ON ZERO-SUM SEQUENCES IN $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$

Weidong Gao [1]
Department of Computer Science and Technology, University of Petroleum, Beijing, Shuiku Road, Changping, Beijing 102200, P.R. China
wdgao@public.fhnet.cn.net

Alfred Geroldinger [2]
Institut für Mathematik, Karl-FranzensUniversität, Heinrichstrasse 36, 8010 Graz, Austria
alfred.geroldinger@uni-graz.at

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Abstract

It is well known that the maximal possible length of a minimal zero-sum sequence S in the group $\mathbb{Z}/n\mathbb{Z}\oplus\mathbb{Z}/n\mathbb{Z}$ equals $2n-1$, and we investigate the structure of such sequences. We say that some integer $n \ge 2$ has Property B, if every minimal zero-sum sequence S in $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ with length $2n-1$ contains some element with multiplicity $n-1$. If some $n \ge 2$ has Property B, then the structure of such sequences is completely determined. We conjecture that every $n \ge 2$ has Property B, and we compare Property B with several other, already well-studied properties of zero-sum sequences in $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. Among others, we show that if some integer $n \ge 6$ has Property B, then $2n$ has Property B.

1. Introduction

In 1961, P. Erdös, A. Ginzburg and A. Ziv proved that every sequence S in $\mathbb{Z}/n\mathbb{Z}$ with length $|S| \ge 2n-1$ contains a zero-sum subsequence with length n [EGZ61]. Some years later, P. Erdös (for the special group $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$), H. Davenport (for general finite abelian groups) and P.C. Baayen formulated the following problem (see [MO67], [vEBK67]).

Problem 1: For a finite abelian group G, determine the smallest integer $l \in \mathbb{N}$ such that every sequence S in G with length $|S| \ge l$ contains a zero-sum subsequence.

In subsequent literature, the integer l in Problem 1 has come to be known as the Davenport constant of G, and we will denote it by $\mathsf{D}(G)$. J.E. Olson and D. Kruyswijk

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determined independently its precise value for p-groups and for groups with rank at most two ([Ols69a], [Ols69b], [vEB69b]). In particular, we have $\mathsf{D}(\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}) = 2n - 1$, which implies the Theorem of Erdös-Ginzburg-Ziv. However, for general finite abelian groups, even for groups with rank three or for groups of the form $(\mathbb{Z}/n\mathbb{Z})^r$, $\mathsf{D}(G)$ is still unknown (cf. [Gao00a], [GG03] [CFGS02] for recent developments).

The result of P. Erdös, A. Ginzburg and A. Ziv was also the starting point for much recent research devoted to the more general problem of studying subsequences of given sequences that have sum zero and satisfy some given additional property (see [Ham96], [Car96b], [HOO98], [GGH$^+$02], [Tha02a], [Tha02b], [Sch01] and the literature cited there). We give a precise formulation of some key questions of this type.

Problem 2: For a finite abelian group G, determine the smallest integer $l \in \mathbb{N}$ such that every sequence S in G with length $|S| \ge l$ contains a zero-sum subsequence T such that

1. $|T| \le \exp(G)$,
2. $|T| = \exp(G)$,
3. $|T| = |G|$.

For general finite abelian groups only Problem 2.3 is solved ($|G| + \mathsf{D}(G) - 1$ is the required integer (see [Car96a] and [Gao96a]). For finite cyclic groups 2.1 is obvious and 2.2 (resp. 2.3) is answered by the Erdös-Ginzburg-Ziv-Theorem. Now, suppose $G = \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ with $n \ge 2$. Then $3n-2$ is the required integer in Problem 2.1 ([Ols69b], [GG99], Lemma 4.4). In 1983, A. Kemnitz conjectured that $4n-3$ is the required integer in Problem 2.2. Recent progress on this topic was made by L. Ronyai and W. Gao, but the conjecture is still open (see [Har73], [Kem83], [AD93], [Ron00], [Gao01a], [Els] and the literature cited there).

Let us consider the inverse questions associated with Problem 1 and Problem 2. Let G be a finite abelian group.

Problem 1:* Determine the structure of a sequence S with maximal length (i.e., $|S| = \mathsf{D}(G) - 1$) which has no zero-sum subsequence.

Problem 2:* Determine the structure of a sequence S with maximal length which has no zero-sum subsequence T such that

1. $|T| \le \exp(G)$,
2. $|T| = \exp(G)$,
3. $|T| = |G|$.

Let $G = \mathbb{Z}/n\mathbb{Z}$ with $n \ge 2$. Then, obviously, a sequence S in G with maximal length which contains no zero-sum subsequence has the form $S = (a + n\mathbb{Z})^{n-1}$ for some $a \in \mathbb{Z}$ with $\gcd\{a, n\} = 1$. This answers Problem 1* and Problem 2*.1. The structure of a sequence S in G with length $|S| = 2n - k$ for "small" $k \ge 2$ which does not contain a zero-sum subsequence with length n was studied successfully by several authors (cf. [BD92], [Car92], [FO96], [Car96b], [Gao97]).

Let $G = \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ with $n \geq 2$. Problem 2*.1 was first tackled by P. van Emde Boas who asked for the structure of sequences S with length $|S| = 3n - 3$ which have no zero-sum subsequences with length at most n. This was motivated by investigations of Davenport's constant for groups having rank three (see [vEB69b] and [Gao00a], Lemma 4.7). Problem 1* appears naturally in the theory of non-unique factorizations and it was first addressed in [GG99]. Problem 2*.2 was first considered by W. Gao in [Gao00b]. All three problems (1*, 2*.1 2*.2) are open; there are conjectures which would provide complete answers to these problems and some partial results supporting these conjectures (cf. the discussion after Definition 3.2).

This paper concentrates on Problem 1* (for sequences in $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$). We say that an integer $n \geq 2$ has Property B, if every minimal zero-sum sequence S in $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ with length $|S| = \mathsf{D}(G)$ contains some element with multiplicity $n - 1$ (cf. Theorem 4.3 for various characterizations of this Property). We conjecture that every integer $n \geq 2$ satisfies Property B. If this holds true, then, by Theorem 4.3, Problem 1* is completely answered. We show that Property B is closely related to (usually stronger than) several other already well-studied properties of sequences in $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ (cf. Theorems 5.3 and 6.2); after having introduced some additional terminology, we give a more detailed preview of our results after Definition 3.2. Among these results, we show that if some integer $n \geq 6$ has Property B, then $2n$ has Property B (Theorem 8.1).

2. Preliminaries

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{P} \subset \mathbb{N}$ the set of prime numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a prime $p \in \mathbb{P}$ let $\mathsf{v}_p : \mathbb{N} \to \mathbb{N}_0$ denote the p-adic exponent whence $n = \prod_{p \in \mathbb{P}} p^{\mathsf{v}_p(n)}$ for every $n \in \mathbb{N}$. For $a, b \in \mathbb{Z}$ we set

$$[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}.$$

Throughout, all abelian groups will be written additively, and for $n \in \mathbb{N}$ let C_n denote the cyclic group with n elements. Let G be a finite abelian group. There are $n_1, \ldots, n_r \in \mathbb{N}$ such that $G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$ where either $r = n_1 = 1$ or $1 < n_1 \mid \ldots \mid n_r$. Then $r = \mathsf{r}(G)$ is the rank of the group and $n_r = \exp(G)$ its exponent.

Elements $e_1, \ldots, e_r \in G$ are called *independent*, if every equation of the form $\sum_{i=1}^r m_i e_i = 0$ with $m_1, \ldots, m_r \in \mathbb{Z}$, implies that $m_1 e_1 = \cdots = m_r e_r = 0$. We say that $(e_1, \ldots, e_r)$ is *a basis* of G, if $e_1, \ldots e_r$ are independent and generate the group (equivalently, $G = \oplus_{i=1}^r \langle e_i \rangle$).

Let $G = C_n \oplus C_n$ with $n \geq 2$ and $e_1, e_2 \in G$. Then (e_1, e_2) is a basis if and only if $(e_1, e_2$ are independent with $\mathrm{ord}(e_1) = \mathrm{ord}(e_2) = n)$ if and only if e_1, e_2 generate G.

Let (e_1, e_2) be a basis of G. An endomorphism $\varphi : G \to G$ with

$$(\varphi(e_1), \varphi(e_2)) = (e_1, e_2) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{where} \qquad a, b, c, d \in \mathbb{Z}$$

is an automorphism if and only if $(\varphi(e_1), \varphi(e_2))$ is a basis which is equivalent to $\gcd\{ad - bc, n\} = 1$. Let $f_1 \in G$ with $\operatorname{ord}(f_1) = n$. Then there are $a, c \in \mathbb{Z}$ with $\gcd\{a, c, n\} = 1$ such that $f_1 = ae_1 + ce_2$ and there are $b, d \in \mathbb{Z}$ with $ad - bc \equiv 1 \mod n$ whence $(f_1, f_2 = be_1 + de_2)$ is a basis of G.

Let $\mathcal{F}(G)$ denote the free abelian monoid with basis G. An element $S \in \mathcal{F}(G)$ is called a *sequence in* G and will be written in the form

$$S = \prod_{i=1}^{l} g_i = \prod_{g \in G} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G) \quad \text{where all} \quad \mathsf{v}_g(S) \in \mathbb{N}_0.$$

For every $g \in G$ we call $\mathsf{v}_g(S)$ the *multiplicity of* g *in* S, and a sequence $T \in \mathcal{F}(G)$ is a *subsequence of* S, if $\mathsf{v}_g(T) \le \mathsf{v}_g(S)$ for every $g \in G$. The unit element $1 \in \mathcal{F}(G)$ is called the *empty sequence*. We denote by

- $|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0$ the *length of* S,
- $\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G$ the *sum of* S,
- $\operatorname{supp}(S) = \{g_i \mid i \in [1, l]\} = \{g \in G \mid \mathsf{v}_g(S) > 0\} \subset G$ the *support of* S, and by
- $\Sigma(S) = \{\sum_{i \in I} g_i \mid \emptyset \ne I \subset [1, l]\} \subset G$ the set of sums of non-empty subsequences of S.

The sequence S is called

- *zero-sumfree*, if $0 \notin \Sigma(S)$,
- a *zero-sum sequence*, if $\sigma(S) = 0$,
- a *minimal zero-sum sequence*, if it is a zero-sum sequence and every proper zero-sum subsequence is zero-sumfree,
- a *short zero-sum sequence*, if it is a zero-sum sequence with length $|S| \in [1, \exp(G)]$.

Every group homomorphism $\varphi : G \to H$ extends in a canonical way to a homomorphism $\varphi : \mathcal{F}(G) \to \mathcal{F}(H)$ where $\varphi(S) = \prod_{i=1}^{l} \varphi(g_i) \in \mathcal{F}(H)$. Obviously, $\varphi(S)$ is a zero-sum sequence if and only if $\sigma(S) \in \ker(\varphi)$. If $\varphi : G \to G$ is an automorphism, then S is a (minimal) zero-sum sequence if and only if $\varphi(S)$ is a (minimal) zero-sum sequence. Suppose $G = C_{mn}^r$ with $r, m, n \in \mathbb{N}_{\ge 2}$. If $\varphi : G \to G$ denotes the multiplication by n, then clearly we have $\ker(\varphi) = \{g \in G \mid ng = 0\} \cong C_n^r$ and $\varphi(G) = nG \cong C_m^r$.

Davenport's constant $\mathsf{D}(G)$ *of* G is defined as the maximal length of a minimal zero-sum sequence in G, equivalently this is the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ with $|S| \ge l$ contains a zero-sum subsequence. It is easy to see that $1 + \sum_{i=1}^{r}(n_i - 1) \le \mathsf{D}(G)$. J.E. Olson and D. Kruyiswijk proved independently that equality holds if $\mathsf{r}(G) \le 2$ or G a p-group (see [Ols69a] and [vEB69b]). If $S \in \mathcal{F}(G)$ is zero-sumfree with length $|S| = \mathsf{D}(G) - 1$, then $\Sigma(S) = G \setminus \{0\}$ whence $G = \langle \mathrm{supp}(S) \rangle$.

We shall frequently use the fact that in a cyclic group G with $n \ge 2$ elements every minimal zero-sum sequence $S \in \mathcal{F}(G)$ with length $|S| = n$ has the form $S = g^n$ for some $g \in G$ with $\mathrm{ord}(g) = n$, and that a sequence $S \in \mathcal{F}(G)$ with length $|S| = n - 1$ is zero-sumfree if and only if $S = g^{n-1}$ for some $g \in G$ with $\mathrm{ord}(g) = n$.

3. Sequences in $C_n \oplus C_n$

In this section we give a key definition of various well-studied properties of sequences in $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ (Definition 3.2) and outline the program of the subsequent sections. Then we prepare the main tools which will be used throughout the whole paper (Lemma 3.3 to Lemma 3.14). Among them Theorem 3.7 may be of its own interest.

Lemma 3.1. *Let $n \ge 2$.*

1. **(Erdős-Ginzburg-Ziv-Theorem)** *Every sequence $S \in \mathcal{F}(C_n)$ with $|S| \ge 2n - 1$ contains a zero-sum subsequence with length n.*
2. *Every sequence $S \in \mathcal{F}(C_n \oplus C_n)$ with $|S| \ge 3n - 2$ contains a short zero-sum subsequence.*

Proof. 1. see [EGZ61] and [AD93] for a variety of proofs.

2. See [GG99], Lemma 4.4. □

Definition 3.2. Let $G = C_n \oplus C_n$ with $n \ge 2$. We say that n has

- *Property B*, if every minimal zero-sum sequence $S \in \mathcal{F}(G)$ with length $|S| = 2n - 1$ contains some element with multiplicity $n - 1$.
- *Property C*, if every sequence $S \in \mathcal{F}(G)$ with length $|S| = 3n - 3$ which contains no short zero-sum subsequence has the form $S = a^{n-1}b^{n-1}c^{n-1}$ with some pairwise distinct elements $a, b, c \in G$ of order n.
- *Property D*, if every sequence $S \in \mathcal{F}(G)$ with length $|S| = 4n - 4$ which contains no zero-sum subsequence of length n has the form $S = a^{n-1}b^{n-1}c^{n-1}d^{n-1}$ with some pairwise distinct elements $a, b, c, d \in G$ of order n.
- *Property E*, if every sequence $S \in \mathcal{F}(G)$ with length $|S| = 4n - 3$ contains a zero-sum subsequence of length n.

We say that Property B (resp. C, D, E) is *multiplicative* if the following holds: if two integers $m, n \in \mathbb{N}$ both satisfy Property B (resp. C, D, E), then so does their product mn.

Let $G = C_n \oplus C_n$ with $n \geq 2$. It has been conjectured, that every integer $n \geq 2$ satisfies each of the above Properties. If n has Property B, then, as we shall see in Theorem 4.3, this answers Problem 1* of the Introduction. If n has Property C, then by Lemma 3.1.2 this answers Problem 2*.1. A. Kemnitz conjectured that n has Property E (which answers Problem 2.2 of the Introduction) and if this holds true, then Property D answers the associated inverse problem.

It is immediately clear that 2 satisfies each of these Properties whence whenever it is convenient we restrict to integers $n \geq 3$. Lemma 3.3 states that Properties C, D and E are multiplicative and that D implies C and E. The Properties C, D and E have been verified for $2, 3, 5$ and 7 ([vEB69b], [vEB69a], [Kem83], [ST02]). Furthermore, E holds true for various classes of composite numbers (cf. [Gao96b], [Gao03], [Gao01b], [Tha01]). We are going to prove that $2, 3, 5$ and 6 have Property B (Proposition 4.2), that (under some weak additional assumption) Property B implies Property C and that if some $n \geq 6$ has Property B, then $2n$ has Property B (Theorem 8.1).

Lemma 3.3. *Let $n \geq 2$.*

1. *The Properties C, D and E are multiplicative.*
2. *Property D implies Properties C and E.*

Proof. We set $G = C_n \oplus C_n$.

1. In [Gao00b] it is proved that Properties C and D are multiplicative. The fact that Property E is multiplicative follows from a more general result of H. Harborth (cf. [Har73], Hilfssatz 2). For convenience we provide a simple proof.

Let $m, n \in \mathbb{N}$ be two integers satisfying Property E. We have to verify that every sequence $S \in G \cong C_{mn} \oplus C_{mn}$ with $|S| \geq 4mn - 3$ has a zero-sum subsequence with length mn Let $\varphi : G \to G$ denote the multiplication by n and let S be a sequence in G with length $|S| = 4mn - 3$. Since every sequence in $\varphi(G) \cong C_m \oplus C_m$ with length $4m - 3$ contains a zero-sum subsequence of length m and since

$$4mn - 3 = (4n - 4)m + (4m - 3)$$

there exist $t = 4n - 3$ disjoint subsequences $S_1, \ldots, S_t$ of S with length $|S_i| = m$ such that $\varphi(S_i)$ has sum zero in $\varphi(G)$ for every $i \in [1, t]$. Thus

$$T = \prod_{i=1}^{t} \sigma(S_i)$$

is a sequence in $\ker(\varphi) \cong C_n \oplus C_n$. Since n has Property E there exists some $I \subset [1, t]$ with $|I| = n$ such that $\prod_{i \in I} \sigma(S_i)$ is a zero-sum subsequence of T. This implies that

$$S' = \prod_{i \in I} S_i \in \mathcal{F}(G)$$

is a zero-sum subsequence of S with length $|S| = \sum_{i \in I} |S_i| = mn$.

2. Suppose that n satisfies Property D and that $n \ge 3$.

We first verify that n has Property C. Let $S \in \mathcal{F}(G)$ be a sequence with length $|S| = 3n-3$ and suppose that S contains no short zero-sum subsequence. We consider the sequence $T = 0^{n-1} \cdot S$. If T has a zero-sum subsequence T' with $|T'| = n$, then $T' = 0^k \cdot S'$ with $k \in [0, n-1]$ and $S' \mid S$ whence S' is a short zero-sum subsequence of S. Thus T has no zero-sum subsequence of length n, and the assertion follows.

Next we show that n satisfies Property E. Let $S \in \mathcal{F}(G)$ with length $|S| = 4n-3$ and assume to the contrary that S contains no zero-sum subsequence of length n. Let $g \in \operatorname{supp}(S)$. Then $|g^{-1} \cdot S| = 4n-4$, and $g^{-1} \cdot S$ contains no zero-sum subsequence of length n whence $g^{-1} \cdot S = a^{n-1} \cdot b^{n-1} \cdot c^{n-1} \cdot d^{n-1}$ for some $a, b, c, d \in G$. Thus there is some $h \in \operatorname{supp}(S)$ with $\mathsf{v}_h(S) \ge 2$. After changing notation if necessary, we suppose that $\mathsf{v}_g(S) \ge 2$ and that $g = a$. Thus a^n is a zero-sum subsequence of S, a contradiction. □

Definition 3.4. Let G be a finite abelian group with $\exp(G) = n \ge 2$. Let $\mathsf{s}(G)$ (resp. $\mathsf{s}_0(G)$) denote the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ with $|S| \ge l$ contains a zero-sum subsequence T with length $|T| = n$ (resp. with length $|T| \equiv 0 \mod n$).

Hence, by definition, an integer $n \ge 2$ has Property E if and only if $\mathsf{s}(C_n \oplus C_n) = 4n-3$.

Lemma 3.5. *Let G be a finite abelian group with $\exp(G) = n \ge 2$. Then*

$$\mathsf{D}(G) + n - 1 \le \mathsf{s}_0(G) \le \min\{\mathsf{s}(G), \mathsf{D}(G \oplus C_n)\}.$$

Proof. If $S \in \mathcal{F}(G)$ is a zero-sumfree sequence with length $|S| = \mathsf{D}(G) - 1$, then the sequence $0^{n-1} \cdot S$ has no zero-sum subsequence with length divisible by n. Thus $\mathsf{D}(G) + n - 2 = |0^{n-1} \cdot S| < \mathsf{s}_0(G)$. By definition we have $\mathsf{s}_0(G) \le \mathsf{s}(G)$.

Suppose that $G \oplus C_n = G \oplus \langle e \rangle$. In order to verify that $\mathsf{s}_0(G) \le \mathsf{D}(G \oplus C_n)$, let $S = \prod_{i=1}^l g_i \in \mathcal{F}(G)$ with $l = \mathsf{D}(G \oplus C_n)$. Then the sequence $\prod_{i=1}^l (g_i + e) \in \mathcal{F}(G \oplus C_n)$ contains a zero-sum subsequence T with length $|T| \equiv 0 \mod n$, and whence the same is true for S. □

Lemma 3.6. *Let $m, n \in \mathbb{N}_{\ge 2}$ and suppose that $\mathsf{s}_0(C_m \oplus C_m) = 3m-2$, $\mathsf{s}(C_m \oplus C_m) \le 4m-2$ and that $\mathsf{D}(C_n^3) = 3n-2$. Then $\mathsf{s}_0(C_{mn} \oplus C_{mn}) = 3mn-2$.*

Proof. By Lemma 3.5 it remains to show that $\mathsf{s}_0(C_{mn} \oplus C_{mn}) \le 3mn-2$. Let $S = \prod_{i=1}^l g_i$ be a sequence in $G = C_{mn} \oplus C_{mn}$ with length $l = 3mn-2$. We set $H = G \oplus C_{mn} = G \oplus \langle e \rangle$ and $S^H = \prod_{i=1}^l (g_i + e)$. It is sufficient to prove that S^H has a non-empty zero-sum subsequence. Let $\varphi : H \to H$ denote the multiplication by n. Since $3mn - 2 = (3n-4)m + (4m-2)$ and $\mathsf{s}(C_m \oplus C_m) \le 4m-2$, there exist $3n-3$ disjoint subsequences $S_1, \ldots, S_{3n-3}$ of S with length m such that all $\varphi(S_i)$ have sum zero. Since

$|\prod_{i=1}^{3n-3} S_i^{-1} \cdot S| = 3m-2 = \mathsf{s}_0(C_m \oplus C_m)$, there exists a subsequence S_{3n-2} of $\prod_{i=1}^{3n-3} S_i^{-1} \cdot S$ such that $\varphi(S_{3n-2})$ has sum zero and with $|S_{3n-2}| \in \{m, 2m\}$. For $i \in [1, 3n-2]$ we denote by S_i^H the subsequence of S^H corresponding to S_i and obtain that $\sigma(S_i^H) \in \ker(\varphi)$. Thus $\prod_{i=1}^{3n-2} \sigma(S_i^H)$ is a sequence in $\ker(\varphi)$ with length $3n-2 = \mathsf{D}(C_n^3)$. Therefore there exists some $\emptyset \ne I \subset [1, 3n-2]$ such that $\sum_{i \in I} \sigma(S_i^H) = 0$ whence $\prod_{i \in I} S_i^H$ is a non-empty zero-sum subsequence of S^H. □

Theorem 3.7. *Let $n \in \mathbb{N}$ with $n \ge 2$.*

1. *If n is divisible by at most two distinct primes, then $\mathsf{s}_0(C_n \oplus C_n) = 3n-2$.*
2. *If Property E holds for all prime divisors of n, then $\mathsf{s}_0(C_n \oplus C_n) = 3n-2$.*

Proof. 1. If n is a prime power, then the assertion follows from Lemma 3.5. If n is a product of two prime powers, then by the previous case and by [Gao01a] the assumptions of Lemma 3.6 are satisfied whence the assertion follows.

