R :_T \mathfrak{a})) \subset R$  is a divisorial ideal, and

$$(R :_T (R :_T \mathfrak{a}))^\bullet \stackrel{(a)}{=} (R^\bullet :_G (R :_T \mathfrak{a})^\bullet) \stackrel{(b)}{=} (R^\bullet :_G (R^\bullet :_G \mathfrak{a})) = \mathfrak{a}.$$

Therefore the map  $I \mapsto I^\bullet$  is bijective, and clearly it is inclusion-preserving.

Finally, (d) is an immediate consequence of (c).  $\square$

2. *The monoid of  $r$ -invertible  $r$ -ideals.* We use the the same terminology as in [20]. Let  $r$  be an ideal system on  $H$ ,  $\mathcal{F}_r(H)$  the set of fractional  $r$ -ideals,  $(\mathcal{F}_r(H)^\times, \cdot_r)$  the group of  $r$ -invertible fractional  $r$ -ideals endowed with  $r$ -multiplication, and  $\mathcal{I}_r^*(H) = \{\mathfrak{a} \in \mathcal{F}_r(H)^\times \mid \mathfrak{a} \subset H\}$  the monoid of  $r$ -invertible (integral)  $r$ -ideals of  $H$ .

- (a) If  $(\mathcal{I}_r^*(H), \cdot_r)$  is  $v$ -noetherian, then  $H$  is  $v$ -noetherian.
- (b) If  $H$  is  $r$ -noetherian, then  $(\mathcal{I}_r^*(H), \cdot_r)$  is  $v$ -noetherian.
- (c) If  $R$  is a noetherian domain, then  $\mathcal{I}^*(R)$ , the monoid of invertible ideals endowed with the usual ideal multiplication, is  $v$ -noetherian.

**Proof.** (a) Obviously, the map  $\partial : H \rightarrow \mathcal{I}_r^*(H)$ ,  $a \mapsto aH$ , is a cofinal divisor homomorphism. Thus if  $\mathcal{I}_r^*(H)$  is  $v$ -noetherian, then  $H$  is  $v$ -noetherian [15, Proposition 2.4.4.2].

(b) Suppose that  $H$  is  $r$ -noetherian and set  $D = \mathcal{I}_r^*(H)$ . Then  $\mathfrak{q}(D) = \mathcal{F}_r(H)^\times$ ,  $\mathcal{F}_v(H) \subset \mathcal{F}_r(H)$  (see [20, Corollary 11.4]), and  $\mathcal{F}_r(H)^\times \subset \mathcal{F}_v(H)^\times$  is a subgroup [20, Theorem 12.1]. In particular,  $H$  is  $v$ -noetherian, and, on  $\mathcal{F}_r(H)^\times$ , the  $r$ -multiplication coincides with the  $v$ -multiplication.

Let  $X = \{\mathfrak{a}_\lambda \in D \mid \lambda \in \Lambda\} \subset D$  be a non-empty subset. We shall prove that there exists a finite set  $E \subset X$  such that  $E^{-1} = X^{-1}$ . Since  $H$  is  $v$ -noetherian, the non-empty set

$$\Omega = \{(\cup_{\lambda \in \Lambda_0} \mathfrak{a}_\lambda)_v \mid \Lambda_0 \subset \Lambda \text{ is finite}\}$$

of  $v$ -ideals of  $H$  has a maximal element  $(\cup_{\lambda \in \Lambda^*} \mathfrak{a}_\lambda)_v$ . Then

$$(\cup_{\lambda \in \Lambda^*} \mathfrak{a}_\lambda)_v = (\cup_{\lambda \in \Lambda} \mathfrak{a}_\lambda)_v,$$

and we obtain

$$\begin{aligned} X^{-1} &= \{\mathfrak{c} \in \mathfrak{q}(D) \mid \mathfrak{c} \cdot_v \mathfrak{a}_\lambda \in D \text{ for all } \lambda \in \Lambda\} \\ &= \{\mathfrak{c} \in \mathcal{F}_v(H)^\times \mid \mathfrak{c} \cdot_v \mathfrak{a}_\lambda \subset H \text{ for all } \lambda \in \Lambda\} \\ &= \{\mathfrak{c} \in \mathcal{F}_v(H)^\times \mid \mathfrak{c} \cdot_v (\cup_{\lambda \in \Lambda} \mathfrak{a}_\lambda)_v \subset H\} \\ &= \{\mathfrak{c} \in \mathcal{F}_v(H)^\times \mid \mathfrak{c} \cdot_v (\cup_{\lambda \in \Lambda^*} \mathfrak{a}_\lambda)_v \subset H\} \\ &= \{\mathfrak{a}_\lambda \in D \mid \lambda \in \Lambda^*\}^{-1}. \end{aligned}$$

(c) This follows from (b) by taking  $r = d$ , where the  $d$ -system is the system of usual ring ideals.  $\square$

Next we recall some basic arithmetical notions from factorization theory. If  $P$  is a set, we denote by  $\mathcal{F}(P)$  the free (abelian) monoid generated by  $P$ . We denote by  $\mathcal{A}(H)$  the set of atoms of  $H$ , and we call  $\mathbf{Z}(H) = \mathcal{F}(\mathcal{A}(H_{\text{red}}))$  the factorization monoid of  $H$ . Further,  $\pi: \mathbf{Z}(H) \rightarrow H_{\text{red}}$  denotes the natural homomorphism. For  $a \in H$  the set

$$\mathbf{Z}(a) = \pi^{-1}(aH^\times) \subset \mathbf{Z}(H) \quad \text{is called the set of factorizations of } a, \text{ and}$$

$$\mathbf{L}(a) = \{|z| \mid z \in \mathbf{Z}(a)\} \subset \mathbb{N}_0 \quad \text{is called the set of lengths of } a.$$

$H$  is said to be

- atomic if  $\mathbf{Z}(a) \neq \emptyset$  for all  $a \in H$ ,
- half-factorial if  $|\mathbf{L}(a)| = 1$  for every  $a \in H$ ,
- a BF-monoid if  $H$  is atomic and  $\mathbf{L}(a)$  is finite for every  $a \in H$ , and
- an FF-monoid if  $H$  is atomic and  $\mathbf{Z}(a)$  is finite for every  $a \in H$ .

For  $k \in \mathbb{N}$  we set  $\rho_k(H) = k$  if  $H = H^\times$ , and

$$\rho_k(H) = \sup \{ \sup \mathbf{L}(a) \mid a \in H, k \in \mathbf{L}(a) \} \in \mathbb{N} \cup \{\infty\} \quad \text{if } H \neq H^\times.$$

Then

$$\rho(H) = \sup \left\{ \frac{\rho_k(H)}{k} \mid k \in \mathbb{N} \right\} = \lim_{k \rightarrow \infty} \frac{\rho_k(H)}{k}$$

is the elasticity of  $H$  (cf. [15, Proposition 1.4.2 and Section 6.3]). By definition,  $H$  is half-factorial if and only if  $\rho(H) = 1$ .

Let  $z, z' \in \mathbf{Z}(H)$ . Then we can write

$$z = u_1 \cdot \dots \cdot u_l v_1 \cdot \dots \cdot v_m \quad \text{and} \quad z' = u_1 \cdot \dots \cdot u_l w_1 \cdot \dots \cdot w_n,$$

where  $l, m, n \in \mathbb{N}_0, u_1, \dots, u_l, v_1, \dots, v_m, w_1, \dots, w_n \in \mathcal{A}(H_{\text{red}})$  such that

$$\{v_1, \dots, v_m\} \cap \{w_1, \dots, w_n\} = \emptyset.$$

We call  $\mathbf{d}(z, z') = \max\{m, n\} \in \mathbb{N}_0$  the distance between  $z$  and  $z'$ .

### 3. Local tameness

In this section we recall the definitions of local tameness and the  $\omega(H, \cdot)$ -invariants, and we introduce the  $\tau(H, \cdot)$ -invariants. In Theorem 3.6 we show that local tameness can be characterized in terms of  $\tau(H, \cdot)$  and  $\omega(H, \cdot)$ . For general information on local tameness and its relevance in factorization theory we refer to [15]. Recent results on this invariant can be found in [1,3,4,17,18].

**Definition 3.1.** Suppose that  $H$  is atomic.

1. For  $a, b \in H$  let  $\omega(a, b)$  denote the smallest  $N \in \mathbb{N}_0 \cup \{\infty\}$  with the following property:  
For all  $n \in \mathbb{N}$  and  $a_1, \dots, a_n \in H$ , if  $a = a_1 \cdot \dots \cdot a_n$  and  $b \mid a$ , then there exists a subset  $\Omega \subset [1, n]$  such that  $|\Omega| \leq N$  and

$$b \mid \prod_{v \in \Omega} a_v.$$

In particular, if  $b \nmid a$ , then  $\omega(a, b) = 0$ . For  $b \in H$  we define

$$\omega(H, b) = \sup \{ \omega(a, b) \mid a \in H \} \in \mathbb{N}_0 \cup \{\infty\}.$$

2. For  $k \in \mathbb{N}$  and  $b \in H$  we set

$$\tau_k(H, b) = \sup \{ \min \mathbf{L}(b^{-1}a) \mid a = u_1 \cdot \dots \cdot u_j \in bH \text{ with } j \in [0, k], u_1, \dots, u_j \in \mathcal{A}(H),$$

$$\text{and } b \nmid u_i^{-1}a \text{ for all } i \in [1, j] \} \in \mathbb{N}_0 \cup \{\infty\}$$

and

$$\tau(H, b) = \sup \{ \tau_k(H, b) \mid k \in \mathbb{N} \} \in \mathbb{N}_0 \cup \{\infty\}.$$

3. For  $a \in H$  and  $x \in Z(H)$  let  $t(a, x) \in \mathbb{N}_0 \cup \{\infty\}$  denote the smallest  $N \in \mathbb{N}_0 \cup \{\infty\}$  with the following property:  
 If  $Z(a) \cap xZ(H) \neq \emptyset$  and  $z \in Z(a)$ , then there exists  $z' \in Z(a) \cap xZ(H)$  such that  $d(z, z') \leq N$ .  
 For subsets  $H' \subset H$  and  $X \subset Z(H)$ , we define

$$t(H', X) = \sup \{t(a, x) \mid a \in H', x \in X\} \in \mathbb{N}_0 \cup \{\infty\}.$$

$H$  is called *locally tame* if  $t(H, u) < \infty$  for all  $u \in \mathcal{A}(H_{\text{red}})$ .

