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RESEARCH ARTICLE

On the Arithmetic of Strongly Primary Monoids

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Abstract

In this paper we study the arithmetic of strongly primary monoids. Numerical monoids and the multiplicative monoids of one-dimensional local Mori domains are main examples of strongly primary monoids. Our investigations focus on local tameness, a basic finiteness property in the theory of non-unique factorizations. It is well-known that locally tame strongly primary monoids have finite catenary degree and finite set of distances.

1. Introduction

Strongly primary monoids occur as multiplicative monoids of certain integral domains and play a crucial role as auxiliary monoids in the theory of non-unique factorizations. Every v-noetherian primary (commutative and cancellative) monoid H is strongly primary (cf. Definition 2.1), and if its conductor $(H : \widehat{H})$ is non-empty, then its complete integral closure $\widehat{H}$ is a Krull monoid (see Lemma 3.1). It is this special situation we mainly have in mind in the present paper. If R is a one-dimensional local Mori domain, then its multiplicative monoid $R \setminus \{0\}$ is strongly primary. Numerical monoids are further examples of strongly primary monoids (see [6], [27] for recent results on numerical monoids, and [4] for their importance in the theory of semigroup algebras). Note that all strongly primary monoids are finitary (cf. [15, Example 3.7]).

In this article we study the arithmetic of strongly primary monoids with a special emphasis on local tameness (cf. Definition 2.3), a basic finiteness property in the theory of non-unique factorizations (see [14] for general information and [8], [9] for recent results on this invariant). In the setting of strongly primary monoids local tameness implies the finiteness of various other arithmetical invariants (see Theorems 2.4 and 4.1).

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Let H be a strongly primary monoid such that $(H : \widehat{H}) \neq \emptyset$ and $\widehat{H}$ is a Krull monoid. In all situations investigated so far local tameness has been proved under the additional assumption that the class group $\mathcal{C}(\widehat{H})$ is finite (cf. the first paragraph of Section 3). In this paper we prove that H is locally tame—without imposing any conditions on the size of the class group $\mathcal{C}(\widehat{H})$. Theorem 3.5 is the main finiteness result for local tameness in the setting of monoids, and Corollary 3.6 shows its consequences for integral domains. If D is a (global) v-noetherian monoid such that the class group $\mathcal{C}(\widehat{D})$ is finite, then D may have a divisor-closed primary submonoid H with infinite class group $\mathcal{C}(\widehat{H})$. Therefore Theorem 3.5 is a crucial tool for arithmetical investigations of (global) v-noetherian monoids D, even if $\mathcal{C}(\widehat{D})$ is finite (for details we refer to [17]).

It turns out that local tameness does not hold for arbitrary strongly primary monoids. In Proposition 3.7 we construct a v-noetherian primary monoid H that is not locally tame, and Example 3.8 shows that such monoids occur as submonoids of certain one-dimensional local noetherian domains. Our construction yields the first known examples of v-noetherian primary monoids that fail to be locally tame.

It is well-known that (long) sets of lengths in locally tame strongly primary monoids have an extremely simple structure (cf. Theorem 4.1). As a counterpart we show that every finite set $L \subset \mathbb{N}_{\geq 2}$ can be realized as a set of lengths in some locally tame strongly primary monoid (Theorem 4.2).

2. Preliminaries

We denote by $\mathbb{N}$ the set of positive integers, and we put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For integers $a, b \in \mathbb{Z}$ we define $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. For a non-empty set $L \subset \mathbb{Z}$ we denote by $\Delta(L)$ the set of all $d \in \mathbb{N}$ for which there exists $l \in L$ such that $L \cap [l, l+d] = \{l, l+d\}$. Clearly, $|\Delta(L)| = 1$ if and only if L is an arithmetical progression. For an abelian group G we denote by $\exp(G)$ its exponent. By a *monoid* we mean a commutative cancellative semigroup with unit element. In the following we briefly recall some algebraic and arithmetic notation for monoids. Our terminology is consistent with [14], and any notion that is not defined in this paper is explained there.

Throughout this paper H denotes a monoid.

Let $H^\times$ denote the set of invertible elements of H, $H_{\mathrm{red}} = \{aH^\times \mid a \in H\}$ the associated reduced monoid, and $\mathsf{q}(H)$ the quotient group of H. We denote by

$$\widetilde{H} = \{x \in \mathsf{q}(H) \mid x^n \in H \text{ for some } n \in \mathbb{N}\}$$

the *root closure* of H, and by

$$\widehat{H} = \{x \in \mathsf{q}(H) \mid \text{there exists } c \in H \text{ such that } cx^n \in H \text{ for all } n \in \mathbb{N}\}$$

the *complete integral closure* of H. We have

$$H \subset \widetilde{H} \subset \widehat{H} \subset \mathsf{q}(H).$$

The monoid H is said to be *completely integrally closed* if $H = \widehat{H}$, and it is said to be *seminormal* if $x^2, x^3 \in H$ implies $x \in H$ for all $x \in \mathsf{q}(H)$ (see, e.g., [7], [25]).

A submonoid $S \subset H$ is called *saturated* if $S = \mathsf{q}(S) \cap H$ and *cofinal* if for all $h \in H$ there exists $s \in S$ such that h divides s in H. Let $X \subset \mathsf{q}(H)$ be a subset. Then X is called an *s-ideal* of H if $X \subset H$ and $XH = X$. We put $X^{-1} = (H : X) = \{a \in \mathsf{q}(H) \mid aX \subset H\}$, and we call $X_v = (X^{-1})^{-1}$ the *v-ideal* generated by X. Every v-ideal $\mathfrak{a} \subset H$ is an s-ideal of H. We denote by s-spec(H) the set of all prime s-ideals of H, and by $\mathfrak{X}(H)$ the set of all minimal non-empty prime s-ideals of H. The monoid H is called *v-noetherian* if it satisfies the ascending chain condition on v-ideals, and it is called a *Krull monoid* if it is v-noetherian and completely integrally closed. If H is a Krull monoid, then its class group is denoted by $\mathcal{C}(H)$ (see [14, Definition 2.4.9]). For all the terminology used in the theory of Krull monoids we refer to one of the monographs [14], [19], [21]. In Definition 2.1 we recall the definition of some less known classes of monoids which play a role in this paper, and then we discuss some of their main properties (for details and proofs see [14, Sections 2.7 and 2.9]).

Definition 2.1.

1. H is called *primary* if $H \neq H^\times$ and s-spec$(H) = \{\emptyset, H \setminus H^\times\}$.

2. H is called *strongly primary* if for every $a \in H \setminus H^\times$ there exists $n \in \mathbb{N}$ such that $(H \setminus H^\times)^n \subset aH$. We denote by $\mathcal{M}(a)$ the smallest n having this property.

3. H is called a G*-monoid* if

$$\bigcap_{\substack{\mathfrak{p} \in s\text{-spec}(H) \\ \mathfrak{p} \neq \emptyset}} \mathfrak{p} \neq \emptyset.$$

4. Let $F = F^\times \times [p_1, \ldots, p_s]$ be a factorial monoid with pairwise non-associated prime elements $p_1, \ldots, p_s$, and suppose that $H \subset F$ is a submonoid such that $H \cap F^\times = H^\times$. Then H is called a C*-monoid* (defined in F with parameter $\alpha \in \mathbb{N}$) if there exist $\alpha \in \mathbb{N}$ and a subgroup $V \subset F^\times$ such that the following two conditions are satisfied:

   (C1) $(F^\times : V) \mid \alpha$ and $V \cdot (H \setminus H^\times) \subset H$.

   (C2) For all $j \in [1, s]$ and $a \in p_j^\alpha F$ we have $a \in H$ if and only if $p_j^\alpha a \in H$.