2. Suppose that $n = \prod_{i=1}^r p_i^{k_i}$ with $r, k_1, \ldots, k_r \in \mathbb{N}$ and primes $p_1, \ldots, p_r$. If Property E holds for $p_1, \ldots, p_r$, then for every divisor $1 < d$ of n we have $\mathsf{s}(C_d \oplus C_d) = 4d-3$ by Lemma 3.3.1. Using Lemma 3.6 we obtain the assertion by induction on r. □

Lemma 3.8. *Let $G = C_n \oplus C_n$ with $n \ge 2$ and $S \in \mathcal{F}(G)$ a minimal zero-sum sequence with $|S| = 2n-1$.*

1. *Then $\operatorname{ord}(g) = n$ for every $g \in \operatorname{supp}(S)$.*
2. *Let n be prime. Then each two distinct elements in $\operatorname{supp}(S)$ are independent and $3 \le |\operatorname{supp}(S)| \le n+1$.*

Proof. See [GG99], Proposition 6.3, Theorem 10.3 and Corollary 10.5. □

Lemma 3.9. *Let $G = C_n \oplus C_n$ with $n \ge 3$ and $e_1, e_2 \in G$ distinct such that the sequence $e_1^{n-2} e_2^{n-2}$ does not contain a short zero-sum subsequence. Then (e_1, e_2) is a basis of G.*

Proof. Obviously, the assertion is true for $n = 3$. Suppose that $n \ge 4$. Since $\operatorname{ord}(e_i) \mid n$ and $\operatorname{ord}(e_i) > n-2$ for $i \in [1,2]$, it follows that $\operatorname{ord}(e_1) = \operatorname{ord}(e_2) = n$. Thus it remains to show that e_1, e_2 are independent. Let $\lambda_1, \lambda_2 \in [0, n-1]$ such that $\lambda_1 e_1 + \lambda_2 e_2 = 0$. We have to verify that $\lambda_1 = \lambda_2 = 0$. Assume to the contrary, that $\lambda_1 + \lambda_2 > n$. Then $\lambda_1, \lambda_2 \in [2, n-1]$ and $T = e_1^{n-\lambda_1} \cdot e_2^{n-\lambda_2}$ is a zero-sum subsequence of S with length $|T| = 2n - (\lambda_1 + \lambda_2) \in [1, n-1]$, a contradiction. Thus $\lambda_1 + \lambda_2 \le n$. If $\lambda_1 = n-1$, then $\lambda_2 = 1$, $e_1 = e_2$ and e_1^n is a short zero-sum subsequence of S, a contradiction. Thus $\lambda_1 \le n-2$, and similarly we obtain that $\lambda_2 \le n-2$. Therefore $T = e_1^{\lambda_1} \cdot e_2^{\lambda_2}$ is a zero-sum subsequence of S, which implies that $\lambda_1 + \lambda_2 = |T| = 0$. □

Lemma 3.10 (Moser-Scherk)**.** *Let G be a finite abelian group and $S \in \mathcal{F}(G)$ a zero-sumfree sequence. If $S = \prod_{i=1}^{l} S_i$, then $|\Sigma(S)| \ge \sum_{i=1}^{l} |\Sigma(S_i)|$.*

Proof. See [MS55]. □

Lemma 3.11. *Let $G = C_m \oplus C_m$ with $m \ge 2$ and $S \in \mathcal{F}(G)$ a zero-sum sequence with $|S| \ge tm$ for some $t \ge 2$. Then S may be written as a product of t non-empty zero-sum subsequences and at least $t-2$ of these sequences are short.*

Proof. We proceed by induction on t. If $t = 2$, then $|S| \ge 2m > \mathsf{D}(G)$ whence S contains a zero-sum subsequence S_1 with $|S_1| \le 2m-1$ and the assertion follows. If $t \ge 3$, then Lemma 3.1.2 implies that S contains a short zero-sum subsequence S_1. Since $S_1^{-1}S$ is a zero-sum sequence with $|S_1^{-1}S| \ge (t-1)m$, the assertion follows by induction hypothesis. □

Lemma 3.12. *Let $G = C_m \oplus C_m$ with $m \ge 2$ and $S \in \mathcal{F}(G)$ a zero-sum sequence with $|S| = tm-1$ for some $t \ge 3$ which cannot be written as a product of t non-empty zero-sum subsequences.*

1. *Every short zero-sum subsequence of S has length m. In particular, we have $0 \notin \mathrm{supp}(S)$.*
2. *S has a product decomposition of the form $S = \prod_{\nu=0}^{t-2} S_\nu$ where S_0 is a minimal zero-sum sequence with length $2m-1$ and $S_1, \ldots, S_{t-2}$ are short zero sum sequences.*
3. *If $S = \prod_{\nu=1}^{t-1} S_\nu$ with zero-sum subsequences $S_1, \ldots, S_{t-1}$, then at most $m-1$ of these sequences are not short.*

Proof. 1. Assume to the contrary that S contains a short zero-sum subsequence T with $|T| \in [1, m-1]$. Then $|T^{-1}S| \ge (t-1)m$ whence Lemma 3.11 implies that $T^{-1}S$ may be written as a product of $t-1$ non-empty zero-sum subsequences. Thus S may be written as a product of t non-empty zero-sum subsequences, a contradiction.

2. Applying Lemma 3.1.2 $(t-2)$-times we see that S may be written in the form

$$S = S_0 \cdot \prod_{\nu=1}^{t-2} S_\nu$$

where $S_1, \ldots, S_{t-2}$ are zero-sum subsequences with length m. Thus S_0 is a zero-sum subsequence with $|S_0| = 2m-1$. Since S is not a product of t zero-sum subsequences, it follows that S_0 is minimal.

3. Assume to the contrary, that $S = \prod_{\nu=1}^{t-1} S_\nu$ where all S_ν are zero-sum subsequences and $S_1, \ldots, S_m$ are not short. Then $T = \prod_{i=1}^{m} S_i$ is a zero-sum subsequence with length $|T| \ge m(m+1)$. Thus by Lemma 3.11 T may be written as a product of $m+1$ zero-sum subsequences whence S is a product of t zero-sum subsequences, a contradiction. □

Lemma 3.13. *Let $G = C_{mn} \oplus C_{mn}$ with $m, n \in \mathbb{N}_{\ge 2}$ and $\varphi : G \to G$ the multiplication by n. If (e_1', e_2') is a basis of $\ker(\varphi)$, then there is a basis (e_1, e_2) of G such that $me_i = e_i'$ for $i \in [1, 2]$.*

Proof. Suppose that $G = \mathbb{Z}/mn\mathbb{Z} \times \mathbb{Z}/mn\mathbb{Z}$ and $e_i' = (a_i' + mn\mathbb{Z}, b_i' + mn\mathbb{Z})$ with $a_i', b_i' \in [0, mn-1] \cap m\mathbb{Z}$ for $i \in [1, 2]$ such that (e_1', e_2') is a basis of $\ker(\varphi) = m\mathbb{Z}/mn\mathbb{Z} \times m\mathbb{Z}/mn\mathbb{Z}$. For $i \in [1, 2]$ we set $a_i = m^{-1}a_i'$, $b_i = m^{-1}b_i'$ and $e_i = (a_i + mn\mathbb{Z}, b_i + mn\mathbb{Z})$. Then $\operatorname{ord}(e_1) = \operatorname{ord}(e_2) = mn$ and e_1, e_2 are independent whence (e_1, e_2) is a basis of G. □

Lemma 3.14. *Let $G = C_{mn} \oplus C_{mn}$ with $m, n \in \mathbb{N}_{\ge 2}$, $\varphi : G \to G$ the multiplication by n and $S \in \mathcal{F}(G)$ a minimal zero-sum sequence with length $|S| = 2mn - 1$.*

1. *$\varphi(S)$ is not a product of $2n$ zero-sum subsequences. Every short zero-sum subsequence of $\varphi(S)$ has length m and $0 \notin \operatorname{supp}(\varphi(S))$.*
2. *S has a product decomposition $S = \prod_{\nu=0}^{2n-2} S_\nu$ where $|S_0| = 2m - 1$, $|S_1| = \ldots = |S_{2n-2}| = m$ and $\sigma(S_0), \ldots, \sigma(S_{2n-2}) \in \ker(\varphi)$.*

Proof. 1. Obviously, $\varphi(S)$ is a zero-sum sequence in nG with length $tm - 1$ where $t = 2n$. Assume to the contrary, that $\varphi(S)$ can be written as a product of t non-empty zero-sum subsequences, say $\varphi(S) = \prod_{\nu=1}^{t} \varphi(S_\nu)$. Then $T = \prod_{\nu=1}^{t} \sigma(S_\nu)$ is a sequence in $\ker(\varphi)$. Since $t = 2n > \mathsf{D}(\ker(\varphi))$, T contains a proper zero-sum subsequence whence S contains a proper zero-sum subsequence, a contradiction. The remaining assertions follow from Lemma 3.12.1.

2. By 1. we may apply Lemma 3.12.2 to the sequence $\varphi(S)$ (with $t = 2n$) whence the assertion follows. □

4. Some characterizations of Property B

After some technical preparation we show that 2, 3, 4, 5 and 6 satisfy Property B and then we give some characterizations of Property B in Theorem 4.3.

Proposition 4.1. *Let $G = C_n \oplus C_n$ with $n \ge 2$.*

1. *If (e_1, e_2) is a basis of G and $a_1, \ldots, a_n \in \mathbb{Z}$ with $\sum_{\nu=1}^{n} a_\nu \equiv 1 \mod n$, then*

$$(*) \qquad S = e_1^{n-1} \cdot \prod_{\nu=1}^{n} (a_\nu e_1 + e_2)$$

   is a minimal zero-sum sequence with $|S| = \mathsf{D}(G)$.
2. *Let $S \in \mathcal{F}(G)$ be a minimal zero-sum sequence with $|S| = \mathsf{D}(G)$ and $e_1 \in G$ with $\mathsf{v}_{e_1}(S) = n - 1$.*

(a) *If* (e_1, e_2') *is a basis of* G*, then there exist some* $b \in [0, n-1]$ *with* $\gcd\{b, n\} = 1$ *and* $a_1', \ldots, a_n' \in [0, n-1]$ *with* $\sum_{\nu=1}^{n} a_\nu' \equiv 1 \mod n$ *such that*

$$S = e_1^{n-1} \cdot \prod_{\nu=1}^{n} (a_\nu' e_1 + b e_2').$$

(b) *There exists a basis* (e_1, e_2) *of* G *such that* S *has the form* $(*)$.

(c) *If* $g, g' \in \mathrm{supp}(S) \setminus \{e_1\}$*, then* $g - g' \in \langle e_1 \rangle$.

Proof. 1. Let S be a sequence of the form $(*)$. Then S has sum zero and length $2n - 1 = \mathsf{D}(G)$. Let T be a non-empty zero-sum subsequence of S with $e_1 \mid T$. Since e_1^{n-1} is zero-sumfree, there exists some $i \in [1, n]$ such that $(a_i e_1 + e_2) \mid T$. This implies that $\prod_{i=1}^{n} (a_i e_1 + e_2)$ divides T. Thus $S = T$ whence S is a minimal zero-sum sequence.

2. Suppose $S = e_1^{n-1} \prod_{i=1}^{n} g_i$.

a) Let (e_1, e_2') be a basis of G. Then for every $i \in [1, n]$ we have

$$g_i = a_i' e_1 + b_i e_2'$$

with $a_i', b_i \in [0, n-1]$. Since the sequence $e_1^{n-1} \cdot (a_i' e_1) \in \mathcal{F}(\langle e_1 \rangle)$ is not zero-sumfree, it follows that $b_i \neq 0$ for every $i \in [1, n]$. Assume to the contrary, that $\prod_{i=1}^{n} b_i e_2' \in \mathcal{F}(\langle e_2' \rangle)$ is not a minimal zero-sum sequence. Then there exists some $\emptyset \neq I \subsetneq [1, n]$ such that $\prod_{i \in I} b_i e_2'$ is a zero-sum sequence and hence

$$e_1^{n-1} \prod_{i \in I} (a_i' e_1 + b_i e_2')$$

contains a zero-sum subsequence, a contradiction. Therefore, $\prod_{i=1}^{n} b_i e_2'$ is a minimal zero-sum sequence and thus it follows that $b_1 e_2' = \cdots = b_n e_2'$ whence $b_1 = \cdots = b_n = b \in [1, n-1]$. Since $G = \langle \mathrm{supp}(S) \rangle$, it follows that $\gcd\{b, n\} = 1$.

b) Clearly, there exists some $e_2' \in G$ such that (e_1, e_2') is a basis of G whence S has the form described in 2. a). Then

$$(e_1, e_2) = (e_1, e_2') \cdot \begin{pmatrix} 1 & a_1' \\ 0 & b \end{pmatrix}$$

is a basis of G and for every $\nu \in [1, n]$ we obtain that

$$a_\nu' e_1 + b e_2' = (a_\nu' - a_1') e_1 + e_2.$$

Since S is a zero-sum sequence, the required congruence is satisfied.

c) This follows immediately from b). □

Proposition 4.2. *The integers* $2, 3, 4, 5$ *and* 6 *have Property B.*

Remark: We have also verified that 7 has Property B, but we do not give this proof here.

Proof. Let $G = C_n \oplus C_n$ with $n \ge 2$ and $S = \prod_{i=1}^{l} g_i^{k_i}$ a minimal zero-sum sequence with length $|S| = \mathsf{D}(G) = 2n-1$, $k_1 \ge \cdots \ge k_l \ge 1$, $g_1, \ldots, g_l$ pairwise distinct and $|\text{supp}(S)| = l$. We have to show that $k_1 = n-1$.

This is obvious for $n = 2$. If $n = 3$, then Lemma 3.8 implies that $l \in [3,4]$ whence $k_1 = 2$.

Let $n = 4$. By Lemma 3.8 all elements in $\text{supp}(S)$ have order 4, and clearly G has exactly 12 elements of order 4. Assume to the contrary that $k_1 \le 2$. If $k_1 = 1$, then $l = |S| = 7$, $\prod_{i=1}^{6} g_i$ is zero-sumfree and $\{-g_i, g_i \mid i \in [1,6]\}$ are the twelve elements of order 4 whence $g_7 \in \{-g_i \mid i \in [1,6]\}$ a contradiction. Thus $k_1 = 2$ and $S = h_1^2 \prod_{i=2}^{6} h_i$ with $h_2, \ldots, h_6 \in G$ not necessarily pairwise distinct. Since by Lemma 3.1.2 every sequence in $C_2 \oplus C_2 \setminus \{0\}$ with length 4 contains a short zero-sum subsequence with length two, every sequence $S \in \mathcal{F}(G \setminus \{0\})$ with $|S| \ge 4$ contains a subsequence S' with $|S'| = 2$ and $\text{ord}(\sigma(S')) = 2$. Thus after renumeration we may suppose that $\text{ord}(h_2 + h_3) = 2$. Then $h_4 + h_5 + h_6$ has order two and no proper subsum has order two. Considering the sequence $h_1 \cdot h_4 \cdot h_5 \cdot h_6$ we may suppose (after renumerating again) that $\text{ord}(h_1 + h_4) = 2$. Therefore we obtain that $h_1 + h_4 \in \{h_1 + h_1, h_2 + h_3, h_4 + h_5 + h_6\}$ whence either $h_1 h_4 h_2 h_3$ or $h_1 h_5 h_6$ is a zero-sum subsequence of S, a contradiction.

Let $n = 5$. Lemma 3.8 implies that $l \in [3,6]$ whence $k_1 \ge 2$. Assume to the contrary that $k_1 \in [2,3]$. By Lemma 3.8 g_1 and g_2 are independent whence (g_1, g_2) is a basis of G.

Case 1: $k_1 = 3$. Since $|S| = 9$ and $l \le 6$, it follows that $k_2 \ge 2$. If $l = 3$, then $k_2 = k_3 = 3$ and $0 = \sigma(S) = 3(g_1 + g_2 + g_3)$ implies that $0 = g_1 + g_2 + g_3$, a contradiction to the fact that S is a minimal zero-sum sequence. Thus we have $l \in [4,6]$. Since for every $i \in [3,l]$ the sequence $g_1^3 g_2^2 g_i$ is zero-sumfree, it follows that

$$\begin{aligned} g_i &\in G \setminus \big(\{0, g_1, g_2\} \cup \Sigma(-(g_1^3 g_2^2))\ \big) \\ &= \{2g_2, g_1 + g_2, g_1 + 2g_2, g_1 + 3g_2, g_1 + 4g_2, \\ &\quad 2g_1 + g_2, 3g_1 + g_2, 4g_1 + g_2, 2g_1 + 2g_2, 3g_1 + 2g_2, 4g_1 + 2g_2\}. \end{aligned}$$

We argue step by step that none of the following elements lies in $\text{supp}(S)$: $g_1 + 2g_2, g_1 + 3g_2, g_1 + 4g_2, 2g_1 + 2g_2, 3g_1 + 2g_2$ and $4g_1 + 2g_2$. To exclude the remaining cases we decide between $k_2 = 2$ and $k_2 = 3$ and obtain a contradiction to $k_1 = 3$.

Case 2: $k_1 = 2$. Since $|S| = 9$ and $l \le 6$, it follows that either $(k_1, \ldots, k_l) = (2,2,2,1,1,1)$ or $(k_1, \ldots, k_l) = (2,2,2,2,1)$ whence

$$\prod_{i=1}^{3} g_i^2 \cdot g_4 \cdot g_5$$

zero-sumfree. Since (g_1, g_2) is a basis of G, there are $a, b, c, d, e, f \in [0,4]$ such that $g_3 = a g_1 + b g_2$, $g_4 = c g_1 + d g_2$ and $g_5 = e g_1 + f g_2$. Then Lemma 3.8 implies that $a, b, c, d, e, f \in [1,4]$. Obviously, none of the pairs $(a,b), (c,d), (e,f)$ lies in $\{(4,4), (3,3), (3,4), (4,3)\}$.

Since $\prod_{i=1}^{3} g_i^2$ is zero-sumfree, it follows that $(a,b) \notin \{(2,4),(4,2),(2,2)\}$. Thus by symmetry it remains to consider the cases $(a,b) \in \{(1,1),(1,2),(1,3),(1,4)(2,3)\}$. Discussing these five possibilities we obtain a contradiction.

Let $n = 6$. We proceed in two steps. First we show that $k_1 \ge 4$ and then we verify that $k_1 \ne 4$ whence $k_1 = 5$ follows.

Assertion 1: $k_1 \ge 4$. Let $\varphi : G \to G$ denote the multiplication by 2 whence $\varphi(G) = 2G \cong C_3 \oplus C_3$ and $\ker(\varphi) \cong C_2 \oplus C_2$. We set $S = \prod_{h \in \varphi(G)} S_h$ where $\varphi(S_h) = h^{|S_h|}$. Since by Lemma 3.14.1 every short zero-sum subsequence of $\varphi(S) \in \mathcal{F}(\varphi(G))$ has length 3, we have $|S_0| = 0$ and if $|S_h| > 0$ for some $h \in \varphi(G)$, then $|S_{-h}| = 0$. Thus $S = \prod_{\nu=1}^{t} S_{h_\nu}$ with $t \le \frac{1}{2}|\varphi(G) \setminus \{0\}| = 4$ and $|S_{h_1}| \ge \ldots \ge |S_{h_t}| \ge 1$.

Frequently we shall use the fact that S does not have a proper subsequence of the form $T = T_1T_2T_3$ with $|T_i| \ge 1$ and $\sigma(T_i) \in \ker(\varphi)$ for $i \in [1,3]$: because $\mathsf{D}(\ker(\varphi)) = 3$ the sequence $\prod_{i=1}^{3} \sigma(T_i) \in \mathcal{F}(\ker(\varphi))$ is not zero-sumfree whence $T_1T_2T_3$ would not be zero-sumfree.

In particular, S does not have disjoint subsequences T_1, T_2, T_3 where each T_i has length 3 and divides some S_{h_j} for some $j \in [1,t]$. This implies that $|S_{h_3}| \le 2$. From this we obtain, because $t \le 4$ and $|S| = 11$, that $|S_{h_1}| \ge 4$.

Next we assert that

$$|\mathrm{supp}(S_{h_1})| \le 2.$$

Assume to the contrary, that there are pairwise distinct elements x, y, z such that $xyz \mid S_{h_1}$. Since $|S_{h_1}| \ge 4$, there is some $w \in G$ such that $wxyz \mid S_{h_1}$. Since $\mathsf{D}(\varphi(G)) = 5$, there exists a subsequence $1 \ne T$ of $(wxyz)^{-1}S$ such that $\varphi(T)$ has sum zero whence $\sigma(T) \in \ker(\varphi) = \{0, w+x+y, w+x+z, w+y+z\}$. Thus we obtain a proper zero-sum subsequence of S, a contradiction.

If $|S_{h_1}| \ge 7$, then $|\mathrm{supp}(S_{h_1})| \le 2$ implies that $k_1 \ge 4$. Hence it remains to consider the cases where $|S_{h_1}| \in [4,6]$. We assume to the contrary that $k_1 < 4$. Then $|\mathrm{supp}(S_{h_1})| = 2$, say $\mathrm{supp}(S_{h_1}) = \{\alpha, \beta\}$. We distinguish two cases.

Case 1: $|S_{h_1}| \in \{5,6\}$, say $S_{h_1} = \alpha^3\beta^3$ or $S_{h_1} = \alpha^3\beta^2$. By Lemma 3.1.2 (applied to the group $\varphi(G)$) the sequence $\alpha^2 \prod_{i=2}^{t} S_{h_i}$ contains a subsequence T_3 with $|T_3| \le 3$ and $\sigma(T_3) \in \ker(\varphi)$, and Lemma 3.14.1 implies that $|T_3| = 3$.

