Also available on SCIENCE DIRECT
www.sciencedirect.com

European Journal of Combinatorics 24 (2003) 531–549

European Journal of Combinatorics

www.elsevier.com/locate/ejc

Zero-sum problems and coverings by proper cosets

Weidong Gao[a], Alfred Geroldinger[b]

[a]*Department of Computer Science and Technology, University of Petroleum, Shuiku Road, Changping, Beijing 102200, People's Republic of China*
[b]*Institut für Mathematik, Karl-Franzens Universität, Heinrichstrasse 36, 8010 Graz, Austria*

Received 5 December 2001; received in revised form 24 February 2003; accepted 25 February 2003

Abstract

Let G be a finite Abelian group and $\mathsf{D}(G)$ its Davenport constant, which is defined as the maximal length of a minimal zero-sum sequence in G. We show that various problems on zero-sum sequences in G may be interpreted as certain covering problems. Using this approach we study the Davenport constant of groups of the form $(\mathbb{Z}/n\mathbb{Z})^r$, with $n \geq 2$ and $r \in \mathbb{N}$. For elementary p-groups G, we derive a result on the structure of minimal zero-sum sequences S having maximal length $|S| = \mathsf{D}(G)$.

1. Introduction

Let G be an additively written finite Abelian group and $S = \prod_{i=1}^{l} g_i$ a sequence in G. Then S is called a zero-sum sequence if $\sum_{i=1}^{l} g_i = 0$ and it is called zero-sumfree if $\sum_{i \in I} g_i \neq 0$ for all $\emptyset \neq I \subset [1, l]$. Key problems in zero-sum theory are to find the maximal possible length $l \in \mathbb{N}$ of zero-sumfree sequences, to determine the structure of such maximal sequences and to find in given sequences zero-sum subsequences satisfying additional properties.

A main aim of this paper is to present a new method in this area. We show that various zero-sum problems may be interpreted and successfully tackled as covering problems in finitely generated, free modules.

Let R be a commutative ring and M an R-module. A subset $C \subset M$ is called a proper coset, if $C = a + N$ for some R-submodule $N < M$ and some $a \in M \setminus N$. For given subsets $A \subset M$ we study the smallest number $s \in \mathbb{N}_0 \cup \{\infty\}$ such that $A \setminus \{0\}$ is contained in the union of s proper cosets. In Section 3 we concentrate on sets of subsums of zero-sumfree sequences in vectorspaces including cubes in vectorspaces. These investigations generalize former work on coverings by affine hyperplanes (resp. coverings by single-valued sets), and they might be of their own interest (see Theorem 3.9

E-mail addresses: wdgao@public.fhnet.cn.net (W. Gao), alfred.geroldinger@uni-graz.at (A. Geroldinger).

0014-5793/03/$ - see front matter
doi:10.1016/S0195-6698(03)00033-7

and the subsequent remark). Section 4 deals with finite Abelian groups M. We show that $\mathsf{s}(M, M) \le \sum_{p\in\mathbb{P}} \mathsf{v}_p(|M|)(p-1)$, and that equality holds, among others, for cyclic groups and elementary groups (see Theorem 4.7).

In Section 5 we build the bridge between covering problems and zero-sum problems. Section 6 contains our two main results on zero-sum sequences. Let $G = (\mathbb{Z}/n\mathbb{Z})^r$ with $r, n \in \mathbb{N}$, $n \ge 2$, and let $\mathsf{D}(G)$ denote the Davenport constant of G, which is defined as the maximal length of a minimal zero-sum sequence in G. Then $1 + r(n-1) \le \mathsf{D}(G)$, and equality holds, if G is a p-group. But even in the case where n is a prime, up to now only very little is known about the structure of minimal zero-sum sequences with maximal length (the theory of non-unique factorizations in Krull monoids naturally leads to questions about the structure of such sequences, cf. [5, 9, 17]). Theorem 6.2 presents a (sharp) structural result on zero-sumfree sequences with maximal length in elementary p-groups (see also Corollary 6.3 and the subsequent discussion). If n is not a prime power, it is still a conjecture that $\mathsf{D}(G) = 1 + r(n-1)$ holds true. In Theorem 6.6 we show that a certain covering condition implies that $\mathsf{D}(G) = 1 + r(n-1)$. In our opinion this result provides some theoretical evidence why the conjecture should be true and opens a way how to tackle it.