Let  $b \in H$  and  $k \in \mathbb{N}$ . Clearly, we have  $\tau_k(H, b) = \tau_k(H_{\text{red}}, bH^\times)$  and  $\omega(H, b) = \omega(H_{\text{red}}, bH^\times)$ . If  $b \in H^\times$ , then  $\omega(H, b) = \tau_k(H, b) = 0$ . If  $b = p_1 \cdot \dots \cdot p_k$  is a product of primes  $p_1, \dots, p_k \in H$ , then  $\omega(H, b) = k$  and  $\tau(H, b) = 0$ .

**Lemma 3.2.** *Suppose that  $H$  is atomic.*

1. Let  $b \in H$ . Then  $0 = \tau_1(H, b) \leq \tau_2(H, b) \leq \dots$ , and if  $\omega(H, b) < \infty$ , then  $\tau_{\omega(H, b)}(H, b) = \tau(H, b)$ .
2. If  $b \in H$ ,  $k \in \mathbb{N}$  and  $m \in L(b)$ , then  $\tau_k(H, b) \leq \max\{0, \rho_k(H) - m\}$ .
3. If  $k \in \mathbb{N}$ ,  $b \in H$  and  $a \in bH$  with  $\min L(a) \leq k$ , then  $\min L(b^{-1}a) \leq \tau_k(H, b) + k$ .
4. If  $\tau_k(H, u) < \infty$  for all  $u \in \mathcal{A}(H)$  and all  $k \in \mathbb{N}$ , then  $\tau_k(H, b) < \infty$  for all  $b \in H$  and all  $k \in \mathbb{N}$ . In particular, if  $\tau(H, u) < \infty$  for all  $u \in \mathcal{A}(H)$ , then  $\tau(H, b) < \infty$  for all  $b \in H$  with  $\omega(H, b) < \infty$ .

**Proof.** Without loss of generality we may suppose that  $H$  is reduced.

1. If  $k, l \in \mathbb{N}$  with  $k \leq l$ , then it follows by the very definition of the  $\tau_k(H, \cdot)$ -quantities that  $\tau_k(H, b) \leq \tau_l(H, b)$ . Thus it remains to verify that  $0 = \tau_1(H, b)$ , and if  $\omega(H, b) < \infty$ , then  $\tau_{\omega(H, b)}(H, b) = \tau(H, b)$ . If  $b = 1$ , then both assertions are clear. Hence suppose that  $b \in H \setminus \{1\}$ ,  $k \in \mathbb{N}$ , and  $a \in bH$  such that  $a = u_1 \cdot \dots \cdot u_j$  with  $j \in [0, k]$ ,  $u_1, \dots, u_j \in \mathcal{A}(H)$ , and  $b \nmid u_i^{-1}a$  for all  $i \in [1, j]$ . Then  $j \leq \omega(H, b)$ , and hence it follows that  $\tau_l(H, b) = \tau(H, b)$  for all  $l \geq \omega(H, b)$ . If  $k = 1$ , then  $j = 1$ ,  $a = u_1 = b$ ,  $b^{-1}a = 1$ ,  $L(b^{-1}a) = \{0\}$ , and thus  $\tau_1(H, b) = 0$ .

2. Suppose there exists  $a \in bH$  such that  $a = u_1 \cdot \dots \cdot u_j$  with  $j \in [0, k]$ ,  $u_1, \dots, u_j \in \mathcal{A}(H)$  and  $b \nmid u_i^{-1}a$  for all  $i \in [1, j]$ . Let  $m \in L(b)$ . Then  $L(b^{-1}a) + L(b) \subset L(a)$ , and hence  $\min L(b^{-1}a) + m \leq \sup L(a) \leq \rho_k(H)$ . If there is no such  $a$ , then  $\tau_k(H, b) = \sup \emptyset = 0$ .

3. Let  $k \in \mathbb{N}$ ,  $b \in H$  and  $a \in bH$  with  $\min L(b) \leq k$ . If  $a = 1$ , then  $L(b^{-1}a) = \{0\}$  and the assertion follows. Otherwise  $a$  has a factorization of the form  $a = u_1 \cdot \dots \cdot u_j$ , where  $j \in [1, k]$  and  $u_i \in \mathcal{A}(H)$  for all  $i \in [1, j]$ . Let  $J \subset [1, j]$  be a subset such that  $b \mid \prod_{i \in J} u_i$  and such that  $b \nmid \prod_{i \in J'} u_i$  for all proper subsets  $J'$  of  $J$ . Put  $a^* = \prod_{i \in J} u_i$ . Then  $a^* \in bH$  and  $a^* u_i^{-1} \notin bH$  for all  $i \in J$ . Therefore we have

$$\min L(b^{-1}a^*) \leq \tau_k(H, b),$$

and we obtain

$$\min L(b^{-1}a) \leq \tau_k(H, b) + j - |J| \leq \tau_k(H, b) + k.$$

4. Suppose that  $\tau_k(H, u) < \infty$  for all  $u \in \mathcal{A}(H)$  and all  $k \in \mathbb{N}$ . We prove the first statement in 4. by induction on  $\min L(b)$ . Let  $b \in H$  and  $k \in \mathbb{N}$ , and suppose that  $a \in bH$  such that  $a$  has a decomposition  $a = u_1 \cdot \dots \cdot u_j$ , with  $j \in [0, k]$  and  $u_i \in \mathcal{A}(H)$  for all  $i \in [1, j]$ , and such that  $b \nmid u_i^{-1}a$  for all  $i \in [1, j]$ . Let  $v_1 \cdot \dots \cdot v_n \in Z(b)$  be a factorization of  $b$  into atoms  $v_i$  of  $H$  such that  $n = \min L(b)$ . Then  $a \in v_1 H$ , and by 3. we obtain

$$\min L(v_1^{-1}a) \leq \tau_k(H, v_1) + k.$$

Put  $c = v_1^{-1}a \in H$ , and define  $l = \tau_k(H, v_1) + k$ . Then  $l$  is finite since  $\tau_k(H, u) < \infty$  for all  $u \in \mathcal{A}(H)$ . Since  $v_1^{-1}a \in cH$ , a second application of 3. yields

$$\min L(b^{-1}a) = \min L(c^{-1}v_1^{-1}a) \leq \tau_l(H, c) + l.$$

By the induction hypothesis it follows that  $\tau_l(H, c)$  is finite. Therefore we obtain  $\tau_k(H, b) \leq \tau_l(H, c) + l < \infty$ . The ‘‘In particular, ...’’ statement now follows by taking into account that  $\tau(H, b) = \tau_{\omega(H, b)}(H, b)$  if  $\omega(H, b)$  is finite.  $\square$

**Lemma 3.3.** *Suppose that  $H$  is atomic.*

1. If  $b_1, b_2 \in H$ , then  $\omega(H, b_1) \leq \omega(H, b_1 b_2) \leq \omega(H, b_1) + \omega(H, b_2)$ .

2. Let  $U \subset H$  be a subset such that  $H = \llbracket U \rrbracket$  and  $\omega(H, u) < \infty$  for all  $u \in U$ .

Then  $\omega(H, b) < \infty$  for all  $b \in H$ .

3. For all  $b \in H$  we have  $\sup \mathbf{L}(b) \leq \omega(H, b)$ . In particular, if  $\omega(H, b) < \infty$  for all  $b \in H$ , then  $H$  is a BF-monoid.

**Proof.** 1. Let  $n \in \mathbb{N}$  and  $b_1, b_2, a_1, \dots, a_n \in H$  such that  $b_1 b_2 \mid a_1 \cdot \dots \cdot a_n$ . Then there exists a subset  $\Omega_1 \subset [1, n]$ , say  $\Omega_1 = [1, m_1]$ , such that  $m_1 \leq \omega(H, b_1)$  and  $b_1 \mid a_1 \cdot \dots \cdot a_{m_1}$ . Then

$$b_2 \mid (b_1^{-1} a_1 \cdot \dots \cdot a_{m_1} a_{m_1+1} \cdot \dots \cdot a_n),$$

and there exists  $\Omega_2 \subset [m_1 + 1, n]$ , say  $\Omega_2 = [m_1 + 1, m_1 + m_2]$ , such that  $m_2 \leq \omega(H, b_2)$  and

$$b_2 \mid (b_1^{-1} a_1 \cdot \dots \cdot a_{m_1} a_{m_1+1} \cdot \dots \cdot a_{m_1+m_2}).$$

Then  $b_1 b_2 \mid a_1 \cdot \dots \cdot a_{m_1+m_2}$ , and the second inequality in 1. follows.

To prove the first inequality in 1. let  $n \in \mathbb{N}$  and  $b_1, a_1, \dots, a_n \in H$  such that  $b_1 \mid a_1 \cdot \dots \cdot a_n$ . Then  $b_1 b_2 \mid a_0 a_1 \cdot \dots \cdot a_n$ , where  $a_0 = b_2$ , and there exists  $\Omega \subset [0, n]$  such that  $|\Omega| \leq \omega(H, b_1 b_2)$  and

$$b_1 b_2 \mid \prod_{v \in \Omega} a_v.$$

Then

$$b_1 \mid \prod_{v \in \Omega \setminus \{0\}} a_v,$$

and we see that  $\omega(H, b_1) \leq |\Omega \setminus \{0\}| \leq |\Omega| \leq \omega(H, b_1 b_2)$ .

