Zero-sum problems and coverings by proper cosets

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Abstract

Let $G$ be a finite Abelian group and $D(G)$ its Davenport constant, which is defined as the maximal length of a minimal zero-sum sequence in $G$. We show that various problems on zero-sum sequences in $G$ may be interpreted as certain covering problems. Using this approach we study the Davenport constant of groups of the form $(\mathbb{Z}/n\mathbb{Z})^r$, with $n \geq 2$ and $r \in \mathbb{N}$. For elementary $p$-groups $G$, we derive a result on the structure of minimal zero-sum sequences $S$ having maximal length $|S| = D(G)$.

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1. Introduction

Let $G$ be an additively written finite Abelian group and $S = \prod_{i=1}^l g_i$ a sequence in $G$. Then $S$ is called a zero-sum sequence if $\sum_{i=1}^l g_i = 0$ and it is called zero-sumfree if $\sum_{i \in I} g_i \neq 0$ for all $\emptyset \neq I \subset [1, l]$. Key problems in zero-sum theory are to find the maximal possible length $l \in \mathbb{N}$ of zero-sumfree sequences, to determine the structure of such maximal sequences and to find in given sequences zero-sum subsequences satisfying additional properties.

A main aim of this paper is to present a new method in this area. We show that various zero-sum problems may be interpreted and successfully tackled as covering problems in finitely generated, free modules.

Let $R$ be a commutative ring and $M$ an $R$-module. A subset $C \subseteq M$ is called a proper coset, if $C = a + N$ for some $R$-submodule $N \subset M$ and some $a \in M \setminus N$. For given subsets $A \subseteq M$ we study the smallest number $s \in \mathbb{N}_0 \cup \{\infty\}$ such that $A \setminus \{0\}$ is contained in the union of $s$ proper cosets. In Section 3 we concentrate on sets of subsums of zero-sumfree sequences in vectorspaces including cubes in vectorspaces. These investigations generalize former work on coverings by affine hyperplanes (resp. coverings by single-valued sets), and they might be of their own interest (see Theorem 3.9).
and the subsequent remark). Section 4 deals with finite Abelian groups $M$. We show that $s(M, M) \leq \sum_{p \in \mathbb{P}} \nu_p(|M|)(p-1)$, and that equality holds, among others, for cyclic groups and elementary groups (see Theorem 4.7).

In Section 5 we build the bridge between covering problems and zero-sum problems. Section 6 contains our two main results on zero-sum sequences. Let $G = (\mathbb{Z}/n\mathbb{Z})^r$ with $r, n \in \mathbb{N}$, $n \geq 2$, and let $D(G)$ denote the Davenport constant of $G$, which is defined as the maximal length of a minimal zero-sum sequence in $G$. Then $1 + r(n - 1) \leq D(G)$, and equality holds, if $G$ is a $p$-group. But even in the case where $n$ is a prime, up to now only very little is known about the structure of minimal zero-sum sequences with maximal length (the theory of non-unique factorizations in Krull monoids naturally leads to questions about the structure of such sequences, cf. [5, 9, 17]). Theorem 6.2 presents a (sharp) structural result on zero-sumfree sequences with maximal length in elementary $p$-groups (see also Corollary 6.3 and the subsequent discussion). If $n$ is not a prime power, it is still a conjecture that $D(G) = 1 + r(n - 1)$ holds true. In Theorem 6.6 we show that a certain covering condition implies that $D(G) = 1 + r(n - 1)$. In our opinion this result provides some theoretical evidence why the conjecture should be true and opens a way how to tackle it.

2. Preliminaries

Let $\mathbb{N}$ denote the positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{P} \subset \mathbb{N}$ the set of prime numbers. For some prime $p \in \mathbb{P}$ let $\nu_p : \mathbb{N} \to \mathbb{N}_0$ denote the $p$-adic exponent whence $n = \prod_{p \in \mathbb{P}} p^{\nu_p(n)}$ for every $n \in \mathbb{N}$. For integers $a, b \in \mathbb{Z}$ we set $[a, b] = \{x \in \mathbb{Z} | a \leq x \leq b\}$.

Throughout, all Abelian groups will be written additively and for $n \in \mathbb{N}$ let $C_n = \mathbb{Z}/n\mathbb{Z}$ denote the cyclic group with $n$ elements. Let $G$ be a finite Abelian group. Then $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 | \cdots | n_r$ if $|G| > 1$ and with $r = n_1 = 1$ if $|G| = 1$. Then $r = r(G)$ is called the rank of $G$ and $n_r = \exp(G)$ is the exponent of $G$. Whenever it is convenient we consider $G$ as an $R$-module for $R = \mathbb{Z}/n_r\mathbb{Z}$. Clearly, the $R$-submodules of $G$ coincide with the subgroups. In particular, if $n_r = p$, then $G$ might be considered as an $r$-dimensional $\mathbb{Z}/p\mathbb{Z}$-vectorspace.

Let $\mathcal{F}(G)$ denote the free Abelian monoid with basis $G$. An element $S \in \mathcal{F}(G)$ is called a sequence in $G$ and will be written in the form

$$S = \prod_{g \in G} g^{\nu_g(S)} = \sum_{i=1}^l g_i \in \mathcal{F}(G).$$

A sequence $T \in \mathcal{F}(G)$ is called a subsequence of $S$, if there exists some $T' \in \mathcal{F}(G)$ such that $S = T \cdot T'$ (equivalently, $\nu_g(T) \leq \nu_g(S)$ for every $g \in G$). As usual

$$\sigma(S) = \sum_{g \in G} \nu_g(S)g = \sum_{i=1}^l g_i \in G.$$
denotes the sum of $S$,
\[ |S| = \sum_{g \in G} v_g(S) = l \in \mathbb{N}_0 \]
denotes the length of $S$ and
\[ \Sigma(S) = \left\{ \sum_{i \in I} g_i \mid \emptyset \neq I \subset [1, l] \right\} \subset G \]
the set of all possible subsums of $S$. Clearly, $|S| = 0$ if and only if $S = \{1\}$ is the empty sequence. We say that the sequence $S$ is
- zero-sumfree, if $0 \not\in \Sigma(S)$,
- a zero-sum sequence, if $\sigma(S) = 0$,
- a minimal zero-sum sequence, if it is a zero-sum sequence and each proper subsequence is zero-sumfree.

All rings are commutative, they are supposed to have a unit element and all $R$-modules are unitary. Let $R$ be a commutative ring, $M$ be a free $R$-module with basis $X_1, \ldots, X_l$ and $C$ an $R$-module. Then $r(M) = l$ denotes its rank, and for every $\theta \in \text{Hom}_R(M, C)$ there exists some $e = (c_1, \ldots, c_l) \in C^l$ such that
\[ \theta = ev_e : M \rightarrow C \]
\[ f = \sum_{i=1}^l \lambda_i X_i \longmapsto \theta(f) = f(e) = \sum_{i=1}^l \lambda_i c_i, \]
whence $\theta$ is the evaluation homomorphism in $e$, and we use the notation $\theta(f) = ev_e(f) = f(e)$ whenever it is convenient.

3. Coverings by proper cosets

**Definition 3.1.** Let $R$ be a commutative ring and $M$ an $R$-module.

1. A subset $C \subset M$ is called a proper coset, if $C = a + N$ for some $R$-submodule $N < M$ and some $a \in M \setminus N$.
2. For a subset $A \subset M$ let $s(A, M)$ denote the smallest integer $s \in \mathbb{N}_0 \cup \{\infty\}$ such that $A \setminus \{0\}$ is contained in the union of $s$ proper cosets.

By definition we have $s(A, M) = 0$ if and only if $A \subset \{0\}$ and $s(A, M) = 1$ if and only if $A$ is contained in a proper coset.

In combinatorics various problems of the following type have been studied: find the minimal number of (proper) affine hyperplanes $H_1, \ldots, H_s$, which cover a given finite set of points $A$ in a (real) finite-dimensional vector space. Of course, this minimal number is the same which is needed by a minimal covering of $A$ by proper cosets, as is shown in the following simple lemma.