A monoid is strongly primary if and only if it is finitary and primary. Every monoid H with s-spec(H) finite is a G-monoid, and therefore every primary monoid is a G-monoid. If H is primary, then its root-closure $\widetilde{H}$ is primary, too. Every C-monoid H defined in a finitely generated factorial monoid is a v-noetherian G-monoid with $(H : \widehat{H}) \neq \emptyset$ (see [14, Theorems 2.9.11 and 2.9.13]). The algebraic properties and the interplay of the various types of primary monoids are investigated in a series of papers [11], [15], [16], [22], [14, Sections 2.7–2.10], and [21, Chapter 15]. Here we need the following result [14, Theorem 2.7.9]:

Theorem 2.2. *Suppose that H is a v-noetherian* G*-monoid. Then H is finitary and s-*spec(H) *is finite. If $(H : \widehat{H}) \neq \emptyset$, then $\widehat{H}$ is a Krull monoid, s-*spec$(\widehat{H})$ *is finite, and $\widehat{H}_{\mathrm{red}}$ is finitely generated.*

Next we recall some basic arithmetical notions from factorization theory. If P is a set, we denote by $\mathcal{F}(P)$ the free abelian monoid generated by P. Clearly, $\mathcal{F}(P)$ is isomorphic to the coproduct of $|P|$ copies of $(\mathbb{N}_0, +)$. We denote by $\mathcal{A}(H)$ the *set of atoms* of H, and we call $\mathsf{Z}(H) = \mathcal{F}(\mathcal{A}(H_{\mathrm{red}}))$ the *factorization monoid* of H. Further, $\pi\colon \mathsf{Z}(H) \to H_{\mathrm{red}}$ denotes the natural homomorphism. For $a \in H$ the set

$\mathsf{Z}(a) = \mathsf{Z}_H(a) = \pi^{-1}(aH^\times) \subset \mathsf{Z}(H)$ is called the *set of factorizations* of a,

and

$\mathsf{L}(a) = \mathsf{L}_H(a) = \{|z| \mid z \in \mathsf{Z}(a)\} \subset \mathbb{N}_0$ is called the *set of lengths* of a.

H is said to be *atomic* if $\mathsf{Z}(a) \neq \emptyset$ for all $a \in H$. Strongly primary monoids and v-noetherian monoids are atomic, and all their sets of lengths are finite (see [14, Theorem 2.2.9]).

For the rest of the section we suppose that H is atomic.

For a prime element $p \in H$ we denote by $\mathsf{v}_p\colon \mathsf{q}(H) \to \mathbb{Z}$ the *p-adic valuation*. The *set of distances* $\Delta(H)$ of H is defined by

$$\Delta(H) = \bigcup_{a \in H} \Delta(\mathsf{L}(a)) \subset \mathbb{N}.$$

By definition, $\Delta(H) = \emptyset$ if and only if $|\mathsf{L}(a)| \le 1$ for all $a \in H$. Let $k \in \mathbb{N}$, and suppose that $H \neq H^\times$. We set

$$\rho_k(H) = \sup\{\sup \mathsf{L}(a) \mid a \in H,\ \min \mathsf{L}(a) \le k\} \in \mathbb{N} \cup \{\infty\}.$$

Then

$$\rho(H) = \sup\left\{\frac{\rho_l(H)}{l} \mid l \in \mathbb{N}\right\} \in \mathbb{R}_{\ge 1} \cup \{\infty\}$$

is the *elasticity* of H (cf. [14, Proposition 1.4.2]).

Let $z,\, z' \in \mathsf{Z}(H)$. Then we can write

$$z = u_1 \cdot \ldots \cdot u_l v_1 \cdot \ldots \cdot v_m \quad \text{and} \quad z' = u_1 \cdot \ldots \cdot u_l w_1 \cdot \ldots \cdot w_n,$$

where $l,\, m,\, n \in \mathbb{N}_0$, $u_1, \ldots, u_l,\, v_1, \ldots, v_m,\, w_1, \ldots, w_n \in \mathcal{A}(H_{\mathrm{red}})$ such that

$$\{v_1, \ldots, v_m\} \cap \{w_1, \ldots, w_n\} = \emptyset.$$

We call $\mathsf{d}(z, z') = \max\{m, n\} \in \mathbb{N}_0$ the *distance* of z and z'. Let $a \in H$ and $N \in \mathbb{N}_0 \cup \{\infty\}$, and suppose that $z, z' \in \mathsf{Z}(a)$. We say that z and z' can be *concatenated by an N-chain* if there exists a finite sequence $z = z_0, z_1, \ldots, z_k = z'$ of factorizations in $\mathsf{Z}(a)$ such that $\mathsf{d}(z_{i-1}, z_i) \le N$ for all $i \in [1, k]$. For an element $a \in H$, we define its *catenary degree* $\mathsf{c}(a)$ to be the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ such that any two factorizations of a can be concatenated by an N-chain, and we call

$$\mathsf{c}(H) = \sup\{\mathsf{c}(a) \mid a \in H\} \in \mathbb{N}_0 \cup \{\infty\}$$

the *catenary degree* of H. Next we recall the definition of local tameness.

Definition 2.3.

1. For $a \in H$ and $x \in \mathsf{Z}(H)$ let $\mathsf{t}(a, x) \in \mathbb{N}_0 \cup \{\infty\}$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property: If $\mathsf{Z}(a) \cap x\mathsf{Z}(H) \neq \emptyset$ and $z \in \mathsf{Z}(a)$, then there exists a factorization $z' \in \mathsf{Z}(a) \cap x\mathsf{Z}(H)$ such that $\mathsf{d}(z, z') \le N$.
2. For subsets $H' \subset H$ and $X \subset \mathsf{Z}(H)$ we define
$$\mathsf{t}(H', X) = \sup\{\mathsf{t}(a, x) \mid a \in H',\ x \in X\} \in \mathbb{N}_0 \cup \{\infty\}.$$
3. H is called *locally tame* if $\mathsf{t}(H, u) < \infty$ for all $u \in \mathcal{A}(H_{\mathrm{red}})$.

Local tameness is a basic finiteness property in the theory of non-unique factorizations. In most settings local tameness has to be proved first, and then the finiteness of more refined arithmetical invariants—such as the catenary degree—can be investigated. For strongly primary monoids, however, local tameness already implies the finiteness of the catenary degree (see Theorem 2.4 below) as well as a strong structural result for sets of lengths (cf. Theorem 4.1). In the sequel we shall make use of the following result (see [14, Theorems 3.1.1 and 3.1.5]).

Theorem 2.4. *Suppose that H is strongly primary.*

1. *Suppose that one of the following conditions is satisfied*:
   - $\sup\{\min \mathsf{L}(c) \mid c \in H\} < \infty$.
   - *There exists $a \in H \setminus H^\times$ such that $\rho_{\mathcal{M}(a)}(H) < \infty$.*
   - *There exists $a \in H \setminus H^\times$ such that $\mathsf{t}(H, \mathsf{Z}(a)) < \infty$.*

   Then H is locally tame.

2. *If H is locally tame, then $\mathsf{c}(H) < \infty$, and $\Delta(H)$ is finite.*

The precise values of the arithmetical invariants occurring in Theorem 2.4 are only known in very special situations, see, e.g., [2], [5], [8].

3. Local tameness of strongly primary monoids

A special type of strongly primary monoids (called finitely primary monoids) was first introduced in [20] as a multiplicative model for one-dimensional local noetherian domains with non-zero conductor. A finitely primary monoid H is locally tame [13, Lemma 5.3], and its complete integral closure $\widehat{H}$ is factorial. In Theorem 3.5 we present a result showing that local tameness of strongly primary monoids holds in more generality.

We start with Lemma 3.1 where we show that important properties such as being v-noetherian or primary are preserved when passing to saturated submonoids. We note that Theorem 3.5 does not only apply to v-noetherian monoids: There exist strongly primary monoids H for which $\widehat{H}$ is factorial, $(H : \widehat{H}) \neq \emptyset$ (hence all assumptions of Theorem 3.5 are satisfied), and which fail to be v-noetherian (see [22, Example 3.7]). Multiplicative monoids of domains will be discussed in Corollary 3.6.