2. Let  $b \in H$  and  $u_1, \dots, u_k \in U$  such that  $b \mid u_1 \cdot \dots \cdot u_k$ . Then 1. implies that

$$\omega(H, b) \leq \omega(H, u_1 \cdot \dots \cdot u_k) \leq \omega(H, u_1) + \dots + \omega(H, u_k) < \infty.$$

3. Assume to the contrary that there is an element  $b \in H$  with  $\sup \mathbf{L}(b) > \omega(H, b)$ . Then there are  $n \in \mathbb{N}$  and  $u_1, \dots, u_n \in \mathcal{A}(H)$  such that  $b = u_1 \cdot \dots \cdot u_n$  and  $n > \omega(H, b)$ . This implies that there is a subset  $\Omega \subset [1, n]$  with  $|\Omega| \leq \omega(H, b) < n$  such that

$$b \mid \prod_{v \in \Omega} u_v,$$

a contradiction.  $\square$

We point out two important special cases where the assumption of Lemma 3.3.2 is satisfied. First, since  $H$  is atomic, we clearly have  $H = \llbracket \mathcal{A}(H) \rrbracket$ . Therefore, if  $\omega(H, u) < \infty$  for all atoms  $u \in \mathcal{A}(H)$ , then  $\omega(H, b) < \infty$  for all  $b \in H$ . Second, suppose that  $H$  is a G-monoid (see [15, Definition 2.7.6]). Then there exists  $a \in H$  such that  $H = \llbracket a \rrbracket$ , and then  $\omega(H, a) < \infty$  implies that  $\omega(H, b) < \infty$  for every  $b \in H$ .

**Lemma 3.4.** Suppose that  $H$  and  $D$  are atomic monoids with  $H \subset D$ , and let  $b \in H$ .

1. If  $H \subset D$  is saturated, then  $\omega(H, b) \leq \omega(D, b)$ .

2. Suppose that  $f \in (H : D)$ .

(a) If  $n \in \mathbb{N}$ ,  $a_1, \dots, a_n \in H$  and  $c \in D$  such that  $ca_1 \cdot \dots \cdot a_n \in H$ , then there exists  $\Omega \subset [1, n]$  such that  $|\Omega| \leq \omega(H, f)$  and

$$c \prod_{v \in \Omega} a_v \in H.$$

(b)  $\omega(H, b) \leq \omega(D, b) + \omega(H, f)$ .

(c) Let  $F = F^\times \times \mathcal{F}(P)$  be a factorial monoid. If  $D \subset F$  is saturated, then  $\omega(H, b) \leq \sum_{p \in P} \nu_p(b) + \omega(H, f)$ .

**Proof.** Let  $n \in \mathbb{N}$  and  $a_1, \dots, a_n \in H$  such that  $b \mid a_1 \cdot \dots \cdot a_n$  in  $H$ .

1. There exists  $\Omega \subset [1, n]$ , say  $\Omega = [1, m]$ , with  $m \leq \omega(D, b)$  such that  $b \mid_D a_1 \cdot \dots \cdot a_m$ . Hence  $b \mid_H a_1 \cdot \dots \cdot a_m$ , and thus we have  $\omega(H, b) \leq \omega(D, b)$ .

2.(a) Since  $f \mid_H (fc)a_1 \cdot \dots \cdot a_n$  there exist  $\Omega \subset [1, n]$  with  $|\Omega| \leq \omega(H, f)$  and

$$f \mid_H (fc) \prod_{v \in \Omega} a_v.$$

It follows that  $c \prod_{v \in \Omega} a_v \in H$ .

2.(b) Let  $m$  and  $\Omega$  be as in 1, and set  $a_1 \cdot \dots \cdot a_m = bc$  with  $c \in D$ . Since  $b$  divides  $a_1 \cdot \dots \cdot a_n$  in  $H$  it follows that  $ca_{m+1} \cdot \dots \cdot a_n \in H$ . By 2.(a) there exists  $\Omega \subset [m+1, n]$ , say  $\Omega = [m+1, m+k]$ , with  $k \leq \omega(H, f)$  such that  $b^{-1}a_1 \cdot \dots \cdot a_{m+k} = ca_{m+1} \cdot \dots \cdot a_{m+k} \in H$ . From this the assertion follows.

2.(c) Since  $\omega(D, b) \leq \sum_{p \in P} \nu_p(b)$  the assertion follows from 2.(b).  $\square$

**Lemma 3.5.** Suppose that  $H$  is atomic and reduced. Let  $u \in \mathcal{A}(H)$  and  $a \in H$ .

1. If  $a \in H \setminus uH$ , then  $(\omega(a, u), \mathfrak{t}(a, u)) = (0, 0)$ . If  $a \in uH$ , then either  $(\omega(a, u), \mathfrak{t}(a, u)) = (1, 0)$  or  $2 \leq \omega(a, u) \leq \mathfrak{t}(a, u)$ .

2. If  $u$  is a prime, then  $(\omega(H, u), \mathfrak{t}(H, u)) = (1, 0)$ . Otherwise, we have  $2 \leq \omega(H, u) \leq \mathfrak{t}(H, u)$ .

3. If  $\omega(a, u) < \infty$ , then we have  $\mathfrak{t}(a, u) \leq \max\{\omega(a, u), 1 + \tau_{\omega(a, u)}(H, u)\} \leq \max\{\omega(H, u), 1 + \tau(H, u)\}$ , and hence

$$\mathfrak{t}(H, u) \leq \max\{\omega(H, u), 1 + \tau(H, u)\}.$$

4. If  $u$  is not a prime, then  $1 + \tau(H, u) \leq \mathfrak{t}(H, u)$ .

**Proof.** 1. If  $a \in H \setminus uH$ , then, by definition,  $(\omega(a, u), \mathfrak{t}(a, u)) = (0, 0)$ . Now let  $a \in uH$ . Suppose that in all product decompositions of  $a$  of the form  $a = a_1 \cdot \dots \cdot a_n$ , with  $n \in \mathbb{N}$  and  $a_1, \dots, a_n \in H$ , there is an  $i \in [1, n]$  such that  $u \mid a_i$ . Then we clearly have  $(\omega(a, u), \mathfrak{t}(a, u)) = (1, 0)$ . Conversely, suppose that there exists a product decomposition of  $a$  without this property. Then we have  $\omega(a, u) \geq 2$ , and it remains to show that  $\omega(a, u) \leq \mathfrak{t}(a, u)$ .

Suppose that  $\omega(a, u) = k \geq 2$ . Then  $a$  has a product decomposition  $a = a_1 \cdot \dots \cdot a_n$ , with  $n \in \mathbb{N}$  and  $a_1, \dots, a_n \in H$ , such that  $u$  divides a subproduct of  $k$  factors, but  $u$  does not divide any subproduct of  $k - 1$  factors. For every  $i \in [1, n]$  we choose a factorization  $z_i \in \mathcal{Z}(a_i)$ . Then we obtain a factorization  $z = z_1 \cdot \dots \cdot z_n \in \mathcal{Z}(a)$ . Choose  $z' \in \mathcal{Z}(a) \cap u\mathcal{Z}(H)$  such that  $\mathfrak{d}(z, z')$  is minimal. Then it follows that  $k \leq \mathfrak{d}(z, z') \leq \mathfrak{t}(a, u)$ .

2. If  $u$  is a prime, then  $(\omega(H, u), \mathfrak{t}(H, u)) = (1, 0)$ . Suppose that  $u$  is not a prime. Then there are  $a_1, a_2 \in H$  such that  $u \mid a_1 a_2$  but  $u \nmid a_1$  and  $u \nmid a_2$ . It follows that  $2 \leq \omega(a_1 a_2, u) \leq \omega(H, u)$ , and 1. implies that

$$\begin{aligned} \omega(H, u) &= \sup\{\omega(a, u) \mid a \in H \text{ with } \omega(a, u) \geq 2\} \\ &\leq \sup\{\mathfrak{t}(a, u) \mid a \in H \text{ with } \mathfrak{t}(a, u) \geq 2\} = \mathfrak{t}(H, u). \end{aligned}$$

3. Clearly, it is sufficient to verify the first inequality. If  $a \notin uH$ , then  $\mathfrak{t}(a, u) = 0$ , and the assertion follows. Now let  $a \in uH$  and  $z = u_1 \cdot \dots \cdot u_n \in \mathcal{Z}(a)$  with  $n \in \mathbb{N}$  and  $u_1, \dots, u_n \in \mathcal{A}(H)$ . After a renumbering of the indices if necessary, we may suppose that  $u \mid u_1 \cdot \dots \cdot u_j$ , where  $j \leq \omega(a, u)$  and  $u \nmid u_i^{-1} u_1 \cdot \dots \cdot u_j$  for all  $i \in [1, j]$ . Then  $u_1 \cdot \dots \cdot u_j = uv_1 \cdot \dots \cdot v_l$ , with  $v_1, \dots, v_l \in \mathcal{A}(H)$  and  $l \leq \tau_{\omega(a, u)}(H, u)$ . If we set  $z' = uv_1 \cdot \dots \cdot v_l u_{j+1} \cdot \dots \cdot u_n \in \mathcal{Z}(a) \cap u\mathcal{Z}(H)$ , we obtain

$$\mathfrak{d}(z, z') \leq \max\{j, 1 + l\} \leq \max\{\omega(a, u), 1 + \tau_{\omega(a, u)}(H, u)\}.$$

It follows that  $\mathfrak{t}(H, u) \leq \max\{\omega(H, u), 1 + \tau(H, u)\}$ .