**Lemma 3.2.** Let $R$ be a field, $M$ a free $R$-module of rank $r \in \mathbb{N}$ and $A \subset M$ a subset. Then $s(A, M)$ is the smallest integer $s \in \mathbb{N}_0 \cup \{\infty\}$ such that $A \setminus \{0\} \subset \bigcup_{i=1}^s H_i$ where
$H_1, \ldots, H_s$ are affine hyperplanes (i.e. $H_i = a_i + N_i$ where all $N_i$ are free $R$-submodules of $M$ with rank $r - 1$ and $a_i \in M\setminus N_i$).

**Proof.** If $N < M$ is an $R$-submodule and $a \in M\setminus N,$ then $(a)_R \cap N = \{0\}.$ By base extension we obtain some $R$-submodule $N^*$ with $N < N^* < M,$ $(a)_R \cap N^* = \{0\}$ and with rank $r - 1.$ Thus every proper coset can be blown up to an affine hyperplane and since clearly every affine hyperplane not containing zero is a proper coset, the assertion follows. □

In a series of papers coverings by so-called single-valued sets have been studied: let $R$ be a commutative ring, $M$ a free $R$-module of finite rank and $C$ an $R$-module (mainly the situation $C = R$ was considered). A subset $A \subset M$ is called single-valued if there is some $\theta \in \text{Hom}_R(M, C)$ and some $b \in C\setminus \{0\}$ such that $\theta(A) = \{b\}.$

In the following lemma we point out that in a wide class of rings single-valued sets coincide with proper cosets.

**Proposition 3.3.** Let $R$ be a commutative ring, $M$ an $R$-module and $\emptyset \neq A \subset M\setminus \{0\}.$

1. Suppose there exists some $R$-module $C,$ some $\theta \in \text{Hom}_R(M, C)$ and some $b \in C\setminus \{0\}$ such that $\theta(A) = \{b\}.$ Then $\mathcal{S}(A, M) = 1.$

2. Suppose that $A = A_1 \cup \cdots \cup A_s$ for every $i \in \{1, s\}$ such that $\theta_i(A_i) = \{b_i\}$ for every $i \in \{1, s\}$ and $b_i \in C\setminus \{0\}$ such that $\theta_i(A_i) = \{b_i\}$ for every $i \in \{1, s\}.$ Furthermore, if $R$ is an Artinian ring and an injective $R$-module and $M$ a finitely generated $R$-module, then $C = R$ has the required property.

**Proof.**

1. If $N = \{m \in M \mid \theta(m) = 0\},$ then $N < M$ is a proper $R$-submodule and $A \subset a + N$ for every $a \in A$ whence $\mathcal{S}(A, M) = 1.$

2. Suppose that for every $i \in \{1, s\}$ we have $A_i \subset a_i + N_i$ for some $R$-submodule $N_i < M$ and with $a_i \in M\setminus N_i.$ By standard construction we build an $R$-algebra $C$ out of the $R$-module $B = \bigoplus_{i=1}^s M/N_i;,$ we set $C = R \oplus B,$ define $\epsilon : R \rightarrow C$ by $\epsilon(r) = (r, 0)$ and define multiplication on $C$ by $(r, b)(r', b') = (rr', rb' + r'b)$ for all $r, r' \in R$ and all $b, b' \in B.$ For every $i \in \{1, s\}$

\[\theta_i : M \rightarrow B \hookrightarrow C, \quad m \mapsto (0, \ldots, 0, m + N_i, 0, \ldots, 0) = b \mapsto (0, b)\]

is an $R$-module homomorphism with $\theta(a_i) = b_i \neq 0,$ $\theta_i(N_i) = \{0\}$ whence $\theta_i(A_i) \subset \theta_i(a_i + N_i) = \{b_i\}.$

Suppose that $R$ is an Artinian ring, injective as an $R$-module and $M$ a finitely generated $R$-module. Then $R$ is zero-dimensional, semi-local and Noetherian. Let $N < M$ be an $R$-submodule and $a \in M\setminus N.$ By D. Eisenbud [6, Propositions 21.2 and 21.5]

\[\epsilon : M/N \rightarrow \text{Hom}_R(\text{Hom}_R(M/N, R), R)\]

\[x + N \mapsto (\epsilon_x : \theta \mapsto \theta(x + N))\]

is an $R$-module isomorphism. Since $a + N \neq 0 \in M/N,$ it follows that $\epsilon_a \neq 0$ whence there is some $\theta \in \text{Hom}_R(M/N, R)$ with $\theta(a + N) \neq 0.$ If $\pi : M \rightarrow M/N$ denotes the canonical projection, then $\theta \circ \pi : M \rightarrow R$ satisfies $\theta(N) = \{0\}$ and $\theta(a + N) \neq 0.$ □
In the following lemma we summarize some basic properties of the $s(\cdot, M)$-invariant.

**Lemma 3.4.** Let $R$ be a commutative ring, $M$ an $R$-module and $A, B \subset M$.

1. $s(A, M) \leq |A|$.
2. Let $C$ be an $R$-module and $\theta \in \text{Hom}_R(M, C)$ such that $0 \notin \theta(A \setminus \{0\})$. Then $s(A, M) \leq |	heta(A)|$.
3. $s(A \cup B, M) \leq s(A, M) + s(B, M)$.
4. If $B \subset A$ with $s(B, M) < s(A, M)$, then $A \setminus B \neq \emptyset$.
5. Suppose that $A = A_1 \cup \cdots \cup A_t$ where $s(A_i, M) = 1$ for all $i \in [1, t]$ and $t = s(A, M)$. Then for every non-empty set $I \subset [1, t]$ we have $s(\bigcup_{i \in I} A_i, M) = |I|$.

**Proof.** Without restriction we may suppose that $0 \notin A \cup B$.

1. Since $A = \bigcup_{a \in A} (a + \{0\})$, the assertion follows.
2. Since
   $$A = \bigcup_{b \in \theta(A)} (A \cap \theta^{-1}(b)),$$
   and since by **Proposition 3.3**(1) $s(A \cap \theta^{-1}(b), M) = 1$, it follows that $s(A, M) \leq |	heta(A)|$.
3. If $A = \bigcup_{i=1}^{s(A, M)} A_i$ and $B = \bigcup_{j=1}^{s(B, M)} B_j$ with proper cosets $A_i, B_j$, then $A \cup B$ is the union of the $A_i's$ and $B_j's$ whence $s(A \cup B, M) \leq s(A, M) + s(B, M)$.
4. Suppose that $B \subset A$ and $s(B, M) < s(A, M)$. Then $A = B \cup (A \setminus B)$ and
   $$s(B, M) < s(A, M) \leq s(B, M) + s(A \setminus B, M),$$
   whence $s(A \setminus B, M) \neq 0$ and $A \setminus B \neq \emptyset$.
5. Obvious. □

**Definition 3.5.** Let $R$ be a commutative ring and $M$ a free $R$-module with basis $X_1, \ldots, X_l$ for some $l \in \mathbb{N}$.

1. For every $\theta \neq k \in \mathbb{N}_0^l$ we set
   $$A^l_R(k) = A^l(k) = A(k) = \left\{ \sum_{i=1}^{l} a_i X_i \mid 0 \leq a_i \leq k_i \text{ for every } i \in [1, l] \right\} \subset M.$$
2. Let $G$ be an Abelian group and $S = \prod_{i=1}^{l} g_i \in \mathcal{F}(G)$ a sequence in $G$. We set
   $$A^l_R(S) = A(S) = \left\{ \sum_{i=1}^{l} X_i \mid \emptyset \neq I \subset [1, l], \sum_{i \in I} g_i = 0 \right\} \subset A^l_R(1).$$

In particular, write $A^l_R(1) = A^l_R((1, \ldots, 1))$ and we may interpret $A^l_R(1)$ as the set of vertices of the cube in $M$. Clearly, $A^l_R(1)$ depends on the choice of a basis in $M$ but $s(A^l_R(1), M)$ is independent of the basis whence we simply write $s(A^l_R(1), R^l)$.