We assert that there exists such a sequence T_3 with $\mathsf{v}_\alpha(T_3) > 0$. Assume to the contrary that for all such sequences T_3 we have $\mathsf{v}_\alpha(T_3) = 0$. Then there is no T_3' with $|T_3'| = 3$, $\sigma(T_3) \in \ker(\varphi)$, $T_3' \mid \beta^2 \prod_{i=2}^{t} S_{h_i}$ and $\mathsf{v}_\beta(T_3) > 0$. We show that there exist sequences T_1, T_2 such that $T_1T_2T_3$ is a proper subsequence of S and $\sigma(T_1), \sigma(T_2) \in \ker(\varphi)$, which leads to a contradiction. If $S_{h_1} = \alpha^3\beta^3$, then we set $T_1 = \alpha^3$ and $T_2 = \beta^3$. Suppose

$S_{h_1} = \alpha^3\beta^2$. Then $T_3^{-1}S = \alpha^3\beta^2\gamma_1\gamma_2\gamma_3$ contains a subsequence T_2 with $|T_2| \le 3$ and $\sigma(T_2) \in \ker(\varphi)$. Since $\mathsf{v}_\alpha(T_2) = \mathsf{v}_\beta(T_2) = 0$ we obtain $T_2 = \gamma_1\gamma_2\gamma_3$ and we set $T_1 = \alpha^3$.

Thus we have some sequence $T_3 = \alpha\gamma\delta \in \mathcal{F}(\ker(\varphi))$. Clearly, $2\alpha+\beta$ and $2\beta+\alpha$ are distinct non-zero elements of $\ker(\varphi)$. If $\alpha+\gamma+\delta = 2\beta+\alpha$, then $\alpha^2\beta^2\gamma\delta$ is a proper zero-sum subsequence of S. Hence $\alpha+\gamma+\delta \ne 2\beta+\alpha$ and similarly $\alpha+\gamma+\delta \ne 2\alpha+\beta$ whence $\ker(\varphi)\setminus\{0\} = \{2\alpha+\beta, 2\beta+\alpha, \alpha+\gamma+\delta\}$. Since $\alpha \ne \beta$ but $\varphi(\alpha+\gamma+\delta) = \varphi(\beta+\gamma+\delta)$, we have $\beta+\gamma+\delta \in \{2\beta+\alpha, 2\alpha+\beta\}$. If $\beta+\gamma+\delta = 2\alpha+\beta$, then $\alpha^2\beta^2\gamma\delta$ is a proper zero-sum subsequence of S whence we infer that $\beta+\gamma+\delta = 2\beta+\alpha = 3\alpha \in \ker(\varphi)$ (note that $2\alpha = \varphi(\alpha) = \varphi(\beta) = 2\beta = h_1$) whence $\alpha^3\beta\gamma\delta$ is a proper zero-sum subsequence of S, a contradiction.

Case 2: $|S_{h_1}| = 4$. First we suppose that $|S_{h_2}| = 4$. If $|\mathrm{supp}(S_{h_2})| = 1$, then we obtain $k_1 \ge 4$. Assume that $|\mathrm{supp}(S_{h_2})| > 1$. Arguing as for $\mathrm{supp}(S_{h_1})$ we obtain that $|\mathrm{supp}(S_{h_2})| = 2$, say $\mathrm{supp}(S_{h_2}) = \{\gamma, \delta\}$. Then we may suppose without restriction that $S_{h_1} \in \{\alpha^3\beta, \alpha^2\beta^2\}$ and $S_{h_2} \in \{\gamma^3\delta, \gamma^2\delta^2\}$. Thus we have either $\{3\alpha, 2\alpha+\beta\} \subset \ker(\varphi)\setminus\{0\}$ or $\{2\alpha+\beta, \alpha+2\beta\} \subset \ker(\varphi)\setminus\{0\}$; and similarly, either $\{3\gamma, 2\gamma+\delta\} \subset \ker(\varphi)\setminus\{0\}$ or $\{2\gamma+\delta, \gamma+2\delta\} \subset \ker(\varphi)\setminus\{0\}$. Therefore we obtain a proper zero-sum subsequence of S, a contradiction.

Suppose that $|S_{h_2}| \le 3$. Since $t \le 4$ and $|S_{h_3}| \le 2$, it follows that $|S_{h_2}| = 3$ and $|S_{h_3}| = |S_{h_4}| = 2$. Since $|\mathrm{supp}(S_{h_1})| = 2$, S_{h_1} has two (not disjoint) subsequences V, V' with $|V| = |V'| = 3$ and $\sigma(V), \sigma(V') \in \ker(\varphi)\setminus\{0\}$ distinct. Thus for every subsequence T of $S_{h_2}S_{h_3}S_{h_4}$ with $\sigma(T) \in \ker(\varphi)$, we obtain $\sigma(T) = \sigma(V)+\sigma(V')$. Since $\mathsf{D}(\varphi(G)) = 5$, $h_2^2h_3^2h_4$ contains a zero-sum subsequence whence $S_{h_2}S_{h_3}S_{h_4}$ contains a subsequence T such that $\varphi(T) \mid h_2^2h_3^2h_4$ and $\sigma(T) \in \ker(\varphi)\setminus\{0\}$. Since $h_2, h_3, h_4 \in \varphi(G) \cong C_3 \oplus C_3$ are pairwise distinct, we have $h_2h_3h_4 \mid \varphi(T)$. Since $\sigma(T)$ has the same value for all such T, we infer that

$$S_{h_2} = \gamma^3,\ S_{h_3} = \delta^2 \quad\text{and}\quad S_{h_4} = \epsilon^2.$$

By Lemma 3.1.2 the sequence $h_1^2h_2^2h_3^2h_4^2 \in \mathcal{F}(\varphi(G))$ contains a short zero-sum subsequence whence S has a subsequence T_3 such that $|T_3| = 3$ and $\varphi(T_3) \mid h_1^2h_2^2h_3^2h_4^2$. If $\mathsf{v}_{h_2}(\varphi(T_3)) = 0$, then $\varphi(T_3) = h_1h_3h_4$ and we set T_1, T_2 such that $\varphi(T_2) = \varphi(T_3)$ and $\varphi(T_1) = h_2^3$, which leads to a contradiction. If $\mathsf{v}_{h_1}(\varphi(T_3)) = 0$, then $\varphi(T_3) = h_2h_3h_4$ and we set T_1, T_2 such that $\varphi(T_1) = h_1^3$ and $\varphi(T_2) = \varphi(T_3)$, which leads to a contradiction.

Thus $h_1h_2 \mid \varphi(T_3)$ and after a suitable renumeration we may suppose that $\varphi(T_3) = h_1h_2h_3$. Since $\sigma(T_3) \in \{\alpha+\gamma+\delta, \beta+\gamma+\delta\} \subset \ker(\varphi)\setminus\{0\} = \{\sigma(V), \sigma(V'), 3\gamma\}$, we may suppose that

$$\alpha+\gamma+\delta \in \{\sigma(V), \sigma(V')\}.$$

If $S_{h_1} = \alpha\beta^3$, then $\{\sigma(V), \sigma(V')\} = \{\alpha+2\beta, 3\beta\}$, $\gamma+\delta = 2\beta$ or $\alpha+\gamma+\delta = 3\beta$ whence either $\gamma^2\delta^2\beta^2$ or $\alpha\gamma\delta\beta^3$ is a proper zero-sum subsequence of S, a contradiction.

If $S_{h_1} = \alpha^2\beta^2$, then $\{\sigma(V), \sigma(V')\} = \{\alpha + 2\beta, 2\alpha + \beta\}$, $\gamma + \delta = 2\beta$ or $\gamma + \delta = \alpha + \beta$ whence either $\gamma^2\delta^2\beta^2$ or $\gamma^2\delta^2\alpha\beta$ is a proper zero-sum subsequence of S, a contradiction.

If $S_{h_1} = \alpha^3\beta$, then $\{\sigma(V), \sigma(V')\} = \{3\alpha, 2\alpha + \beta\}$, $\gamma + \delta = 2\alpha$ or $\gamma + \delta = \alpha + \beta$ whence either $\gamma^2\delta^2\alpha^2$ or $\gamma^2\delta^2\alpha\beta$ is a proper zero-sum subsequence of S, a contradiction.

Assertion 2: $k_1 \neq 4$. Assume to the contrary that $k_1 = 4$. Let $e_2 \in G$ such that $(g_1 = e_1, e_2)$ is a basis of G. Then

$$S = e_1^4 \prod_{i=1}^{7} (x_i e_1 + y_i e_2)$$

with $x_i, y_i \in [0, 5]$ and $(x_i, y_i) \neq (1, 0)$ for all $i \in [1, 7]$. Since S is a minimal zero-sum sequence, it follows that for every $\emptyset \neq I \subsetneq [1, 7]$

$$(*) \qquad \sum_{i \in I} y_i \equiv 0 \mod 6 \quad \text{implies that} \quad \sum_{i \in I} x_i \equiv 1 \mod 6.$$

In particular, this implies that $y_1, \ldots, y_7 \in [1, 5]$. Next we assert that for each two distinct $i, j \in [1, 7]$ we have

$$(**) \qquad y_i + y_j \not\equiv 0 \mod 6.$$

Assume to the contrary, that this does not hold, say $y_6 + y_7 \equiv 0 \mod 6$. Then $x_6 + x_7 \equiv 1 \mod 6$. Clearly, $\prod_{i=1}^{5} y_i e_2$ is a zero-sum sequence and $(*)$ implies that it is minimal. Thus it follows that $\prod_{i=1}^{5} y_i e_2 = (y e_2)^4 \cdot (2 y e_2)$ for some $y \in [1, 5]$ with $\gcd\{y, 6\} = 1$ (cf. for example [Ger90], Lemma 13). Thus $y \in \{1, 5\}$ and without restriction we may suppose that $y = y_1 = \ldots = y_4 = 1$ and $y_5 = 2$. Therefore we have

$$\prod_{i=1}^{7} y_i e_2 = e_2^4 \cdot (2e_2) \cdot (y_6 e_2) \cdot (y_7 e_2).$$

Since $y_6 + y_7 \equiv 0 \mod 6$, it follows that $\{y_6, y_7\} \cap [3, 5] \neq \emptyset$, say $y_6 \in [3, 5]$. Since for every $I \subset [1, 4]$ with $|I| = 6 - y_6$ we have $|I| \cdot 1 + y_6 \equiv 0 \mod 6$, $(*)$ implies that $x_6 + \sum_{i \in I} x_i \equiv 1 \mod 6$ whence $x_1 = x_2 = x_3 = x_4 = x$. If $y_6 = 5$, then $y_7 = 1$ whence $x_7 = x$, a contradiction to $4 = k_1 \geq \ldots \geq k_l \geq 1$. Thus $y_6 \in [3, 4]$. Then $(*)$ implies that $x_6 + (6 - y_6)x \equiv 1 \mod 6$ and $x_7 + (6 - y_7)x \equiv 1 \mod 6$ whence $2 \equiv x_6 + x_7 \mod 6$, a contradiction.

We consider the sequence

$$T = \prod_{i=1}^{7} y_i e_2.$$

By $(**)$ it follows that $\mathsf{v}_{3e_2}(T) \leq 1$, $\mathsf{v}_{e_2}(T)\mathsf{v}_{5e_2}(T) = 0$ and $\mathsf{v}_{2e_2}(T)\mathsf{v}_{4e_2}(T) = 0$. Without restriction we may suppose that $\mathsf{v}_{5e_2}(T) = 0$. If $\mathsf{v}_{2e_2}(T) \geq 4$, say $y_1 = \ldots = y_4 = 2$, then by $(*)$ we infer that $x_1 = \ldots = x_4$ and $3x_1 = x_1 + x_2 + x_3 \equiv 1 \mod 6$, a contradiction. Thus $\mathsf{v}_{2e_2}(T) \leq 3$ and by a similar argument we obtain that $\mathsf{v}_{4e_2}(T) \leq 3$. Since $\mathsf{v}_{2e_2}(T)\mathsf{v}_{4e_2}(T) = 0$, it follows that $\mathsf{v}_{2e_2}(T) + \mathsf{v}_{4e_2}(T) \leq 3$. This implies that

$$\mathsf{v}_{e_2}(T) \geq 3.$$

Suppose $\mathsf{v}_{2e_2}(T) = 3$, say $y_1 = y_2 = y_3 = 1$ and $y_4 = y_5 = y_6 = 2$. Since $1+1+2+2 \equiv 0 \mod 6$, $(*)$ implies that $x_4 = x_5 = x_6$. Since $y_4 + y_5 + y_6 \equiv 0 \mod 6$, $(*)$ implies that $3x_4 = x_4 + x_5 + x_6 \equiv 1 \mod 6$, a contradiction. Thus we get $\mathsf{v}_{2e_2}(T) \le 2$ and similarly $\mathsf{v}_{4e_2}(T) \le 2$. Thus $\mathsf{v}_{2e_2}(T) + \mathsf{v}_{4e_2}(T) \le 2$, which implies that

$$\mathsf{v}_{e_2}(T) \ge 4.$$

Suppose $\mathsf{v}_{e_2}(T) = 4$. If $\mathsf{v}_{2e_2}(T) = 2$, then $\sum_{i=1}^{7} y_i \equiv 0 \mod 6$ implies that $\mathsf{v}_{4e_2}(T) = 1$, a contradiction. If $\mathsf{v}_{4e_2}(T) = 2$, then $\sum_{i=1}^{7} y_i \equiv 0 \mod 6$ implies that $\mathsf{v}_0(T) = 1$, a contradiction. Thus $\mathsf{v}_{2e_2}(T) + \mathsf{v}_{4e_2}(T) \le 1$ whence $7 = \sum_{i=1}^{5} \mathsf{v}_{ie_2}(T) \le 6$, a contradiction. So we finally obtain that

$$\mathsf{v}_{e_2}(T) \ge 5, \quad \text{say} \quad T = e_2^5 \cdot (y_6 e_2) \cdot (y_7 e_2).$$

If $y_6 \in [2,5]$ and $I \subset [1,5]$ with $|I| = 6 - y_6$, then $y_6 + |I|.1 \equiv 0 \mod 6$ whence $(*)$ implies that $x_6 + \sum_{i \in I} x_i \equiv 1 \mod 6$ whence $x_1 = \ldots = x_5$, a contradiction to $k_1 = 4$. Thus $y_6 = 1$ and similarly $y_7 = 1$. Thus $\mathsf{v}_{e_2}(T) = 7$ and $(*)$ implies that $x_1 = \ldots = x_7$, a contradiction to $4 = k_1 \ge \ldots \ge k_l \ge 1$. □

Theorem 4.3 (Characterization of Property B)**.** *Let $G = C_n \oplus C_n$ with $n \ge 2$. Then the following statements are equivalent:*

1. *Every sequence $S \in \mathcal{F}(G)$ with length $|S| = 3n-3$, which contains no zero-sum subsequence of length greater than or equal to n, has a subsequence of the form $0^{n-1}a^{n-2}$ for some $a \in G$.*
2. *Every zero-sumfree sequence $S \in \mathcal{F}(G)$ with length $|S| = 2n-2$ contains some element with multiplicity at least $n-2$.*
3. *Every minimal zero-sum sequence $S \in \mathcal{F}(G)$ with length $|S| = 2n-1$ contains some element with multiplicity $n-1$.*
4. *For every minimal zero-sum sequence $S \in \mathcal{F}(G)$ with length $|S| = 2n-1$ there exists some basis (e_1, e_2) of G and integers $a_1, \ldots, a_n \in [0, n-1]$ with $\sum_{\nu=1}^{n} a_\nu \equiv 1 \mod n$ such that $S = e_1^{n-1} \prod_{\nu=1}^{n} (a_\nu e_1 + e_2)$.*

Proof. 1. $\Longrightarrow$ 2. Let $S \in \mathcal{F}(G)$ be a zero-sumfree sequence with length $|S| = 2n-2$. Then the sequence $0^{n-1}S$ contains no zero-sum subsequence of length greater than or equal to n. By assumption there exists some $a \in G$ such that $0^{n-1}a^{n-2}$ divides $0^{n-1}S$ and the assertion follows.

2. $\Longrightarrow$ 3. Let $S = \prod_{i=1}^{2n-1} g_i \in \mathcal{F}(G)$ be a minimal zero-sum sequence. Then there are $a, b \in G$ such that $a^{n-2} \mid g_{2n-1}^{-1} \cdot S$ and $b^{n-2} \mid a^{-1} \cdot S$. Assume to the contrary that $\mathsf{v}_g(S) < n-1$ for all $g \in G$. Then $a \ne b$ and $a^{n-2}b^{n-2}$ is a zero-sumfree subsequence of S. By Lemma 3.9 (a,b) is a basis of G whence S has the form

$$S = a^{n-2}b^{n-2} \cdot (x_1 a + y_1 b) \cdot (x_2 a + y_2 b) \cdot (x_3 a + y_3 b)$$

with all $x_i, y_i \in [0, n-1]$. Since S is a minimal zero-sum sequence, there exists some $i \in [1,3]$ such that $x_i a + y_i b \in \{a, b\}$ and the assertion follows.

3. $\Longrightarrow$ 4. This follows from Proposition 4.1.

4. $\Longrightarrow$ 1. Let $S \in \mathcal{F}(G)$ be a sequence with length $|S| = 3n-3$ containing no zero-sum subsequence of length greater than or equal to n. Since $|S| > \mathsf{D}(G)$, S has a zero-sum subsequence. Let T denote a maximal zero-sum subsequence of S. Then $U = T^{-1}S$ is zero-sumfree whence in particular we have $|U| \le \mathsf{D}(G)-1 = 2n-2$. Therefore $|T| \ge n-1$ whence $|T| = n-1$ and $|U| = 2n-2$. Therefore $-\sigma(U)\cdot U$ is a minimal zero-sum sequence with length $2n-1$ whence by assumption there exists a basis (e_1, e_2) of G such that

$$U = e_1^r \cdot \prod_{i=1}^{2n-2-r} (x_i e_1 + e_2)$$

with $r \in \{n-1, n-2\}$ and all $x_i \in [0, n-1]$. Then T has the form

$$T = 0^k \cdot \prod_{i=1}^{n-1-k} (u_i e_1 + v_i e_2)$$

with $k \in [0, n-1]$ and all $u_i, v_i \in [0, n-1]$ such that $(u_i, v_i) \ne (0,0)$. If $k = n-1$, then the assertion follows. Assume to the contrary, that $k < n-1$.

We first show that

$$(*) \qquad \text{if } \emptyset \ne I \subset [1, n-1-k] \text{ and } \sum_{i\in I} v_i e_2 = 0, \text{ then } \sum_{i \in I} u_i e_1 = 0.$$

Assume to the contrary, that $\emptyset \ne I \subset [1, n-1-k]$, $\sum_{i\in I} v_i \equiv 0 \mod n$ and $\sum_{i\in I} u_i \not\equiv 0 \mod n$. Let $a \in [1, n-1]$ such that $a \equiv \sum_{i \in I} u_i \mod n$. We construct a zero-sum subsequence S' of S with length $|S'| \ge n$, which contradicts our assumption on S. Suppose $r = n-1$. If $a \le |I|$, then set $S' = e_1^{n-a} \prod_{i\in I}(u_i e_1 + v_i e_2)$, and if $a > |I|$, then set $S' = T \prod_{i \in I}(u_i e_1 + v_i e_2)^{-1} \cdot e_1^a$. Suppose $r = n-2$. Since U is zero-sumfree, it follows that $\sum_{i=1}^n x_i \equiv 1 \mod n$ whence

$$S' = e_1^{n-a-1} \cdot \prod_{i\in I}(u_i e_1 + v_i e_2) \prod_{i=1}^{n} (x_i e_1 + e_2)$$

is the required sequence.

Since T is a zero-sum sequence, there exists some $J \subset [1, n-1-k]$ with $|J| \ge 2$ such that $\prod_{j \in J} v_j e_2$ is a minimal zero-sum sequence in $\langle e_2 \rangle$. We assert that there exists some $\emptyset \ne I \subset J$ such that

$$(**) \qquad 1 \le \sum_{i \in I} v_i \le n - |J|.$$

This obviously holds in case $|J| = 2$. Suppose that $|J| \ge 3$. First we consider the case that at least two of the v_i are distinct, say $J = [1, t]$ with $3 \le t \le n-1-k$ and $v_1 \ne v_2$. Then $\prod_{i=1}^{t-1} v_i e_2$ is zero-sumfree and Lemma 3.10 implies that

$$|\Sigma(\prod_{i=1}^{t-1} v_i e_2)| \ge |\Sigma(v_1 e_2 \cdot v_2 e_2)| + \sum_{i=3}^{t-1} |\Sigma(v_i e_2)| = 3 + (t-3) = t = |J|,$$

whence $(**)$ holds. It remains to consider the case where there exists some $v \in [0, n-1]$ such that $v_j = v$ for all $j \in J$. Then $|J|ve_2 = 0$, $\operatorname{ord}(ve_2) < n$, $-e_2 \notin \Sigma((ve_2)^{|J|-1})$ and $|\Sigma((ve_2)^{|J|-1})| = |J| - 1$ whence $(**)$ holds.

Let $\emptyset \neq I \subset J$ such that $(**)$ holds and let $a = \sum_{i \in I} v_i$. For every $Z \subset [1, 2n-2-r]$ with $|Z| = n - a$ let $b = b_Z \in [1, n]$ such that $b \equiv \sum_{i \in I} u_i + \sum_{i \in Z} x_i \mod n$. If $r = n-2$ and for all such sets Z we have $b_Z = 1$, then $x_1 = \ldots = x_n$, a contradiction to U zero-sumfree. Thus in case $r = n-2$ we may choose Z such that $b = b_Z \neq 1$. If $r = n-1$, we choose any subset Z. Then, in both cases,

$$S' = T \cdot \prod_{j \in J} (u_j e_1 + v_j e_2)^{-1} \prod_{i \in I} (u_i e_1 + v_i e_2) \prod_{i \in Z} (x_i e_1 + e_2) \cdot e_1^{n-b}$$

is a zero-sum subsequence of S with length

$$|S'| = n - 1 - |J| + |I| + n - a + n - b \geq n - 1 - |J| + 1 + n - a \overset{(**)}{\geq} n,$$

a contradiction. □

5. Property B and $\nu(G)$

The invariant $\nu(G)$ (see Definition 5.1) was introduced by van Emde Boas in 1969. It plays a key role in all investigations of Davenport's constant of groups with rank three (see [vEB69b] and also [Gao00a], section 5 where for groups G of rank two $\nu(G)$ is studied in detail). The relationship between zero-sum problems in finite abelian groups and covering problems by proper cosets was recently investigated in [**?**]. The proof of the inequalities in Proposition 5.2.1 is straightforward. Up to now there is known no group G such that $\mathsf{D}(G) < \nu(G) + 2$.