4. Since  $u$  is not a prime, we have  $2 \leq \omega(H, u) \leq \mathfrak{t}(H, u)$ . Further,  $\tau_1(H, u) = 0$ . Thus it suffices to verify that

$$1 + \min L(u^{-1}a) \leq \mathfrak{t}(a, u) \leq \mathfrak{t}(H, u)$$

for all  $a \in uH$  having a factorization of the form  $a = u_1 \cdot \dots \cdot u_j$  such that  $j \in \mathbb{N}_{\geq 2}$  and  $u \nmid u_i^{-1} a$  for all  $i \in [1, j]$ . We pick such an  $a \in uH$ . Then  $z = u_1 \cdot \dots \cdot u_j \in \mathcal{Z}(a)$ . By the definition of  $\mathfrak{t}(a, u)$  there exists  $z' \in \mathcal{Z}(a) \cap u\mathcal{Z}(H)$ , say  $z' = uv_1 \cdot \dots \cdot v_l$  with  $l \in \mathbb{N}$  and  $v_1, \dots, v_l \in \mathcal{A}(H)$ , such that  $\mathfrak{d}(z, z') \leq \mathfrak{t}(a, u)$ . Since  $u \nmid u_i^{-1} a$  for all  $i \in [1, j]$ , we infer that  $\{u_1, \dots, u_j\} \cap \{v_1, \dots, v_l\} = \emptyset$ . Therefore we obtain

$$1 + \min L(u^{-1}a) \leq 1 + l \leq \max\{j, 1 + l\} = \mathfrak{d}(z, z') \leq \mathfrak{t}(a, u). \quad \square$$



**Theorem 3.6.** *Suppose that  $H$  is atomic and  $u \in \mathcal{A}(H)$ . If  $u$  is prime, then  $(\mathfrak{t}(H, uH^\times), \omega(H, u), \tau_1(H, u)) = (0, 1, 0)$ , and otherwise*

$$\mathfrak{t}(H, uH^\times) = \max \{ \omega(H, u), 1 + \tau(H, u) \} \in \mathbb{N}_{\geq 2} \cup \{ \infty \}.$$

*In particular,  $H$  is locally tame if and only if  $\omega(H, v) < \infty$  and  $\tau(H, v) < \infty$  for all  $v \in \mathcal{A}(H)$ .*

**Proof.** This follows immediately from Lemma 3.5.  $\square$

In Theorem 4.2 we show that for  $v$ -noetherian monoids the  $\omega(H, \cdot)$ -invariants are finite, and in Theorem 4.4 we show that in Krull monoids for which every class contains a prime the tame degree depends only on the  $\tau(H, \cdot)$ -invariants. However, these  $\tau(H, \cdot)$ -invariants are very difficult to study (see Remark 4.5 and the investigations in [17]). Therefore we introduce  $\tau^*(H, \cdot)$ -invariants which, in general, are larger than the  $\tau(H, \cdot)$ -invariants, but they are easier to study and they still control local tameness (see Proposition 3.8.3). Apart from these facts, they are of interest in their own right. Note that they are defined in the style of the elasticities  $\rho(H)$  and  $\rho_k(H)$  (cf. [15, Section 1.4]) which are among the best investigated invariants in the theory of non-unique factorizations.

**Definition 3.7.** Suppose that  $H$  is atomic, and let  $b \in H$ .

1. For  $k \in \mathbb{N}$  we set

$$\tau_k^*(H, b) = \sup \left\{ \min \mathsf{L}(b^{-1}a) \mid a \in bH, \min \mathsf{L}(a) \leq k \right\} \in \mathbb{N}_0 \cup \{ \infty \}.$$

2.

$$\tau^*(H, b) = \sup \left\{ \frac{\min \mathsf{L}(b^{-1}a)}{\min \mathsf{L}(a)} \mid a \in bH \right\} \in \mathbb{R}_{\geq 0} \cup \{ \infty \}.$$

Suppose that  $H$  is atomic,  $H \neq H^\times$ ,  $b \in H$ , and  $k \in \mathbb{N}$ . Clearly, we have  $\tau_k^*(H, b) = \tau_k^*(H_{\text{red}}, bH^\times)$  and  $\tau^*(H, b) = \tau^*(H_{\text{red}}, bH^\times)$ . If  $b \in H^\times$ , then  $\tau_k^*(H, b) = k$  and  $\tau^*(H, b) = 1$ . If  $b = p_1 \cdots p_k$  is a product of primes  $p_1, \dots, p_k \in H$ ,  $l \geq k$ , and  $a \in bH$ , then  $k + \min \mathsf{L}(b^{-1}a) = \min \mathsf{L}(a)$ ,  $\tau_l^*(H, b) = l - k$ , and  $\tau^*(H, b) = 1$ .

**Proposition 3.8.** *Suppose that  $H$  is atomic, and let  $u \in \mathcal{A}(H)$ .*

1. *We have  $\tau_1^*(H, u) = 0$ ,  $\tau_2^*(H, u) \geq 1$ , and*

$$\tau^*(H, u) = \sup \left\{ \frac{\tau_k^*(H, u)}{k} \mid k \in \mathbb{N} \right\}.$$

2. *For every  $k \in \mathbb{N}$  we have  $\tau_k(H, u) \leq \tau_k^*(H, u) \leq \tau_k(H, u) + k$ .*

3. *For every  $a \in H$  we have  $\mathfrak{t}(a, u) \leq \max \{ \omega(a, u), \omega(a, u)\tau^*(H, u) + 1 \}$ , and further  $\mathfrak{t}(H, u) \leq \max \{ \omega(H, u), \omega(H, u)\tau^*(H, u) + 1 \}$ . In particular,  $H$  is locally tame if and only if  $\omega(H, v) < \infty$  and  $\tau^*(H, v) < \infty$  for all  $v \in \mathcal{A}(H)$ .*

**Proof.** Without loss of generality we may suppose that  $H$  is reduced.

1. If  $a \in uH$  with  $\min \mathsf{L}(a) \leq 1$ , then  $a = u$ ,  $\mathsf{L}(u^{-1}a) = \{0\}$ , and hence  $\tau_1^*(H, u) = 0$ . Since  $\min \mathsf{L}(u^2) \leq 2$ , it follows that  $\tau_2^*(H, u) \geq 1$ . If  $a \in uH$  with  $\min \mathsf{L}(a) = l$ , then

$$\frac{\min \mathsf{L}(u^{-1}a)}{\min \mathsf{L}(a)} \leq \frac{\tau_l^*(H, u)}{l} \leq \sup \left\{ \frac{\tau_k^*(H, u)}{k} \mid k \in \mathbb{N} \right\},$$

and hence  $\tau^*(H, u)$  is bounded from above by the supremum on the right-hand side.

Conversely, let  $k \in \mathbb{N}$ . If  $\tau_k^*(H, u) = \infty$ , then it follows that  $\tau^*(H, u) = \infty$ . If  $\tau_k^*(H, u) < \infty$ , then there exists  $a \in uH$  such that  $\min \mathsf{L}(a) \leq k$  and  $\tau_k^*(H, u) = \min \mathsf{L}(u^{-1}a)$ , and then

$$\frac{\tau_k^*(H, u)}{k} = \frac{\min \mathsf{L}(u^{-1}a)}{k} \leq \frac{\min \mathsf{L}(u^{-1}a)}{\min \mathsf{L}(a)} \leq \tau^*(H, u).$$

Thus the reverse inequality follows.

2. Let  $k \in \mathbb{N}$ . By definition, we have  $\tau_k(H, u) \leq \tau_k^*(H, u)$ , and Lemma 3.2.3 implies that  $\tau_k^*(H, u) \leq \tau_k(H, u) + k$ .

3. It suffices to verify that  $t(a, u) \leq \max \{ \omega(a, u), \omega(a, u)\tau^*(H, u) + 1 \}$ . From this the second inequality follows. The second inequality implies, together with 2 and Theorem 3.6, the purported characterization of local tameness. Let  $a \in H$ . If  $a \notin uH$ , then  $t(a, u) = 0$ . Thus suppose that  $a \in uH$ , and let  $z = u_1 \cdot \dots \cdot u_n \in Z(a)$ , with  $n \in \mathbb{N}$  and  $u_1, \dots, u_n \in \mathcal{A}(H)$ . After a renumbering of the indices, we may suppose that  $u \mid c$ , where  $c = u_1 \cdot \dots \cdot u_m$  with  $m \leq \omega(a, u)$ . Then  $c$  has a factorization of the form  $c = uv_1 \cdot \dots \cdot v_l$ , with  $l = \min L(u^{-1}c)$  and  $v_1, \dots, v_l \in \mathcal{A}(H)$ . It follows that  $z' = uv_1 \cdot \dots \cdot v_l u_{m+1} \cdot \dots \cdot u_n \in Z(a) \cap uZ(H)$ ,  $d(z, z') \leq \max\{m, l + 1\}$ , and

$$l \leq \frac{m \min L(u^{-1}c)}{\min L(c)} \leq \omega(a, u)\tau^*(H, u).$$

Hence  $d(z, z') \leq \max \{ \omega(a, u), \omega(a, u)\tau^*(H, u) + 1 \}$ , and we obtain the inequality

$$t(a, u) \leq \max\{\omega(a, u), \omega(a, u)\tau^*(H, u) + 1\}. \quad \square$$

#### 4. $v$ -noetherian monoids satisfy $\omega(H, \cdot) < \infty$

For a subset  $X \subset H$  we denote by  $\mathcal{V}(X) = \{ \mathfrak{p} \in v\text{-spec}(H) \mid X \subset \mathfrak{p} \}$  the set of prime  $v$ -ideals of  $H$  which contain  $X$ , and we denote by  $\mathcal{P}(X)$  the set of minimal elements (with respect to inclusion) of  $\mathcal{V}(X)$ .