Whenever for a sequence $S$ one has $s(A^l_R(1) \setminus A^l_R(S), R^l) < s(A^l_R(1), R^l)$, then $A^l_R(S) \neq \emptyset$ whence $S$ is not zero-sumfree. In this way we shall give a new proof that the Davenport constant of $C_p^r$ equals $r(p - 1) + 1$ (see the discussion after **Theorem 6.6**).
Lemma 3.6. Let $R$ be a commutative ring, $M$ an $R$-module, $\{X_1, \ldots, X_l\} \subset M$ an independent subset and $1 \neq S = \prod_{v=1}^{l} X_v^{m_v} \in F(M)$.

1. By definition we have $\Sigma(S) = A_R'(m) \setminus \{0\} \subset (X_1, \ldots, X_l)_R$ and $S$ is zero-sumfree if and only if either $\text{char}(R) = 0$ or $m \in [0, \text{char}(R) - 1]'.$

2. Suppose that $S$ is zero-sumfree and let $0 \neq k \leq m$ and $I \subset [1, l]$. Then

$$1 \leq s \left( \Sigma \left( \prod_{v=1}^{l} X_v^{m_v} \right), M \right) \leq s(\Sigma(S), M) \leq s \left( \Sigma \left( \prod_{v \in [1, l] \setminus I} X_v^{m_v} \right), M \right) + \sum_{i \in I} m_i \leq \sum_{i = 1}^{l} m_i = |S|.$$

3. If $\text{char}(R) = n$ and $p$ is a prime divisor of $n$ with $p < n$ and $p < l$, then $s(A_R'(1), R^l) \leq l - 1$.

Proof. 1. By definition we have $\Sigma(S) = A_R'(m) \setminus \{0\}$. $S$ is zero-sumfree if and only if $0 \neq \Sigma(S)$ if and only if for every $k \leq m$ the equation $\sum_{v=1}^{l} k_v X_v = 0$ implies that $k = 0$. Since $X_1, \ldots, X_l$ are independent elements, the assertion follows.

2. Since $\Sigma(X_i) \setminus \{0\} = \{X_i\} = X_i + \{0\} \subset M$ is a proper coset, it follows that $s(\Sigma(X_i), M) = 1$ for every $i \in [1, l]$. Since $0 \neq k \leq m$, we have $\Sigma \left( \prod_{v=1}^{l} X_v^{k_v} \right) \subset \Sigma(S)$ and Lemma 3.4(3) implies that $s \left( \Sigma \left( \prod_{v=1}^{l} X_v^{k_v} \right), M \right) \leq s(\Sigma(S), M)$.

Next we show that

$$s(\Sigma(S), M) \leq s \left( \Sigma \left( \prod_{v=1}^{l-1} X_v^{m_v} \right), M \right) + m_l$$

which implies the remaining inequalities by an inductive argument.

Let $v \in [1, m_l]$. Since $S$ is zero-sumfree and $\{X_1, \ldots, X_l\}$ is independent, we obtain that

$$0 \neq -vX_l \notin \Sigma \left( \prod_{v=1}^{l-1} X_v^{m_v} \right) \subset \langle X_1, \ldots, X_{l-1} \rangle_R$$

and that $vX_l \notin \langle X_1, \ldots, X_{l-1} \rangle_R$. Thus for

$$B_v = \{vX_l\} + \left( \Sigma \left( \prod_{v=1}^{l-1} X_v^{m_v} \right) \cup \{0\} \right) \subset \{vX_l\} + \langle X_1, \ldots, X_{l-1} \rangle_R$$

we obtain that $s(B_v, R^l) = 1$. Since $\Sigma(S) = \Sigma \left( \prod_{v=1}^{l-1} X_v^{m_v} \right) \cup \bigcup_{v=l-1}^{l} B_v$, Lemma 3.4 implies the assertion.

3. Suppose $\{X_1, \ldots, X_l\}$ is a basis of $R^l$, $\text{char}(R) = n$ and $p$ a prime divisor of $n$ with $p < \min\{n, l\}$. For $i \in \mathbb{N}$ we set

$$A_i = \left\{ \sum_{j \in I} X_j : I \subset [1, l] \text{ with } |I| = t \right\}.$$
Lemma 3.7. Let R be a commutative ring. Using Lemma 3.8 with X = 1 + n\mathbb{Z}, \ldots, 1 + n\mathbb{Z}
whence s(A_i, R^l) = 1. Furthermore, s(A_1 \cup A_{p+1}, R^l) = 1, because
ev_c(A_1) = \frac{n}{p} + n\mathbb{Z} = ev_c(A_{p+1}) for c = \left(\frac{n}{p} + n\mathbb{Z}, \ldots, \frac{n}{p} + n\mathbb{Z}\right).
Thus we infer that
\[ A_R^l(1) = \bigcup_{i \in [1,l]} A_i = (A_1 \cup A_{p+1}) \cup \bigcup_{i \in [1,l] \setminus [1,p+1]} A_i \]
which implies that
\[ s(A_R^l(1), R^l) \leq l - 1. \]

Lemma 3.7. Let R be a commutative ring, l \in \mathbb{N} and k \in [1, l - 1]. In R[X, Y_{i, j} | i \in [1, k], j \in [1, l]] we have the following polynomial identity:
\[ \sum_{i \neq j \subset [1, l]} (-1)^{|J|} \prod_{i=1}^{l} \left( X - \sum_{j \in J} Y_{i, j} \right) = -X^k. \]

Proof. see Lemma 9.3 in [16].

Proposition 3.8. Let R be a commutative ring, M an R-module, C an R-algebra and S a zero-sumfree sequence in M. Suppose \( \Sigma(S) = A_1 \cup \cdots \cup A_k \) where k < |S| and \( \theta_i(A_i) = \{b_i\} \) with \( \theta_i \in \text{Hom}_R(M, C) \) and \( b_i \in C \setminus \{0\} \) for all \( i \in [1, k] \). Then \( \prod_{i=1}^{k} b_i^k = 0. \)

Proof. Let \( b = \prod_{i=1}^{k} b_i \) and for \( i \in [1, k] \) we set \( \theta'_i = \frac{b}{b_i} \cdot \theta_i \in \text{Hom}_R(M, C) \) whence \( \theta'_i(A_i) = \{b\}. \)
Suppose that \( S = \prod_{i=1}^{k} f_i \) and let \( \emptyset \neq J \subset [1, |S|]. \) Since \( \Sigma(S) = A_1 \cup \cdots \cup A_k, \) there exists some \( \lambda \in [1, k] \) such that \( \sum_{j \in J} f_j \in A_\lambda. \) This implies that
\[ \sum_{j \in J} \theta'_i(f_j) = \theta'_\lambda \left( \sum_{j \in J} f_j \right) = b \]
whence
\[ \prod_{i=1}^{k} \left( b - \sum_{j \in J} \theta'_i(f_j) \right) = 0. \]
Using Lemma 3.8 with \( X = b \) and \( Y_{i, j} = \theta'_i(f_j) \) we infer that \( 0 = -b^k. \)

Theorem 3.9. Let R be a field, l \in \mathbb{N} and S be a zero-sumfree sequence in R^l.

(1) \( s(\Sigma(S), R^l) \geq |S|. \)
(2) If \( \text{supp}(S) \subset R^l \) is independent, then \( s(\Sigma(S), R^l) = |S|. \)
(3) Let \( k \in \mathbb{N}^d \) if \( \text{char}(R) = 0 \), and \( k \in \{ 0, \text{char}(R)−1 \}^d \) otherwise. Then \( s(A_R^I(k), R^I) = \sum_{i=1}^l k_i \).