Definition 5.1. Let G be a finite abelian group. Let $\nu(G)$ denote the smallest integer $l \in \mathbb{N}_0$ such that for every zero-sumfree sequence $S \in \mathcal{F}(G)$ with $|S| \geq l$ there exists a subgroup $H < G$ and some $a \in G \setminus H$ such that $G \setminus (\Sigma(S) \cup \{0\}) \subset a + H$.

Proposition 5.2. *Let G be a finite abelian group.*

1. $\nu(G) + 1 \leq \mathsf{D}(G) \leq \nu(G) + 2$.
2. *If G is cyclic or a p-group, then* $\mathsf{D}(G) = \nu(G) + 2$.

Proof. 1. see [Gao00a], Lemma 3.3.

2. see [vEB69b], Proposition 1.19 and Theorem 2.8. □

Theorem 5.3. *Let $G = C_n \oplus C_n$ with $n \geq 2$. If n satisfies Property B, then* $\mathsf{D}(G) = \nu(G) + 2$.

Proof. If $n \in [2,3]$, then G is a p-group whence the assertion follows from Proposition 5.2.2. Suppose that $n \geq 4$. By Proposition 5.2.1 we have $\nu(G) \geq \mathsf{D}(G) - 2$. Hence it remains to show that for every zero-sum free sequence $S \in \mathcal{F}(G)$ with $|S| \geq \mathsf{D}(G) - 2$ there exists a subgroup $H < G$ and some $a \in G \setminus H$ such that $G \setminus (\Sigma(S) \cup \{0\}) \subset a + H$.

Let S be such a sequence. If $\Sigma(S) = G \setminus \{0\}$, then the assertion is clear. Suppose that there exists some $b \in G \setminus \{0\}$ such that $-b \notin \Sigma(S)$. Thus bS is a zero-sumfree sequence of length $\mathsf{D}(G) - 1 \geq |bS| \geq 1 + (\mathsf{D}(G) - 2) = 2n - 2$ and hence there is some $a \in G$, such that $a \cdot b \cdot S$ is a minimal zero-sum sequence of length $2n - 1$. By Proposition 4.1.2 there exists a basis (e_1, e_2) of G and $a_1, \ldots, a_n \in [0, n-1]$ with $\sum_{i=1}^{n} a_i \equiv 1 \mod n$ such that

$$a \cdot b \cdot S = e_1^{n-1} \prod_{i=1}^{n} (a_i e_1 + e_2).$$

Hence, up to enumeration, there are the following three possibilities for S.

Case 1: $S = e_1^{n-1} \prod_{i=1}^{n-2} (a_i e_1 + e_2)$. We assert that $G \setminus (\Sigma(S) \cup \{0\}) \subset -e_2 + \langle e_1 \rangle$. Let $g \in G \setminus (-e_2 + \langle e_1 \rangle)$. We have to verify that $g \in \Sigma(S) \cup \{0\}$. There are $\lambda_1 \in [0, n-1]$ and $\lambda_2 \in [0, n-2]$ such that $g = \lambda_1 e_1 + \lambda_2 e_2$, and obviously we have

$$g \in \sum_{i=1}^{\lambda_2} (a_i e_1 + e_2) + \langle e_1 \rangle \subset \Sigma(S) \cup \{0\}.$$

Case 2: $S = e_1^{n-2} \prod_{i=1}^{n-1} (a_i e_1 + e_2)$. We distinguish two subcases.

Case 2.1: $a_1 = \ldots = a_{n-1} = a$. Clearly, we obtain $\Sigma(S) \cup \{0\} = \bigcup_{i=0}^{n-2} (ie_1 + \langle ae_1 + e_2 \rangle)$ whence $G \setminus (\Sigma(S) \cup \{0\}) \subset -e_1 + \langle ae_1 + e_2 \rangle$.

Case 2.2: $|\{a_1, \ldots, a_{n-1}\}| \geq 2$, say $a_1 \neq a_2$. We set $a = \sum_{i=1}^{n-1} a_i$ and assert that

$$\bigcup_{i=0}^{n-2} (ie_1 + \langle ae_1 - e_2 \rangle) = G \setminus (-e_1 + \langle ae_1 - e_2 \rangle) \subset \Sigma(S) \cup \{0\}.$$

Let $i \in [0, n-2]$ and $\lambda \in [0, n-1]$. We have to verify that there exists some $\Lambda \subset [1, n-1]$ with $|\Lambda| = n - \lambda$ and some $\theta \in [0, n-2]$ such that

$$ie_1 + \lambda(ae_1 - e_2) = \theta e_1 + \sum_{j \in \Lambda} (a_j e_1 + e_2).$$

If $\lambda = 1$, then $\Lambda = [1, n-1]$ and $\theta = i$ fulfill the requirements. Suppose $\lambda > 1$. We choose some $\Lambda \subset [2, n-1]$ with $2 \in \Lambda$ and $|\Lambda| = n - \lambda$. If $i + \sum_{j \in \Lambda} (a - a_j) \not\equiv n - 1 \mod n$, then $\theta \in [0, n-2]$ with $\theta \equiv i + \sum_{j \in \Lambda} (a - a_j) \mod n$ fulfills the requirements. If $i + \sum_{j \in \Lambda} (a - a_j) \equiv n - 1 \mod n$, we set $\Lambda' = (\Lambda \setminus \{2\}) \cup \{1\}$ whence $i + \sum_{j \in \Lambda'} (a - a_j) \not\equiv n - 1 \mod n$ and there exists some θ having the required properties.

Case 3: $S = e_1^{n-3} \prod_{i=1}^{n} (a_i e_1 + e_2)$ with $n \geq 3$. We distinguish two subcases.

Case 3.1: There exist $i, j \in [1, n]$ such that $a_j - a_i \ge 2$, say $a_2 - a_1 \ge 2$. We assert that $G \setminus (\Sigma(S) \cup \{0\}) \subset -e_1 + \{0\}$. Let $g \in G \setminus \{0, -e_1\}$. We have to verify that $g \in \Sigma(S)$. Let $\lambda_1, \lambda_2 \in [0, n-1]$ with $g = \lambda_1 e_1 + \lambda_2 e_2$ whence $(\lambda_1, \lambda_2) \notin \{(0,0), (n-1, 0)\}$. If $\lambda_2 = 0$, then $g \in \Sigma(e_1^{n-2}) \subset \Sigma(S)$ because $\sum_{i=1}^n (a_i e_1 + e_2) = e_1$. Suppose that $\lambda_2 \in [1, n-1]$. We choose some $\Lambda \subset [1, n]$ with $|\Lambda| = \lambda_2$, $1 \in \Lambda$ and $2 \notin \Lambda$. If $\Lambda' = (\Lambda \setminus \{1\}) \cup \{2\}$, then

$$g \in \{(\sum_{j \in \Lambda} a_j e_1 + e_2) + i e_1 \mid i \in [0, n-3]\} \cup \{(\sum_{j \in \Lambda'} a_j e_1 + e_2) + i e_1 \mid i \in [0, n-3]\} \subset \Sigma(S).$$

Case 3.2: $\{a_1, \ldots, a_n\} = \{a, a+1\}$ for some $a \in [0, n-2]$. Then $S = e_1^{n-3}(ae_1 + e_2)^k((a+1)e_1 + e_2)^{n-k}$ for some $k \in [1, n-1]$, and since $ka + (n-k)(a+1) \equiv 1 \mod n$, it follows that $k = n-1$. We assert that $G \setminus (-e_1 + \langle ae_1 + e_2 \rangle) \subset \Sigma(S) \cup \{0\}$. Since obviously, $\bigcup_{i=0}^{n-3} (ie_1 + \langle ae_1 + e_2 \rangle) \subset \Sigma(S) \cup \{0\}$, it remains to check that $(n-2)e_1 + \langle ae_1 + e_2 \rangle \subset \Sigma(S) \cup \{0\}$. Let $g = (n-2)e_1 + \lambda(ae_1 + e_2)$ with $\lambda \in [0, n-1]$. If $\lambda = 0$, then $g = (n-3)e_1 + (n-1)(ae_1 + e_2) + ((a+1)e_1 + e_2) \in \Sigma(S)$. If $\lambda > 0$, then $g - ((a+1)e_1 + e_2) = (n-3)e_1 + (\lambda - 1)(ae_1 + e_2) \in \Sigma(e_1^{n-3}(ae_1 + e_2)^{n-1})$ whence the assertion follows. □

6. Property B implies Property C

In this section we show that, under some additional weak condition, Property B implies Property C. This was first done for prime numbers in [GG99]. For $a \in \mathbb{Z}$ we denote by $|a|_n$ the positive integer in $[1, n]$ such that $a \equiv |a|_n \mod n$.

Proposition 6.1. *Let $G = C_n \oplus C_n$ with $n \ge 2$ and $S = a^{n-1} b^{n-1} \prod_{i=1}^{n-1} c_i \in \mathcal{F}(G)$ a sequence which does not contain a short zero-sum subsequence. If n satisfies Property B, then $c_1 = \cdots = c_{n-1}$.*

Proof. For $n = 2$ there is nothing to do. Suppose that $n \ge 3$ and let S be as above. Since $a \ne b$, Lemma 3.9 implies that $(e_1 = a, e_2 = b)$ is a basis of G whence S has the form

$$S = e_1^{n-1} e_2^{n-1} \prod_{i=1}^{n-1} (x_i e_1 + y_i e_2)$$

with $x_i, y_i \in [1, n]$. Since S has no short zero-sum subsequence, it follows that $x_i, y_i \in [1, n-1]$. Furthermore, S has no zero-sum subsequence of length n or $2n > \mathsf{D}(G)$ and the same is true for

$$S_{e_2} = (e_1 - e_2)^{n-1} 0^{n-1} \prod_{i=1}^{n-1} (x_i e_1 + (y_i - 1) e_2).$$

Therefore

$$(e_1 - e_2)^{n-1} \prod_{i=1}^{n-1} (x_i(e_1 - e_2) + (x_i + y_i - 1) e_2)$$

is zero-sumfree whence $\prod_{i=1}^{n-1}(x_i+y_i-1)e_2$ is zero-sumfree in $\langle e_2\rangle \cong C_n$ which implies that

$$x_1+y_1 \equiv \cdots \equiv x_{n-1}+y_{n-1} \mod n.$$

Since for every $i \in [1, n-1]$

$$e_1^{n-x_i} e_2^{n-y_i}(x_i e_1 + y_i e_2)$$

is a zero-sum subsequence of S of length $2n+1-(x_i+y_i)$, it follows that $x_i+y_i \le n$. Thus

$$x_1+y_1 = \cdots = x_{n-1}+y_{n-1} = m$$

for some $m \in [2, n]$. Since $\prod_{i=1}^{n-1}(x_i+y_i-1)e_2 = ((m-1)e_2)^{n-1}$ is zero-sumfree, it follows that $\gcd\{m-1, n\} = 1$.

If $m = 2$, then $x_1 = y_1 = \cdots = x_{n-1} = y_{n-1} = 1$ and the assertion is proved.

Suppose $m = n$. If $\prod_{i\in I} x_i e_1$ is a zero-sum sequence for some $\emptyset \ne I \subset [1, n-1]$, then the same is true for $\prod_{i\in I} y_i e_2$ and thus $\prod_{i\in I}(x_i e_1 + y_i e_2)$ would be a zero-sum sequence. Since S contains no short zero-sum subsequence, $\prod_{i=1}^{n-1} x_i e_1$ is zero-sumfree whence $x_1 = \cdots = x_{n-1}$. Therefore $y_1 = \cdots = y_{n-1}$ and the assertion is proved.

It remains to consider the case where $m \in [3, n-1]$. Since $\gcd\{m-1, n\} = 1$, there is a unique $t \in [1, n]$ such that $t(m-1) \equiv 1 \mod n$. Since $m \in [3, n-1]$, it follows that $t \in [2, n-2]$ whence $|tm|_n = t+1$. Since $t \ge 2$, it suffices to show that for every subset $I \subset [1, n-1]$ with $|I| = t$ all x_i with $i \in I$ are equal.

Let $I \subset [1, n-1]$ with $|I| = t$ and consider the sequence

$$S_I = e_1^{n-|\Sigma_{i\in I} x_i|_n} e_2^{n-|\Sigma_{i\in I} y_i|_n} \prod_{i\in I}(x_i e_1 + y_i e_2)\,.$$

Clearly, S_I is a zero-sum subsequence of S of length

$$\begin{aligned} |S_I| &= 2n+t-\mid \Sigma_{i\in I} x_i \mid_n - \mid \Sigma_{i\in I} y_i \mid_n \\ &= 2n+t-\mid \Sigma_{i\in I} x_i \mid_n - \mid tm - \Sigma_{i\in I} x_i \mid_n \\ &= \begin{cases} 2n+t-\mid tm \mid_n = 2n-1\,, & \mid tm \mid_n > \mid \Sigma_{i\in I} x_i \mid_n \\ 2n+t-(n+\mid tm \mid_n) = n-1\,, & \mid tm \mid_n \le \mid \Sigma_{i\in I} x_i \mid_n \end{cases} \end{aligned}$$

Since S has no short zero-sum subsequence, we infer that $|S_I| = 2n-1$ and that S_I is a minimal zero-sum sequence.

Since $t \le n-2$ and $\{x_i e_1 + y_i e_2 \mid i \in I\} \cap \{e_1, e_2\} = \emptyset$, Property B implies that either

$$n-\mid \Sigma_{i\in I} x_i \mid_n = n-1 \quad \text{or} \quad n-\mid \Sigma_{i\in I} y_i \mid = n-1\,.$$

Therefore by Proposition 4.1.2.a) either ($y_i = 1$ for all $i \in I$) or ($x_i = 1$ for all $i \in I$). $\square$

Theorem 6.2. *Let $G = C_n \oplus C_n$ with $n \ge 2$. Suppose that n satisfies Property B and that every sequence $S \in \mathcal{F}(G)$ with $|S| \ge 3n-2$ has a zero-sum subsequence of length n or $2n$. Then n satisfies Property C.*

Remark: If n has at most two distinct prime divisors or if Property E holds for all prime divisors of n, then every sequence $S \in \mathcal{F}(G)$ with $|S| \ge 3n-2$ has a zero-sum subsequence of length n or $2n$ (see Theorem 3.7)

Proof. Since 2 satisfies Property C, we may suppose that $n \ge 3$. Let $S \in \mathcal{F}(G)$ be a sequence with length $|S| = 3n-3$ which does not contain a short zero-sum subsequence. By assumption the sequence $0.S$ contains a zero-sum subsequence of length n or $2n$ whence S contains a zero-sum subsequence T of length $|T| \in \{n-1, n, 2n-1, 2n\}$. Therefore $|T| = 2n-1$ and T is a minimal zero-sum sequence. Hence by Property B there is some $b \in G$ with $b^{n-1} \mid T$ and thus

$$S = b^{n-1} \prod_{i=1}^{2n-2} c_i \,.$$

Since S has no zero-sum subsequence of length n or $2n$, the same is true for

$$S_b = 0^{n-1} \prod_{i=1}^{2n-2} (c_i - b).$$

Therefore $\prod_{i=1}^{2n-2}(c_i - b)$ is zero-sumfree and thus $c\prod_{i=1}^{2n-2}(c_i - b)$ is a minimal zero-sum sequence where $c = -\sum_{i=1}^{2n-2}(c_i - b)$. Since n satisfies Property B, there are two possiblities. If there is some $g \in G$ such that $g^{n-1} \mid \prod_{i=1}^{2n-2}(c_i - b)$, then $b^{n-1}(g+b)^{n-1} \mid S$ and the assertion follows from Proposition 6.1. Otherwise it follows that c^{n-2} divides $\prod_{i=1}^{2n-2}(c_i - b)$, say $c = c_1 - b$. Setting $e_1 = c_1 = c + b$ and $e_2 = b$ we obtain that $e_1^{n-2} e_2^{n-1}$ is a subsequence of S. By Lemma 3.9 (e_1, e_2) is a basis of G whence S has the form

$$S = e_1^{n-2} e_2^{n-1} \prod_{i=1}^{n} (x_i e_1 + y_i e_2)$$

with $x_i, y_i \in [1, n]$. Setting

$$S_{e_2} = 0^{n-1} (e_1 - e_2)^{n-2} \prod_{i=1}^{n} (x_i e_1 + (y_i - 1) e_2)$$

and arguing as above we infer that

$$(e_1 - e_2)^{n-2} \prod_{i=1}^{n} (x_i e_1 + (y_i - 1) e_2)$$

is zero-sumfree. Since

$$\begin{aligned} 0 &= c_1 - b + \sum_{i=1}^{2n-2}(c_i - b) = c_1 + b + \sum_{i=1}^{2n-2} c_i \\ &= e_1 + e_2 + (n-2)e_1 + \sum_{i=1}^{n}(x_ie_1 + y_ie_2) = (n-1)(e_1 - e_2) + \sum_{i=1}^{n}(x_ie_1 + y_ie_2), \end{aligned}$$

we obtain that

$$(e_1 - e_2)^{n-1}\prod_{i=1}^{n}(x_i(e_1 - e_2) + (x_i + y_i - 1)e_2)$$

is a minimal zero-sum sequence.

Clearly, $(e_1 - e_2, e_2)$ is a basis of G whence $\prod_{i=1}^{n}(x_i + y_i - 1)e_2$ is a minimal zero-sum sequence in $\langle e_2\rangle$ which implies that

$$x_1 + y_1 \equiv \cdots \equiv x_n + y_n \mod n.$$

If for some $i \in [1, n]$ we have $x_i = 1$ and $y_i = n$, then $e_1^{n-1}e_2^{n-1} \mid S$ and the assertion follows from Proposition 6.1. Suppose that all $(x_i, y_i) \neq (1, n)$. If $x_i + y_i \geq n + 1$ for some $i \in [1, n]$, then $x_i \geq 2$ and

$$e_1^{n-x_i}e_2^{n-y_i}(x_ie_1 + y_ie_2)$$

is a zero-sum subsequence of S with length $2n + 1 - (x_i + y_i) \leq n$, a contradiction. Thus

$$x_1 + y_1 = \cdots = x_n + y_n = m$$

for some $m \in [2, n]$. Since $\prod_{i=1}^{n}(x_i + y_i - 1)e_2$ is a minimal zero-sum sequence, we infer that $\gcd\{m - 1, n\} = 1$.

Suppose that $m = n$. There is some $\emptyset \neq I \subset [1, n]$ such that $\Sigma_{i\in I}x_ie_1 = 0$. This implies that $\Sigma_{i\in I}y_ie_2 = 0$ whence $\prod_{i\in I}(x_ie_1 + y_ie_2)$ is a short zero-sum subsequence of S, a contradiction.

If $m = 2$, then $x_1 = y_1 = \ldots x_n = y_n = 1$ whence $\prod_{i=1}^{n}(x_ie_1 + y_ie_2)$ is a short zero-sum subsequence of S, a contradiction.

Therefore we obtain that $m \in [3, n - 1]$. Let $t \in [2, n]$ such that $t(m - 1) \equiv 1 \mod n$ and $I \subset [1, n]$ be a subset with $|I| = t$ and $\Sigma_{i\in I}x_i \not\equiv 1 \mod n$. Then $|\Sigma_{i\in I}x_i|_n \in [2, n]$, and arguing as in Proposition 6.1 we infer that

$$S_I = e_1^{n-|\sum_{i\in I}x_i|_n}e_2^{n-|\sum_{i\in I}y_i|_n}\prod_{i\in I}(x_ie_1 + y_ie_2)$$

is a minimal zero-sum subsequence of S with length $|S_I| = 2n - 1$. As in Propositon 6.1 we argue that either all x_i are equal to 1 or all y_i are equal to 1.

Therefore, for every subset $I \subset [1, n]$ with $|I| = t$ we have:

(∗)

$$\text{either}\quad (\sum_{i\in I} x_i \equiv 1 \mod n) \quad\text{or}\quad (\text{all } x_i \text{ are equal to } 1) \quad\text{or}\quad (\text{all } x_i \text{ are equal to } m-1).$$

Assume to the contrary, that $|\{x_1,\dots,x_n\}| \ge 3$, say $|\{x_{n-2},x_{n-1},x_n\}| = 3$. Since $t-1 \le n-3$, it follows that $|\{x_j + \sum_{i=1}^{t-1} x_i \mid n-2 \le j \le n\}| = 3$, a contradiction to (∗).

Therefore, $\prod_{i=1}^n x_i e_1 = (xe_1)^u (x'e_1)^v$ with $x, x' \in [1,n]$, $x \ne x'$, $u+v=n$ and $0 \le v \le u$. If $v \le 1$, then $u \ge n-1$ and Proposition 6.1 implies the assertion.

Assume to the contrary, that $v \ge 2$. If $t \ge 3$, one can choose $u_0 \in [2, u-1]$ and $v_0 \in [1, v-1]$ such that $u_0 + v_0 = t$ because $t \le n-2 = u+v-2$. However,

$$u_0 x + v_0 x' \ne (u_0 - 1)x + (v_0+1)x'$$

which contradicts (∗). Hence we have $t=2$, and (∗) implies that $x + x' \equiv 1 \mod n$. Thus $x + x \not\equiv x + x' \equiv 1 \mod n$ whence (∗) implies that $x \in \{1, m-1\}$. We argue in a similar way for x' and obtain $\{x,x'\} = \{1, m-1\}$. Therefore $m = x + x' \equiv 1 \mod n$, a contradiction to $m \in [3, n-1]$. □

7. Zero-sum sequences S in $C_m \oplus C_m$ with length $|S| = tm - 1$

Let $G = C_{mn} \oplus C_{mn}$ with $m, n \in \mathbb{N}_{\ge 2}$, $\varphi : G \to G$ the multiplication by n and $S \in \mathcal{F}(G)$ a minimal zero-sum sequence with length $|S| = \mathsf{D}(G) = tm - 1$ where $t = 2n$. Then by Lemma 3.14 $\varphi(S)$ is a zero-sum sequence in $nG \cong C_m \oplus C_m$ which is not a product of $t = 2n$ zero-sum subsequences. It is the aim of this section to determine the structure of such sequences under the assumption that $C_m \oplus C_m$ has Property B.