**Lemma 4.1.** *Suppose that  $H$  is  $v$ -noetherian and  $\mathfrak{a} \subset H$  is a non-empty  $v$ -ideal of  $H$ . Then there exists  $K(\mathfrak{a}) \in \mathbb{N}$  such that  $(\sqrt{b})^{K(\mathfrak{a})} \subset \mathfrak{a}$  for all  $v$ -ideals  $b$  of  $H$  which contain  $\mathfrak{a}$ .*

**Proof.** For  $\mathfrak{a} = H$  the assertion is clear. Suppose that  $\mathfrak{a} \subsetneq H$ , and let  $\tau$  be a radical  $v$ -ideal of  $H$  containing  $\mathfrak{a}$ . For  $k \in \mathbb{N}$  define  $\mathfrak{h}_k(\tau, \mathfrak{a}) = (\tau^k \cup \mathfrak{a})_v$ . Then  $\mathfrak{h}_1(\tau, \mathfrak{a}) \supset \mathfrak{h}_2(\tau, \mathfrak{a}) \supset \dots$  is a descending chain of  $v$ -ideals containing  $\mathfrak{a}$ . By [15, Proposition 2.1.10] this chain eventually becomes stationary, say,  $\mathfrak{h}_k(\tau, \mathfrak{a}) = \mathfrak{h}_{k+1}(\tau, \mathfrak{a})$  for all  $k \geq K(\tau, \mathfrak{a})$ .

Since  $H$  is  $v$ -noetherian, the set  $\mathcal{V}(\mathfrak{a})$  is finite, non-empty, and every radical  $v$ -ideal  $\tau$  of  $H$  is the intersection of the (minimal) prime  $v$ -ideals containing it [15, Theorem 2.2.5]. It follows that the number of  $v$ -radical ideals of  $H$  containing  $\mathfrak{a}$  is finite, and we define

$$K(\mathfrak{a}) = \max \{ K(\tau, \mathfrak{a}) \mid \tau \subset H \text{ is a radical } v\text{-ideal containing } \mathfrak{a} \}.$$

Suppose now that  $b$  is a  $v$ -ideal which contains  $\mathfrak{a}$ . By [15, Theorem 2.2.5] there exists  $k \in \mathbb{N}$  such that  $\sqrt{b}^k \subset b$ . By the definition of the constant  $K(\mathfrak{a})$  it follows that  $\mathfrak{h}_{K(\mathfrak{a})}(\sqrt{b}, \mathfrak{a}) \subset \mathfrak{h}_k(\sqrt{b}, \mathfrak{a})$ . Hence  $(\sqrt{b})^{K(\mathfrak{a})} \subset \mathfrak{h}_{K(\mathfrak{a})}(\sqrt{b}, \mathfrak{a}) \subset \mathfrak{h}_k(\sqrt{b}, \mathfrak{a}) \subset b$ , and the assertion follows.  $\square$

**Theorem 4.2.** *Suppose that  $H$  is  $v$ -noetherian.*

1. For every  $v$ -ideal  $\mathfrak{a} \subset H$  there exists a constant  $\omega(\mathfrak{a}) \in \mathbb{N}$  having the following property:  
For all  $n \in \mathbb{N}$  and  $c, a_1, \dots, a_n \in H$  with  $ca_1 \cdot \dots \cdot a_n \in \mathfrak{a}$  there exists a subset  $\Omega \subset [1, n]$  such that  $|\Omega| \leq \omega(\mathfrak{a})$  and

$$c \prod_{v \in \Omega} a_v \in \mathfrak{a}.$$

2.  $\omega(H, b) < \infty$  for all  $b \in H$ .

**Proof.** 1. Let  $\mathfrak{a} \subset H$  be a  $v$ -ideal. If  $\mathfrak{a} \in \{\emptyset, H\}$ , then the assertion holds with  $\omega(\mathfrak{a}) = 1$ . Hence suppose that  $\mathfrak{a} \notin \{\emptyset, H\}$ . For  $k \in \mathbb{N}_0$  we set

$$\Gamma_k(\mathfrak{a}) = \{ c \in H \mid |\mathcal{V}((\mathfrak{a} : c))| \leq k \},$$

and we define a sequence  $(\omega_i(\mathfrak{a}))_{i \geq 0}$  of integers inductively by

$$\omega_0(\mathfrak{a}) = 0, \quad \omega_i(\mathfrak{a}) = (2^i - 1)\omega_{i-1}(\mathfrak{a}) + K(\mathfrak{a}),$$

where  $K(\mathfrak{a})$  is the constant from Lemma 4.1. We consider the following assertion:

- A. Let  $k \in \mathbb{N}_0$ . If  $c \in \Gamma_k(\mathfrak{a})$ ,  $n \in \mathbb{N}$  and  $a_1, \dots, a_n \in H$  with  $ca_1 \cdot \dots \cdot a_n \in \mathfrak{a}$ , then there exists a subset  $\Omega \subset [1, n]$  such that  $|\Omega| \leq \omega_k(\mathfrak{a})$  and  $c \prod_{i \in \Omega} a_i \in \mathfrak{a}$ .

Suppose that A holds. If  $c \in H$  and  $\mathfrak{p} \in \mathcal{V}((\mathfrak{a} : c))$ , then  $\mathfrak{p} \supset (\mathfrak{a} : c) \supset \mathfrak{a}$ . Therefore  $\mathfrak{p} \in \mathcal{V}(\mathfrak{a})$ , and we see that  $|\mathcal{V}((\mathfrak{a} : c))| \leq |\mathcal{V}(\mathfrak{a})|$ . Thus  $H = \Gamma_k(\mathfrak{a})$  for all  $k \geq |\mathcal{V}(\mathfrak{a})|$ , and the assertion of the theorem follows if we set  $\omega(\mathfrak{a}) = \omega_{|\mathcal{V}(\mathfrak{a})|}(\mathfrak{a})$ .

*Proof of A.* We proceed by induction on  $k$ . If  $c \in \Gamma_0(\mathfrak{a})$ , then  $c \in \mathfrak{a}$ , and  $\Omega = \emptyset$  does the job. Thus let  $k \in \mathbb{N}$ ,  $c \in \Gamma_k(\mathfrak{a})$ ,  $n \in \mathbb{N}$ , and  $a_1, \dots, a_n \in H$  such that  $ca_1 \cdots a_n \in \mathfrak{a}$  and  $c \prod_{i \in \Omega} a_i \notin \mathfrak{a}$  for all  $\Omega \subsetneq [1, n]$ . We must prove that  $n \leq \omega_k(\mathfrak{a})$ . We define an equivalence relation  $\sim$  on  $[1, n]$  by setting  $i \sim j$  if  $\mathcal{V}(a_i) \cap \mathcal{P}((\mathfrak{a} : c)) = \mathcal{V}(a_j) \cap \mathcal{P}((\mathfrak{a} : c))$ . Let  $Z_1 \cup \dots \cup Z_m$  be the corresponding partition of  $[1, n]$  into equivalence classes. Then  $|\mathcal{P}((\mathfrak{a} : c))| \leq k$  and  $m \leq 2^{|\mathcal{P}((\mathfrak{a} : c))|} \leq 2^k$ . Let

$$Y = \{i \in [1, n] \mid \mathcal{P}((\mathfrak{a} : c)) \subset \mathcal{V}(a_i)\} = \left\{i \in [1, n] \mid a_i \in \sqrt{(\mathfrak{a} : c)}\right\}.$$

Of course, if  $Y \neq \emptyset$ , then  $Y = Z_q$  for some index  $q \in [1, m]$ . If  $Y = \emptyset$ , we define  $q = 0$ . By Lemma 4.1 it follows that  $(\sqrt{(\mathfrak{a} : c)})^{K(\mathfrak{a})} \subset (\mathfrak{a} : c)$ , and the minimal choice of  $n$  implies that  $|Y| \leq K(\mathfrak{a})$ . If we can show that  $|Z_j| \leq \omega_{k-1}(\mathfrak{a})$  for all  $j \in [1, m] \setminus \{q\}$ , we are done (note that  $m \leq 2^k - 1$  if  $Y = \emptyset$ ). Suppose to the contrary that  $|Z_j| > \omega_{k-1}(\mathfrak{a})$  for some  $j \in [1, m] \setminus \{q\}$ , and put

$$d = \prod_{i \in [1, n] \setminus Z_j} a_i.$$

Then  $cd \prod_{i \in Z_j} a_i \in \mathfrak{a}$ , whence  $\prod_{i \in Z_j} a_i \in (\mathfrak{a} : cd)$ . Since  $j \in [1, m] \setminus \{q\}$ , we have  $\mathcal{P}((\mathfrak{a} : c)) \not\subset \mathcal{V}(a_i)$  for all  $i \in Z_j$ , and there exists

$$\mathfrak{p} \in \mathcal{P}((\mathfrak{a} : c)) \setminus (\mathcal{V}(a_i) \cap \mathcal{P}((\mathfrak{a} : c))),$$

where  $i \in Z_j$ . In other words,  $\mathfrak{p}$  is a prime  $v$ -ideal which contains  $(\mathfrak{a} : c)$  and which does not contain  $a_i$  for all  $i \in Z_j$ . Therefore  $\prod_{i \in Z_j} a_i \notin \mathfrak{p}$ . Since  $\prod_{i \in Z_j} a_i \in (\mathfrak{a} : cd)$ , we see that  $\mathcal{V}((\mathfrak{a} : cd))$  must be properly contained in  $\mathcal{V}((\mathfrak{a} : c))$ . Hence  $cd \in \Gamma_{k-1}(\mathfrak{a})$ . By the induction hypothesis it now follows from  $|Z_j| > \omega_{k-1}(\mathfrak{a})$  that there is a proper subset  $\Omega_j \subsetneq Z_j$  such that

$$cd \prod_{i \in \Omega_j} a_i \in \mathfrak{a}.$$

But this contradicts the minimal choice of  $n$ .