Lemma 3.1. *Suppose that H is v-noetherian and primary.*

1. *H is strongly primary and $v\text{-spec}(H) = \{\emptyset, H \setminus H^\times\}$. In particular H is v-local.*
2. *If $(H : \widehat{H}) \neq \emptyset$, then $\widehat{H}$ is a Krull monoid.*
3. *If $\widetilde{H}$ is v-noetherian, then $\widehat{\widetilde{H}}$ is a Krull monoid.*
4. *Let $S \subset H$ be a saturated submonoid.*
   (a) *S is v-noetherian and primary.*
   (b) *$\widehat{S} \subset \widehat{H}$ is saturated, and if $\widehat{H}$ is a Krull monoid, then $\widehat{S}$ is a Krull monoid.*
   (c) *If $(H : \widehat{H}) \neq \emptyset$, then $(S : \widehat{S}) \neq \emptyset$.*

Proof. 1. Theorem 2.2 implies that H is finitary, and hence it is strongly primary. We have $v\text{-spec}(H) \subset s\text{-spec}(H) = \{\emptyset, H \setminus H^\times\}$, and thus it remains to show that $H \setminus H^\times$ is a v-ideal. If $a \in H \setminus H^\times$, then aH is contained in some maximal v-ideal [14, Proposition 2.2.4]. Therefore $H \setminus H^\times$ is a maximal v-ideal.

2. This follows from Theorem 2.2.

3. Since $\widetilde{H}$ is primary and $(\widetilde{H} : \widehat{\widetilde{H}}) \neq \emptyset$ [15, Proposition 4.8], the assertion follows from 2.

4.(a) This follows from [14, Corollary 2.4.3.3.(b)] and [14, Proposition 2.4.4.2.(b)].

4.(b) The first assertion follows from [18, Lemma 3.3], and thus $\widehat{S}$ inherits the Krull monoid property from $\widehat{H}$ (see [14, Proposition 2.4.4.3]).

4.(c) Let $f \in (H : \widehat{H})$ and pick $s \in S \setminus S^\times$. Since H is primary there exists $n \in \mathbb{N}$ such that f divides s^n in H. Thus $s^n \in (H : \widehat{H}) \cap S$, whence $s^n \widehat{S} \subset \mathsf{q}(S) \cap H = S$. ∎

Lemma 3.2. *Suppose that H is strongly primary, and let $a \in H$ and $b \in H \setminus H^\times$ such that $\sup\{\min \mathsf{L}(ab^k) \mid k \in \mathbb{N}\} < \infty$. Then $\sup\{\min \mathsf{L}(c) \mid c \in H\} < \infty$.*

Proof. Let $c \in H$. If $c \in H^\times$, then $\mathsf{L}(c) = \{0\}$. Thus suppose that $c \in H \setminus H^\times$. If $a \nmid c$, then $a \notin H^\times$ and $\max \mathsf{L}(c) < \mathcal{M}(a)$. Hence assume that $a \mid c$, and let $n \in \mathbb{N}_0$ be maximal such that $b^n \mid a^{-1}c$. Then $c = ab^n d$ with $d \in H$ and $b \nmid d$, and it follows that

$$\min \mathsf{L}(c) \le \min \mathsf{L}(ab^n) + \min \mathsf{L}(d) \le \sup\{\min \mathsf{L}(ab^k) \mid k \in \mathbb{N}\} + \mathcal{M}(b)$$ ∎

Lemma 3.3. *Suppose that H is primary and assume that $F = F^\times \times [p_1, \ldots, p_s]$, with $s \in \mathbb{N}$ and non-associated primes $p_1, \ldots, p_s$, is a factorial monoid such that $H \subset F$ is cofinal.*

1. *H is a* BF*-monoid with $H \cap F^\times = H^\times$. For every $a \in H \setminus H^\times$ we have* $\operatorname{supp}(a) = \{p_1, \ldots, p_s\}$ *and* $\max \mathsf{L}(a) \le \min\{\mathsf{v}_{p_i}(a) \mid i \in [1, s]\}$.
2. *If there exist $i \in [1, s]$ and $M \in \mathbb{N}$ such that M is an upper bound for $\{\mathsf{v}_{p_i}(u) \mid u \in \mathcal{A}(H)\}$, then $\rho_k(H) \le kM$ for all $k \in \mathbb{N}$. In particular, $\rho(H) \le M$.*

Proof. 1. Since $H \subset F$ is cofinal there exists $b \in H$ such that $p_1 \cdot \ldots \cdot p_s \mid b$ in F. Let $a \in H \setminus H^\times$. Since H is primary there exists $n \in \mathbb{N}$ such that $b \mid a^n$. This implies that $\mathsf{v}_{p_i}(a) \ge 1$ for every $i \in [1, s]$. Therefore $H \cap F^\times = H^\times$, and H is a BF-monoid [14, Corollary 1.3.3]. If $a = u_1 \cdot \ldots \cdot u_k$ with $u_1, \ldots, u_k \in \mathcal{A}(H)$, then

$$k \le \sum_{j=1}^{k} \mathsf{v}_{p_i}(u_j) = \mathsf{v}_{p_i}(a)$$

for all $i \in [1, s]$. This yields the purported upper bound for $\max \mathsf{L}(a)$.

2. If $k \in \mathbb{N}$ and $a = u_1 \cdot \ldots \cdot u_k \in H$ with $u_1, \ldots, u_k \in \mathcal{A}(H)$, then 1. implies that

$$\max \mathsf{L}(a) \le \mathsf{v}_{p_i}(a) = \sum_{j=1}^{k} \mathsf{v}_{p_i}(u_j) \le kM.$$

From this inequality we obtain

$$\rho_k(H) \le kM \quad \text{and} \quad \rho(H) = \sup\left\{\frac{\rho_m(H)}{m} \mid m \in \mathbb{N}\right\} \le M.$$ ∎

Lemma 3.4. *Suppose that H is a* G*-monoid.*

1. $\widehat{\widehat{H}}$ *is completely integrally closed.*
2. $(H : \widetilde{H}) \neq \emptyset$ *if and only if* $(H : \widehat{H}) \neq \emptyset$.
3. *If* $(H : \widehat{H}) \neq \emptyset$, *then* $\widehat{H} = \widehat{\widehat{H}}$.

Proof. 1. See [11, Theorem 4].

2. Since $\widetilde{H} \subset \widehat{H}$ it follows that $(H : \widehat{H}) \subset (H : \widetilde{H})$. Since $\widetilde{H}$ is a seminormal G-monoid, $(\widetilde{H} : \widehat{\widehat{H}}) \neq \emptyset$ by [15, Proposition 4.8]), and thus

$$\emptyset \neq (H : \widetilde{H})(\widetilde{H} : \widehat{\widehat{H}}) \subset (H : \widehat{\widehat{H}}) \subset (H : \widehat{H}).$$

3. If $(H : \widehat{H}) \neq \emptyset$, then $\widehat{H}$ is completely integrally closed [21, Theorem 14.1.(v)], and hence $\widehat{H} \subset \widehat{\widehat{H}} \subset \widehat{\widehat{H}} = \widehat{H}$. ■

Suppose that H is strongly primary. If there exists a sequence of monoids $H = D_0 \subset D_1 \subset \cdots \subset D_k \subset \mathsf{q}(H)$ such that $D_i = \widehat{D_{i-1}}$ for all $i \in [1, k]$ and D_k is a Krull monoid, then $D_k = \widehat{\widehat{H}}$. This follows from Lemma 3.4.1, and it is this situation which we consider in the next theorem.

Theorem 3.5. *Suppose H is strongly primary and $\widehat{\widehat{H}}$ is a Krull monoid.*

1. *There exists a factorial monoid $F = F^\times \times \mathcal{F}(P)$ containing $\widehat{\widehat{H}}$ such that*

   $F^\times = \widehat{\widehat{H}}^\times$, $|P| = |\mathfrak{X}(\widehat{\widehat{H}})| < \infty$, *and* $\widehat{\widehat{H}}_{\mathrm{red}} \hookrightarrow \mathcal{F}(P)$ *is a divisor theory.*

   Moreover $H \subset F$ is cofinal, $\widetilde{H} \cap F^\times = \widetilde{H}^\times$, and $H \cap F^\times = H^\times$.
2. $\widetilde{H} = \{a \in \widehat{\widehat{H}} \mid \operatorname{supp}(a) = P\} \cup \widetilde{H}^\times$, *and $\widetilde{H}$ is a* C*-monoid if and only if $\mathcal{C}(\widehat{\widehat{H}})$ is finite.*
3. *If $\widehat{H}$ is a Krull monoid and $|\mathfrak{X}(\widehat{H})| \geq 2$, then* $\sup\{\min \mathsf{L}(c) \mid c \in H\} < \infty$.
4. *If $(H : \widetilde{H}) \neq \emptyset$ and $|\mathfrak{X}(\widehat{\widehat{H}})| = 1$, then $\rho(H) < \infty$ and $\rho_k(H) < \infty$ for all $k \in \mathbb{N}$.*

In particular, if the assumptions of 3. or 4. are satisfied, then H is locally tame, $\mathsf{c}(H) < \infty$ and $\Delta(H)$ is finite.