Theorem 7.1. *Let $G = C_m \oplus C_m$ with $m \ge 2$. Suppose that m satisfies Property B and that every sequence $T \in \mathcal{F}(G)$ with $|T| \ge 3m-2$ has a zero-sum subsequence of length m or $2m$. Let $S \in \mathcal{F}(G)$ be a zero-sum sequence with $|S| = tm-1$ for some $t \ge 3$ which cannot be written as a product of t non-empty zero-sum subsequences. Then there exists a basis (e_1, e_2) of G such that either*

$$S = e_1^{sm-1} \cdot \prod_{\nu=1}^{(t-s)m} (a_\nu e_1 + e_2)$$

where $a_1, \dots, a_{(t-s)m} \in [0, m-1]$ and $s \in [1, t-1]$ or

$$S = e_1^{s_1 m} \cdot (be_1 + e_2)^{s_2 m - 1} \cdot e_2^{s_3 m - 1} \cdot (be_1 + 2e_2)$$

where $b \in [0, m-1]$ with $\gcd\{b, m\} = 1$ and $s_1, s_2, s_3 \in \mathbb{N}$ with $s_1 + s_2 + s_3 = t$.

We are going to prove Theorem 7.1 by induction on t. Throughout this section, let all notations be as in Theorem 7.1.

Lemma 7.2. *The assertion of Theorem 7.1 holds for $t = 3$.*

Proof. Suppose that

$$S = \prod_{\nu=1}^{l} g_\nu^{k_\nu}$$

where $g_1, \ldots, g_l \in G$ are pairwise distinct and $k_1 \geq k_2 \geq \ldots \geq k_l \geq 1$. Since S does not contain three disjoint nonempty zero-sum subsequences, it follows that $k_2 \leq m-1$. By Lemma 3.12.1 every short zero-sum subsequence of S has length m. Suppose there is some $j \in [1, l-1]$ such that $k_j \geq m-1$ and $k_{j+1} \geq m-1$.

We assert that either((g_j, g_{j+1}) is a basis of G) or ($k_1 = m-1$ and (g_1, g_j)) is a basis of G). This is obviously true for $m = 2$. Suppose that $m \geq 3$ and that (g_j, g_{j+1}) is not a basis of G. Then by Lemma 3.9 $g_j^{m-1} g_{j+1}^{m-1}$ contains a short zero-sum subsequence T. Then $T^{-1}S$ is a minimal zero-sum subsequence with length $2m-1$ containing some element g with multiplicity $m-1$, say $g = g_i$ with $i \in [1, l]$ minimal. Note that $g_j g_{j+1} \mid T^{-1}S$. If $g' \in \mathrm{supp}(T^{-1}S) \setminus \{g\}$, then by Proposition 4.1(g, g') is a basis of G whence the assertion follows.

We distinguish several cases.

Case 1: $k_2 < m-1$. By Lemma 3.12.2 S has a product decomposition of the form $S = S_0 S_1$ where S_0 is a minimal zero-sum sequence with length $2m-1$ and S_1 is a short zero-sum sequence. Since m has Property B, Theorem 4.3 implies that there exists a basis (e_1, e_2) of G such that

$$S = e_1^{m-1} \cdot \prod_{\nu=1}^{m} (a_\nu e_1 + e_2) \cdot \prod_{\nu=1}^{m} (x_\nu e_1 + y_\nu e_2)$$

with all $x_\nu, y_\nu, a_\nu \in [0, m-1]$ and $\sum_{\nu=1}^{m} a_\nu \equiv 1 \mod m$. Let $\nu \in [1, m]$. It remains to verify that $y_\nu = 1$. The sequence $(x_\nu e_1 + y_\nu e_2)^{-1} \cdot S$ contains a short zero-sum subsequence W (clearly, $W \neq \prod_{\nu=1}^{m} (a_\nu e_1 + e_2)$) and $W^{-1} \cdot S$ is a minimal zero-sum sequence with length $2m-1$. Since $\max\{\mathsf{v}_g(W^{-1} \cdot S) \mid g \in G\} = m-1 > k_2$, it follows that

$$W^{-1} \cdot S = e_1^{m-1} \cdot (a_\mu e_1 + e_2) \cdot (x_\nu e_1 + y_\nu e_2) \cdot \prod_{\lambda=1}^{m-2} (u_\lambda e_1 + v_\lambda e_2)$$

for some $\mu \in [1, m]$ and all $u_\lambda, v_\lambda \in [0, m-1]$. Since $W^{-1} \cdot S$ is a minimal zero-sum sequence, Proposition 4.1.2.a) implies that $y_\nu = v_1 = \ldots = v_{m-2} = 1$.

Case 2: $k_2 = m-1$ and $k_3 < m-1$. Then $(g_1 = e_1, g_2 = e_2)$ is a basis of G, and we distinguish three subcases.

Case 2.1: $k_1 \geq m+1$. The sequence

$$g_1^{-m} \cdot S = e_2^{m-1} \cdot e_1^{k_1 - m} \cdot \prod_{\nu=1}^{2m-k_1} (x_\nu e_1 + y_\nu e_2)$$

is a minimal zero-sum sequence whence $x_1 = \cdots = x_{2m-k_1} = 1$ by Proposition 4.1.

Case 2.2: $k_1 = m$. We have

$$S = e_1^m \cdot e_2^{m-1} \cdot \prod_{\nu=1}^{m} (x_\nu e_1 + y_\nu e_2)$$

with all $x_\nu, y_\nu \in [0, m-1]$, $\sum_{\nu=1}^m x_\nu \equiv 0 \mod m$ and $\sum_{\nu=1}^m y_\nu \equiv 1 \mod m$, say $y_1 \ne 1$. Since $e_1^{-m} \cdot S$ is a minimal zero-sum sequence, it follows that $x_1 = \cdots = x_m$. We verify that $x_1 = 1$ which implies the assertion. Since $k_3 < m-1$, Theorem 6.2 implies that $e_1^{-1} \cdot (x_1 e_1 + y_1 e_2)^{-1} \cdot S$ contains a short zero-sum subsequence W, and clearly we have $|W| = m$ and $e_2^{m-1} \nmid W$. Then

$$W^{-1} \cdot S = e_1 \cdot (x_1 e_1 + y_1 e_2) \cdot e_2 \cdot T \qquad \text{for some} \qquad T \in \mathcal{F}(G)$$

is a minimal zero-sum sequence whence either $\mathsf{v}_{e_1}(W^{-1} \cdot S) = m-1$ or $\mathsf{v}_{e_2}(W^{-1} \cdot S) = m-1$. Since $y_1 \ne 1$, Proposition 4.1 implies that $\mathsf{v}_{e_2}(W^{-1} \cdot S) = m-1$ and $x_1 = 1$.

Case 2.3: $k_1 = m-1$. We have

$$S = e_1^{m-1} \cdot e_2^{m-1} \cdot \prod_{\nu=1}^{m+1} (x_\nu e_1 + y_\nu e_2)$$

with all $x_\nu, y_\nu \in [0, m-1]$, $\sum_{\nu=1}^{m+1} x_\nu \equiv \sum_{\nu=1}^{m+1} y_\nu \equiv 1 \mod m$. If $x_1 = x_2 = \cdots = x_{m+1} = 1$ or $y_1 = y_2 = \cdots = y_{m+1} = 1$, then we are done. Assume to the contrary that this does not hold. Then there are $i < j$ with $x_i \ne 1$ and $x_j \ne 1$ and there are $i' < j'$ such that $y_{i'} \ne 1$ and $y_{j'} \ne 1$, say $x_1 \ne 1$ and $y_2 \ne 1$. Since $k_3 < m-1$, Theorem 6.2 implies that $S \cdot (x_1 e_1 + y_1 e_2)^{-1} \cdot (x_2 e_1 + y_2 e_2)^{-1}$ contains a short zero-sum subsequence W. Then $|W| = m$ and $W^{-1} \cdot S$ is a minimal zero-sum subsequence with length $2m-1$ which contains the sequence $e_1 \cdot e_2 \cdot (x_1 e_1 + y_1 e_2) \cdot (x_2 e_1 + y_2 e_2)$. Since $k_l \le \cdots \le k_3 < m-1$, either $\mathsf{v}_{e_1}(W^{-1} \cdot S) = m-1$ or $\mathsf{v}_{e_2}(W^{-1} \cdot S) = m-1$. If $\mathsf{v}_{e_1}(W^{-1} \cdot S) = m-1$, then Proposition 4.1 implies that $y_1 = y_2 = 1$, a contradiction. If $\mathsf{v}_{e_2}(W^{-1} \cdot S) = m-1$, then Proposition 4.1 implies that $x_1 = x_2 = 1$, a contradiction.

Case 3: $k_2 = m-1$ and $k_3 = m-1$. Then $k_1 \in [m-1, m+1]$. We distinguish two subcases.

Case 3.1: $k_1 \in \{m, m+1\}$. Then $(g_2 = e_1, g_3 = e_2)$ is a basis of G. Thus

$$S = e_1^{m-1} \cdot e_2^{m-1} \cdot (a e_1 + b e_2)^{k_1} \cdot \prod_{\nu=1}^{m+1-k_1} (x_\nu e_1 + y_\nu e_2)$$

where a, b and all $x_\nu, y_\nu \in [0, m-1]$. If $k_1 = m+1$, then $(m+1)a + (m-1) \equiv 0 \mod m$ implies that $a = 1$, whence the assertion is proved. Suppose that $k_1 = m$. Then $ma + x_1 + m - 1 \equiv 0 \mod m$ and $mb + y_1 + m - 1 \equiv 0 \mod m$ whence $x_1 = y_1 = 1$. If $a = 1$ or $b = 1$, then the assertion follows. Suppose that both a and b are distinct to 1. The sequence $(a e_1 + b e_2)^{-1} \cdot S$ contains a short zero-sum subsequence W and clearly $e_1^{m-1} \nmid W$ and $e_2^{m-1} \nmid W$. Thus $W^{-1} \cdot S$ is a minimal zero-sum sequence containing e_1, e_2

and $ae_1 + be_2$. Since $a \neq 1$ and $b \neq 1$, Proposition 4.1 implies that $\mathsf{v}_{e_i}(W^{-1} \cdot S) < m-1$ for $i \in [1,2]$ whence $\mathsf{v}_{ae_1+be_2}(W^{-1} \cdot S) = m-1$ and $e_1 - e_2 \in \langle ae_1 + be_2 \rangle$. This implies that $b = m-a$ and $\gcd\{b, m\} = 1$. If $c \in [0, m-1]$ with $-ac \equiv 1 \mod m$ and

$$(f_1, f_2) = (e_1, e_2) \cdot \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix},$$

then

$$S = f_1^{m-1} \cdot (f_1 + cf_2)^{m-1} \cdot f_2^m \cdot (2f_1 + cf_2)$$

has form 2 (with basis (f_2, f_1) and $s_1 = 1$).

Case 3.2: $k_1 = m-1$. Then $\{g_1, g_2, g_3\}$ contains a basis $\{e_1, e_2\} \subset G$. Therefore we have

$$S = e_1^{m-1} \cdot e_2^{m-1} \cdot (ae_1 + be_2)^{m-1} \cdot (x_1e_1 + y_1e_2) \cdot (x_2e_1 + y_2e_2)$$

with $a, b, x_1, x_2, y_1, y_2 \in [0, m-1]$ such that $-1-a+x_1+x_2 \equiv 0 \mod m$ and $-1-b+y_1+y_2 \equiv 0 \mod m$. The sequence $S \cdot (x_1e_1 + y_1e_2)^{-1}$ contains a short zero-sum subsequence W. Then $W^{-1} \cdot S$ is a minimal zero-sum sequence with contains the sequence

$$e_1 \cdot e_2 \cdot (ae_1 + be_2) \cdot (x_1e_1 + y_1e_2).$$

If $\mathsf{v}_{e_1}(W^{-1} \cdot S) = m-1$, then Proposition 4.1 implies that $1 = b = y_1$ whence $y_2 = 1$ and we are done. If $\mathsf{v}_{e_2}(W^{-1} \cdot S) = m-1$, then Proposition 4.1 implies that $1 = a = x_1$ whence $x_2 = 1$ and we are done. Suppose that $\mathsf{v}_{ae_1+be_2}(W^{-1} \cdot S) = m-1$. Then Proposition 4.1 implies that $e_1 - e_2 \in \langle ae_1 + be_2 \rangle$ whence $b = m-a$ and $\gcd\{b, m\} = 1$. Furthermore, we have $(1-x_1)e_1 - y_1e_2 = e_1 - (x_1 + y_1e_2) \in \langle a(e_1 - e_2) \rangle$ which implies that $y_1 \equiv 1 - x_1 \mod m$. We deal with the sequence $S \cdot (x_2e_1 + y_2e_2)^{-1}$ in a similar way. In the only remaining case we have $b = m-a$, $y_1 \equiv 1 - x_1 \mod m$ and $y_2 \equiv 1 - x_2 \mod m$ whence

$$S = e_1^{m-1} \cdot e_2^{m-1} \cdot (ae_1 - ae_2)^{m-1} \cdot (x_1e_1 + (1-x_1)e_2) \cdot (x_2e_1 + (1-x_2)e_2)$$

If $c \in [0, m-1]$ such that $-ac \equiv 1 \mod m$ and

$$(f_1, f_2) = (e_1, e_2) \cdot \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix},$$

then

$$S = f_1^{m-1} \cdot (f_1 + cf_2)^{m-1} \cdot f_2^{m-1} \cdot (f_1 + (1-x_1)cf_2) \cdot (f_1 + (1-x_2)cf_2)$$

has form 1 (with basis (f_2, f_1) and $s = 1$). □

Lemma 7.3. *Suppose $t \geq 4$ and let T be a subsequence of S with length $|T| = m+1$. Then there exists a zero-sum sequence W with $T \mid W \mid S$ and a basis (e_1, e_2) of G such that*

$$W = e_1^{sm-1} \cdot \prod_{\nu=1}^{(3-s)m} (a_\nu e_1 + e_2)$$

where $s \in [1,2]$ and $a_1, \ldots, a_{(3-s)m} \in [0, m-1]$ or

$$W = e_1^m \cdot (be_1 + e_2)^{m-1} \cdot e_2^{m-1} \cdot (be_1 + 2e_2)$$

where $b \in [0, m-1]$ *with* $\gcd\{b, m\} = 1$.

Proof. Since the sequence $T^{-1}S$ has length $|T^{-1}S| = (t-1)m - 2$, Lemma 3.1.2 implies that $T^{-1}S$ has $t-3$ disjoint short zero-sum subsequences $S_1, \cdots, S_{t-3}$, and by Lemma 3.12 all of them have length m. Clearly, $W = (S_1 \cdot \ldots \cdot S_{t-3})^{-1} \cdot S$ does not contain three disjoint non-empty zero-sum subsequences and has length $|W| = 3m - 1$. Now the assertion follows from Lemma 7.2. □

Proof of Theorem 7.1. The case $t = 3$ was handled in Lemma 7.2 whence we may suppose that $t \geq 4$. We distinguish three cases.

Case 1: $|\mathrm{supp}(S)| \geq 5$. This implies that $m \geq 3$. If $m = 3$, then there is some $g \in G$ such that $\{-g, g\} \subset \mathrm{supp}(S)$ whence S contains a nonempty zero-sum subsequence of length 2, a contradiction to Lemma 3.12.1. So it follows that $m \geq 4$. Let T be a subsequence of S with $|T| = m + 1$ and $|\mathrm{supp}(T)| \geq 5$. By Lemma 7.3 there exist a zero-sum sequence W with $T \mid W \mid S$ and a basis (e_1, e_2) of G such that

$$W = e_1^{s'm-1} \cdot \prod_{\nu=1}^{(3-s')m} (a_\nu e_1 + e_2)$$

with $s' \in [1, 2]$, $a_1, \ldots, a_{(3-s')m} \in [0, m-1]$, a_1, a_2, a_3 pairwise distinct and $\sum_{\nu=1}^{(3-s')m} a_\nu \equiv 1 \mod m$. This implies that

$$S = e_1^{s'm-1} \cdot \prod_{\nu=1}^{(3-s')m} (a_\nu e_1 + e_2) \cdot \prod_{\nu=1}^{l} (x_\nu e_1 + y_\nu e_2)$$

with all $x_\nu, y_\nu \in [0, m-1]$.

Assume to the contrary that there exists some $y_\nu \notin \{0, 1\}$, say $y_1 \notin \{0, 1\}$. Then the sequence

$$U = e_1 \cdot (a_1 e_1 + e_2) \cdot (a_2 e_1 + e_2) \cdot (a_3 e_1 + e_2) \cdot (x_1 e_1 + y_1 e_2)$$

has length $|U| = 5 \leq m + 1$ and $|\mathrm{supp}(U)| \geq 5$. The sequence $U^{-1} \cdot S$ contains $(t-3)$ disjoint short zero-sum subsequences and let T denote their product. Then $V = T^{-1} \cdot S$ has length $3m - 1$, contains the sequence U and cannot be written as a product of three proper zero-sum subsequences. Since $|\mathrm{supp}(V)| \geq |\mathrm{supp}(U)| \geq 5$, Lemma 7.2 implies that there exists a basis (f_1, f_2) of G such that

$$V = f_1^{m-1} \cdot \prod_{\nu=1}^{2m} (b_\nu f_1 + f_2).$$

If we can verify that $e_1 = f_1$, then $(x_1 e_1 + y_1 e_2) - (a_1 e_1 + e_2) \in \langle f_1 \rangle$ whence $(y_1 - 1)e_2 = 0$ and thus $y_1 = 1$ gives the required contradiction. Assume to the contrary that $e_1 \neq f_1$. Since a_1, a_2, a_3 are pairwise distinct, we may suppose that $f_1 \notin \{a_1 e_1 + e_2, a_2 e_1 + e_2\}$ whence $(a_1 - a_2)e_1 \in \langle f_1 \rangle$ and $e_1 - (a_1 e_1 + e_2) = (1 - a_1)e_1 + e_2 \in \langle f_1 \rangle$. This implies that $f_1 = z_1 e_1 + z_2 e_2$ with $\gcd\{z_2, m\} = 1$ whence $a_1 = a_2$, a contradiction.

Therefore $y_\nu \in \{0,1\}$ for all $\nu \in [1,l]$. If $y_\nu = 0$, then $(x_\nu e_1) \cdot e_1^{m-x_\nu}$ is a short zero-sum subsequence of S with length $m - x_\nu + 1$ whence $x_\nu = 1$. Therefore $(x_\nu, y_\nu) = (1,0)$ or $(x_\nu, y_\nu) = (x_\nu, 1)$ which implies that

$$S = e_1^{s''} \cdot \prod_{\nu=1}^{|S|-s''} (a_\nu e_1 + e_2)$$

for some $s'' \ge m-1$. Since S is a zero-sum sequence, it follows that $|S| - s'' \equiv 0 \mod m$ whence $s'' = sm - 1$ for some $s \in [1, t-1]$.

Case 2: $|\mathrm{supp}(S)| = 3$. By Lemma 7.3 there exists a zero-sum subsequence W of S and a basis (e_1, e_2) of G such that

$$W = e_1^{s'm-1} \cdot \prod_{\nu=1}^{(3-s')m} (a_\nu e_1 + e_2)$$

with $s' \in [1,2]$ and $a_1, \ldots, a_{(3-s')m} \in [0, m-1]$. Since $3 \le |\mathrm{supp}(W)| \le |\mathrm{supp}(S)| = 3$, it follows that $\mathrm{supp}(S) = \mathrm{supp}(W)$ whence

$$S = e_1^{s''} \cdot \prod_{\nu=1}^{|S|-s''} (a_\nu e_1 + e_2)$$

where all $a_\nu \in [0, m-1]$ and $s'' \ge m-1$. Since S is a zero-sum sequence, it follows that $|S| - s'' \equiv 0 \mod m$ whence $s'' = sm - 1$ for some $s \in [1, t-1]$.

Case 3: $|\mathrm{supp}(S)| = 4$. This implies that $m \ge 3$. Let T be a subsequence of S with $|T| = 4$ and $|\mathrm{supp}(T)| = 4$. By Lemma 7.3 there exists a zero-sum sequence W with $T \mid W \mid S$ and a basis (e_1, e_2) of G such that either

$$W = e_1^{s'm-1} \cdot \prod_{\nu=1}^{(3-s')m} (a_\nu e_1 + e_2)$$

where $s' \in [1,2]$ and $a_1, \ldots, a_{(3-s')m} \in [0, m-1]$ or

$$W = e_1^m \cdot (be_1 + e_2)^{m-1} \cdot e_2^{m-1} \cdot (be_1 + 2e_2)$$

where $b \in [0, m-1]$ with $\gcd\{b, m\} = 1$. Clearly we have $\mathrm{supp}(S) = \mathrm{supp}(W)$. Hence in the first case the assertion follows as in Case 2, and it remains to consider the case where

$$S = e_1^u \cdot (be_1 + e_2)^v \cdot e_2^w \cdot (be_1 + 2e_2)^q$$

with $u \ge m$, $v \ge m-1$, $w \ge m-1$, $q \ge 1$ and $u + v + w + q = tm - 1$.

Assume to the contrary that $q \ge 2$. If $b = m - 1$ and

$$(f_1, f_2) = (e_1, e_2) \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

then

$$S = f_1^u \cdot f_2^v \cdot (f_1 + f_2)^w \cdot (f_1 + 2f_2)^q$$

whence we may suppose that $b \in [1, m-2]$. If $b = 1$ and

$$(f_1, f_2) = (e_1, e_2) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then

$$S = f_2^u \cdot (f_1 + f_2)^v \cdot f_1^w \cdot (2f_1 + f_2)^q$$

whence we may suppose that $b \in [2, m-2]$. Thus there is some $b' \in [2, m-2]$ such that $b' \cdot b \equiv -1 \mod m$ whence

$$e_1 \cdot (be_1 + e_2)^{b'-2} \cdot e_2^{m-b'-2} \cdot (be_1 + 2e_2)^2$$

is a short zero-sum subsequence of S with length $m-1$, a contradiction.

Therefore we infer that $q = 1$. We write $u = u_1 m + u_0$, $v = v_1 m + v_0$ and $w = w_1 m + w_0$ with $u_0, v_0, w_0 \in [0, m-1]$ and set

$$M = e_1^{u_0} \cdot (be_1 + e_2)^{v_0} \cdot e_2^{w_0} \cdot (be_1 + 2e_2).$$

Clearly, we have $|M| = u_0 + v_0 + w_0 + 1 \le 3m - 2$ and $|M| \equiv |S| \equiv -1 \mod m$ which implies that $|M| = 2m - 1$ and M is a minimal zero-sum sequence. If $u_0 = 0$ then $v_0 = w_0 = m - 1$ and we are done. Assume to the contrary that $u_0 \in [1, m-1]$. The sequence

$$N = e_1^{u_0} \cdot (be_1 + e_2)^{v_0} \cdot e_2^{w_0+1}$$

contains a zero-sum subsequence

$$1 \ne N' = e_1^{u'} \cdot (be_1 + e_2)^{v'} \cdot e_2^{w'}.$$

Since M is a minimal zero-sum subsequence, it follows that $w' = w_0 + 1$. If $v' \ge 1$, then

$$e_1^{u'} \cdot (be_1 + e_2)^{v'-1} \cdot e_2^{w'-1} \cdot (be_1 + 2e_2)$$

is a proper zero-sum subsequence of M, a contradiction. Thus $v' = 0$ whence $\sigma(N') = 0 = u'e_1 + (w_0 + 1)e_2$. This implies that $u' = 0$ and $w_0 = m - 1$. Since M is a zero-sum sequence, it follows that $v_0 = m - 1$ and $|M| = u_0 + v_0 + w_0 + 1 = 2m - 1$ implies that $u_0 = 0$, a contradiction. □

8. If n has Property B, then $2n$ has Property B

It is the aim of this section to prove the following theorem.