2. Preliminaries

Let $\mathbb{N}$ denote the positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{P} \subset \mathbb{N}$ the set of prime numbers. For some prime $p \in \mathbb{P}$ let $\mathsf{v}_p : \mathbb{N} \to \mathbb{N}_0$ denote the p-adic exponent whence $n = \prod_{p\in\mathbb{P}} p^{\mathsf{v}_p(n)}$ for every $n \in \mathbb{N}$. For integers $a, b \in \mathbb{Z}$ we set $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$.

Throughout, all Abelian groups will be written additively and for $n \in \mathbb{N}$ let $C_n = \mathbb{Z}/n\mathbb{Z}$ denote the cyclic group with n elements. Let G be a finite Abelian group. Then $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r$ if $|G| > 1$ and with $r = n_1 = 1$ if $|G| = 1$. Then $r = \mathsf{r}(G)$ is called the rank of G and $n_r = \exp(G)$ is the exponent of G. Whenever it is convenient we consider G as an R-module for $R = \mathbb{Z}/n_r\mathbb{Z}$. Clearly, the R-submodules of G coincide with the subgroups. In particular, if $n_r = p$, then G might be considered as an r-dimensional $\mathbb{Z}/p\mathbb{Z}$-vectorspace.

Let $\mathcal{F}(G)$ denote the free Abelian monoid with basis G. An element $S \in \mathcal{F}(G)$ is called a *sequence in G* and will be written in the form

$$S = \prod_{g\in G} g^{\mathsf{v}_g(S)} = \prod_{i=1}^{l} g_i \in \mathcal{F}(G).$$

A sequence $T \in \mathcal{F}(G)$ is called a *subsequence of S*, if there exists some $T' \in \mathcal{F}(G)$ such that $S = T \cdot T'$ (equivalently, $\mathsf{v}_g(T) \le \mathsf{v}_g(S)$ for every $g \in G$). As usual

$$\sigma(S) = \sum_{g\in G} \mathsf{v}_g(S) g = \sum_{i=1}^{l} g_i \in G$$

denotes the *sum of* S,

$$|S| = \sum_{g \in G} \mathsf{v}_g(S) = l \in \mathbb{N}_0$$

denotes the *length of* S and

$$\Sigma(S) = \left\{ \sum_{i \in I} g_i \mid \emptyset \neq I \subset [1, l] \right\} \subset G$$

the set of all possible subsums of S. Clearly, $|S| = 0$ if and only if $S = 1$ is the empty sequence. We say that the sequence S is

- *zero-sumfree*, if $0 \notin \Sigma(S)$,
- *a zero-sum sequence*, if $\sigma(S) = 0$,
- *a minimal zero-sum sequence*, if it is a zero-sum sequence and each proper subsequence is zero-sumfree.

All rings are commutative, they are supposed to have a unit element and all R-modules are unitary. Let R be a commutative ring, M be a free R-module with basis $X_1, \ldots, X_l$ and C an R-module. Then $\mathsf{r}(M) = l$ denotes its rank, and for every $\theta \in \mathrm{Hom}_R(M, C)$ there exists some $\boldsymbol{c} = (c_1, \ldots, c_l) \in C^l$ such that

$$\theta = \mathrm{ev}_{\boldsymbol{c}} : M \longrightarrow C$$
$$f = \sum_{i=1}^{l} \lambda_i X_i \longmapsto \theta(f) = f(\boldsymbol{c}) = \sum_{i=1}^{l} \lambda_i c_i,$$

whence θ is the evaluation homomorphism in $\boldsymbol{c}$, and we use the notation $\theta(f) = \mathrm{ev}_{\boldsymbol{c}}(f) = f(\boldsymbol{c})$ whenever it is convenient.

3. Coverings by proper cosets

Definition 3.1. Let R be a commutative ring and M an R-module.

(1) A subset $C \subset M$ is called a *proper coset*, if $C = a + N$ for some R-submodule $N < M$ and some $a \in M \setminus N$.
(2) For a subset $A \subset M$ let $\mathsf{s}(A, M)$ denote the smallest integer $s \in \mathbb{N}_0 \cup \{\infty\}$ such that $A \setminus \{0\}$ is contained in the union of s proper cosets.

By definition we have $\mathsf{s}(A, M) = 0$ if and only if $A \subset \{0\}$ and $\mathsf{s}(A, M) = 1$ if and only if A is contained in a proper coset.