Proof. The statement "In particular…" follows from Theorem 2.4.

1. Since H is a G-monoid, the overmonoid $\widehat{\widehat{H}}$ is again a G-monoid, and Theorem 2.2 (applied to $\widehat{\widehat{H}}$) implies that s-spec$(\widehat{\widehat{H}})$ is finite. Thus $\mathfrak{X}(\widehat{\widehat{H}})$

is finite, and the first assertion follows from [14, Theorem 2.7.14]. Since all inclusions

$$H \subset \widetilde{H}, \quad \widetilde{H} \subset \widehat{\widetilde{H}} \quad \text{and} \quad \widehat{\widetilde{H}} \subset F$$

are cofinal, $H \subset F$ is cofinal. Since H is primary, $\widetilde{H}$ is primary, and [12, Propositions 1 and 2] implies that

$$H \cap \widetilde{H}^\times = H^\times \quad \text{and} \quad \widetilde{H} \cap \widehat{\widetilde{H}}^\times = \widetilde{H}^\times,$$

whence

$$H \cap F^\times = H \cap \widetilde{H} \cap \widehat{\widetilde{H}}^\times = H \cap \widetilde{H}^\times = H^\times.$$

2. Let $a \in \widehat{\widetilde{H}}$ with $\operatorname{supp}(a) = P$. By [15, Proposition 4.8] there exists $f \in (\widetilde{H} : \widehat{\widetilde{H}})$. Clearly, there exists $n \in \mathbb{N}$ such that f divides a^n in F and hence in $\widehat{\widetilde{H}}$. This implies that

$$a^n = f(f^{-1}a^n) \in f\widehat{\widetilde{H}} \subset \widetilde{H},$$

and thus $a \in \widetilde{H}$. Conversely, let $a \in \widetilde{H} \setminus \widetilde{H}^\times$. Since $\widehat{\widetilde{H}} \subset F$ is cofinal and $(\widetilde{H} : \widehat{\widetilde{H}})\widehat{\widetilde{H}} \subset \widetilde{H}$ there exists $c \in \widetilde{H}$ such that $\operatorname{supp}(c) = P$. Since $\widetilde{H}$ is primary there exists $n \in \mathbb{N}$ such that $c \mid a^n$ in $\widetilde{H}$, whence $\operatorname{supp}(a) = P$.

If $\widetilde{H}$ is a C-monoid, then [14, Theorem 2.9.11] implies that $\mathcal{C}(\widehat{\widetilde{H}})$ is finite. Conversely, suppose that $\mathcal{C}(\widehat{\widetilde{H}})$ is finite. In order to show that $\widetilde{H}$ is a C-monoid (defined in F), we verify conditions (**C1**) and (**C2**) of Definition 2.1 with $V = F^\times$ and $\alpha = \exp(\mathcal{C}(\widehat{\widetilde{H}}))$. By 1. we have $\widetilde{H} \cap F^\times = \widetilde{H}^\times$. If $\varepsilon \in F^\times$ and $a \in \widetilde{H} \setminus \widetilde{H}^\times$, then $\operatorname{supp}(a) = P = \operatorname{supp}(\varepsilon a)$, whence $\varepsilon a \in \widetilde{H}$. Therefore we obtain $F^\times(\widetilde{H} \setminus \widetilde{H}^\times) \subset \widetilde{H}$. Let $p \in P$ and $a \in p^\alpha F$. Then $\operatorname{supp}(a) = \operatorname{supp}(p^\alpha a)$. We know that $a \in \widetilde{H}$ if and only if

$$a \in \widehat{\widetilde{H}} \quad \text{and} \quad \operatorname{supp}(a) = P,$$

and $p^\alpha a \in \widetilde{H}$ if and only if

$$p^\alpha a \in \widehat{\widetilde{H}} \quad \text{and} \quad \operatorname{supp}(p^\alpha a) = P.$$

By the choice of α we have $p^\alpha \in \widehat{\widetilde{H}}$, and hence we conclude that $a \in \widetilde{H}$ if and only if $p^\alpha a \in \widetilde{H}$.

3. Since $\widehat{H}$ is a Krull monoid it follows that $\widehat{H} = \widehat{\widetilde{H}}$, and we write $P = \{p_1, \ldots, p_s\}$. Since F is a divisor theory of $\widehat{H}$ there exist finite subsets $E, E' \subset \widehat{H}$ such that

$$p_1 \cdot \ldots \cdot p_{s-1} = \gcd(E) \quad \text{and} \quad p_2 \cdot \ldots \cdot p_s = \gcd(E').$$

Thus there exists $a \in E$ with $p_s \nmid a$. Hence $\mathrm{supp}(a) = \{p_1, \dots, p_{s-1}\}$. Similarly, there is some $b \in E'$ with $p_1 \nmid b$ and $\mathrm{supp}(b) = \{p_2, \dots, p_s\}$. Then 2. implies that $ab \in \widetilde{H}$. After replacing a and b by a suitable power if necessary, we obtain that $a, b \in \widehat{H}$, $ab \in H$, $\mathrm{supp}(a) = \{p_1, \dots, p_{s-1}\}$ and $\mathrm{supp}(b) = \{p_2, \dots, p_s\}$. Let $c \in H$ such that $ca^k \in H$ and $cb^k \in H$ for all $k \in \mathbb{N}$. Then Lemma 3.3.1 implies that

$$\min \mathsf{L}(c^2(ab)^k) \le \min \mathsf{L}(ca^k) + \min \mathsf{L}(cb^k) \le \mathsf{v}_{p_s}(c) + \mathsf{v}_{p_1}(c),$$

and hence the assertion follows from Lemma 3.2.

4. Since $(H : \widetilde{H}) \ne \emptyset$ Lemma 3.4 implies that $\widehat{H} = \widehat{\widetilde{H}}$ and that there exists $f \in (H : \widehat{H})$. Let $P = \{p\}$, and suppose that $u \in \mathcal{A}(H)$. We assert that $\mathsf{v}_p(u) < 2\mathsf{v}_p(f)$. Then the assertion follows from Lemma 3.3.2. Assume to the contrary that $\mathsf{v}_p(u) \ge 2\mathsf{v}_p(f)$. Then $f^2 \mid u$ in F and hence in $\widehat{H}$. Then $f(f^{-2}u) \in H \setminus H^\times$ and $u = f(f^{-1}u)$, a contradiction to $u \in \mathcal{A}(H)$. ∎

First we outline how Theorem 3.5 applies to integral domains, and then we discuss the additional assumptions in 3. and 4. of Theorem 3.5.

Let R be an integral domain and $R^\bullet = R \setminus \{0\}$ its multiplicative monoid. Then $R^\bullet$ is primary if and only if R is one-dimensional and local. R is a Mori domain if and only if $R^\bullet$ is v-noetherian, and R is a Krull domain if and only if $R^\bullet$ is a Krull monoid (see [14, Section 2.10]). We denote by $\widehat{R}$ the complete integral closure of R. Clearly, we have $\widehat{R} = \widehat{R^\bullet} \cup \{0\}$. Suppose that R is a one-dimensional local Mori domain. In each of the following situations its complete integral closure is a Krull domain:

- If R is noetherian, then $\widehat{R}$ is a Krull domain by the Mori-Nagata Integral Closure Theorem (see [10, Chapter 1]).
- If $(R : \widehat{R}) \ne \{0\}$, then $\widehat{R}$ is a Krull domain [14, Theorem 2.10.9].
- If R is seminormal, then $\widehat{R}$ is a Krull domain [3, Theorem 2.9].