2. This follows from 1 with  $\mathfrak{a} = bH$  and  $c = 1$ .  $\square$

**Corollary 4.3.** *Suppose that  $H$  is  $v$ -noetherian.*

1.  $H$  is a BF-monoid.
2. If  $(H : \widehat{H}) \neq \emptyset$ , then  $\widehat{H}$  is a Krull monoid. If  $f \in (H : \widehat{H})$  and  $F = F^\times \times \mathcal{F}(P)$  is a factorial monoid such that  $\widehat{H} \subset F$  is saturated, then

$$\omega(H, b) \leq \sum_{p \in P} v_p(b) + \omega(H, f) < \infty \quad \text{for all } b \in H.$$

3. If either  $\sup\{\min L(c) \mid c \in H\} < \infty$  or  $\rho_k(H) < \infty$  for all  $k \in \mathbb{N}$ , then  $H$  is locally tame.

**Proof.** 1. Since  $H$  is  $v$ -noetherian,  $H$  satisfies the ascending chain condition on principal ideals. Therefore  $H$  is atomic. Now 1. follows from Theorem 4.2.2 and Lemma 3.3.3.

2.  $\widehat{H}$  is a Krull monoid by [15, Theorem 2.3.5.3]. The second statement in 2 follows from Lemma 3.4.2.(c) and Theorem 4.2.2.

3. Using Lemma 3.2.2 and Theorem 4.2.2 the assertion follows by Theorem 3.6.  $\square$

It is well known that  $v$ -noetherian monoids are BF-monoids [15, Theorem 2.2.9]. Theorem 4.2.2 together with Lemma 3.3.3 provides yet another proof of this fact. Suppose that  $H$  is  $v$ -noetherian with  $(H : \widehat{H}) \neq \emptyset$  (this includes multiplicative monoids of noetherian domains  $R$  whose integral closure  $\overline{R}$  is a finitely generated  $R$ -module). Then Corollary 4.3.2 states that, up to a constant,  $\omega(H, b)$  is bounded by the total valuation of  $b$ . This upper bound is important for the investigation of the catenary degree in weakly C-monoids (see [17, Theorem 6.3]).

Monoids with the property that  $\sup\{\min L(c) \mid c \in H\} < \infty$  are discussed in [18] and in [15, Section 3.1]. Every monoid with finite elasticity (and thus every half-factorial monoid) satisfies  $\rho_k(H) < \infty$  for all  $k \in \mathbb{N}$ . The elasticity and the concept of half-factoriality received a great deal of attention in the literature (for recent progress and surveys see [2,5,29,26]). By the corollary above it follows that  $\nu$ -noetherian monoids with finite elasticity are locally tame. On the other hand, it is well known that these monoids may have infinite catenary degree and even an infinite set of distances (see [15, Example 4.8.11] for a Dedekind domain having a prescribed elasticity but an infinite set of distances).

Suppose  $H$  is a Krull monoid. Then the multiplicative properties of  $H$  depend on its class group and on the distribution of the primes in the classes. We discuss two special situations.

First, suppose that  $\rho_k(H) < \infty$  for all  $k \in \mathbb{N}$ . This condition holds whenever the number of classes containing primes is finite (see [15, Theorem 3.4.10.3 and Corollary 3.4.13]), and of course it holds for half-factorial monoids. It is an open problem [28,13] whether for every abelian group  $G$  there exists a half-factorial Krull monoid (equivalently, a half-factorial Dedekind domain) whose class group is isomorphic to  $G$ . By Corollary 4.3.3 such monoids (and domains) are locally tame.

Second, suppose that every class contains a prime (examples of such Krull monoids can be found in [15, 2.10.4, 7.4.2 and 8.9.5]). This situation is studied in the next theorem. We recall the definition of the Davenport constant  $D(G)$  in the proof of Theorem 4.4.

**Theorem 4.4.** *Suppose that  $H$  is a Krull monoid with class group  $G$  such that every class contains a prime, and let  $D(G)$  be the Davenport constant of  $G$ .*

1. Let  $\varphi: H \rightarrow \mathcal{F}(P)$  be a divisor theory. Suppose that  $u \in \mathcal{A}(H)$  such that  $\varphi(u) = p_1 \cdot \dots \cdot p_k$ , where  $k \geq 2$  and  $p_1, \dots, p_k \in P$ .
  - (a)  $\omega(H, u) \leq k$ .
  - (b) If  $G$  is infinite, then  $\tau_k(H, u) = \tau(H, u) = \mathfrak{t}(H, u) = \infty$ .
2. If  $|G| > 1$ , then the following statements are equivalent:
  - (a)  $G$  is finite.
  - (b)  $H$  is locally tame.
  - (c)  $\mathfrak{t}(H, u) < \infty$  for some  $u \in \mathcal{A}(H_{\text{red}})$  that is not prime.
3. If  $|G| = 1$ , then  $\mathfrak{t}(H, uH^\times) = 0$  for all  $u \in \mathcal{A}(H)$ . If  $|G| > 1$ , then

$$\sup\{\mathfrak{t}(H, uH^\times) \mid u \in \mathcal{A}(H)\} = \sup\{D(G), 1 + \tau(H, u) \mid u \in \mathcal{A}(H)\},$$

and these suprema are finite if and only if  $G$  is finite.

**Remark 4.5.** The supremum given in Theorem 4.4.3 is called the *tame degree* of  $H$ . Its precise value is known only in very special cases (see [15, Section 6.5]), and Theorem 4.4.3 reduces the computation of the tame degree to the computation of the supremum of the  $\tau(H, \cdot)$ -invariants.

Consider the inequality in Theorem 4.4.1.(a). Conditions which imply equality are described in [15, Proposition 7.1.9].

**Proof of Theorem 4.4.** Without restriction we may suppose that  $H$  is reduced and that  $\varphi: H \rightarrow F = \mathcal{F}(P)$  is an embedding.

1. Note that  $H \subset F$  is saturated and  $u = p_1 \cdot \dots \cdot p_k$ .

1.(a) Lemma 3.4.1 implies that  $\omega(H, u) \leq \omega(F, u) = k$ .

1.(b) Suppose that  $G$  is infinite. By Theorem 3.6 it suffices to show that  $\tau_k(H, u) = \infty$ . We first introduce some terminology “on the fly” (see [15, Section 5.1]). Let  $\mathcal{F}(G)$  be the free (multiplicative) monoid with basis  $G$ . The elements of  $\mathcal{F}(G)$  are called *sequences* over  $G$ . Let  $S = g_1 \cdot \dots \cdot g_l$  be a sequence over  $G$ . Then  $|S| = l \in \mathbb{N}_0$  is the *length* of  $S$ ,  $\text{supp}(S) = \{g_i \mid i \in [1, l]\} \subset G$  is the *support* of  $S$ ,  $\sigma(S) = g_1 + \dots + g_l \in G$  is the *sum* of  $S$ , and we set  $-S = (-g_1) \cdot \dots \cdot (-g_l) \in \mathcal{F}(G)$ .  $S$  is called a *zero-sum sequence* if  $\sigma(S) = 0$ , and it is called *zero-sumfree* if  $\sum_{i \in I} g_i \neq 0$  for all  $\emptyset \neq I \subset [1, l]$ . The set of all zero-sum sequences is denoted by  $\mathcal{B}(G)$ , and  $\mathcal{B}(G) \subset \mathcal{F}(G)$  is a saturated submonoid. Thus  $\mathcal{B}(G)$  is a Krull monoid whose set of atoms is denoted by  $\mathcal{A}(G)$ . The quantity  $D(G) = \sup\{|U| \mid U \in \mathcal{A}(G)\} \in \mathbb{N}_0 \cup \{\infty\}$  is called the *Davenport constant* of  $G$ . Let  $\beta: H \rightarrow \mathcal{B}(G)$  be the block homomorphism of  $H \subset F$ . It is defined as the restriction of the homomorphism  $F \rightarrow \mathcal{F}(G)$  which sends

a prime  $p \in P$  onto the class  $[p] \in G$ . By [15, Section 3.2] we have  $\tau_k(H, u) \geq \tau_k(\mathcal{B}(G), \beta(u))$  for all  $u \in \mathcal{A}(H)$ . Therefore it suffices to show that  $\tau_k(\mathcal{B}(G), U) = \infty$  for all  $U \in \mathcal{A}(G)$  with  $|U| \geq 2$ .

Let  $U \in \mathcal{A}(G)$  with  $|U| = k \geq 2$ . Suppose there exist  $g \in \text{supp}(U)$  and a zero-sumfree sequence  $S \in \mathcal{F}(G)$  such that  $\sigma(S) = -g$  and  $\text{supp}(S) \cap G_0 = \emptyset$ , where  $G_0 = \text{supp}(U) \cup \text{supp}(-U)$ . Put  $T = g^{-1}U \in \mathcal{F}(G)$ . Then  $U_1 = gS \in \mathcal{A}(G)$ , and  $T(-S) \in \mathcal{B}(G)$  has a factorization of the form  $T(-S) = U_2 \cdot \dots \cdot U_j$ , where  $U_2 = X_2Y_2, \dots, U_j = X_jY_j$  with  $T = X_2 \cdot \dots \cdot X_j, -S = Y_2 \cdot \dots \cdot Y_j$  and  $X_2, \dots, Y_j \in \mathcal{F}(G)$ . Since  $T$  and  $-S$  are both zero-sumfree, it follows that  $|X_i| \geq 1$  and  $|Y_i| \geq 1$  for all  $i \in [2, j]$ , and hence  $j - 1 \leq |T| = k - 1$ . Thus  $A = U(-S)S = U_1 \cdot \dots \cdot U_j$  with  $j \in [1, k]$ . Since  $\text{supp}(U) \cap (\text{supp}(-S) \cup \text{supp}(S)) = \emptyset$ , it follows that  $U \nmid U_i^{-1}A$  for all  $i \in [1, j]$ .