Theorem 8.1. *Let $n \in \mathbb{N}$ with $n \ge 6$. If n satisfies Property B, then $2n$ satisfies Property B.*

We start with two lemmata, which rest on Lemmata 3.11 to 3.14. Let $S \in \mathcal{F}(C_{mn} \oplus C_{mn})$, where $m, n \in \mathbb{N}_{\ge 2}$, be a minimal zero-sum sequence with length $|S| = 2mn - 1$. A product decomposition $S = \prod_{\nu=0}^{2n-2} S_\nu$ having the properties described in Lemma 3.14.2

will be called a *canonical product decomposition of* S. If not stated otherwise, we always numerate the sequences in such a way that $|S_0| = 2m-1$ and $|S_1| = \ldots = |S_{2n-2}| = m$.

Lemma 8.2. *Let* $G = C_{mn} \oplus C_{mn}$ *with* $m, n \in \mathbb{N}_{\geq 2}$, $\varphi : G \to G$ *the multiplication by* n *and* $S \in \mathcal{F}(G)$ *a minimal zero-sum sequence with length* $|S| = 2mn-1$. *Suppose that* n *has Property B and let* $S = \prod_{\nu=0}^{2n-2} S_\nu$ *be a canonical product decomposition of* S. *Then* $\prod_{\nu=0}^{2n-2} \sigma(S_\nu)$ *is a minimal zero-sum sequence in* $\ker(\varphi)$ *and there exists a basis* (e_1, e_2) *of* G *such that*

$$\prod_{\nu=0}^{2n-2} \sigma(S_\nu) = (me_1)^{n-1} \cdot \prod_{i=1}^{r} (a_i m e_1 + m e_2)^{t_i}$$

where $r \in [1, n]$, $t_1 \geq \ldots \geq t_r \geq 1$, $\sum_{i=1}^{r} t_i = n$, $a_1, \ldots, a_r \in [0, n-1]$ *and* $\sum_{i=1}^{r} t_i a_i \equiv 1 \mod n$.

Proof. Clearly, $\prod_{\nu=0}^{2n-2} \sigma(S_\nu)$ is a minimal zero-sum sequence in $\ker(\varphi) \cong C_n \oplus C_n$. By Theorem 4.3 there exists a basis (e_1', e_2') of $\ker(\varphi)$ such that

$$\prod_{\nu=0}^{2n-2} \sigma(S_\nu) = {e_1'}^{n-1} \cdot \prod_{i=1}^{r} (a_i e_1' + e_2')^{t_i}$$

where $r \in [1, n]$, $t_1 \geq \ldots \geq t_r \geq 1$, $\sum_{i=1}^{r} t_i = n$, $a_1, \ldots, a_r \in [0, n-1]$ and $\sum_{i=1}^{r} t_i a_i \equiv 1 \mod n$. Thus the assertion follows from Lemma 3.13. □

Lemma 8.3. *Let* $G = C_{mn} \oplus C_{mn}$ *with* $m, n \in \mathbb{N}_{\geq 2}$, $\varphi : G \to G$ *the multiplication by* n *and* $S \in \mathcal{F}(G)$ *a minimal zero-sum sequence with length* $|S| = 2mn-1$. *Suppose that* n *has Property B and let* $S = \prod_{\nu=0}^{2n-2} S_\nu$ *be a canonical product decomposition such that in all decompositions*

$$\prod_{\nu=0}^{2n-2} \sigma(S_\nu) = (me_1)^{n-1} \cdot \prod_{i=1}^{r} (a_i m e_1 + m e_2)^{t_i},$$

derived in Lemma 8.2, t_1 *is minimal possible. Then we have*

1. *If* ($t_1 = n-1$ *and* $n \geq m+3$) *or* ($t_1 \leq n-m-1$), *then for every subsequence* T *of* S *with* $\sigma(T) \in \ker(\varphi)$ *and* $|T| = m$, *we have either* $\sigma(T) = me_1$ *or* $\sigma(T) = ame_1 + me_2$ *for some* $a \in [0, n-1]$.
2. *If* ($t_1 \in [n-m, n-2]$ *and* $n > 3m$), *then there exists some* $\lambda \in [0, 2n-2]$ *such that for every subsequence* T *of* $S_\lambda^{-1}S$ *with* $\sigma(T) \in \ker(\varphi)$ *and* $|T| = m$, *we have either* $\sigma(T) = me_1$ *or* $\sigma(T) = ame_1 + me_2$ *for some* $a \in [0, n-1]$.
3. *If* $n \geq 6$ *and* $m = 2$, *then for every subsequence* T *of* S *with* $\sigma(T) \in \ker(\varphi)$ *and* $|T| = 2$, *we have either* $\sigma(T) = 2e_1$ *or* $\sigma(T) = 2ae_1 + 2e_2$ *for some* $a \in [0, n-1]$.

Proof. Let T be a subsequence of S (resp. of $S_\lambda^{-1}S$ for some $\lambda \in [0, 2n-2]$) with $\sigma(T) \in \ker(\varphi)$ and $|T| = m$. Without restriction we may suppose that $T \notin \{S_1, \ldots, S_{2n-2}\}$. Let $\Gamma_1 \subset [0, 2n-2]$ (resp. $\Gamma_1 \subset [0, 2n-2] \setminus \{\lambda\}$) be a minimal subset such that T divides

$\prod_{i\in\Gamma_1} S_i$. We set $\Gamma_2 = [0, 2n-2]\setminus\Gamma_1$, $W = T^{-1}\prod_{i\in\Gamma_1} S_i$ and $l = |\Gamma_1|$. By the minimality of Γ_1 we obtain that $l = |\Gamma_1| \le |T| = m$. Furthermore, $\varphi(W)$ is a zero-sum sequence with length

$$|W| = \sum_{i\in\Gamma_1} |S_i| - |T| \ge |\Gamma_1| \cdot m - m = (l-1)m.$$

By Lemma 3.11 $W = W_1 \cdot \ldots \cdot W_{l-3} \cdot W'$ where $\varphi(W_1), \ldots, \varphi(W_{l-3})$ are short zero-sum sequences (in case $l \le 3$ we have $W' = W$). Since $S = \prod_{i\in\Gamma_2} S_i \cdot T \cdot W$ and since by Lemma 3.12.1 all short zero-sum sequences of $\varphi(S)$ have length m, $\varphi(W_1), \ldots, \varphi(W_{l-3})$ have length m.

Now we distinguish two cases. Firstly, we suppose that $0 \notin \Gamma_1$. Then $0 \in \Gamma_2$, $|W| = (l-1)m$ and $\varphi(W')$ is a zero-sum sequence of length $2m$. Hence $W' = W_{l-2}W_{l-1}$ where $\varphi(W_{l-2})$ and $\varphi(W_{l-1})$ are zero-sum sequences with length m. Secondly, we suppose that $0 \in \Gamma_1$. Then $|W| = lm - 1$ whence $|W'| = |W| - (l-3)m = 3m - 1$. Thus by Lemma 3.1.2 $W' = W_{l-2}W_{l-1}$ where $\varphi(W_{l-2})$ is a short zero-sum sequence of length m. Since $\varphi(S)$ is not a product of $2n$ zero-sum subsequences, it follows that $\varphi(W_l)$ is a minimal zero-sum sequence of length $2m-1$.

Therefore in both cases

$$S = \prod_{i\in\Gamma_2} S_i \cdot T \cdot \prod_{i=1}^{l-1} W_i$$

is a canonical product decomposition and

$$\overline{S} = \left(\prod_{i\in\Gamma_2} \sigma(S_i)\right) \sigma(T)\sigma(W_1) \cdot \ldots \cdot \sigma(W_{l-1})$$

is a minimal zero-sum sequence in $\ker(\varphi)$.

1. (i) Suppose that $t_1 = n-1$ and $n \ge m+3$. By the minimality of t_1 there are two distinct elements $\alpha, \beta \in \ker(\varphi)$ each occuring exactly $(n-1)$-times in the sequence $\overline{S}$. Assume to the contrary that $\{me_1, a_1me_1 + me_2\} \ne \{\alpha, \beta\}$. If $\gamma \in \{me_1, a_1me_1 + me_2\} \setminus \{\alpha, \beta\}$ then we infer that

$$\begin{aligned} 2n-1 = |\overline{S}| &\ge \mathsf{v}_\gamma(\overline{S}) + \mathsf{v}_\alpha(\overline{S}) + \mathsf{v}_\beta(\overline{S}) \\ &\ge (n-1-|\Gamma_1|) + (n-1) + (n-1) \ge (n-1-m) + (2n-2) \\ &> 2n-1, \end{aligned}$$

a contradiction. Therefore, $\{me_1, a_1me_1 + me_2\} = \{\alpha, \beta\}$ and the assertion follows.

1. (ii) Suppose that $t_1 \le n-m-1$. Then every element distinct to me_1 occurs at most

$$t_1 + (l-1) \le n-m-1 + (l-1) \le n-2$$

times in $\overline{S}$. Since n satisfies Property B, there is some element α occuring $(n-1)$-times in $\overline{S}$ whence $\alpha = me_1$. Thus either $\sigma(T) = me_1$ or, by Proposition 4.1, $\sigma(T) = ame_1 + me_2$ for some $a \in [0, n-1]$.

2. Suppose that $t_1 \in [n-m, n-2]$ and $n > 3m$. First we discuss how to choose a suitable $\lambda \in [0, 2n-2]$. Since $\sum_{j=1}^r t_j a_j \equiv 1 \mod n$, $\sum_{j=1}^r t_j = n$ and $t_1 \le n-2$, it follows that there exists some $j \in [2, r]$ such that $a_j \not\equiv a_1 + 1 \mod n$, say $j = r$. Choose $\lambda \in [0, 2n-2]$ such that $\sigma(S_\lambda) = a_r m e_1 + m e_2$.

Let T be a subsequence of $S_\lambda^{-1} \cdot S$ with $\sigma(T) \in \ker(\varphi)$ and $|T| = m$. Since n satisfies Property B and by the minimality of t_1, there exist two elements α, β such that α occurs $(n-1)$-times and β occurs at least $t_1 \ge n-m$ times in the sequence $\overline{S}$. Assume to the contrary, that $\{\alpha, \beta\} \ne \{me_1, a_1 m e_1 + m e_2\}$. Then we infer that

$$\begin{aligned} 2n-1 = |\overline{S}| &\ge \mathsf{v}_\alpha(\overline{S}) + \mathsf{v}_\beta(\overline{S}) + \min\{\mathsf{v}_{me_1}(\overline{S}), \mathsf{v}_{a_1 m e_1 + m e_2}(\overline{S})\} \\ &\ge (n-1) + (n-m) + \min\{n-1-|\Gamma_1|, t_1 - |\Gamma_1|\} \\ &\ge (n-1) + (n-m) + (n-m-|\Gamma_1|) \\ &\ge (n-1) + (n-m) + (n-2m) \\ &> 2n-1 \end{aligned}$$

a contradiction, since $n > 3m$. Thus we obtain that $\{\alpha, \beta\} = \{me_1, a_1 m e_1 + m e_2\}$. Assume to the contrary that $\alpha = a_1 m e_1 + m e_2$ and $\beta = m e_1$. Since

$$(\alpha, \beta) = (me_1, me_2) \cdot \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}$$

and $\gcd\{a_1 \cdot 0 - 1, n\} = 1$, it follows that $\{\alpha, \beta\}$ is a basis of $\ker(\varphi) \cong C_n \oplus C_n$. Since $\sigma(S_\lambda)$ occurs in $\overline{S}$, Proposition 4.1.2.a implies that there exist $a, b \in [0, n-1]$ with $\gcd\{b, n\} = 1$ and $\sigma(S_\lambda) = a\alpha + b\beta$. Since β occurs in $\overline{S}$, it follows that $b = 1$ and we obtain that

$$a_r m e_1 + m e_2 = \sigma(S_\lambda) = a\alpha + b\beta = a(a_1 m e_1 + m e_2) + m e_1 = (a a_1 + 1) m e_1 + a m e_2$$

whence $a \equiv 1 \mod n$ and $a_r \equiv a_1 + 1 \mod n$, a contradiction. Thus $\alpha = m e_1$ whence Proposition 4.1 implies that $\sigma(T)$ has the required form.

3. Suppose $n \ge 6$ and $m = 2$. If $t_1 = n-1$ or $t_1 \le n-3$, the assertion follows from 1. Suppose that $t_1 = n-2$. Then

$$\prod_{\nu=0}^{2n-2} \sigma(S_\nu) = (2e_1)^{n-1} \cdot (2a_1 e_1 + 2e_2)^{n-2} \cdot (2a_2 e_1 + 2e_2) \cdot (2a_3 e_1 + 2e_2)$$

where $a_1, a_2, a_3 \in [0, n-1]$, $a_2 \ne a_1 \ne a_3$, and

$$\overline{S} = \left(\prod_{i \in \Gamma_2} \sigma(S_i) \right) \sigma(T) \sigma(W_1) = b_0^{n-1} b_1^{t_1'} \cdot B$$

where $b_0, b_1 \in \ker(\varphi)$, $B \in \mathcal{F}(\ker(\varphi))$ with $|B| \le 2$ and $t_1' \ge n-2$. We set $\Gamma_1 = \{\lambda, \mu\}$ whence $T W_1 = S_\lambda S_\mu$.

If $2e_1 \notin \{\sigma(S_\lambda), \sigma(S_\mu)\}$, then $(2e_1)^{n-1} \mid \overline{S}$ and the assertion follows by Proposition 4.1. If $2e_1 = \sigma(S_\lambda) = \sigma(S_\nu)$ and $\sigma(T) \ne 2e_1$, then $\sigma(W_1) \ne 2e_1$, $\mathsf{v}_{2e_1}(\overline{S}) = n-3 \ge 3$ whence $b_1 = 2e_1$, a contradiction to $t_1' \ge n-2$. If, say, $\sigma(S_\lambda) = 2e_1$, $\sigma(S_\mu) = 2a_i e_1 + 2e_2$ for some $i \in [1, 3]$ and $\sigma(T) \notin \{2e_1, 2ae_1 + 2e_2\}$ for some $a \in [0, n-1]$, then $\sigma(W_1) \notin$

$\{2e_1, 2ae_1+2e_2\}$ for any $a \in [0, n-1]$ whence $\mathsf{v}_{2e_1}(\overline{S}) = n-2$, $n-3 \le \mathsf{v}_{2a_1e_1+2e_2}(\overline{S}) \le n-2$, a contradiction to $\max\{\mathsf{v}_g(\overline{S}) \mid g \in \ker(\varphi)\} = n-1$. $\square$

Proof of Theorem 8.1. Let $G = C_{2n} \oplus C_{2n}$ with $n \ge 6$ and suppose that n satisfies Property B. Let $S \in \mathcal{F}(G)$ be a minimal zero-sum sequence with length $|S| = 4n-1$. We have to show that S contains some element with multiplicity $2n-1$.

Let $\varphi : G \to G$ denote the multiplication by n. By Lemmata 3.14 and 8.2 (with $m = 2$) S has a canonical product decomposition $S = \prod_{\nu=0}^{2n-2} S_\nu$ where $|S_0| = 3$, $|S_1| = \ldots = |S_{2n-2}| = 2$, and there exists a basis (f_1, f_2) of G such that

$$\prod_{\nu=0}^{2n-2} \sigma(S_\nu) = (2f_1)^{n-1} \cdot \prod_{i=1}^{r} (a_i 2f_1 + 2f_2)^{t_i} \quad \in \ \mathcal{F}(\ \ker(\varphi)\ )$$

where $r \in [1, n]$, $t_1 \ge \ldots \ge t_r \ge 1$, $\sum_{i=1}^{r} t_i = n$, $a_1, \ldots, a_r \in [0, n-1]$ and $\sum_{i=1}^{r} t_i a_i \equiv 1 \mod n$. Suppose that t_1 is minimal possible under all decompositions of this type.

Let (e_1, e_2) be any basis of G such that $2e_1 = 2f_1$ and $2e_2 \in 2f_2 + \langle 2f_1 \rangle$. A basis having these properties will be called suitable. For $i \in [1, 2]$ we denote by $\mathsf{p}_i : G = \langle e_1 \rangle \oplus \langle e_2 \rangle \to \langle e_i \rangle$ the canonical projection, and we set $nG = \{0, \alpha, \beta, \gamma\} \cong C_2 \oplus C_2$.

By Lemma 3.14.1 we have $0 \notin \operatorname{supp}(\varphi(S))$ whence S has the form $S = S_\alpha S_\beta S_\gamma$ where $\varphi(S_\delta) = \delta^{|S_\delta|}$ for every $\delta \in \{\alpha, \beta, \gamma\}$. Clearly, if $\nu \in [1, 2n-2]$, then S_ν divides S_δ for some $\delta \in \{\alpha, \beta, \gamma\}$, $\varphi(S_0) = \alpha\beta\gamma$ and $|S_\delta| \equiv 1 \mod 2$ for every $\delta \in \{\alpha, \beta, \gamma\}$. Let $k, l, m \in \mathbb{N}_0$ such that $|S_\alpha| = 2k+1$, $|S_\beta| = 2l+1$ and $|S_\gamma| = 2m+1$.

Lemma 8.3.3 implies that for every subsequence T of S with $\sigma(T) \in \ker(\varphi)$ and $|T| = 2$ we have

(1) $\qquad$ either $\quad \sigma(T) = 2e_1 \quad$ or $\quad \sigma(T) = 2ae_1 + 2e_2 \quad$ for some $a \in [0, n-1]$.

Let $\delta \in \{\alpha, \beta, \gamma\}$ and $S_\delta = \prod_{i=1}^{|S_\delta|} (x_i e_1 + u_i e_2)$ with all $x_i, u_i \in [0, 2n-1]$. We assert that

(2) $$|\{u_i \mid i \in [1, |S_\delta|]\}| = 2.$$

Assume to the contrary, that (2) does not hold. Then we may suppose without restriction that $|\{u_1, u_2, u_3\}| = 3$. Then u_1+u_2, u_1+u_3 and u_2+u_3 are pairwise distinct. However, (1) implies that $u_1+u_2+2n\mathbb{Z}, u_1+u_3+2n\mathbb{Z}, u_2+u_3+2n\mathbb{Z} \in \{2n\mathbb{Z}, 2+2n\mathbb{Z}\}$, a contradiction.

Therefore we obtain that

$$S_\alpha = \prod_{i=1}^{k_1}(x_ie_1+ue_2)\prod_{i=1}^{k_2}(x_{k_1+i}e_1+u'e_2) \quad \text{where} \quad k_1 \ge k_2 \ge 0,\ k_1+k_2=2k+1,$$
$$S_\beta = \prod_{i=1}^{l_1}(y_ie_1+ve_2)\prod_{i=1}^{l_2}(y_{l_1+i}e_1+v'e_2) \quad \text{where} \quad l_1 \ge l_2 \ge 0,\ l_1+l_2=2l+1,$$
$$S_\gamma = \prod_{i=1}^{m_1}(z_ie_1+we_2)\prod_{i=1}^{m_2}(z_{m_1+i}e_1+w'e_2) \quad \text{where} \quad m_1 \ge m_2 \ge 0,\ m_1+m_2=2m+1,$$

and all $x_i, y_i, z_i, u, u', v, v', w, w' \in [0, 2n-1]$. Obviously, k_1, l_1 and m_1 are non-zero.

We assert that

$$k_2, l_2, m_2 \in \{0,1\}. \tag{3}$$

Assume to the contrary that $k_2 \ge 2$. Then $k_1 \ge k_2 \ge 2$ and (1) implies that $2u + 2n\mathbb{Z}, 2u' + 2n\mathbb{Z} \in \{2n\mathbb{Z}, 2+2n\mathbb{Z}\}$ whence $u, u' \in \{0, 1, n, n+1\}$. Since $u \ne u'$, it follows that $u + u' \in \{1, n, n+1, n+2, 2n+1\}$ whence $u + u' + 2n\mathbb{Z} \notin \{2n\mathbb{Z}, 2+2n\mathbb{Z}\}$, a contradiction to (1). Similarly, we argue for l_2 and m_2.

Assume to the contrary, that at least two elements of $\{k_1, l_1, m_1\}$ are equal to 1, say $l_1 = m_1 = 1$. This implies that $l_2 = m_2 = 0$, $k_1 = 4n - 1 - (k_2 + l_1 + l_2 + m_1 + m_2) \ge 4n-4 \ge 2$, and by (1) we have $2u+2n\mathbb{Z} \in \{2n\mathbb{Z}, 2+2n\mathbb{Z}\}$. The number of $\nu \in [0, 2n-2]$ for which $\mathsf{p}_2(S_\nu) \ne (ue_2)^2$ is at most two. If $2u \equiv 0 \mod 2n$, then the multiplicity of $2e_1$ in the sequence $\prod_{\nu=0}^{2n-2} \sigma(S_\nu)$ is at least $(2n-1)-2 > n-1$, a contradiction. If $2u \equiv 2 \mod 2n$, then the multiplicity of $2e_1$ in the sequence $\prod_{\nu=0}^{2n-2} \sigma(S_\nu)$ is at most two, a contradiction.

Next we assert that

$$2n\mathbb{Z} \in \{2u+2n\mathbb{Z}, 2v+2n\mathbb{Z}, 2w+2n\mathbb{Z}\} \ne \{2n\mathbb{Z}\}. \tag{4}$$

Since for every $\nu \in [1, 2n-2]$ S_ν divides S_δ for some $\delta \in \{\alpha, \beta, \gamma\}$ and because of (3) , the number of $\nu \in [0, 2n-2]$ for which $\mathsf{p}_2(S_\nu) \notin \{(ue_2)^2, (ve_2)^2, (we_2)^2\}$ is at most four. Since the number of $\nu \in [0, 2n-2]$ for which $\sigma(S_\nu) = 2e_1$ equals to $n - 1 \ge 5$, it follows that $2n\mathbb{Z} \in \{2u+2n\mathbb{Z}, 2v+2n\mathbb{Z}, 2w+2n\mathbb{Z}\}$.