On the other hand there exist one-dimensional local Mori domains R whose complete integral closure is not a Krull domain (see [24, Example 9] where $\widetilde{R} = \widehat{R}$ is completely integrally closed but not a Mori domain).

Corollary 3.6. *Let R be a one-dimensional local Mori domain whose complete integral closure is a Krull domain. Then its multiplicative monoid $H = R^\bullet$ is strongly primary, and in each of the following situations H is locally tame*:

1. $|\mathfrak{X}(\widehat{H})| \ge 2$.
2. $|\mathfrak{X}(\widehat{H})| = 1$ *and* $(R : \widehat{R}) \ne \{0\}$.
3. R *is noetherian.*

Proof. $R^\bullet$ is strongly primary by [14, Proposition 2.10.7.1]. If $|\mathfrak{X}(\widehat{H})| \ge 2$ or ($|\mathfrak{X}(\widehat{H})| = 1$ and $(R : \widehat{R}) \ne \{0\}$), then the assertion follows from Theorem 3.5. The case when $(R, \mathfrak{m})$ is local noetherian and $(R : \widehat{R}) = \{0\}$ was treated by the second-named author in [23, Theorem 3.5]. ∎

Let R be as in Corollary 3.6. The case when $|\mathfrak{X}(\widehat{H})| = 1$ and $(R : \widehat{R}) = \{0\}$ remains open, and we conjecture that also in that case $R^\bullet$ is locally tame. However, this does not remain true for monoids. In Proposition 3.7 we construct a strongly primary monoid H such that $\widehat{H}$ is Krull, $|\mathfrak{X}(\widehat{H})| = 1$, $(H : \widehat{H}) = \emptyset$, and H is not locally tame. In Example 3.8 we show that such a monoid H can be obtained as a saturated submonoid of Nagata's example of a one-dimensional analytically ramified local domain [26, E 3.2].

Suppose now that H is strongly primary and $\widehat{\widehat{H}}$ is a Krull monoid with $|\mathfrak{X}(\widehat{\widehat{H}})| \ge 2$. Theorem 3.5.3 shows that H is locally tame if $\widehat{H} = \widehat{\widehat{H}}$. It is tempting to speculate that this is also true if $\widehat{H} \ne \widehat{\widehat{H}}$, and we build an example (Example 3.9) to support this conjecture. The assumption that H is *strongly* primary is essential: There exists a primary BF-monoid H which is neither strongly primary nor locally tame [18, Example 4.6].

Proposition 3.7. *Let $F = F^\times \times [\pi]$ be a factorial monoid, where π is a prime element of F. Suppose $D \subset F$ is a primary submonoid such that $D^\times = D \cap F^\times$, $\exp(F^\times / D^\times) < \infty$, and $A = \{\mathsf{v}_\pi(u) \mid u \in \mathcal{A}(D)\}$ is infinite.*

1. *If $\Delta(A)$ is finite, then $\sup\{\min \mathsf{L}(c) \mid c \in D\} < \infty$.*
2. *Suppose that $\pi \in D$, $\widehat{D} = F$, and $\exp(F^\times / D^\times) = 2$. Then there exists a saturated submonoid $H \subset D$ having the following properties: H is primary, $\pi \in H$, $\widetilde{H} = \widehat{H}$, and $\Delta(H)$ is infinite. Moreover, if D is v-noetherian, then H is strongly primary but not locally tame, $(H : \widehat{H}) = \emptyset$, $\widehat{H}_{\mathrm{red}} \cong (\mathbb{N}_0, +)$, and H has infinite catenary degree.*

Proof. Taking [14, Proposition 2.3.4.1] into account it is easy to see that D may assumed to be reduced. Since $D \cap F^\times = D^\times = \{1\}$ it follows that D is a BF-monoid.

1. Since $D \ne \{1\}$ there exist $\varepsilon \in F^\times$ and $l \in \mathbb{N}$ such that $a = \varepsilon \pi^l \in D$. Then $a^{\exp(F^\times)} = \pi^\alpha \in D$, where $\alpha = l \exp(F^\times)$. If $\eta \in F^\times$ and $k \in \mathbb{N}$, then $(\eta \pi^k)^\alpha = (\pi^\alpha)^k \in D$, whence $F^\alpha \subset D$.

Let $k \in \mathbb{N}$. Then there exists $u = \eta \pi^n \in \mathcal{A}(D)$, with $\eta \in F^\times$ and $n \in \mathbb{N}$, such that $k - n \in [0, \max \Delta(A) - 1]$. We can write $\pi^{\alpha k} = u^\alpha \pi^{\alpha(k-n)}$ and obtain

$$\min \mathsf{L}(\pi^{\alpha k}) \le \min \mathsf{L}(u^\alpha) + \min \mathsf{L}(\pi^{\alpha(k-n)}) \le \alpha \max \Delta(A).$$

Thus the assertion follows from Lemma 3.2 (with $a = 1$ and $b = \pi^\alpha$).

2. Let $H \subset D$ be any saturated submonoid with $\pi \in H$. Then H is primary by [14, Corollary 2.4.3.3.(b)]. Since $F^\times$ is a torsion group and $\pi \in H$

it follows that for all $x \in F \supset \widehat{H}$ there exist $n \in \mathbb{N}$ such that $x^n \in H$. Thus we obtain $\widehat{H} = \widetilde{H}$.

Suppose that $\Delta(H)$ is infinite and D is v-noetherian. Then H is strongly primary by Lemma 3.1. Thus Theorem 2.4.2 implies that H is not locally tame, and by [14, Theorem 1.6.3] we infer that H has infinite catenary degree. By Theorems 3.5.4 and 2.4.1 it follows that $(H : \widehat{H}) = \emptyset$. Since $\widehat{H} \subset \widehat{D}$ is saturated (see Lemma 3.1.4) and $\widehat{D}$ is a discrete valuation monoid, it follows that $\widehat{H}$ is a discrete valuation monoid, too. To prove 2. it thus remains to construct a saturated submonoid $H \subset D$ with $\pi \in H$ such that $\Delta(H)$ is infinite.

For any $r \geq 2$ we choose a number $C_r \geq 3$ such that the set $\{C_r \mid r \geq 2\}$ is unbounded. Set $q_1 = \pi$, $t_1 = 1$, and choose a sequence of atoms $(q_r)_{r \in \mathbb{N}}$ of D with $\mathsf{v}_\pi(q_r) = t_r \in \mathbb{N}$ such that

$$t_r \geq C_r \sum_{i=1}^{r-1} t_i \tag{1}$$

for all $r \geq 2$. For all $r \in \mathbb{N}$ we have

$$q_r = \varepsilon_r \pi^{t_r} \in \mathcal{A}(D), \quad \text{with } \varepsilon_r \in F^\times \text{ and } \ q_r^2 = q_1^{2t_r} = \pi^{2t_r}.$$

Denote by $H_r \subset D$ the smallest saturated submonoid of D containing $\{q_1, \ldots, q_r\}$, i.e., $H_r = \langle q_1, \ldots, q_r \rangle \cap D$, and put

$$H = \bigcup_{r=1}^{\infty} H_r.$$

We continue with three assertions.