**Proof.** (1) Assume to the contrary that \( s(\Sigma(S), R^I) < |S| \). Then \( \Sigma(S) = A_1 \cup \cdots \cup A_k \) with \( k < |S| \) and \( s(A_1, R^I) = \cdots = s(A_k, R^I) = 1 \). By Proposition 3.3 there exist \( \theta_i \in \text{Hom}_R(R^I, R) \) and \( b_i \in R \setminus \{ 0 \} \) such that \( \theta_i(A_i) = \{ b_i \} \) for every \( i \in [1, k] \). Thus Proposition 3.8 implies that \( \prod_{i=1}^k b_i^k = 0 \) whence \( b_v = 0 \) for some \( v \in [1, k] \), a contradiction.

(2) Lemma 3.6(2) implies that \( s(\Sigma(S), R^I) \leq |S| \) whence the assertion follows from (1).

(3) If \( \{ X_1, \ldots, X_l \} \) is a basis of \( R^I \), then by Lemma 3.6(1) \( T = \prod_{i=1}^l X_i^{k_i} \) is zero-sumfree and \( \Sigma(T) = A_R^I(k) \setminus \{ 0 \} \). Hence the assertion follows from (2).

**Remark 3.10.** For \( k = 1 \) and \( R = \mathbb{Z}/p\mathbb{Z} \) the result on \( s(A_R^I(k), R^I) \) was proved by Gao in [11] and for \( k = 1 \) and \( R \) the real numbers a first proof was given by Alon and Füredi in [1]. For further results of this type see also [4] and [16].

4. **On \( s(M, M) \) for finite Abelian groups \( M \)**

Let \( M \) be a finite Abelian group with exponent \( \exp(M) = n \). Then \( M \) may be considered as an \( R \)-module with \( R = \mathbb{Z}/n\mathbb{Z} \) and the \( R \)-submodules coincide with the subgroups of \( M \). Since \( M \) is finite, Lemma 3.4 shows that

\[
s(M, M) \leq |M| < \infty.
\]

In this section we study \( s(M, M) \) and for simplicity we set \( s(M) = s(M, M) \).

**Definition 4.1.** We define a homomorphism \( L : (\mathbb{N}, \cdot) \rightarrow (\mathbb{N}_0, +) \) by

\[
L : \mathbb{N} \rightarrow \mathbb{N}_0
n \mapsto \sum_{p \in \mathbb{P}} v_p(n)(p - 1).
\]

**Lemma 4.2.** Let \( M \) be a finite Abelian group.

(1) If \( N < M \) is a subgroup, then

\[
s(N) = s(N, M) \leq s(M) \leq s(N) + s(M/N).
\]

(2) \( s(M) \leq L(|M|) \).

(3) If \( s(M) = L(|M|) \), then \( s(N) = L(|N|) \) for all subgroups \( N < M \).

**Proof.** (1) Let \( N < M, N \setminus \{ 0 \} = \bigcup_{i=1}^{s(N)} (g_i + N_i) \), with all \( N_i < M \) and all \( g_i \in M \setminus N_i \), and let \( M/N \setminus \{ N \} = \bigcup_{i=1}^{s(M/N)} (a_i + N) + H_i/N \) with all \( N < H_i < M \) and all \( a_i \in M \setminus H_i \). If \( x \in M \setminus N \), then there is some \( i \in [1, s(M/N)] \) such that
Lemma 4.3. Let $M$ be a finite Abelian group and $\theta < M$ a subgroup.

(1) Let $M \setminus \{0\} = \bigcup_{i=1}^{s(M)} (g_i + N_i)$ where, for every $i \in [1, s(M)]$, $N_i < M$ is a subgroup and $g_i \in M \setminus N_i$, and let $I$ consist of those $i \in [1, s(M)]$ such that $(g_i + N_i) \cap \theta \neq \emptyset$. If $x + \theta \notin \bigcup_{i \in I} (g_i + N_i)$ for every $x \in M \setminus \theta$, then $s(M) \geq s(\theta) + s(M/\theta)$.

(2) If $s(M) \geq s(\theta) + s(M/\theta)$, $s(\theta) = L(|\theta|)$ and $s(M/\theta) = L(|M/\theta|)$, then $s(M) = L(|\theta|)$. 

Proof. 1. We set $J = [1, s(M)] \setminus I$. For every $i \in I$ there are $h_i \in N_i$ and $t_i \in \theta$ such that $g_i + h_i = t_i$ whence $g_i + N_i = t_i + N_i$, and since $N_i \neq g_i + N_i$, it follows that $t_i \notin N_i$. Thus we obtain that

$$\theta \setminus \{0\} = \bigcup_{i \in I} ((t_i + N_i) \cap \theta) = \bigcup_{i \in I} (t_i + (N_i \cap \theta)).$$

We assert that
Proposition 4.4. Let $M$ be a finite Abelian group.

(1) Suppose that $M/\theta \setminus \{\theta\} \subseteq \bigcup_{j \in J}((g_j + \theta) + (N_j + \theta)/\theta)$

is a covering by proper cosets. Then (\star) and (\star\star) imply that

$$S(M) = |I| + |J| \geq S(\theta) + S(M/\theta).$$

Let $i \in [1, S(M)]$. If $g_i + \theta \in (N_i + \theta)/\theta$, then there is some $h_i \in N_i$ with $g_i + \theta = h_i + \theta$ whence $g_i \in N_i + \theta$ and $(g_i + N_i) \cap \theta \neq \emptyset$. Thus it follows that $(g_j + \theta) + (N_j + \theta)/\theta$ is a proper coset of $M/\theta$ for every $j \in J$.

To verify equality, let $x \in M\setminus\theta$. Since $x + \theta \notin \bigcup_{i \in I}(g_i + N_i)$, there exists some $y \in x + \theta$ and some $j \in J$ such that $y = g_j + N_j$. Then $x + \theta = y + \theta \subset g_j + N_j + \theta$ whence $x + \theta \in (g_j + \theta) + (N_j + \theta)/\theta$.

2. Lemma 4.2 implies that

$$L(|M|) \geq s(M) \geq S(\theta) + S(M/\theta) = L(|\theta|) + L(|M/\theta|) = L(|M|)$$

whence the assertion follows. \(\Box\)

Proposition 4.4. Let $M$ be a finite Abelian group.

(1) If $M = M_1 \oplus M_2$ with $\gcd(|M_1|, |M_2|) = 1$, then $S(M) \geq S(M_1) + S(M_2)$.

(2) If $M = \bigoplus_{i=1}^k M_i$ a direct decomposition into subgroups with $\gcd(|M_1|, |M_j|) = 1$ for all $1 \leq i < j \leq k$ and $S(M_i) = L(|M_i|)$ for every $i \in [1, k]$, then $S(M) = L(|M|)$.

(3) If $\exp(M) = \prod_{i=1}^k p_i^{m_i}$ and $S((C_{p_i^{m_i}})^{r_i}) = L(|(C_{p_i^{m_i}})^{r_i}|)$ where $r_i$ is the $p_i$-rank of $M$, then $S(M) = L(|M|)$.

Proof. (1) Suppose that $M = M_1 \oplus M_2$. We verify the assumption of Lemma 4.3 with $\theta = M_1$. Then the assertion follows. With all notations as in Lemma 4.3, let $x \in M\setminus M_1$ whence $x + M_1 = b + M_1$ for some $b \in M_2\setminus\{0\}$. Then there exists some $\lambda \in [1, S(M)]$ such that $b \in g_\lambda + N_\lambda$. It suffices to verify that

$$(g_\lambda + N_\lambda) \cap M_1 = \emptyset$$

(whence $\lambda \notin I$ and $b \notin \bigcup_{i \in I}(g_i + N_i)$).

We set $N_\lambda = H$ and since $\gcd(|M_1|, |M_2|) = 1$, it follows that $H = H_1 \oplus H_2$ with $H_i < M_i$. Assume to the contrary that

$$(b + H) \cap M_1 = (g_\lambda + H) \cap M_1 \neq \emptyset.$$ 

Then there are $h_1 \in H_1, h_2 \in H_2$ and $m_1 \in M_1$ such that $b + h_1 + h_2 = m_1$ whence $b + h_2 = m_1 - h_1 \in M_1 \cap M_2 = \{0\}$. Therefore $b = -h_2 \in H_2 < H$ and $g_\lambda + H = b + H = H$, a contradiction.