If $2u \equiv 2v \equiv 2w \equiv 0 \mod 2n$, then the number of $\nu \in [0, 2n-2]$ for which $\sigma(\mathsf{p}_2(S_\nu)) = 2e_2$, is at most four, whence $n \le 4$ a contradiction.

Thus (4) holds and (1) implies the following facts: ($k_1 \ge 2 \Rightarrow 2u + 2n\mathbb{Z} \in \{2n\mathbb{Z}, 2+2n\mathbb{Z}\}$), ($l_1 \ge 2 \Rightarrow 2v + 2n\mathbb{Z} \in \{2n\mathbb{Z}, 2+2n\mathbb{Z}\}$) and ($m_1 \ge 2 \Rightarrow 2w + 2n\mathbb{Z} \in \{2n\mathbb{Z}, 2+2n\mathbb{Z}\}$). Assume to the contrary, that $k_1 \ge 2, l_1 \ge 2, m_1 \ge 2$ and that exactly two of the values $2u, 2v, 2w$ are congruent to zero modulo $2n$, say $2u \equiv 2v \equiv 0 \mod 2n$ and $2w \equiv 2$

mod $2n$. Since, by (1),

$$\begin{aligned} k_2 = 0 \quad &\text{or} \quad (k_2 = 1 \text{ and } u + u' \equiv 2 \mod 2n) \\ l_2 = 0 \quad &\text{or} \quad (l_2 = 1 \text{ and } v + v' \equiv 2 \mod 2n) \\ m_2 = 0 \quad &\text{or} \quad (m_2 = 1 \text{ and } w + w' \equiv 0 \mod 2n) \end{aligned}$$

it follows that

$$\begin{aligned} k_2 = 0 \quad &\text{or} \quad (2u' \equiv 4 \mod 2n) \\ l_2 = 0 \quad &\text{or} \quad (2v' \equiv 4 \mod 2n) \\ m_2 = 0 \quad &\text{or} \quad (2w' \equiv -2 \mod 2n) \end{aligned}$$

whence $2\overline{u}+2\overline{v}+2\overline{w}+2n\mathbb{Z} \in \{0,4\}+\{0,4\}+\{2,-2\}+2n\mathbb{Z} = \{2,6,10,-2\}+2n\mathbb{Z}$ where $\overline{u} \in \{u,u'\}, \overline{v} \in \{v,v'\}$ and $\overline{w} \in \{w,w'\}$. Therefore $2\sigma(\mathsf{p}_2(S_0)) \in \{2,6,10,-2\}e_2$. On the other hand we have $\sigma(S_0) \in \{2f_1, a_i 2f_1 + 2f_2 \mid i \in [1,r]\}$ whence $\sigma(\mathsf{p}_2(S_0)) \in \{0, 2e_2\}$ and thus $\{0,4\} + 2n\mathbb{Z} \cap \{2,6,10,-2\} + 2n\mathbb{Z} \neq \emptyset$, a contradiction to $2n \geq 12$.

Assume to the contrary that $1 \in \{k_1, l_1, m_1\}$, say $m_1 = 1$, and $2u \equiv 2v \mod 2n$. If $2u \equiv 2v \equiv 2 \mod 2n$, then the number of $\nu \in [0, 2n-2]$ with $\sigma(S_\nu) = 2e_1$ is at most three, a contradiction. If $2u \equiv 2v \equiv 0 \mod 2n$, then the number of $\nu \in [0, 2n-2]$ for which $\sigma(\mathsf{p}_2(S_\nu)) = 2e_2$ is at most four, a contradiction.

All these considerations show that we may suppose without restriction that $k_1 \geq 2$, $l_1 \geq 2$, $2u \equiv 0 \mod 2n$, $2v \equiv 2 \mod 2n$ and (either $2w \equiv 2 \mod 2n$ or $m_1 = 1$).

Our next aim is to choose a special suitable basis $(\widetilde{e_1}, \widetilde{e_2})$. The number of $\nu \in [0, 2n-2]$ with $\mathsf{p}_2(S_\nu) \neq (ue_2)^2$ but $\sigma(\mathsf{p}_2(S_\nu)) = 0$ is at most three whence $k_1 \geq 2(n-1-3) = 2n-8 \geq 4$. Since $2u \equiv 0 \mod 2n$, (1) implies that $x_i + x_j \equiv 2 \mod 2n$ for each two distinct $i, j \in [1, k_1]$. This implies that $x_1 = \ldots = x_{k_1} = x \in [0, 2n-1]$. Lemma 3.8.1 implies that $2n = \operatorname{ord}(xe_1 + ue_2)$ whence $\gcd\{x, u, 2n\} = 1$. Since $(2xe_1)$ occurs in the sequence $\prod_{i=0}^{2n-2} \sigma(S_\nu)$, it follows that $2xe_1 = 2e_1$ whence $x \in \{1, n+1\}$.

If $u = 0$, then

$$(\widetilde{e_1}, \widetilde{e_2}) = (xe_1, e_2) = (e_1, e_2) \cdot \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

is a basis of G with $2\widetilde{e_i} = 2e_i$ for $i \in [1,2]$. Suppose $u = n$. Then $\gcd\{x, n\} = 1$ whence there are $x', n' \in \mathbb{Z}$ such that $xx' - nn' = 1$ and

$$(\widetilde{e_1}, \widetilde{e_2}) = (xe_1 + ne_2, n'e_1 + x'e_2) = (e_1, e_2) \cdot \begin{pmatrix} x & n' \\ n & x' \end{pmatrix}$$

is a basis of G with $2\widetilde{e_1} = 2e_1 = 2f_1$ and $2\widetilde{e_2} \in 2e_2 + \langle 2e_1 \rangle \in 2f_2 + \langle 2f_1 \rangle$.

Thus $(\widetilde{e_1}, \widetilde{e_2})$ is a suitable basis and we may write all elements of $S = S_\alpha S_\beta S_\gamma$ with this new basis. We get new coordinates $\widetilde{x}_i, \widetilde{y}_i, \widetilde{z}_i, \widetilde{u} = 0, \widetilde{u'}, \widetilde{v}, \widetilde{v'}, \widetilde{w}$ and $\widetilde{w'}$. For simplicity of notation we omit all $\widetilde{*}$, write (e_1, e_2) instead of $(\widetilde{e_1}, \widetilde{e_2})$ and so on. In new notation we obtain that

$$S = e_1^{k_1} \cdot \prod_{i=1}^{k_2} (x_{k_1+i} e_1 + u' e_2) \cdot S_\beta \cdot S_\gamma. \tag{5}$$

We distinguish the cases $m_1 \geq 2$ and $m_1 = 1$.

Case 1: $k_1 \geq 2, l_1 \geq 2, m_1 \geq 2$. Without restriction we suppose that $l_2 \geq m_2$. Recall that $u = 0$ and $2v \equiv 2w \equiv 2 \mod 2n$ whence $v, w \in \{1, n+1\}$.

We assert that

$$v = w. \tag{6}$$

Assume to the contrary that $v \neq w$. Since $2v \equiv 2w \equiv 2 \mod 2n$, it follows that $\{v, w\} = \{1, n+1\}$. Since $u' \neq u = 0$, $v \neq v'$, $w \neq w'$, (1) implies that ($k_2 = 0$ or $u' = 2$), ($l_2 = 0$ or $v + v' \equiv 0 \mod 2n$) and ($m_2 = 0$ or $w + w' \equiv 0 \mod 2n$). Thus if $\overline{u} \in \{u, u'\}$, $\overline{v} \in \{v, v'\}$ and $\overline{w} \in \{w, w'\}$, then $\overline{u} + \overline{v} + \overline{w} \in \{0, 2\} + \overline{v} + \overline{w} \in \{0, 2\} + \{v+w, v+w', v'+w, v'+w'\} = \{0, 2\} + \{n+2, n, n-2\} = \{n-2, n, n+2, n+4\}$. Thus $\sigma(\mathsf{p}_2(S_0)) \in \{n-2, n, n+2, n+4\}e_2$. On the other hand we have $\sigma(\mathsf{p}_2(S_0)) \in \{0, 2e_2\}$ whence $\{0, 2\} + 2n\mathbb{Z} \cap \{n-2, n, n+2, n+4\} + 2n\mathbb{Z} \neq \emptyset$, a contradiction to $n \geq 6$.

We distinguish six cases.

Case 1.1: $k_2 = l_2 = m_2 = 0$. Then $l_1 + m_1$ is even.

We have

$$S = e_1^{k_1} \cdot \prod_{i=1}^{l_1} (y_i e_1 + v e_2) \cdot \prod_{i=1}^{m_1} (z_i e_1 + v e_2).$$

Since S is a zero-sum sequence, it follows that $(l_1 + m_1)v \equiv 0 \mod 2n$ whence $l_1 + m_1 \equiv 0 \mod 2n$ and $l_1 + m_1 = 2n$. Thus $k_1 = 2n - 1$ and the assertion is proved.

Case 1.2: $k_2 = 0, l_2 = m_2 = 1$. Then $l_1 + m_1$ is even.

Since $v \neq v'$, $w \neq w'$ and $2v \equiv 2w \equiv 2 \mod 2n$, (1) implies that $v + v' \equiv w + w' \equiv 0 \mod 2n$. Since $v = w$, we infer that either

$$(v = w = 1 \text{ and } v' = w' = 2n-1) \quad \text{or} \quad (v = w = n+1 \text{ and } v' = w' = n-1).$$

Since S is a zero-sum sequence, we have $l_1 v + m_1 w + v' + w' \equiv 0 \mod 2n$. Therefore we obtain that $(l_1 + m_1)v - 2 \equiv 0 \mod 2n$, $l_1 + m_1 \equiv 2 \mod 2n$ and $l_1 + m_1 \in \{2, 2n+2\}$.

Since $k_1 = 4n-1-(l_1+m_1+l_2+m_2)$, we have $l_1+m_1 = 2n+2$ and $k_1 = 2n-5$. Therefore we obtain that either

$$S = e_1^{2n-5} \prod_{i=1}^{2n+2} (y_i e_1 + e_2) \cdot (d_1 e_1 - e_2) \cdot (d_2 e_1 - e_2)$$

or

$$S = e_1^{2n-5} \prod_{i=1}^{2n+2} (y_i e_1 + (n+1)e_2) \cdot (d_1 e_1 + (n-1)e_2) \cdot (d_2 e_1 + (n-1)e_2)$$

where in both cases $d_1 = y_{l_1+1}$ and $d_2 = z_{m_1+1} \in [0, 2n-1]$ whence $d_1 \neq d_2$.

We consider the first case. If T is a non-empty proper subsequence of $\prod_{i=1}^{2n+2}(y_i e_1 + e_2) \cdot (d_1 e_1 - e_2) \cdot (d_2 e_1 - e_2)$ such that $\sigma(\mathsf{p}_2(T)) = 0$, then $\sigma(\mathsf{p}_1(T)) \in \{1,2,3,4\}e_1$. This implies that for every $i \in \{1,2\}$ and every $j \in [1, 2n+2]$ we have $d_i + y_j + 2n\mathbb{Z} \in \{1,2,3,4\} + 2n\mathbb{Z}$. Since $d_1 + d_2 + y_{j'} + y_j + 2n\mathbb{Z} \in \{1,2,3,4\} + 2n\mathbb{Z}$ and $n \geq 6$, it follows that $d_i + y_j + 2n\mathbb{Z} \in \{1,2,3\} + 2n\mathbb{Z}$. If $d_1 + y_j \equiv 3 \mod 2n$, then $d_2 + y_i \equiv 1 \mod 2n$ for all $i \in [1, 2n+2] \setminus \{j\}$ whence $y_1 = \ldots = y_{j-1} = y_{j+1} = \ldots = y_{2n+2}$ and $(y_1 e_1 + e_2)^{2n}$ is a zero-sum subsequence of S, a contradiction. The same

argument works for $d_2 + y_j$.

Thus $d_i + y_j + 2n\mathbb{Z} \in \{1,2\} + 2n\mathbb{Z}$ for all $i \in [1,2]$ and all $j \in [1, 2n+2]$. This implies that $|\{y_1, \ldots, y_{2n+2}\}| \leq 2$, say $\prod_{i=1}^{2n+2} y_i = y_1^{h_1} y_2^{h_2}$ with $h_1 \geq h_2 \geq 0$. Since S is a minimal zero-sum sequence, it follows that $h_2 \geq 3$. After a suitable renumeration we may suppose that $d_1 + y_1 \equiv 1 \mod 2n$. Then it follows that $d_2 + y_1 \equiv 2 \mod 2n$, $d_1 + y_2 \equiv 2 \mod 2n$ and $d_2 + y_2 \equiv 1 \mod 2n$ whence $2y_1 \equiv 3 - d_1 - d_2 \equiv 2y_2 \mod 2n$. For $i \in [1,2]$ we choose even $h_i' \in [0, h_i]$ with $h_1' + h_2' = 2n$. Then

$$h_1' y_1 + h_2' y_2 \equiv h_1' y_1 + \frac{h_2'}{2}(2y_1) \equiv y_1(h_1' + h_2') \equiv 0 \mod 2n$$

whence $(y_1 e_1 + e_2)^{h_1'} \cdot (y_2 e_1 + e_2)^{h_2'}$ is a zero-sum subsequence of S, a contradiction.

Arguing in a similar way in the second case we obtain again a contradiction.

Case 1.3: $k_2 = 0, l_2 = 1, m_2 = 0$. Then $l_1 + m_1$ is odd.

As in Case 1.2 we have $v \neq v', 2v \equiv 2 \mod 2n$ and $v + v' \equiv 0 \mod 2n$. Since S is a zero-sum sequence, we have $0 \equiv l_1 v + m_1 w + v' \equiv (l_1 + m_1)v - v \mod 2n$ whence $l_1 + m_1 \equiv 1 \mod 2n$ and thus $l_1 + m_1 = 2n+1$. Therefore we obtain that either

$$S = e_1^{2n-3} \prod_{i=1}^{2n+1} (y_i e_1 + e_2) \cdot (d e_1 - e_2)$$

or

$$S = e_1^{2n-3} \prod_{i=1}^{2n+1} (y_i e_1 + (n+1)e_2) \cdot (d e_1 + (n-1)e_2)$$

where $d = y_{l_1+1} \in [0, 2n-1]$.

We consider the first case. If T is a non-empty proper subsequence of $\prod_{i=1}^{2n+1}(y_ie_1 + e_2) \cdot (de_1 - e_2)$ such that $\sigma(\mathsf{p}_2(T)) = 0$, then $\sigma(\mathsf{p}_1(T)) \in \{1, 2\}e_1$. Thus $d + y_i + 2n\mathbb{Z} \in \{1, 2\} + 2n\mathbb{Z}$ for every $i \in [1, 2n+1]$. This implies that $|\{y_1, \ldots , y_{2n+1}\}| = 2$, and we set $\prod_{i=1}^{2n+1} y_i = y_1^{h_1} y_2^{h_2}$ with $h_1 + h_2 = 2n + 1$. Since S is a minimal zero-sum sequence, it follows that $h_1, h_2 \in [2, 2n-1]$. After a suitable renumeration we may suppose that $d + y_1 \equiv 1 \mod 2n$, and clearly we have $h_1y_1 + h_2y_2 + d \equiv 3 \mod 2n$. Therefore, $d + y_2 \equiv 2 \mod 2n$, $h_1y_1 + h_2y_2 - y_1 \equiv 2 \mod 2n$, $h_1y_1 + h_2y_2 - y_2 \equiv 1 \mod 2n$, $y_2 - y_1 \equiv 1 \mod 2n$, $h_1y_1 + (2n - h_1)y_2 \equiv 1 \mod 2n$, $h_1(y_1 - y_2) \equiv 1 \mod 2n$ whence $h_1 \equiv -1 \mod 2n$. This implies that $h_1 = 2n - 1$ and the assertion is proved.

Arguing in a similar way in the second case we obtain again the assertion.

Case 1.4: $k_2 = l_2 = m_2 = 1$. Then $l_1 + m_1$ is even.

Since $0 = u \neq u'$ and ($u + u' \equiv 0$ or $u + u' \equiv 2 \mod 2n$), it follows that $u' = 2$. As in Case 1.2 we infer that either

$$(v = w = 1 \text{ and } v' = w' = 2n-1) \quad \text{or} \quad (v = w = n+1 \text{ and } v' = w' = n-1).$$

Since S is a zero-sum sequence, we have $l_1v+m_1w+u'+v'+w' \equiv 0 \mod 2n$, $(l_1+m_1)v \equiv 0 \mod 2n$ and $l_1 + m_1 = 2n$. Thus $k_1 = 4n - 1 - (l_1 + m_1 + k_2 + l_2 + m_2) = 2n - 4$ and

$$S = e_1^{2n-4} \prod_{i=1}^{2n}(y_ie_1 + ve_2) \cdot (d_1e_1 - ve_2) \cdot (d_2e_1 - ve_2) \cdot (d_3e_1 + 2e_2)$$

where $d_1, d_2, d_3 \in [0, 2n-1]$ and $d_1 \neq d_2$. Arguing as in Case 1.2 we obtain a contradiction.

Case 1.5: $k_2 = l_2 = 1, m_2 = 0$. Then $l_1 + m_1$ is odd.

As in Case 1.4 we conclude that $u' = 2$, $v + v' \equiv 0 \mod 2n$ and either

$$v = w = 1 \quad \text{or} \quad v = w = n + 1.$$

Since S is a zero-sum sequence, we have $u' + l_1v + m_1w + v' = 2 + (l_1 + m_1 - 1)v \equiv 0 \mod 2n$ whence $l_1 + m_1 = 2n - 1$ and

$$S = e_1^{2n-2} \prod_{i=1}^{2n-1} (y_ie_1 + ve_2) \cdot (d_1e_1 - ve_2) \cdot (d_2e_1 + 2e_2)$$

where $d_1 = x_{k_1+1}$ and $d_2 = y_{l_1+1} \in [0, 2n-1]$. For every $i \in [1, 2n-1]$ we have $d_1 + y_i \equiv 1 \mod 2n$. This implies that $y_1 = \ldots = y_{2n-1}$, and the assertion follows.

Case 1.6: $k_2 = 1, l_2 = m_2 = 0$. Then $l_1 + m_1$ is even.

As in Case 1.4. we conclude that $u' = 2$. Since S is a zero-sum sequence, we infer that $(l_1 + m_1)v + 2 \equiv 0 \mod 2n$ whence $l_1 + m_1 = 2n - 2$. This implies that $k_1 = 2n$, a contradiction.

Case 2: $k_1 \geq 2, l_1 \geq 2, m_1 = 1$.

Since $m_1 \geq m_2$ and $m_1 + m_2$ is odd, it follows that $m_2 = 0$. Recall that $l_1 + l_2$ is odd and that $2v \equiv 2 \mod 2n$ whence $v \in \{1, n+1\}$. We distinguish four cases.

Case 2.1: $k_2 = l_2 = 0$. Then l_1 is odd.

We have

$$S = e_1^{k_1} \cdot \prod_{i=1}^{l_1} (y_i e_1 + v e_2) \cdot (z_1 e_1 + w e_2),$$

$\mathsf{p}_2(S_0) = 0 \cdot (ve_2) \cdot (we_2)$, $\sigma(\mathsf{p}_2(S_0)) \in \{0, 2e_2\}$, and since S is a zero-sum sequence, we infer that $l_1 v + w \equiv 0 \mod 2n$.

Firstly, we suppose that $\sigma(\mathsf{p}_2(S_0)) = 2e_2$. Then $v + w \equiv 2 \mod 2n$ and $(l_1 - 1)v + 2 \equiv 0 \mod 2n$. Thus it follows that $l_1 + 1 \equiv 0 \mod 2n$ whence $l_1 = 2n - 1$. This implies that $k_1 = 2n - 1$ and the assertion is proved.

Secondly, we suppose that $\sigma(\mathsf{p}_2(S_0)) = 0$. Then $v + w \equiv 0 \mod 2n$, $(l_1 - 1)v \equiv 0 \mod 2n$ whence $l_1 = 2n + 1$ and $k_1 = 2n - 3$. Then $z_1 e_1 + y_i e_1 \in \{e_1, 2e_1\}$ for all $i \in [1, 2n+1]$ and, after renumeration, $\prod_{i=1}^{2n+1} y_i = y_1^{h_1} y_2^{h_2}$ with $h_1, h_2 \in [2, 2n-1]$ and $h_1 + h_2 = 2n + 1$. Without restriction we suppose that $z_1 + y_1 \equiv 1 \mod 2n$. Then $z_1 + y_2 \equiv 2 \mod 2n$, $h_1 y_1 + h_2 y_2 - y_1 \equiv 2 \mod 2n$, $h_1 y_1 + h_2 y_2 - y_2 \equiv 1 \mod 2n$, $y_2 - y_1 \equiv 1 \mod 2n$, $h_1 y_1 + (2n - h_1) y_2 \equiv 1 \mod 2n$ and $h_1(y_1 - y_2) \equiv 1 \mod 2n$. Thus $h_1 \equiv -1 \mod 2n$, $h_1 = 2n - 1$ and the assertion is proved.

Case 2.2: $k_2 = 0$ and $l_2 = 1$. Then l_1 is even.

Then $v + v' \equiv 0 \mod 2n$, and we have either $\mathsf{p}_2(S_0) = 0 \cdot (ve_2) \cdot (we_2)$ or $\mathsf{p}_2(S_0) = 0 \cdot (v'e_2) \cdot (we_2)$. Since S is a zero-sum sequence, we have $0 \equiv k_1 u + l_1 v + v' + w \equiv (l_1 - 1)v + w \mod 2n$.

Case 2.2.1: $\mathsf{p}_2(S_0) = 0 \cdot (ve_2) \cdot (we_2)$. Then $v + w + 2n\mathbb{Z} \in \{2n\mathbb{Z}, 2 + 2n\mathbb{Z}\}$.

Firstly, we suppose that $v + w \equiv 0 \mod 2n$. Then $0 \equiv (l_1 - 2)v \mod 2n$, $l_1 - 2 \equiv 0 \mod 2n$ whence $l_1 = 2n + 2$. Therefore

$$S = e_1^{2n-5} \cdot \prod_{i=1}^{2n+2} (y_i e_1 + v e_2) \cdot (y_{2n+3} e_1 - v e_2) \cdot (z_1 e_1 - v e_2).$$

Since $y_{2n+3} \neq z_1$, we may argue as in Case 1.2 and obtain a contradiction.