To prove the assertion it is sufficient to show that, for every  $N \in \mathbb{N}$ , there exist  $g \in \text{supp}(U)$  and a zero-sumfree sequence  $S \in \mathcal{F}(G)$  with  $\text{supp}(S) \cap G_0 = \emptyset$  and  $\sigma(S) = -g$  such that  $\min L((-S)S) \geq N$ . Then  $\tau_k(\mathcal{B}(G), U) \geq N$ , and it follows that  $\tau_k(\mathcal{B}(G), U) = \infty$ .

Recall [10] that non-zero elements  $g_1, \dots, g_k$  in an abelian group are called *independent* if  $m_1g_1 + \dots + m_kg_k = 0$  implies that  $m_1g_1 = \dots = m_kg_k = 0$  for all  $m_1, \dots, m_k \in \mathbb{Z}$ . The *total rank* of an abelian group is the cardinal number of a maximal system of independent elements containing only elements of infinite and prime power order (cf. [10, Section 16]).

Let  $N \in \mathbb{N}$  be given. We distinguish four cases.

CASE 1: There exist  $g \in \text{supp}(U)$  with  $\text{ord}(g) = \infty$ .

Let  $m_1, \dots, m_s \in \mathbb{Z}$  such that  $G_0 \cap \langle g \rangle = \{m_1g, \dots, m_s g\}$ , and pick  $m \in \mathbb{N}$  such that  $m$  is strictly larger than  $\max\{|m_1|, \dots, |m_s|\}$ . Put  $h = mg$ . Then the sequence  $S = h^N(-Nh - g)$  is zero-sumfree,  $\text{supp}(S) \cap G_0 = \emptyset$ ,  $\sigma(S) = -g$  and

$$L((-S)S) = \{N + 1\}.$$

CASE 2: There exist  $h \in G$  with  $\text{ord}(h) > N$  and  $\langle \text{supp}(U) \rangle \cap \langle h \rangle = \{0\}$ .

For any  $g \in \text{supp}(U)$ , the sequence  $S = h^N(-Nh - g)$  is zero-sumfree,  $\text{supp}(S) \cap G_0 = \emptyset$ ,  $\sigma(S) = -g$  and

$$L((-S)S) = \{N + 1\}.$$

CASE 3: There exist independent elements  $e_1, \dots, e_N \in G \setminus \{0\}$  such that  $\langle \text{supp}(U) \rangle \cap \langle e_1, \dots, e_N \rangle = \{0\}$ .

If  $g \in \text{supp}(U)$  and  $e_{N+1} = -e_1 - \dots - e_N - g$ , then the sequence

$$S = \prod_{i=1}^{N+1} e_i$$

is zero-sumfree,  $\text{supp}(S) \cap G_0 = \emptyset$ ,  $\sigma(S) = -g$  and

$$L((-S)S) = \{N + 1\}.$$

CASE 4: None of the conditions in CASES 1–3 hold.

We first show that  $G$  is a torsion group. Assume to the contrary that there exists  $a \in G$  with  $\text{ord}(a) = \infty$ . Since the condition in CASE 1 does not hold,  $\langle \text{supp}(U) \rangle$  is a finite group. Since  $a$  has infinite order it follows that  $\langle \text{supp}(U) \rangle \cap \langle a \rangle = \{0\}$ , and therefore the assumption in CASE 2 is fulfilled, a contradiction.

Since  $G$  is a torsion group and since CASE 3 does not hold,  $G$  has finite total rank  $s$ . Thus its divisible hull also has total rank  $s$  [10, paragraph after Theorem 24.4], and we get

$$G \subset \overline{G} = \mathbb{Z}(p_1^\infty) \oplus \dots \oplus \mathbb{Z}(p_s^\infty),$$

where, for  $i \in [1, s]$ ,

$$\mathbb{Z}(p_i^\infty) = \left\{ \frac{m}{p_i^k} + \mathbb{Z} \mid m \in \mathbb{Z}, k \in \mathbb{N} \right\} \subset \mathbb{Q}/\mathbb{Z}$$

is the Prüfer group of type  $p_i^\infty$  (see [10, Sections 23 and 24]). For  $i \in [1, s]$  let  $\pi_i : \overline{G} \rightarrow \mathbb{Z}(p_i^\infty)$  denote the projection. After a suitable renumbering of the indices there exists  $r \in [1, s]$  such that  $\pi_i(G)$  is infinite for all  $i \in [1, r]$  and  $\pi_i(G)$  is finite for all  $i \in [r+1, s]$ . Since proper subgroups of a Prüfer group of type  $p^\infty$  are finite, we obtain  $\pi_i(G) = \mathbb{Z}(p_i^\infty)$  for all  $i \in [1, r]$ . We continue with the following assertion:

A. There exist  $g \in \text{supp}(U)$  and  $i \in [1, r]$  such that  $\pi_i(g) \neq 0$ .

Assume to the contrary that this does not hold. Then  $\text{supp}(U) \subset \pi_{r+1}(G) \oplus \dots \oplus \pi_s(G)$ . The latter group is finite, and we denote by  $e \in \mathbb{N}$  its exponent. There exists  $h' = h'_1 + \dots + h'_s \in G$ , with  $h'_i \in \mathbb{Z}(p_i^\infty)$  for all  $i \in [1, s]$ , such that  $\text{ord}(h'_i) > Ne$ . Then  $eh'_i = 0$  for all  $i \in [r + 1, s]$ , and

$$\text{ord}(eh'_1) = \frac{\text{ord}(h'_1)}{\text{gcd}(e, \text{ord}(h'_1))} > N.$$

If we put  $h = eh'$ , then  $\text{ord}(h) \geq \text{ord}(eh'_1) > N$  and  $\langle \text{supp}(U) \rangle \cap \langle h \rangle = \{0\}$ . Thus the condition in CASE 2 holds, a contradiction.

Let  $g \in \text{supp}(U)$  and  $i \in [1, r]$ , say  $i = 1$ , such that  $\pi_1(g) \neq 0$ . We set  $G_1 = \mathbb{Z}(p_1^\infty)$  and  $G_2 = \mathbb{Z}(p_2^\infty) \oplus \dots \oplus \mathbb{Z}(p_s^\infty)$ . Then  $g = g_1 + g_2$ , where  $g_i \in G_i$  for  $i \in [1, 2]$ . Since  $G_1$  is divisible, there exist  $h_1 \in G_1$  and  $N' \in \mathbb{N}$  such that  $N' + 1$  is a  $p_1$ -power,  $(N' + 1)h_1 = -g_1$ , and  $\text{ord}(h_1) > N' + 1 > \max\{N, \exp(\langle G_0 \rangle)\}$ . Let  $h \in G$  with  $\pi_1(h) = h_1$ , say  $h = h_1 + h_2$ , with  $h_2 \in G_2$ .

The sequence

$$S = h^{N'}(h_1 - N'h_2 - g_2)$$

has sum  $\sigma(S) = N'(h_1 + h_2) + h_1 - N'h_2 - g_2 = -g_1 - g_2 = -g$ , and, since  $h_1 - N'h_2 - g_2 = \sigma(S) - N'h \in G$ , we have  $S \in \mathcal{F}(G)$ . Since  $\text{ord}(h) > \exp(\langle G_0 \rangle)$  and  $\text{ord}(h_1 - N'h_2 - g_2) \geq \text{ord}(h_1) > \exp(\langle G_0 \rangle)$ , it follows that  $\text{supp}(S) \cap G_0 = \emptyset$ . We assert that  $S$  is zero-sumfree. Assume to the contrary that this does not hold. Since  $h^{N'}$  is zero-sumfree, there exists  $m \in [1, N']$  such that  $mh = -h_1 + N'h_2 + g_2$ . Then

$$-h_1 + (N'h_2 + g_2) = mh = mh_1 + mh_2 \in G = G_1 \oplus G_2,$$

and we obtain  $(m + 1)h_1 = 0$ , a contradiction. Now a similar argument shows that

$$L((-S)S) = \{N' + 1\}.$$

2. Let  $|G| > 1$ . Then  $H$  is not factorial and hence  $H$  has atoms which are not prime.

(a)  $\Rightarrow$  (b) follows from [15, 3.4.10.6], and (b)  $\Rightarrow$  (c) is obvious.

(c)  $\Rightarrow$  (a) Assume to the contrary that  $G$  is infinite, and let  $u \in \mathcal{A}(H)$  be not prime. Then  $u = p_1 \cdot \dots \cdot p_k$  with  $p_1, \dots, p_k \in P$  and  $k \geq 2$ . Thus 1.(b) and Lemma 3.5 imply that  $t(H, u) = \infty$ , a contradiction.