(2) Lemmas 4.2 and 4.3 imply that

$$L(|M|) = \sum_{i=1}^k L(|M_i|) = \sum_{i=1}^k S(M_i) \leq S(M) \leq L(|M|).$$

(3) If for $i \in [1, s]$ $M_i$ denotes the $p_i$-subgroup of $M$, then $M_i < (C_{p_i^{m_i}})^{r_i}$ and Lemma 4.2 implies that $S(M_i) = L(|M_i|)$. Thus the assertion follows from (1). \(\Box\)
Let $M$ be a finite Abelian $p$-group. A subset $\{e_1, \ldots, e_t\} \subseteq M$ is called independent, if $\sum_{i=1}^{t} m_i e_i = 0$, with $m_1, \ldots, m_t \in \mathbb{Z}$, implies that $m_1 e_1 = \cdots = m_t e_t = 0$. Every independent subset is contained in a maximal independent subset, and each two maximal independent subsets have the same number of elements, which is denoted by $r(M)$ and is called the rank of $M$. Let $\soc(M) = \{x \in M \mid px = 0\}$ denote the socle of $M$. Then $\soc(M)$ is an $\mathbb{F}_p$-vector space with dim$_{\mathbb{F}_p}(\soc(M)) = r(\soc(M)) = r(M)$.

**Lemma 4.5.** Let $M$ be a finite Abelian $p$-group, $N < M$ a subgroup, $g \in M \setminus N$ and $\theta = \soc(M)$.

1. If $r(N) = r(M)$, then $\theta < N$.
2. If $g \in M \setminus N$, then there exists some $N^* < M$ such that $N < N^*$, $pg \in N^*$, $g \notin N^*$ and $r(N^* + \langle g \rangle) = r(M)$.
3. If $r(N + \langle g \rangle) = r(M)$, $pg \in N$ and $(g + N) \cap \theta = \emptyset$, then $r(N) = r(M)$.
4. If $M = (C_{p^n})^\ast$ with $r, n \in \mathbb{N}$ and $(g + N) \cap \theta \neq \emptyset$, then there are $e^* \in M$ and $N^* < M$ such that $M = (e^*) \oplus N^*$, $N < N^*$ and $p^{r-1}e^* + N = g + N$.

**Proof.**

1. Clearly, we have $\soc(N) < \soc(M)$, if $r(N) = r(M)$, then $\soc(N) = \soc(M)$ are $\mathbb{F}_p$-vector spaces with the same dimension whence $\soc(M) = \soc(N) < N$.

2. Let $g \in M \setminus N$ and $N_1 = \langle N, pg \rangle$. Assume to the contrary, that $g \in N_1$. Then there are $a \in \mathbb{Z}$ and $h \in N$ such that $g = -a(pg) + h$ whence $(1 + ap)g = h$. If $x, y \in \mathbb{Z}$ with $x \ord(g) + y(1+ap) = 1$, then $g = (1-x\ord(g))g = yh \in N$, a contradiction.

If $r(M) = r(N_1 + \langle g \rangle)$, we set $N^* = N_1$. Suppose that $r(M) > r(N_1 + \langle g \rangle)$ and set $N_1 + \langle g \rangle = \bigoplus_{i=1}^{t} \langle e_i \rangle$ with $t = r(N_1 + \langle g \rangle)$. Then $\{e_1, \ldots, e_t\} \subset M$ is contained in a maximal independent subset $E \subset M$ whence $|E| = r(M)$. We set $Q = \langle E \setminus \{e_1, \ldots, e_t\} \rangle$ and $N^* = N_1 + Q$. Then $N < N_1 < N^*$, $pg \in N^*$ and $r(N^* + \langle g \rangle) = r(M)$. Assume to the contrary that $g \in N^*$. Then there are $n \in N_1$ and $q \in Q$ such that $g = n + q$ whence $q = g - n \in N_1 + \langle g \rangle \cap Q = \emptyset$ and $g = n \in N_1$, a contradiction.

3. Suppose that $(g + N) \cap \theta = \emptyset$ and $r(N + \langle g \rangle) = r(M)$. We assert that $\soc(N) = \soc(N + \langle g \rangle)$ which implies that $r(N) = r(M)$. Obviously, $\soc(N) < \soc(N + \langle g \rangle)$, and we choose some $x \in \soc(N + \langle g \rangle)$. Since $pg \in N$, we have $x = ag + n$ with $n \in N$ and $a \in [0, p-1]$. Assume to the contrary that $a > 0$. Then there is some $a' \in [1, p-1]$ and some $k \in \mathbb{Z}$ such that $aa' = 1 + kp$. Then $a'x = g + n'$ with $n' = kpg + a'n \in N$. Then $0 = a'px$, but $g + n' \notin g + N$ implies that $p(g + n') \neq 0$, a contradiction.

4. Let $M = (C_{p^n})^\ast$ with $r, n \in \mathbb{N}$ and $(g + N) \cap \theta \neq \emptyset$. Then there is some $e \in \theta$ and some $n \in N$ such that $e = g - n$ whence $g + N = e + N$. Since $e \notin N$ and $pe = 0$, it follows that $\langle e \rangle \cap N = \emptyset$. There is some $e^* \in M$ with $p^{r-1}e^* = e$ and obviously $\langle e^* \rangle \cap N = \emptyset$. Then $N$ is contained in a maximal subset $N^*$ such that $\langle e^* \rangle \cap N^* = \emptyset$. Thus we obtain that $M = \langle e^* \rangle \oplus N^*$ (cf. [22], 4.2.7). \(\square\)
Proposition 4.6. Let $M$ be a finite Abelian $p$-group.

1. If $M$ is elementary, then $s(M) = L(|M|)$.
2. If $M$ is cyclic, then $s(M) = L(|M|)$.

Proof. (1) If $M$ is elementary with basis $X_1, \ldots, X_l$, then $M = A_{Z/pZ}((p-1, \ldots, p-1))$ whence the assertion follows from Theorem 3.9.

(2) Let $M$ be a cyclic group. We proceed by induction on $|M|$. If $|M| = p$, then $M$ is elementary and the assertion follows from (1). To do the induction step, we set $\theta = \text{soc}(M)$. If we can verify the assumption of Lemma 4.3, then the assertion follows.

Let $M\{0\} = \bigcup_{i=1}^{s(M)} (g_i + N_i)$ where, for every $i \in [1, s(M)]$, $N_i < M$ is a subgroup and $g_i \in M/\theta$. Let $I$ consist of those $i \in [1, s(M)]$ such that $(g_i + N_i) \cap \theta \neq \emptyset$. By Lemma 4.5 we may suppose that $pg_i \in N_i$ for every $i \in [1, s(M)]$.

Let $x \in M\theta$. We have to verify that

$$x + \theta \not\subset \bigcup_{i \in I} (g_i + N_i).$$

If $\lambda \in [1, s(M)]$ with $x \in g_\lambda + N_\lambda$, then $0 \neq px \in N_\lambda$, whence $1 = t(N_\lambda) = t(M)$. Thus $\theta < N_\lambda$, $(g_\lambda + N_\lambda) \cap \theta \subset (g_\lambda + N_\lambda) \cap N_\lambda = \emptyset$ whence $\lambda \not\in I$ and $x \not\in \bigcup_{i \in I} (g_i + N_i)$. □

Theorem 4.7. Let $M$ be a finite Abelian group. If $M = M_1 \oplus M_2$, where $M_1$ is cyclic, $\exp(M_2)$ squarefree and $\gcd(|M_1|, |M_2|) = 1$, then $s(M) = L(|M|)$.