Secondly, we suppose that $v + w \equiv 2 \mod 2n$. Thus $v + w \equiv 2 \equiv 2v \mod 2n$ whence $v = w$. Thus we obtain that $0 \equiv l_1 v \mod 2n$ and $l_1 = 2n$. Therefore

$$S = e_1^{2n-3} \cdot \prod_{i=1}^{2n} (y_i e_1 + v e_2) \cdot (y_{2n+1} e_1 - v e_2) \cdot (z_1 e_1 + v e_2).$$

Thus S has the same form as in the second part of Case 2.1 and the assertion follows.

Case 2.2.2: $\mathsf{p}_2(S_0) = 0 \cdot (v'e_2) \cdot (we_2)$. Then $v' + w + 2n\mathbb{Z} \in \{2n\mathbb{Z}, 2 + 2n\mathbb{Z}\}$.

Firstly, we suppose that $v' + w \equiv 0 \mod 2n$. Since $v + v' \equiv 0 \mod 2n$, we obtain $v = w$, $l_1 v \equiv 0 \mod 2n$ and $l_1 = 2n$. Thus we come to a situation which we have already discussed.

Secondly, we suppose that $v' + w \equiv 2 \mod 2n$. Thus $2v \equiv 2 \equiv w - v \mod 2n$, $w \equiv 3v \mod 2n$, $0 \equiv (l_1+2)v \mod 2n$, $l_1 = 2n-2$ which implies $k_1 = 4n-1-(l_1+l_2+m_1+m_2) = 4n-1-(2n-2+1+1) = 2n-1$ and the assertion is proved.

Case 2.3: $k_2 = 1$ and $l_2 = 0$. Then l_1 is odd.

Since $0 = u \neq u'$ and ($u+u' \equiv 0$ or $u+u' \equiv 2 \mod 2n$), it follows that $u' = 2$. Since S is a zero-sum sequence and $2v \equiv 2 \mod 2n$, we infer that $0 \equiv u' + l_1 v + w \equiv (l_1+2)v + w \mod 2n$.

Case 2.3.1: $\mathsf{p}_2(S_0) = 0 \cdot (ve_2) \cdot (we_2)$. Then $v + w + 2n\mathbb{Z} \in \{2n\mathbb{Z}, 2 + 2n\mathbb{Z}\}$.

Firstly, we suppose that $v + w \equiv 0 \mod 2n$. Then $0 \equiv (l_1+1)v \mod 2n$, $0 \equiv l_1 + 1 \mod 2n$ and $l_1 = 2n-1$. Then $k_1 = 2n-2$ and

$$S = e_1^{2n-2} \cdot (x_{k_1+1}e_1 + 2e_2) \cdot \prod_{i=1}^{2n-1} (y_i e_1 + ve_2) \cdot (z_1 - ve_2).$$

Since S is a minimal zero-sum sequence, it follows that $z_1 + y_i \equiv 1 \mod 2n$ for every $i \in [1, 2n-1]$ whence $y_1 = \ldots = y_{2n-1}$ and the assertion is proved.

Secondly, we suppose that $v + w \equiv 2 \mod 2n$. Thus $v + w \equiv 2 \equiv 2v \mod 2n$ whence $v = w$, and we obtain that $(l_1+3)v \equiv 0 \mod 2n$, $l_1 + 3 \equiv 0 \mod 2n$ and $l_1 = 2n-3$. Then $k_1 = 4n-1-(k_2+l_1+l_2+m_1+m_2) = 2n$, a contradiction.

Case 2.3.2: $\mathsf{p}_2(S_0) = (2e_2) \cdot (ve_2) \cdot (we_2)$. Then $2 + v + w + 2n\mathbb{Z} \in \{2n\mathbb{Z}, 2 + 2n\mathbb{Z}\}$.

Firstly, we suppose that $2+v+w \equiv 2 \mod 2n$. Then $(l_1+1)v \equiv 0 \mod 2n$, $l_1+1 \equiv 0 \mod 2n$ and $l_1 = 2n-1$. Then $k_1 = 2n-2$ and

$$S = e_1^{2n-2} \cdot (x_{k_1+1}e_1 + 2e_2) \cdot \prod_{i=1}^{2n-1} (y_i e_1 + ve_2) \cdot (z_1 - ve_2).$$

Now the assertion follows as in Case 2.3.1.

Secondly, we suppose that $2+v+w \equiv 0 \mod 2n$. Then $w \equiv -3v \mod 2n$, $(l_1-1)v \equiv 0 \mod 2n$ and $l_1 = 2n+1$. Then $k_1 = 2n-4$ and

$$S = e_1^{2n-4} \cdot (x_{k_1+1}e_1 + 2e_2) \cdot \prod_{i=1}^{2n+1} (y_i e_1 + ve_2) \cdot (z_1 e_1 - 3ve_2).$$

Since S is a minimal zero-sum sequence, it follows that $y_i + x_{k_1+1} + z_1 + 2n\mathbb{Z} \in \{1,2,3\} + 2n\mathbb{Z}$ for every $i \in [1, 2n+1]$ whence $|\{y_1, \ldots, y_{2n+1}\}| \le 3$. Since S is a minimal zero-sum sequence, it follows that $|\{y_1, \ldots, y_{2n+1}\}| > 1$.

Suppose that $|\{y_1, \ldots, y_{2n+1}\}| = 2$, say $\prod_{i=1}^{2n+1} y_i = y_1^{h_1} y_2^{h_2}$ with $h_1, h_2 \in [1, 2n]$ and $h_1 + h_2 = 2n+1$. Since S is a minimal zero-sum sequence, we have $h_1, h_2 \in [2, 2n-1]$. If $\{h_1, h_2\} = \{2, 2n-1\}$, then we are done. Assume to the contrary that $h_1, h_2 \in [3, 2n-2]$. Since $z_1 + 3y_1, z_1 + 2y_1 + y_2, z_1 + y_1 + 2y_2, z_1 + 3y_2$ are congruent to $1, 2$ or 3 modulo $2n$, it follows that either $2y_1 \equiv 2y_2 \mod 2n$ or $3y_1 \equiv 3y_2 \mod 2n$. On the other hand, we have distinct $i, j \in \{1, 2\}$ such that $y_i - y_j + 2n\mathbb{Z} = (y_i + x_{k_1+1} + z_1) - (y_j + x_{k_1+1} + z_1) + 2n\mathbb{Z} \in \{1, 2\} + 2n\mathbb{Z}$, a contradiction.

Suppose that $|\{y_1, \ldots, y_{2n+1}\}| = 3$, say $\prod_{i=1}^{2n+1} y_i = y_1^{h_1} y_2^{h_2} y_3^{h_3}$ with $h_1, h_2, h_3 \in [1, 2n-1]$ and $h_1+h_2+h_3 = 2n+1$. After renumerating if necessary we obtain that $x_{k_1+1}+z_1+y_i \equiv i \mod 2n$ for every $i \in \{1,2,3\}$ whence $y_2 \equiv y_1+1 \mod 2n$ and $y_3 \equiv y_1+2 \mod 2n$. Since S is a minimal zero-sum sequence, we obtain that $z_1+y_{\nu_1}+y_{\nu_2}+y_{\nu_3}+2n\mathbb{Z} \in \{1,2,3\}+2n\mathbb{Z}$ for every subsequence $y_{\nu_1}y_{\nu_2}y_{\nu_3}$ of $\prod_{\imath=1}^{2n+1} y_i$. If $h_1 \ge 3$, then $y_1^3, y_1^2 y_2, y_1^2 y_3$ and $y_1 y_2 y_3$ are subsequences of $\prod_{\imath=1}^{2n+1} y_i$, but their sums $3y_1, 2y_1+y_2, 2y_1+y_3$ and $y_1+y_2+y_3$ are pairwise incongruent modulo $2n$, a contradiction. Thus $h_1 \le 2$. Similarly, if $h_3 \ge 3$, then then we get sums $3y_3, 2y_3 + y_2, 2y_3 + y_1$ and $y_1 + y_2 + y_3$ which are pairwise incongruent modulo $2n$, a contradiction. Thus $h_1, h_3 \in [1, 2]$ and $h_2 \ge 2n - 3 > 3$. If $h_1 \ge 2$, then we get sums $2y_1 + y_2, 2y_1 + y_3, y_1 + y_2 + y_3, 3y_2$ which are pairwise incongruent modulo $2n$, a contradiction whence $h_1 = 1$. Similarly, we obtain that $h_3 = 1$. Thus $h_2 = 2n - 1$ and the assertion is proved.

Case 2.4: $k_2 = l_2 = 1$. Then l_1 is even.

As in Case 2.3 we have $u' = 2 \equiv 2v \mod 2n$. Since $v + v' \equiv 0 \mod 2n$ and S is a zero-sum sequence, we infer that $0 \equiv u' + l_1 v + v' + w \equiv (l_1 + 1)v + w \mod 2n$. Then

$$S = e_1^{k_1} \cdot (x_{k_1+1}e_1 + 2ve_2) \cdot \prod_{i=1}^{l_1} (y_i e_1 + ve_2) \cdot (y_{l_1+1}e_1 - ve_2) \cdot (z_1 e_1 + we_2)$$

whence $\mathsf{p}_2(S_0)$ has one of the following four forms: $0 \cdot (ve_2) \cdot (we_2), 0 \cdot (-ve_2) \cdot (we_2), (2e_2) \cdot (ve_2) \cdot (we_2), (2e_2) \cdot (-ve_2) \cdot (we_2)$. We distinguish four cases.

Case 2.4.1: $\mathsf{p}_2(S_0) = 0 \cdot (ve_2) \cdot (we_2)$. Then $v + w + 2n\mathbb{Z} \in \{0, 2\} + 2n\mathbb{Z}$.

Suppose $v + w \equiv 0 \mod 2n$. Then $l_1 v \equiv 0 \mod 2n$ whence $l_1 = 2n$ and

$$S = e_1^{2n-4} \cdot (x_{2n-3}e_1 + 2ve_2) \cdot \prod_{i=1}^{2n} (y_i e_1 + ve_2) \cdot (y_{2n+1}e_1 - ve_2) \cdot (z_1 e_1 - ve_2).$$

Since $y_{2n+1} \neq z_1$, this situation has already been discussed in Case 1.4.

Suppose $v + w \equiv 2 \mod 2n$. Then $v = w$, $0 \equiv (l_1 + 2)v \mod 2n$ and $l_1 = 2n - 2$. Thus

$$S = e_1^{2n-2} \cdot (x_{2n-1}e_1 + 2ve_2) \cdot \prod_{i=1}^{2n-2} (y_i e_1 + ve_2) \cdot (y_{2n-1}e_1 - ve_2) \cdot (z_1 e_1 + ve_2)$$

with $z_1 \notin \{y_1, \dots, y_{2n-2}\}$. Thus either $y_{2n-1} + y_1 \not\equiv 1 \mod 2n$ or $y_{2n-1} + z_1 \not\equiv 1 \mod 2n$, and S has a proper zero-sum subsequence, a contradiction.

Case 2.4.2: $\mathsf{p}_2(S_0) = 0 \cdot (-ve_2) \cdot (we_2)$. Then $-v + w + 2n\mathbb{Z} \in \{0, 2\} + 2n\mathbb{Z}$.

If $v = w$, then we obtain a contradiction as in the second part of Case 2.4.1.

Suppose that $-v + w \equiv 2 \mod 2n$. Then $w \equiv 3v \mod 2n$, $0 \equiv (l_1 + 4)v \mod 2n$, $l_1 = 2n - 4$ and $k_1 = 4n - 1 - (k_2 + l_1 + l_2 + m_1 + m_2) = 2n$, a contradiction.

Case 2.4.3: $\mathsf{p}_2(S_0) = (2e_2) \cdot (ve_2) \cdot (we_2)$. Then $2 + v + w + 2n\mathbb{Z} \in \{0, 2\} + 2n\mathbb{Z}$.

If $2 + v + w \equiv 2 \mod 2n$, then we argue as in the first part of Case 2.4.1.

Suppose $2 + v + w \equiv 0 \mod 2n$. Then $w \equiv -3v \mod 2n$, $0 \equiv (l_1 - 2)v \mod 2n$, $l_1 = 2n + 2$ and

$$S = e_1^{2n-6} \cdot (x_{2n-5}e_1 + 2ve_2) \cdot \prod_{i=1}^{2n+2} (y_i e_1 + ve_2) \cdot (y_{2n+3}e_1 - ve_2) \cdot (z_1 e_1 - 3ve_2).$$

Since S is a minimal zero-sum sequence, it follows that $|\{y_1, \dots, y_{2n+2}\}| > 1$.

Suppose that $|\{y_1, \dots, y_{2n+2}\}| \geq 3$. Then without restriction we may suppose that $y_{2n+3} + y_{2n+2} + 2n\mathbb{Z} \in \{3, 4, 5\} + 2n\mathbb{Z}$. If $|\{y_1, \dots, y_{2n+1}\}| \geq 3$, say $|\{y_1, y_2, y_3\}| = 3$, then $z_1 + y_1 + y_4 + y_5, z_1 + y_2 + y_4 + y_5, z_1 + y_3 + y_4 + y_5$ are pairwise distinct whence $z_1 + y_j + y_4 + y_5 + 2n\mathbb{Z} \in \{3, 4, 5\} + 2n\mathbb{Z}$ for some $j \in [1, 3]$ and $y_{2n+3} + z_1 + y_{2n+2} + y_j + y_4 + y_5 \in \{6, 7, 8, 9, 10\} + 2n\mathbb{Z}$ whence S contains a proper zero-sum subsequence, a contradiction. Thus we may suppose that $\prod_{i=1}^{2n+1} y_i = y_1^{h_1} y_2^{h_2}$ with $h_1, h_2 \in [2, 2n - 1]$. Assume to the contrary that $h_1, h_2 \in [3, 2n - 2]$. If $3y_1, 2y_1 + y_2, y_1 + 2y_2$ are pairwise distinct, we obtain a contradiction as before. Hence $2y_1 \equiv 2y_2 \mod 2n$. Since $2[\frac{h_1}{2}] + 2[\frac{h_2}{2}] = 2n$, it follows that $2[\frac{h_1}{2}]y_1 + 2[\frac{h_2}{2}]y_2 \equiv 2ny_1 \equiv 0 \mod 2n$ whence $(y_1 e_1 + ve_2)^{2[\frac{h_1}{2}]}) \cdot (y_2 e_1 + ve_2)^{2[\frac{h_2}{2}]})$ is a zero-sum subsequence of S, a contradiction.

Suppose that $|\{y_1, \dots, y_{2n+2}\}| = 2$, say $\prod_{i=1}^{2n+2} y_i = y_1^{h_1} y_2^{h_2}$ with $h_1, h_2 \in [3, 2n - 1]$. Assume to the contrary that $h_1, h_2 \in [4, 2n-2]$. Since $y_1 + y_{2n+3} + 2n\mathbb{Z}, y_2 + y_{2n+3} + 2n\mathbb{Z} \in [1, 5] + 2n\mathbb{Z}$ are distinct, we may suppose that $y_{2n+3} + y_1 + 2n\mathbb{Z} \in [2, 5] + 2n\mathbb{Z}$. Then the four numbers $z_1 + 3y_1, z_1 + 2y_1 + y_2, z_1 + y_1 + 2y_2, z_1 + 3y_2$ are congruent to $1, 2$ or 3 modulo $2n$ (otherwise, the sum of one of these elements and $y_{2n+3} + y_1$ would not lie in $[1, 5]$ modulo $2n$). Thus $2y_1 \equiv 2y_2 \mod 2n$ or $3y_1 \equiv 3y_2 \mod 2n$. If $2y_1 \equiv 2y_2 \mod 2n$, we obtain a contradiction as above. Suppose $3y_1 \equiv 3y_2 \mod 2n$. Then $3 \mid n$, $3[\frac{h_1}{3}] + 3[\frac{h_2}{3}] = 3n$,

$3[\frac{h_1}{3}]y_1 + 3[\frac{h_2}{3}]y_2 \equiv 2ny_1 \equiv 0 \mod 2n$ whence $(y_1e_1 + ve_2)^{3[\frac{h_1}{3}]}) \cdot (y_2e_1 + ve_2)^{3[\frac{h_2}{3}]})$ is a zero-sum subsequence of S, a contradiction.

Case 2.4.4: $\mathsf{p}_2(S_0) = (2e_2) \cdot (-ve_2) \cdot (we_2)$. Then $2 - v + w + 2n\mathbb{Z} \in \{0, 2\} + 2n\mathbb{Z}$.

If $2 - v + w \equiv 2 \mod 2n$, then $v = w$ and we obtain a contradiction as in the second part of Case 2.4.1.

Suppose that $2 - v + w \equiv 0 \mod 2n$. Then $0 \equiv 2v - v + w \equiv v + w \mod 2n$ and we argue as in part one of Case 2.4.1.

□

References

[AD93] N. Alon and M. Dubiner. Zero-sum sets of prescribed size. In *Combinatorics, Paul Erdös is Eighty, Vol. 1*, pages 33 – 50. J. Bolyai Math. Soc., 1993.

[BD92] A. Bialostocki and P. Dierker. On the Erdös-Ginzburg-Ziv Theorem and the Ramsey numbers for starts and matchings. *Discrete Mathematics*, 110:1 – 8, 1992.

[Car92] Y. Caro. Zero-sum Ramsey numbers-stars. *Discrete Mathematics*, 104:1 – 6, 1992.

[Car96a] Y. Caro. Remarks on a zero-sum theorem. *J. Comb. Th. Ser. A*, 76:315 – 322, 1996.

[Car96b] Y. Caro. Zero-sum problems - A survey. *Discrete Math.*, 152:93 – 113, 1996.

[CFGS02] S. Chapman, M. Freeze, W. Gao, and W. Smith. On Davenport's constant of finite abelian groups. *Far East J. Math. Sci.*, 2:47 – 54, 2002.

[EGZ61] P. Erdös, A. Ginzburg, and A. Ziv. Theorem in the additive number theory. *Bull. Research Council Israel*, 10:41 – 43, 1961.

[Els] C. Elsholtz. Lower bounds for multidimensional zero sums. *Combinatorica*.

[FO96] C. Flores and O. Ordaz. On the Erdös-Ginzburg-Ziv theorem. *Discrete Mathematics*, 152:321 – 324, 1996.

[Gao96a] W. Gao. A combinatorial problem on finite abelian groups. *J. Number Theory*, 58:100 – 103, 1996.

[Gao96b] W. Gao. On zero-sum subsequences of restricted size. *J. Number Theory*, 61:97 – 102, 1996.

[Gao97] W. Gao. An addition theorem for finite cyclic groups. *Discrete Mathematics*, 163:257 – 265, 1997.

[Gao00a] W. Gao. On Davenport's constant of finite abelian groups with rank three. *Discrete Math.*, 222:111 – 124, 2000.

[Gao00b] W. Gao. Two zero sum problems and multiple properties. *J. Number Theory*, 81:254 – 265, 2000.

[Gao01a] W. Gao. Note on a Zero-Sum Problem. *J. Combinatorial Th. Ser. A*, 95:387 – 389, 2001.

[Gao01b] W. Gao. On zero sum subsequences of restricted size III. *Ars Combinatoria*, 61:65 – 72, 2001.

[Gao03] W. Gao. On zero sum subsequences of restricted size II. *Discrete Mathematics*, 2003.

[Ger90] A. Geroldinger. Systeme von Längenmengen. *Abh. Math. Sem. Univ. Hamburg*, 60:115 – 130, 1990.

[GG99] W. Gao and A. Geroldinger. On long minimal zero sequences in finite abelian groups. *Periodica Math. Hungarica*, 38:179 – 211, 1999.

[GG03] W. Gao and A. Geroldinger. Zero-sum problems and coverings by proper cosets. *Eur. J. Comb.*, 24:531 – 549, 2003.

[GGH+02] L. Gallardo, G. Grekos, L. Habsieger, F. Hennecart, B. Landreau, and A. Plagne. Restricted addition in Z/nZ and an application to the Erdös-Ginzburg-Ziv problem. *J. London Math. Soc.*, 65:513 – 523, 2002.

[Ham96] Y.O. Hamidoune. On weighted sums in abelian groups. *Discrete Mathematics*, 162:127 – 132, 1996.

[Har73] H. Harborth. Ein Extremalproblem für Gitterpunkte. *J. Reine Angew. Math.*, 262:356 – 360, 1973.

[HOO98] Y.O. Hamidoune, O. Ordaz, and A. Ortunio. On a combinatorial theorem of Erdös, Ginzburg and Ziv. *Combinatorics, Probability and Computing*, 7:403 – 412, 1998.

[Kem83] A. Kemnitz. On a lattice point problem. *Ars Combinatoria*, 16:151 – 160, 1983.

[MO67] H.B. Mann and J.E. Olson. Sums of sets in the elementary abelian group of type (p, p). *J. Comb. Theory A*, 2:275 – 284, 1967.

[MS55] L. Moser and P. Scherk. Distinct elements in a set of sums. *Amer. Math. Monthly*, 62:46 – 47, 1955.

[Ols69a] J.E. Olson. A combinatorial problem on finite abelian groups I. *J. Number Th.*, 1:8 – 10, 1969.

[Ols69b] J.E. Olson. A combinatorial problem on finite abelian groups II. *J. Number Th.*, 1:195 – 199, 1969.

[Ron00] L. Ronyai. On a conjecture of Kemnitz. *Combinatorica*, 20:569 – 573, 2000.

[Sch01] W. Schmid. On zero-sum subsequences in finite abelian groups. *Integers: Electronic Journal Comb. Number Th.*, 1:A1, 2001.

[ST02] B. Sury and R. Thangadurai. Gao's conjecture on zero-sum sequences. *Proc. Indian Acad. Sci. (Math. Sci.)*, 112:399 – 414, 2002.

[Tha01] R. Thangadurai. On a conjecture of Kemnitz. *C.R. Math. Rep. Acad. Sci. Canada*, 23:39 – 45, 2001.

[Tha02a] R. Thangadurai. Interplay between four conjectures on certain zero-sum problems. *Expo. Math.*, 20:215 – 228, 2002.

[Tha02b] R. Thangadurai. Non-canonical extensions of Erdös-Ginzburg-Ziv Theorem. *Integers*, 2:1 – 14, 2002.

[vEB69a] P. van Emde Boas. An Algol-60 algorithm for the verification of a combinatorial conjecture on a finite abelian group. 1969.

[vEB69b] P. van Emde Boas. A combinatorial problem on finite abelian groups II. In *Reports ZW-1969-007*. Math. Centre, Amsterdam, 1969.

[vEBK67] P. van Emde Boas and D. Kruyswijk. A combinatorial problem on finite abelian groups. In *Reports ZW-1967-009*. Math. Centre, Amsterdam, 1967.