A1. Let $r \geq 2$ and $p \in \mathcal{A}(H_r) \setminus \mathcal{A}(H_{r-1})$. Then

$$|\mathsf{v}_\pi(p) - t_r| \leq \sum_{i=1}^{r-1} t_i \tag{2}$$

and

$$2t_{r-1} \leq \mathsf{v}_\pi(p) < 2t_r. \tag{3}$$

Proof of A1. Since $p \in H_r \setminus H_{r-1}$, we have $p = q_1^{2k+m_1} q_2^{m_2} \cdot \ldots \cdot q_{r-1}^{m_{r-1}} q_r$, with $k \in \mathbb{Z}$ and $m_i \in \{0, 1\}$. Since p is an atom of H_r it follows that $k \leq 0$ and $\mathsf{v}_\pi(p) \leq \sum_{i=1}^{r-1} m_i t_i + t_r \leq t_r + \sum_{i=1}^{r-1} t_i$. Now we consider $p \prod_{i=1}^{r-1} q_i^{m_i} = q_1^{2(k+m_1)} q_2^{2m_2} \cdot \ldots \cdot q_{r-1}^{2m_{r-1}} q_r = q_1^n q_r$, with $n \in \mathbb{Z}$. Since q_r is an atom, we have $n \geq 0$, and therefore $\mathsf{v}_\pi(p) + \sum_{i=1}^{r-1} m_i t_i \geq t_r$, which yields $\mathsf{v}_\pi(p) \geq t_r - \sum_{i=1}^{r-1} t_i$. Thus (2) is proved. ∎

Together with (1), (2) gives $(C_r-1)t_{r-1} \le (C_r-1)\sum_{i=1}^{r-1} t_i \le \mathsf{v}_\pi(p) \le (1+C_r^{-1})t_r$. This implies (3) since $C_r \ge 3$. For any $r \ge 2$ we pick $p_r \in \mathcal{A}(H)$ with

$$\mathsf{v}_\pi(p_r) = \min\left(\{\mathsf{v}_\pi(q) \mid q \in \mathcal{A}(H)\} \cap \left[t_r - \sum_{i=1}^{r-1} t_i, t_r + \sum_{i=1}^{r-1} t_i\right]\right).$$

Since $q_r \in \mathcal{A}(H)$, we can find such a p_r, and furthermore $\mathsf{v}_\pi(p_r) \le t_r$.

A2. There is no $q \in \mathcal{A}(H)$ and no $r \ge 2$ such that

$$\sum_{i=1}^{r-1} t_i < \mathsf{v}_\pi(q) < \mathsf{v}_\pi(p_r). \tag{4}$$

Proof of A2. Assume to the contrary that there exist $r \ge 2$ and $q \in \mathcal{A}(H)$ such that (4) is satisfied. It follows that $1 = t_1 < \mathsf{v}_\pi(q)$, and therefore $q \notin H_1 = [\pi]$. Let $m \in \mathbb{N}$ be minimal such that $q \in H_m$. Then $m \ge 2$ and $q \in \mathcal{A}(H_m) \setminus H_{m-1}$. From (2) and (3) we obtain the inequality

$$2t_{m-1} \le \mathsf{v}_\pi(q) \le \sum_{i=1}^{m} t_i.$$

Combining this with (4) gives $\sum_{i=1}^{r-1} t_i < \mathsf{v}_\pi(q) \le \sum_{i=1}^{m} t_i$, and it follows that $m \ge r$. The inequality $2t_{m-1} \le \mathsf{v}_\pi(q) < \mathsf{v}_\pi(p_r) < 2t_r$, on the other hand, yields $m \le r$. Thus $m = r$, contradicting the choice of p_r.

A3. Let $r \ge 2$ and $q \in \mathcal{A}(H)$ with $q \mid_H p_r^2$. Then $q \in H_r$, and if furthermore $\mathsf{v}_\pi(q) \ge \mathsf{v}_\pi(p_r)$, then $q \in H_r \setminus H_{r-1}$.

Proof of A3. Since $\mathsf{v}_\pi(q) < 2\mathsf{v}_\pi(p_r) \le 2t_r$, (3) yields $q \in H_r$. If $\mathsf{v}_\pi(q) \ge \mathsf{v}_\pi(p_r) \ge 2t_{r-1}$, (3) yields $q \notin H_{r-1}$.

Now we show that $\Delta(H)$ is infinite. Let $r \ge 2$ and $z \in \mathsf{Z}_H(p_r^2)$. Suppose first that there exists $q \in \mathcal{A}(H)$ such that $q \mid_{\mathsf{Z}(H)} z$ and $\mathsf{v}_\pi(q) \ge \mathsf{v}_\pi(p_r)$. From **A3** we know that $q \in H_r \setminus H_{r-1}$. Put $y = p_r^2 q^{-1} \in H$. Using (2), we obtain

$$\mathsf{v}_\pi(y) = 2\mathsf{v}_\pi(p_r) - \mathsf{v}_\pi(q) \ge 2\left(t_r - \sum_{i=1}^{r-1} t_i\right) - \left(t_r + \sum_{i=1}^{r-1} t_i\right) \ge (C_r - 3)\sum_{i=1}^{r-1} t_i. \tag{5}$$

If y is irreducible, then $|z| = 2$. Suppose y is not irreducible. Then, for every $u \in \mathcal{A}(H)$ with $u \mid_{\mathsf{Z}(H)} y$, we have $\mathsf{v}_\pi(u) < \mathsf{v}_\pi(y) \le \mathsf{v}_\pi(p_r)$, and **A2** yields

$$\mathsf{v}_\pi(u) \le \sum_{i=1}^{r-1} t_i.$$

Combining this with (5) gives $|z| \ge C_r - 2$.

Suppose now that for all $q \in \mathcal{A}(H)$ with $q \mid_{\mathsf{Z}(H)} z$ we have $\mathsf{v}_\pi(q) < \mathsf{v}_\pi(p_r)$. Thus, by **A2**, we have $\mathsf{v}_\pi(q) \le \sum_{i=1}^{r-1} t_i$ for all such q. By the definition of p_r we obtain

$$|z| \ge \frac{2(t_r - \sum_{i=1}^{r-1} t_i)}{\sum_{i=1}^{r-1} t_i} \ge 2(C_r - 1).$$

Since 2 and $\mathsf{v}_\pi(p_r^2) = 2t_r \ge 2C_r$ are both contained in $\mathsf{L}_H(p_r^2)$, the minimum of $\mathsf{L}_H(p_r^2)\setminus\{2\}$ is at least $C_r - 3$. By the choice of the sequence C_r it follows that $|\Delta(H)| = \infty$. ■

Example 3.8. There exists a one-dimensional local noetherian domain $(R, \mathfrak{m})$ having the following properties:

- The integral closure $\overline{R}$ of R is a discrete valuation domain with maximal ideal $\overline{\mathfrak{m}}$.
- $(R : \overline{R}) = \{0\}$.
- $\mathfrak{m}\overline{R} = \overline{\mathfrak{m}}$.
- $\exp(\overline{R}^\times / R^\times) = 2$.
- The monoid $D = R^\bullet$ is v-noetherian, primary and locally tame. There exists an element $\pi \in D$ which is a prime element of $F = \overline{R}^\bullet = F^\times \times [\pi] = \widehat{D}$, $A = \{\mathsf{v}_\pi(u) \mid u \in \mathcal{A}(D)\}$ is infinite, and $\Delta(A)$ is finite.

Proof. We take for R the domain built in [26, E 3.2], where the field K has characteristic 2. For the convenience of the reader we briefly recall the main steps of the construction. Let K be a field with characteristic 2 such that $[K : K^{[2]}] = \infty$, where $K^{[2]} = \{x^2 \mid x \in K\}$, and let $(b_i)_{i\in\mathbb{N}}$ be an infinite sequence of 2-independent elements of K (for the notion of p-independence see [26, p. 195, fourth paragraph]). Let X be an indeterminate over K, and set $c = \sum_{i\ge 1} b_i X^i \in K[[X]]$. We claim that $R = K^{[2]}[[X]][K][c]$ has the required properties.

By [26, E3.2] R is a one-dimensional local noetherian domain whose integral closure $\overline{R}$ fails to be a finitely generated R-module. Thus $(R : \widehat{R}) = \{0\}$. In order to show that $\overline{R}$ is a discrete valuation domain put $V = K^{[2]}[[X]][K]$. By [26, E3.1], V is a discrete valuation domain. Since c is integral over V, the integral closure of R is equal to the integral closure of V in $\mathsf{q}(V)(c)$. Since $\mathsf{q}(V)(c)/\mathsf{q}(V)$ is a purely inseparable extension of degree 2 it follows that $\overline{R}$ is local, and that $\overline{R}^\times / R^\times$ has exponent 2. Finally, we see easily that $X \in R$ and that X is a prime element of $\overline{R}$.