Proof. If $M_1$ is cyclic, then $M_1$ is a direct sum of cyclic groups of prime power order. If $\exp(M_2)$ is squarefree, then $M_2$ is a direct sum of elementary $p$-groups. Thus the assertion follows from Propositions 4.4 and 4.6. □

5. Zero sets

Zero sets play a crucial part in establishing the connection between covering problems and zero-sum problems.

Definition 5.1. Let $R$ be a commutative ring and $C$ an $R$-module. A subset $A$ of a finitely generated free $R$-module $M$ is called a zero set over $C$, if $0 \in \theta(A)$ for every $\theta \in \text{Hom}_R(M, C)$.

We continue with a characterization of zero sets in the case where $C$ is a direct sum of submodules. If $R$ is a field and $C$ an $R$-vector space, then zero sets allow a very simple characterization.

Proposition 5.2. Let $R$ be a commutative ring, $k \in \mathbb{N}$ and $C = \bigoplus_{i=1}^{k} C_i$ an $R$-module.

1. For a subset $A$ of some finitely generated free $R$-module $M$ the following conditions are equivalent:

   a. $A$ is a zero set over $C$.
(b) For every partition (resp. for every decomposition) \( A = A_1 \cup \cdots \cup A_k \) there is some \( i \in [1,k] \) such that \( A_i \) is a zero set over \( C_i \).

(2) Suppose that \( R \) is a field and \( C_1 = \cdots = C_k = R \). For a subset \( A \subset R^l \) the following conditions are equivalent:

(a) \( A \) is a zero set over \( C \).

(b) \( A \cap H \neq \emptyset \) for all submodules \( H < R^l \) with \( r(H) \geq l - k \).

(3) Suppose that \( R = \mathbb{Z}/p\mathbb{Z} \) for some prime \( p \in \mathbb{P} \) and let \( C = \mathbb{F}_q \) be the field with \( q = p^k \) elements. For a subset \( A \subset \langle X_1, \ldots, X_l \rangle_R \subset R[X_1, \ldots, X_l] \) the following conditions are equivalent:

(a) \( A \) is a zero set over \( C \).

(b) \( \prod_{f \in A} f \in \langle X_i - X_j \mid i \in [1,l] \rangle_R \).

**Proof.**

1. \((a) \implies (b)\) Assume to the contrary that \( A = A_1 \cup \cdots \cup A_k \) and no \( A_i \) is a zero set over \( C_i \). Hence for every \( i \in [1,k] \) there is some \( \theta_i \in \text{Hom}_R(M,C_i) \) such that \( \theta_i(A_i) \subset C_i \setminus \{0\} \). Therefore \( \theta = (\theta_1, \ldots, \theta_k) \in \text{Hom}_R(M,C) \) and \( \theta(A) \subset C \setminus \{0\} \), a contradiction.

\((b) \implies (a)\) Assume to the contrary that \( A \) is not a zero set over \( C \). Then there is some \( \theta = (\theta_1, \ldots, \theta_k) \in \text{Hom}_R(M,C) \) such that \( \theta(A) \subset C \setminus \{0\} \). For \( i \in [1,k] \) we set \( A_i = \{a \in A \mid \theta_i(a) \neq 0\} \) and obtain that \( A = A_1 \cup \cdots \cup A_k \) and no \( A_i \) is a zero set over \( C_i \), a contradiction.

2. Every submodule \( H < R^l \) with \( r(H) \geq l - k \) is the intersection of \( k \) (not necessarily different) hyperplanes, say \( H = H_1 \cap \cdots \cap H_k \), and for every \( H_i \) there is some \( \theta_i \in \text{Hom}_R(R^l,R) \) such that \( H_i = \ker(\theta_i) \). Thus \( A \cap H \neq \emptyset \) if and only if there is some \( a \in A \) such that \( \theta = (\theta_1, \ldots, \theta_k) \in \text{Hom}_R(R^l,R^k) \) we have \( \theta(a) = 0 \) whence the assertion follows.

3. \( A \) is a zero set over \( \mathbb{F}_q \) if and only if for all \( \theta \in \text{Hom}_R(R[X_1, \ldots, X_l], \mathbb{F}_q) \) we have \( 0 \in \theta(A) \), which holds if and only if for all \( c \in \mathbb{F}_q^l \) there is some \( f \in A \) such that \( f(c) = 0 \). This is equivalent to the fact that for all \( c \in \mathbb{F}_q^l \) we have \( \prod_{f \in A} f(c) = 0 \) and the assertion follows. \( \square \)

In the following we want to point out that many classical problems in zero-sum theory allow a straightforward formulation in terms of zero sets.

Let \( G \) be a finite Abelian group with exponent \( n \). A first problem, which is still unsolved for general \( G \), is to determine the Davenport constant \( D(G) \) of \( G \) which is defined as the maximal length of a minimal zero-sum sequence in \( G \) (equivalently, \( D(G) \) is the smallest integer \( l \in \mathbb{N} \) such that every sequence \( S \in \mathcal{F}(G) \) with \( |S| \geq l \) contains a zero-sum subsequence). The paper of Erdős–Ginzburg–Ziv [10] was a starting point for investigations of subsequences of given sequences which have sum zero and satisfy certain additional properties. For a subset \( A \subset \mathbb{N} \) let \( \eta_A(G) \) denote the smallest integer \( l \in \mathbb{N} \) such that every sequence \( S \in \mathcal{F}(G) \) has a zero-sum subsequence \( T \) with \( |T| \in A \).

Clearly, \( \eta_G(G) \) is just the Davenport constant \( D(G) \) and the invariants \( \eta_A(G) \) for \( \Lambda = [G] \), \( \Lambda = [n] \) and \( \Lambda = [1,n] \) have found considerable attention in the literature (cf. [8, 12, 19, 24] and the references given there). If \( \Lambda = \{\lambda\} \), then we set \( \eta_{\lambda}(G) = \eta_{\lambda}(G) \).
Main Lemma 5.3. Let $G$ be a finite Abelian group with exponent $n$, $R = \mathbb{Z}/n\mathbb{Z}$ and $\Lambda \subset \mathbb{N}$ a subset. Then $\eta_\Lambda(G)$ is the smallest integer $l \in \mathbb{N}$ such that the subset 
\[ A = \left\{ \sum_{i \in I} X_i \mid I \subset [1, l], |I| \in \Lambda \right\} \subset A^l_\Lambda \subset (X_1, \ldots, X_l)_R \]
is a zero set over the $R$-module $G$.

Proof. Recall that for every $\theta \in \text{Hom}_R(R^l, G)$ there is some $c \in G^l$ such that $\theta = \text{ev}_c : R^l \to G$. A sequence $S = \prod_{i=1}^l c_i \in F(G)$ has a zero-sum subsequence $T = \prod_{i \in I} c_i$ with $|T| = |I| \in \Lambda$ if and only if there exists some $f \in A$ such that $\text{ev}_c(f) = f((c_1, \ldots, c_l)) = 0$. This implies the assertion. □

Hence in this interpretation of zero-sum problems we fix the $R$-module $C$ (here $C = \mathbb{Z}/n\mathbb{Z}$) and vary over the ranks of the free $R$-modules $M$. This motivates the following definition.

Definition 5.4. Let $R$ be a commutative ring. For an $R$-module $C$ we set
\[ s^*(C) = \sup\{ s(A, M) \mid A \text{ is a subset of a free } R\text{-module } M \text{ with finite rank and } A \text{ is not a zero set over } C \} \in \mathbb{N}_0 \cup \{\infty\}. \]

Proposition 5.5. Let $R$ be a commutative ring and $C$ an $R$-module.

(1) $s^*(C) \leq |C| - 1$.

(2) A subset $A$ of some free $R$-module $M$ with finite rank, which satisfies $s(A, M) \geq k s^*(C) + 1$, is a zero set over $C^k$.

Proof. (1) If a subset $A$ of some free $R$-module $M$ with finite rank is not a zero set over $C$ and $\theta \in \text{Hom}_R(M, C)$ such that $\theta(A) \subset C \setminus \{0\}$, then Lemma 3.4(2) implies that
\[ s(A, M) \leq |\theta(A)| \leq |C| - 1. \]
Thus we obtain that $s^*(C) \leq |C| - 1$.