3. If  $|G| = 1$ , then  $H$  is factorial, all atoms are primes, and hence  $t(H, u) = 0$  for all  $u \in \mathcal{A}(H)$ . Suppose that  $|G| > 1$ , and recall that  $G$  is finite if and only if  $D(G)$  is finite. Furthermore,  $G$  is finite if and only if the tame degree  $\sup\{t(H, u) \mid u \in \mathcal{A}(H)\}$  of  $H$  is finite (see [15, Lemma 1.4.9 and Theorem 3.4.10.6]). Now we invoke Theorem 3.6. If  $G$  is infinite, then the assertion follows from 1.(b). If  $G$  is finite and  $u = p_1 \cdot \dots \cdot p_k$  with  $k \geq 2$ ,  $p_1, \dots, p_k \in P$ , then 1.(a) and [15, Theorem 5.1.5] imply that  $\omega(H, u) \leq k \leq D(G)$ . If  $|G| = 2$ , then  $D(G) = 2 \leq 1 + \tau(H, u)$ . Suppose that  $|G| \geq 3$ . Then  $D(G) \geq 3$ . We choose  $k$  to be equal to  $D(G)$ , and we pick  $p_1, \dots, p_k \in P$  such that  $u = p_1 \cdot \dots \cdot p_k \in H$  is irreducible. Since every class in  $G$  contains a prime, we can find  $q_1, \dots, q_k \in P$  such that  $u_1 = p_1q_1, \dots, u_k = p_kq_k \in \mathcal{A}(H)$ . Since  $k \geq 3$  and since  $u$  is an atom of  $H$  an easy argument shows that  $\omega(u_1 \cdot \dots \cdot u_k, u) = k = D(G)$ . Thus the assertion follows.  $\square$

For  $m \in \mathbb{N}$  we denote by  $C_m$  a cyclic group with  $m$  elements. Let  $G$  be a finite abelian group. Then

$$G \cong C_{n_1} \oplus \dots \oplus C_{n_r},$$

where  $r = r(G) \in \mathbb{N}_0$  is the rank of  $G$  and  $n_1, \dots, n_r \in \mathbb{N}$  are integers with  $1 < n_1 \mid \dots \mid n_r$ , and we define

$$d^*(G) = \sum_{i=1}^r (n_i - 1).$$

Then  $D(G) \geq d^*(G) + 1$ , and equality holds (among others) for  $p$ -groups and if  $r(G) \leq 2$  (see [11, Section 3] or [15, Chapter 5]).

**Corollary 4.6.** *Suppose that  $H$  is a Krull monoid with finite class group  $G$  such that every class contains a prime, and suppose that  $D(G) = d^*(G) + 1 > 1$ . Then*

$$\max\{t(H, uH^\times) \mid u \in \mathcal{A}(H)\} = 1 + \max\{\tau(H, u) \mid u \in \mathcal{A}(H)\}.$$

**Proof.** Let the notation be as in the proof of Theorem 4.4. By Theorem 4.4 it suffices to show that there exists a  $u \in \mathcal{A}(H)$  such that  $\tau(H, u) \geq \mathfrak{d}^*(G)$ . Let  $G$  be as above and  $(e_1, \dots, e_r)$  be a basis of  $G$  with  $\text{ord}(e_i) = n_i$  for all  $i \in [1, r]$ . Then, for  $e_0 = e_1 + \dots + e_r$ ,

$$V = e_0 \prod_{i=1}^r e_i^{n_i-1} \in \mathcal{A}(G) \quad \text{with } |V| = \mathfrak{d}^*(G) + 1.$$

For  $i \in [0, r]$  we pick primes  $p_i \in e_i$  and primes  $q_i \in -e_i$ . Then the elements

$$v = p_0 \prod_{i=1}^r p_i^{n_i-1}, \quad v' = q_0 \prod_{i=1}^r q_i^{n_i-1} \quad \text{and} \quad u = p_0 q_0$$

are atoms of  $H$ . Since  $L(u^{-1}vv') = \{\mathfrak{d}^*(G)\}$ , it follows that

$$\tau(H, u) \geq \min L(u^{-1}vv') = \mathfrak{d}^*(G). \quad \square$$

We conclude this paper with an example of a primary BF-monoid such that  $\omega(H, b) = \infty$  for all  $b \in H \setminus H^\times$ . Recall that a monoid  $H$  is said to be *primary* if  $H \neq H^\times$  and  $s\text{-spec}(H) = \{\emptyset, H \setminus H^\times\}$ . Important examples of primary monoids are the multiplicative monoids of one-dimensional local domains [15, Proposition 2.10.7]. A monoid  $H$  is called *strongly primary* if, for every  $b \in H \setminus H^\times$ , there exists  $n \in \mathbb{N}$  such that  $(H \setminus H^\times)^n \subset bH$ . The smallest  $n$  having this property is denoted by  $\mathcal{M}(b)$ . Every  $v$ -noetherian primary monoid is strongly primary [15, Lemma 2.7.7 and Theorem 2.7.9]. Furthermore, every strongly primary monoid is a primary BF-monoid, and, by definition, we have  $\omega(H, b) \leq \mathcal{M}(b)$  for every  $b \in H \setminus H^\times$ . For more information on primary and strongly primary monoids we refer to [15, Section 2.7] and [18]. In the latter paper the arithmetic of strongly primary monoids is investigated in detail. A monoid  $H$  is called *root-closed* if  $x^n \in H$  implies that  $x \in H$  for every  $x \in \mathfrak{q}(H)$  and every  $n \in \mathbb{N}$ .

**Example 4.7.** Let  $\alpha \in \mathbb{R}_{>1} \setminus \mathbb{Q}$  and put

$$H = \left\{ (x, y) \in \mathbb{N}^2 \mid y < \alpha x \right\} \cup \{(0, 0)\} \subset (\mathbb{N}_0^2, +).$$

Then  $H$  is a root-closed primary FF-monoid with  $\mathfrak{q}(H) = \mathbb{Z}^2$ ,  $\widehat{H} = \{(x, y) \in \mathbb{N}_0^2 \mid y \leq \alpha x\}$  and  $\omega(H, b) = \infty$  for all  $b \in H \setminus \{(0, 0)\}$ .

**Proof.** Since  $H$  is a reduced submonoid of the factorial monoid  $(\mathbb{N}_0^2, +)$ , it is an FF-monoid by [15, Theorem 1.5.6]. If  $(a, b) \in \mathbb{Z}^2$ , then there are  $x, x', y, y' \in \mathbb{N}$  such that  $a = x - x'$  and  $b = y - y'$ . If  $k \in \mathbb{N}$  with  $y < \alpha(x + k)$  and  $y' < \alpha(x' + k)$ , then  $(a, b) = (x + k, y) - (x' + k, y')$ . Thus it follows that  $\mathfrak{q}(H) = \mathbb{Z}^2$ . If  $(a, b) \in \mathfrak{q}(H) \setminus \{(0, 0)\}$  and  $n \in \mathbb{N}$  such that  $n(a, b) \in H$ , then  $nb < \alpha na$  and hence  $(a, b) \in H$ . Therefore  $H$  is root-closed. If  $(a, b), (a', b') \in H \setminus \{(0, 0)\}$ , then there is some  $m \in \mathbb{N}$  such that

$$\frac{mb' - b}{ma' - a} < \alpha,$$

whence  $(a, b) \mid m(a', b')$ . Thus  $H$  is primary, and therefore  $H = \llbracket u \rrbracket$  for all  $u \in H \setminus \{(0, 0)\}$ . A straightforward calculation shows that  $\widehat{H} = \mathbb{N}_0^2$ . For an alternate geometric argument showing that  $\widehat{H}$  has the asserted form see [16, Theorem 2].

We set  $u = (1, 1)$  and show that  $\omega(H, u) = \infty$ . Then Lemma 3.3.2 implies that  $\omega(H, b) = \infty$  for all  $b \in H \setminus \{(0, 0)\}$ . Let  $n \in \mathbb{N}$ . We construct some  $a_n \in H$  such that  $u \nmid na_n$ . Since  $H$  is primary, there exists  $N \in \mathbb{N}$  such that  $u \mid Na_n$ . This implies that  $\omega(H, u) \geq \omega(Na_n, u) > n$ .

Let  $a = (q, p) \in \mathbb{N}^2$ . Then  $(1, 1) \nmid na$  if and only if  $(nq, np) - (1, 1) \notin H$  if and only if

$$\alpha \leq \frac{np - 1}{nq - 1}.$$

We construct such an element  $a_n \in H$  by methods from diophantine approximation. We consider the continued fraction expansion of  $\alpha$ , say

$$\alpha = [c_0; c_1, \dots],$$

and we define  $p_0 = c_0$ ,  $q_0 = 1$ ,  $p_1 = c_1c_0 + 1$ ,  $q_1 = c_1$  and, for all  $k \geq 2$ ,

$$p_k = c_k p_{k-1} + p_{k-2} \quad \text{and} \quad q_k = c_k q_{k-1} + q_{k-2}.$$

Then, for all  $k \geq 0$ , we have (by definition)  $q_k \geq k$  and (for example by [24, Satz XIV.2])

$$\frac{p_{2k}}{q_{2k}} < \alpha \quad \text{and} \quad \left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}}.$$

Let  $m = m(n) \in \mathbb{N}$  be even such that

$$\frac{n}{\alpha - 1} \leq q_{m+1},$$

and set  $a_n = (q_m, p_m)$ . Then

$$\frac{p_m}{q_m} < \alpha < \frac{p_m}{q_m} + \frac{1}{q_m q_{m+1}},$$

$a_n \in H$ , and it suffices to show that

$$\frac{p_m}{q_m} + \frac{1}{q_m q_{m+1}} \leq \frac{np_m - 1}{nq_m - 1}. \quad (*)$$

We have

$$\begin{aligned} n &\leq q_{m+1} \left( \alpha - 1 - \frac{1}{q_m q_{m+1}} \right) + \frac{1}{q_m} \\ &\leq q_{m+1} \left( (\alpha - 1) - \left( \alpha - \frac{p_m}{q_m} \right) \right) + \frac{1}{q_m} \\ &= q_{m+1} \left( \frac{p_m}{q_m} - 1 \right) + \frac{1}{q_m}, \end{aligned}$$

and now (\*) follows by a simple calculation.  $\square$

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