Since R is a one-dimensional local noetherian domain, D is v-noetherian and primary. Since $(R : \widehat{R}) = \{0\}$, we have $\rho(R) = \rho(D) = \infty$ by [1, Theorem 2.12], and thus A is infinite by Lemma 3.3.2. Furthermore, $\Delta(A)$ is finite by [23, Theorem 3.8], and thus D is locally tame by Proposition 3.7.1 and Theorem 2.4. ■

Example 3.9. Consider the monoid

$$H = \{(x_1, x_2) \in \mathbb{N}^2 \mid x_2 \le x_1^2\} \cup \{(0,0)\} \subset (\mathbb{N}_0^2, +).$$

1. H is v-noetherian and primary with $(H : \widehat{H}) = \emptyset$. Moreover, $\mathsf{q}(H) = \mathbb{Z}^2$, $\widetilde{H} = \mathbb{N}^2 \cup \{(0,0)\}$ (hence $\widetilde{H}$ is a C-monoid), and $\widehat{H} \subsetneq \widehat{\widehat{H}} = \mathbb{N}_0^2$ (hence $\widehat{H}$ is not completely integrally closed).
2. $\mathcal{A}(H) = \bigcup_{n \ge 1} \{(n,k) \mid k \in \{1\} \cup [2 + (n-1)^2, n^2]\}$.
3. For all $a \in H$ we have $\min \mathsf{L}(a) \le 3$.
4. H is locally tame, $\mathsf{c}(H) < \infty$, and $\Delta(H)$ is finite.

Proof. 1. This monoid has already been studied in [16, Theorem 3] and in [21, Exercise 14.3]. In particular, it has been proved that H is primary, $\widehat{H} \subsetneq \widehat{\widehat{H}}$, and that $\widetilde{H}, \widehat{\widehat{H}}$ and $\mathsf{q}(H)$ have the asserted form. It remains to show that H is v-noetherian. Then Theorem 2.2 implies that $(H : \widehat{H}) = \emptyset$ because $\widehat{H}$ is not a Krull monoid.

Let $X \subset H$ be a subset. We prove that there exists a finite subset $E \subset X$ such that $E^{-1} \subset X^{-1}$. Then H is v-noetherian by [14, Proposition 2.1.10]. Without loss of generality we may assume that X is an infinite set.

(a) *Construction of* E. Put

$$m = \min\{x_1^2 - x_2 \mid (x_1, x_2) \in X \setminus \{(0,0)\}\} \in \mathbb{N}_0,$$

and choose $e = (e_1, e_2) \in X \setminus \{(0,0)\}$ with ${e_1}^2 - e_2 = m$. Let

$$E = E_1 \cup E_2 \cup E_3,$$

where

$$E_1 = \{(x_1, x_2) \in X \mid x_1 \le e_1, x_2 \ge e_2\} \ni e$$

and E_2, E_3 are defined as follows.

For $i \in \mathbb{N}_0$ let $X^{(i)} = \{(x_1, x_2) \in X \mid x_2 = i\}$. If $X^{(i)} \ne \emptyset$ put $x_1^{(i)} = \min\{x_1 \mid (x_1, i) \in X^{(i)}\}$ and $x^{(i)} = (x_1^{(i)}, i) \in X^{(i)}$. We define

$$E_2 = \{x^{(i)} \mid i \in [0, e_2] \text{ and } X^{(i)} \ne \emptyset\}.$$

If $z \in Q = [-e_1, -1] \times [-e_2, m-1] \subset \mathbb{Z}^2$ with $z + X \not\subset H$, we choose $x_{(z)} \in X$ such that $z + x_{(z)} \notin H$, and we define

$$E_3 = \{x_{(z)} \mid z \in Q \text{ with } z + X \not\subset H\}.$$

(b) *Proof that* $E^{-1} \subset X^{-1}$. Let $z = (z_1, z_2) \in \mathsf{q}(H) = \mathbb{Z}^2$ with $z + E \subset H$. We have to show that $z + X \subset H$. Choose any $x \in X \setminus E$. We shall verify that $z + x \in H$.

Suppose first that $z_1 < 0$. Since $e \in E$, we have $z + e \in H \subset \mathbb{N}_0^2$, and an easy calculation yields $z \in Q$. Since $z + E_3 \subset z + E \subset H$ we infer, by the definition of E_3, that $z + x \in z + X \subset H$. Suppose from now on that $z_1 \geq 0$. Since $x \notin E_1$, we either have $x_2 \leq e_2$ or $e + (1,1) \leq x$.

CASE 1: $i = x_2 \leq e_2$. Then $x \in X^{(i)} \setminus E_2$, say $x = (x_1, i)$, with $x_1 > x_1^{(i)}$. Since $|X^{(i)}| \geq 2$ we have $i \geq 1$, and from $z + (x_1^{(i)}, i) \in z + E_2 \subset H$ it follows that $z_2 + i \leq (z_1 + x_1^{(i)})^2 < (z_1 + x_1)^2$. Thus $z + x \in H$.

CASE 2: $e + (1,1) \leq x$. From $z + e \in H \subset \mathbb{N}_0^2$ and from $e + (1,1) \leq x$ we infer that $z + x \in \mathbb{N}^2$. Using $x_1 \geq e_1 + 1$, we obtain

$$\begin{aligned} z_2 + e_2 &\leq (z_1 + e_1)^2 \\ z_2 - z_1^2 - 2z_1e_1 &\leq e_1^2 - e_2 = m \\ z_2 - z_1^2 - 2x_1z_1 \leq z_2 - z_1^2 - 2z_1e_1 &\leq m \leq x_1^2 - x_2 \\ z_2 + x_2 &\leq (x_1 + z_1)^2. \end{aligned}$$

This proves that $z + x \in H$.

2. Let $n, k \in \mathbb{N}$ such that $(n,k) \in \mathcal{A}(H)$. Then $k \in [1, n^2]$. If $k \in [2, 1 + (n-1)^2]$, then $(n-1, k-1) \in H$ and $(n,k) = (1,1) + (n-1, k-1)$, a contradiction.

To verify the converse inclusion, let $n \in \mathbb{N}$. Clearly, we have $(n,1) \in \mathcal{A}(H)$. Let $k \in [2 + (n-1)^2, n^2]$ and assume to the contrary that

$$(n,k) = (a,b) + (n-a, k-b), \quad \text{where} \quad (a,b), (n-a, k-b) \in H \setminus \{(0,0)\}.$$

Then it follows that $b \leq a^2$ and $k - b \leq (n-a)^2$ whence

$$\begin{aligned} 2 + (n-1)^2 \leq k &\leq (n-a)^2 + a^2 \\ 1 + (2n - a - 1)(a-1) &\leq (a+1)(a-1) \\ 2n - a - 1 &\leq a \\ 2(n-a) &\leq 1, \end{aligned}$$

contradicting $(n-a, k-b) \in H \setminus \{(0,0)\}$.

3. Let $a = (n,k) \in H$ with $n, k \in \mathbb{N}_0$. We may suppose that $a \notin \mathcal{A}(H) \cup \{(0,0)\}$, whence $n \geq 2$ and $k \in [2, 1 + (n-1)^2]$. If $k = 2$, then $(n,k) = (n-1, 1) + (1,1)$ is the sum of two atoms. Thus suppose that $k > 2$. We distinguish two cases:

CASE 1: $k - 2$ is not a square.

Then there exists a unique $m \in \mathbb{N}$ such that $k - 1 \in [2 + (m-1)^2, m^2]$, and 2. implies that $(m, k-1) \in \mathcal{A}(H)$. Since $k \in [2, 1 + (n-1)^2]$, it follows that $m < n$, whence $(n - m, 1) \in \mathcal{A}(H)$. Thus $(n,k) = (m, k-1) + (n-m, 1)$ is a factorization into two atoms.

CASE 2: $k-2$ is a square.

Let $m \in \mathbb{N}$ such that $k-2=m^2$. Since $k < 2+(n-1)^2$, it follows that $m^2 = k-2 < (n-1)^2$, whence $m \le n-2$. Thus $(n,k) = (m, k-2) + (1,1) + (n-m-1, 1)$ is a factorization into three atoms.

4. This follows from 3. and from Theorem 2.4. ■

4. Sets of lengths in strongly primary monoids

Sets of lengths in a locally tame, strongly primary monoid are arithmetical progressions, apart from some gaps in their initial and end parts. A proof of the following result can be found in [14, Theorem 4.3.6].