(2) Let $A$ be a set having the above properties and assume to the contrary, that $A$ is not a zero set over $C^k$. Then by Proposition 5.2(1) there exists a partition $A = A_1 \cup \cdots \cup A_k$ such that no $A_i$ is a zero set over $C$. This implies that
\[ s(A, M) \leq \sum_{i=1}^k s(A_i, M) \leq k s^*(C), \]
a contradiction. □

At the end of this section we want to show in an explicit example how zero-sum problems can be attacked via zero sets (see also Theorem 6.6).

Let $n \in \mathbb{N}$ be a positive integer with $n \geq 2$. An old conjecture, going back to Kemnitz, states that
\[ \eta_n(C_n \oplus C_n) = 4n - 3. \]
It is easy to see that $\eta_n(C_n \oplus C_n) \geq 4n - 3$ and quite recently it was proved that for prime powers we have $\eta_n(C_n \oplus C_n) \leq 4n - 2$ (see [14, 21, 23]).
Main Lemma 5.6. Suppose that for every prime \( p \in \mathbb{P} \) and for
\[
A = \left\{ \sum_{i \in I} X_i \mid I \subset [1, l], |I| = p \right\} \subset R^l = (X_1, \ldots, X_l)_R,
\]
where \( R = \mathbb{Z}/p\mathbb{Z} \) and \( l = 4p - 3 \), we have \( \mathfrak{s}(A, R^l) \geq 2p - 1 \). Then \( \eta_n(C_n \oplus C_n) = 4n - 3 \) for every \( n \in \mathbb{N} \).

Proof. It suffices to verify that
\[
(*) \quad \eta_n(C_n \oplus C_n) \leq 4n - 3.
\]
Since \( \eta_n(\cdot) \) is multiplicative, it suffices to show \((*)\) for prime numbers.

Let \( p \in \mathbb{P} \) be a prime number and \( R = \mathbb{Z}/p\mathbb{Z} \). Using Proposition 5.5 we infer that
\[
\mathfrak{s}(A, R^p) \leq p - 1 \quad \text{and} \quad \mathfrak{s}(A, R^{p-3}) \geq 2p - 1 \geq 2(\mathfrak{s}(A, R^p)) + 1
\]
whence \( A \) is a zero set over \( R \oplus R \). Thus Main Lemma 5.3 implies that \( \eta_p(R \oplus R) \leq 4p - 3 \). \( \square \)

6. The case \( G = C_n^r \)

In this final section we concentrate on groups \( G \) of the form \( G = C_n^r \) and study the maximal possible length of minimal zero-sum sequences in \( G \) and consider the structure of such sequences. To begin with, let \( G = C_{n_1} \oplus \cdots \oplus C_{n_r} \) with \( 1 < n_1 | \cdots | n_r \). Then it is easy to see that \( D(G) \geq 1 + \sum_{i=1}^{r}(n_i - 1) \). Equality holds for \( p \)-groups and for groups with rank \( r \leq 2 \), but for every \( r \geq 4 \) there are infinitely many groups for which the above inequality is strict (see [13], Theorem 3.3 in [16, 18] and the references cited there). Although for \( p \)-groups the precise value of the Davenport constant is known, we have almost no information about the structure of minimal zero-sum sequences \( S \) with \( |S| = D(G) \). We start with a structural result for such sequences in elementary \( p \)-groups (Theorem 6.2 and Corollary 6.3). Then we consider the Davenport constant for groups \( G = C_n^r \) where \( n \) is not a prime power.

Lemma 6.1. Let \( G = C_n^r \) with \( n, r \in \mathbb{N}, n \geq 2 \) and \( S = \prod_{i=1}^{r} g_i \in \mathcal{F}(G) \). Then
\[
\mathfrak{s}(A_{R}^l(1) \setminus A_{R}^l(S), R^l) \leq r(n - 1) \quad \text{where} \quad R = \mathbb{Z}/n\mathbb{Z}.
\]

Proof. Let \( \{e_1, \ldots, e_r\} \) be a basis of \( G \). For every \( i \in [1, l] \) we set \( g_i = \sum_{j} c_{v,j} e_j \) with \( c_{v,j} \in \mathbb{Z} \). For \( v \in [1, r] \) and \( m \in [1, n - 1] \) let
\[
A_{v,m} = \left\{ \sum_{i \in I} X_i \in A_{R}^l(1) \mid I \subset [1, l], \sum_{i \in I}(c_{v,i} + n\mathbb{Z}) = m + n\mathbb{Z} \right\} \subset (X_1, \ldots, X_l)_R = R^l.
\]

Then \( \mathfrak{s}(A_{v,m}, R^l) = 1 \), since for \( e_v = (c_{v,1} + n\mathbb{Z}, \ldots, c_{v,l} + n\mathbb{Z}) \in R^l \) we have
\[
\mathfrak{ev}_{e_v}(A_{v,m}) = \sum_{i \in I} (c_{v,i} + n\mathbb{Z}) = m + n\mathbb{Z} \in R\setminus\{0\}.
\]
Hence it suffices to prove that
\[ A^l_R(1) \setminus A^l_R(S) \subset \bigcup_{v=1}^{r-1} A_{v,m}. \]

To verify the inclusion, let \( f = \sum_{i \in I} X_i \in A^l_R(1) \setminus A^l_R(S) \subset R^l \). Then \( \sum_{i \in I} g_i \neq 0 \) whence there exists some \( v \in [1, r] \) with \( \sum_{i \in I} c_{v,i} e_v \neq 0 \). Therefore, \( \sum_{i \in I} c_{v,i} e_v = m + n \mathbb{Z} \) for some \( m \in [1, n-1] \) i.e. \( f \in A_{v,m}. \) \( \Box \)

**Theorem 6.2.** Let \( G \) be an elementary \( p \)-group and \( S \in \mathcal{F}(G) \) a zero-sumfree sequence with maximal length. Then for every subsequence \( T \) of \( S \) and every cyclic subgroup \( H \) of \( G \) we have \( |\Sigma(T) \cap H| \leq |T| \).

**Proof.** Let \( R = \mathbb{Z}/p\mathbb{Z} \), \( r \in \mathbb{N} \) and \( H < G = (\mathbb{Z}/p\mathbb{Z})^r \) a cyclic subgroup. For \( H = \{0\} \) the assertion is obvious whence we suppose that \( |H| = p \) and set \( G = H' \oplus H \). Suppose that
\[ S = \prod_{v=1}^{r(p-1)} a_v \quad \text{and} \quad T = \prod_{v=r(p-1)+1-t}^{r(p-1)} a_v \]
with \( t = |T| \in \mathbb{N} \). If \( t \geq p \), the assertion is obvious. So we suppose that \( t \leq p-1 \).

For every \( i \in [1, |S|] \) we write \( a_i = b_i + c_i \) with \( b_i \in H' \) and \( c_i \in H \) and we set
\[ U' = \prod_{v=1}^{l} b_v \in \mathcal{F}(H') \quad \text{where} \quad l = r(p-1) - t. \]

**Theorem 3.9** implies that \( s(A^l_R(1), R') = l \) and **Lemma 6.1** yields that \( s(A^l_R(1) \setminus A^l_R(U'), R') \leq (r-1)(p-1) \). We have
\[ A^l_R(U') = \left\{ \sum_{i \in I} X_i \mid I \subset [1, l], \sum_{i \in I} b_i = 0 \right\} \]
\[ = \left\{ \sum_{i \in I} X_i \mid I \subset [1, l], \sum_{i \in I} a_i = \sum_{i \in I} c_i \in H \right\} \subset R^l = \langle X_1, \ldots, X_l \rangle_R. \]

Since \( 0 \notin \Sigma(S) \), it follows that
\[ 0 \notin \left\{ \sum_{i \in I} c_i = \sum_{i \in I} a_i \mid I \subset [1, l], \sum_{i \in I} b_i = 0 \right\} = \text{ev}_c(A^l_R(U')). \]

Using **Lemma 3.4** we infer that
\[ |\text{ev}_c(A^l_R(U'))| \geq s(A^l_R(U'), R') \]
\[ \geq s(A^l_R(1), R') - s(A^l_R(1) \setminus A^l_R(U'), R') \]
\[ \geq r(p-1) - t - (r-1)(p-1) \]
\[ = p - 1 - t. \]
Therefore we obtain that

Then \( A' + B' = A \cup B \cup (A + B) \cup \{0\}, A \cup B \cup (A + B) \subset \Sigma(S) \cap H \subset H \setminus \{0\} \) and \(|A \cup B \cup (A + B)| = |A' + B'| - 1. \) Since 0 has exactly one representation of the form \( 0 = a + b \) for some \( a \in A' \) and some \( b \in B' \), a theorem of Kemperman ([20], Theorem 3.2) implies that

\[
|A' + B'| \geq |A'| + |B'| - 1.
\]

Therefore we obtain that

\[
p - 1 = |H \setminus \{0\}| \geq |A \cup B \cup (A + B)| \geq |A'| + |B'| - 2 = |A| + |B| \geq p - 1 + (|\Sigma(T) \cap H| - |T|)
\]

whence \(|\Sigma(T) \cap H| \leq |T|\). \( \square \)

**Corollary 6.3.** Let \( G \) be an elementary \( p \)-group and \( S \in \mathcal{F}(G) \) a zero-sumfree sequence with maximal length. Then each two distinct elements of \( S \) are independent.

**Proof.** Let \( g_1, g_2 \) be two elements occurring in the sequence \( S \) and suppose that they are dependent. We have to show that \( g_1 = g_2 \). Clearly \( H = \langle g_1 \rangle = \langle g_2 \rangle \) is a cyclic subgroup of \( G \) and \( T = g_1 \cdot g_2 \) is a subsequence of \( S \) with \( \Sigma(T) = \{g_1, g_2, g_1 + g_2\} \subset H \). Then Theorem 6.2 implies that \(|\Sigma(T) \cap H| = |T| = 2\) whence \( g_1 = g_2 \). \( \square \)

**Remark 6.4.** Let \( G \) be an elementary \( p \)-group with rank \( r \).

1. Let \( S \in \mathcal{F}(G) \) be a zero-sumfree sequence with length \( r(p - 1) \) and \( g \in G \) with \( \nu_g(S) = i \in [1, p - 1] \). If \( H = \langle g \rangle \) and \( T = g^i \), then \( \Sigma(T) \cap H = \{vg \mid v \in [1, i]\} \) whence \(|\Sigma(T) \cap H| = i = |T| \). Thus Theorem 6.2 is sharp in this case.

2. We briefly discuss what is known about the structure of a minimal zero-sum sequence \( S \in \mathcal{F}(G) \) with maximal length i.e. with \(|S| = D(G) = r(p - 1) + 1\).

   a) If \( r = 1 \), then it is obvious that \( S \) has the form \( S = g^p \) for some \( 0 \neq g \in G \).
   b) If \( r = 2 \), it is conjectured that there exists some \( g \in G \) which occurs \( p - 1 \) times in \( S \) (i.e. with \( \nu_g(S) = p - 1 \); cf. Section 4 [13], [16] and [15]).
   c) If \( r \geq 2p - 1 \), then there exists some minimal zero-sum sequence \( T \in \mathcal{F}(G) \) with \(|T| = D(G)\), which is squarefree (i.e. \( \nu_g(S) \leq 1 \) for all \( g \in G \); cf. Theorem 7.3 in [16]).

Finally we study the Davenport constant for groups \( G = C_n^r \) where \( n \) is not necessarily a prime power. It is still conjectured that for every \( n \geq 2 \) and every \( r \geq 1 \) we have

\[
D(C_n^r) = r(n - 1) + 1
\]

(see [2]) but up to now there is no strong evidence why this should be true (cf. [7], page 462).

We conjecture that for \( R = \mathbb{Z}/n\mathbb{Z} \) and all \( r \in \mathbb{N} \)

\[
S(A_R^{(r-1)+1}(1), R^{(r-1)+1}) = rL(n) + 1.
\]

After a further lemma we show in our final result that \((**\)\) implies \((*)\).
Lemma 5.3. A result follows from Theorem 6.6 and Proposition 6.5 (3). For a second proof, let

\[ \text{Assume to the contrary that} \]

Proof. 1. Theorem 4.7 implies that \( s(R, R) = L(|R|) = L(n) \).

Let \( X_1, \ldots, X_l \) be a basis of \( R^l \) and for \( c \in R^l \) and \( f \in A \) let \( f(c) = ev_c(f) \).

Suppose that \( A \) is not a zero set over \( R \). Then there exists some \( c \in R^l \) such that

\[ \{ f(c) \mid f \in A \} \subset R \setminus \{0\}. \]

Suppose that

\[ R \setminus \{0\} \subset \bigcup_{i=1}^{t}(a_i + H_i) \]

where \( t = s(R, R) \) and for all \( i \in [1, t] \) let

\[ H_i = \langle m_i + n\mathbb{Z} \rangle \text{ with } 1 < m_i \mid n \text{ and } a_i \in R \setminus H_i. \]

For \( i \in [1, t] \) let

\[ A_i = \{ f \in A \mid f(c) \in a_i + H_i \} \]

and since for every \( f \in A_i \) we have \( f(\frac{a_i}{m_i}c) = \frac{a_i}{m_i}c \neq 0 \in \mathbb{Z}/n\mathbb{Z} \), it follows that

\[ s(A_i, R^l) = 1. \]

Since \( A = A_1 \cup \cdots \cup A_t \), we finally infer that \( s(A, R^l) \leq t. \)

2. By definition we have \( A^l_R((n-1, \ldots, n-1)) = R^l \) whence Theorem 4.7 implies that \( \sum_{i=1}^{r} s(A_i, R^l) = L(|R^l|) = L(n^l) = lL(n) \).

3. This follows from Theorem 3.9 and the definition of \( L(\cdot) \). \( \square \)

Theorem 6.5. Let \( G = C_n^l \) with \( n, r \in \mathbb{N}, n \geq 2 \) and suppose that

\[ s(A^{r(n-1)+1}_R(1), R^{r(n-1)+1}) = rL(n) + 1 \text{ where } R = \mathbb{Z}/n\mathbb{Z}. \]

Then \( D(G) > r(n-1) + 1. \)

Proof. Assume to the contrary that \( D(G) < r(n-1) + 1. \) Then by Lemma 5.3 \( A^{r(n-1)+1}_R(1) \) is not a zero set over \( G \). By Proposition 5.2 there exists a partition \( A^{r(n-1)+1}_R(1) = A_1 \cup \cdots \cup A_t \) such that no \( A_i \) is a zero set over \( R \). Then Lemma 3.4 and Proposition 6.5 imply that

\[ s(A^{r(n-1)+1}_R(1), R^{r(n-1)+1}) \leq \sum_{i=1}^{r} s(A_i, R^{r(n-1)+1}) \leq rL(n), \]

a contradiction. \( \square \)

Finally we point out that our methods give two new proofs of the well-known fact that

\[ D(C_n^l) = r(p - 1) + 1 \] (for a discussion of further proofs see [3], Section 6). Firstly, the result follows from Theorem 6.6 and Proposition 6.5(3). For a second proof, let \( S \in \mathcal{F}(C_n^l) \) be a sequence with \( |S| = l = r(p - 1) + 1. \) We have to show that \( S \) is not zero-sumfree. Lemma 6.1 and Proposition 6.5(3) imply that

\[ s(A^l_R(1) \setminus A^l_R(S), R^l) \leq r(p - 1) < r(p - 1) + 1 = s(A^l_R(1), R^l) \]

whence \( A^l_R(S), R^l \neq \emptyset \) and \( S \) is not zero-sumfree.
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References

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