Theorem 4.1. *Suppose that H is locally tame and strongly primary, and assume that $\Delta(H) \ne \emptyset$. Then there exists $M \in \mathbb{N}$ such that, for every $a \in H$, the set of lengths $\mathsf{L}(a)$ has the form*

$$\mathsf{L}(a) = y + (L' \cup \{\nu d \mid \nu \in [0,l]\} \cup L'') \subset y + d\mathbb{Z},$$

where $d = \min \Delta(H)$, $l \in \mathbb{N}_0$, $L' \subset [-M, -1]$ and $L'' \subset ld + [1, M]$.

The aim in this section is to prove a realization theorem for sets of lengths. Recall that every C-monoid is v-noetherian and locally tame [14, Theorems 2.9.13 and 3.3.4], and thus every primary C-monoid is strongly primary by Lemma 3.1.

Theorem 4.2. *Let $L \subset \mathbb{N}_{\ge 2}$ be a finite set. Then, for all sufficiently large $s \in \mathbb{N}$, there exist a primary C-monoid H defined in $(\mathbb{N}_0^s, +)$ with $\widetilde{H} = \mathbb{N}^s \cup \{\mathbf{0}\}$ and $\widehat{H} = \mathbb{N}_0^s$, and an element $a \in H$ such that $\mathsf{L}(a) = L$.*

For sets of lengths in finitely generated monoids and for sets of lengths in Krull monoids with finite class group much stronger realization results are known (see [14, Section 4.8] and [28]). For the proof of Theorem 4.2 we need two lemmas.

Lemma 4.3. *Let $L \subset \mathbb{N}_{\ge 2}$ be a finite set. Then, for all sufficiently large $s \in \mathbb{N}$, there exist a finitely generated Krull monoid $H \subset (\mathbb{N}_0^s, +)$ and some $a \in H$ such that $\mathsf{L}(a) = L$.*

Proof. By [14, Proposition 4.8.3] there exist a reduced finitely generated Krull monoid H' and $a' \in H'$ such that $\mathsf{L}_{H'}(a') = L$. By [14, Proposition 2.4.5] H' is isomorphic to a saturated submonoid $H'' \subset (\mathbb{N}_0^t, +)$, where $t = |\mathfrak{X}(H'')|$. Thus the assertion holds for all $s \ge |\mathfrak{X}(H'')|$. ■

Let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_s\}$ denote the canonical basis of $\mathbb{Z}^s$. If $\boldsymbol{x} \in \mathbb{Z}^s$, then let $x_1, \ldots, x_s \in \mathbb{Z}$ be defined by $\boldsymbol{x} = \sum_{\nu=1}^s x_\nu \boldsymbol{e}_\nu$.

Lemma 4.4. *Suppose that $H \subset (\mathbb{N}_0^s, +)$, with $s \in \mathbb{N}$, is a finitely generated submonoid.*

1. *There exists a submonoid $H' \subset (\mathbb{N}_0^s, +)$ such that $H' \cong H$ and $H' \setminus \{\mathbf{0}\} \subset \mathbb{N}^s$.*

2. *If $H \setminus \{\mathbf{0}\} \subset \mathbb{N}^s$ and $\alpha \ge \max\{u_\nu \mid \boldsymbol{u} \in \mathcal{A}(H),\ \nu \in [1,s]\}$, then $H^* = H \cup \mathbb{N}_{\ge\alpha}^s$ is a primary* C*-monoid defined in $\mathbb{N}_0^s$ with $\widetilde{H^*} = \mathbb{N}^s \cup \{\mathbf{0}\}$, $\widehat{H^*} = \mathbb{N}_0^s$ and $\mathcal{A}(H) \subset \mathcal{A}(H^*)$.*

Proof. 1. Since the matrix

$$M = \begin{pmatrix} s & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & s \end{pmatrix} \in \mathrm{M}_{s\times s}(\mathbb{Z})$$

has positive determinant, $\varphi_M \colon \mathbb{Z}^s \to \mathbb{Z}^s$, $x \mapsto Mx$ is a monomorphism with $\varphi_M(\mathbb{N}_0^s \setminus \{\mathbf{0}\}) \subset \mathbb{N}^s$. Therefore $H' = \varphi_M(H)$ has the required properties.

2. By definition of α we have $\mathcal{A}(H) \subset \mathcal{A}(H^*)$. If $\boldsymbol{x} \in \mathbb{N}^s$, then $\alpha\boldsymbol{x} \in H^*$ and thus $\boldsymbol{x} \in \widetilde{H^*}$. This implies that $\widetilde{H^*} = \mathbb{N}^s \cup \{\mathbf{0}\}$. Since $(\alpha, \ldots, \alpha) + \mathbb{N}_0^s \subset H^*$ it follows that $\widehat{H^*} = \mathbb{N}_0^s$. Obviously $\widetilde{H^*}$ is primary, and thus H^* is primary [12, Proposition 2].

We assert that H^* is a C-monoid defined in $\mathbb{N}_0^s$ with parameter

$$\beta = \alpha \cdot \max\{u_\nu \mid \boldsymbol{u} \in \mathcal{A}(H),\ \nu \in [1,s]\} \quad \text{and subgroup} \quad V = \{\mathbf{0}\}.$$

We shall verify conditions **(C1)** and **(C2)** of Definition 2.1. **(C1)** is obviously satisfied. To verify condition **(C2)** let $j \in [1,s]$ and $\boldsymbol{x} \in \beta\boldsymbol{e}_j + \mathbb{N}_0^s$. If $\boldsymbol{x} \in \mathbb{N}_{\ge\alpha}^s$, then clearly $\boldsymbol{x} \in H^*$ and $\beta\boldsymbol{e}_j + \boldsymbol{x} \in H^*$.

Suppose that $\boldsymbol{x} \notin \mathbb{N}_{\ge\alpha}^s$. Then there exists $\lambda \in [1,s]$ such that $x_\lambda < \alpha$, and $\boldsymbol{x} \in \beta\boldsymbol{e}_j + \mathbb{N}_0^s$ implies that $\lambda \ne j$. Thus $\beta\boldsymbol{e}_j + \boldsymbol{x} \notin \mathbb{N}_{\ge\alpha}^s$. Let $\boldsymbol{y} \in \{\boldsymbol{x}, \beta\boldsymbol{e}_j + \boldsymbol{x}\}$. We show that $\boldsymbol{y} \notin H^*$. Assume to the contrary that $\boldsymbol{y} \in H^*$. Then $\boldsymbol{y} \in H$ whence $\boldsymbol{y} = \sum_{i=1}^t \boldsymbol{u}_i$ with $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_t \in \mathcal{A}(H)$. Since $\mathcal{A}(H) \subset H \setminus \{\mathbf{0}\} \subset \mathbb{N}^s$ and $y_\lambda < \alpha$ it follows that $t < \alpha$. This implies

$$y_i \le t \cdot \max\{u_\nu \mid \boldsymbol{u} \in \mathcal{A}(H),\ \nu \in [1,s]\} < \beta$$

for all $i \in [1,t]$, a contradiction to $y_j \ge x_j \ge \beta$. ■

Proof of Theorem 4.2. By Lemma 4.3 there exist, for all sufficiently large $s \in \mathbb{N}$, a finitely generated monoid $H' \subset (\mathbb{N}_0^s, +)$ and some $\boldsymbol{a} \in H'$ such that $\mathsf{L}_{H'}(\boldsymbol{a}) = L$. By Lemma 4.4.1 we may suppose without restriction that

$H' \setminus \{\mathbf{0}\} \subset \mathbb{N}^s$. We define

$$\alpha = \max\{a_\nu, u_\nu \mid \boldsymbol{u} \in \mathcal{A}(H'),\ \nu \in [1,s]\} \quad \text{and} \quad H = H' \cup \mathbb{N}^s_{\geq \alpha}.$$

Now Lemma 4.4.2 implies that H is a primary C-monoid with $\mathcal{A}(H') \subset \mathcal{A}(H)$, and that $\widetilde{H}$ and $\widehat{H}$ have the asserted form. Since $\mathcal{A}(H') \subset \mathcal{A}(H)$ and $\alpha \geq \max\{a_\nu \mid \nu \in [1,s]\}$ it follows that

$$\mathsf{L}_H(\boldsymbol{a}) = \mathsf{L}_{H'}(\boldsymbol{a}) = L. \qquad \blacksquare$$

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