In combinatorics various problems of the following type have been studied: find the minimal number of (proper) affine hyperplanes $H_1, \ldots, H_s$, which cover a given finite set of points A in a (real) finite-dimensional vector space. Of course, this minimal number is the same which is needed by a minimal covering of A by proper cosets, as is shown in the following simple lemma.

Lemma 3.2. *Let R be a field, M a free R-module of rank $r \in \mathbb{N}$ and $A \subset M$ a subset. Then $\mathsf{s}(A, M)$ is the smallest integer $s \in \mathbb{N}_0 \cup \{\infty\}$ such that $A \setminus \{0\} \subset \bigcup_{i=1}^{s} H_i$ where*

$H_1, \ldots, H_s$ are affine hyperplanes (i.e. $H_i = a_i + N_i$ where all N_i are free R-submodules of M with rank $r - 1$ and $a_i \in M \backslash N_i$).

Proof. If $N < M$ is an R-submodule and $a \in M \backslash N$, then $\langle a \rangle_R \cap N = \{0\}$. By base extension we obtain some R-submodule N^* with $N < N^* < M$, $\langle a \rangle_R \cap N^* = \{0\}$ and with rank $r - 1$. Thus every proper coset can be blown up to an affine hyperplane and since clearly every affine hyperplane not containing zero is a proper coset, the assertion follows. □

In a series of papers coverings by so-called single-valued sets have been studied: let R be a commutative ring, M a free R-module of finite rank and C an R-module (mainly the situation $C = R$ was considered). A subset $A \subset M$ is called single-valued if there is some $\theta \in \mathrm{Hom}_R(M, C)$ and some $b \in C \backslash \{0\}$ such that $\theta(A) = \{b\}$.

In the following lemma we point out that in a wide class of rings single-valued sets coincide with proper cosets.

Proposition 3.3. *Let R be a commutative ring, M an R-module and $\emptyset \neq A \subset M \backslash \{0\}$.*

(1) *Suppose there exists some R-module C, some $\theta \in \mathrm{Hom}_R(M, C)$ and some $b \in C \backslash \{0\}$ such that $\theta(A) = \{b\}$. Then $\mathsf{s}(A, M) = 1$.*
(2) *Suppose that $A = A_1 \cup \cdots \cup A_s$ with $\mathsf{s}(A_i, M) = 1$ for every $i \in [1, s]$. Then there exist an R-algebra $\epsilon : R \to C$, $\theta_1, \ldots, \theta_s \in \mathrm{Hom}_R(M, C)$ and elements $b_1, \ldots, b_s \in C \backslash \{0\}$ such that $\theta_i(A_i) = \{b_i\}$ for every $i \in [1, s]$. Furthermore, if R is an Artinian ring and an injective R-module and M a finitely generated R-module, then $C = R$ has the required property.*

Proof. 1. If $N = \{m \in M \mid \theta(m) = 0\}$, then $N < M$ is a proper R-submodule and $A \subset a + N$ for every $a \in A$ whence $\mathsf{s}(A, M) = 1$.

2. Suppose that for every $i \in [1, s]$ we have $A_i \subset a_i + N_i$ for some R-submodule $N_i < M$ and with $a_i \in M \backslash N_i$. By standard construction we build an R-algebra C out of the R-module $B = \oplus_{i=1}^s M/N_i$: we set $C = R \oplus B$, define $\epsilon : R \to C$ by $\epsilon(r) = (r, 0)$ and define multiplication on C by $(r, b)(r', b') = (rr', rb' + r'b)$ for all $r, r' \in R$ and all $b, b' \in B$. For every $i \in [1, s]$

$$\begin{aligned} \theta_i : M \to \quad & B \hookrightarrow C \\ m \mapsto (0, \ldots, 0, m + N_i, 0, \ldots, 0) &= b \mapsto (0, b) \end{aligned}$$

is an R-module homomorphism with $\theta(a_i) = b_i \neq 0$, $\theta_i(N_i) = \{0\}$ whence $\theta_i(A_i) \subset \theta_i(a_i + N_i) = \{b_i\}$.

Suppose that R is an Artinian ring, injective as an R-module and M a finitely generated R-module. Then R is zero-dimensional, semi-local and Noetherian. Let $N < M$ be an R-submodule and $a \in M \backslash N$. By D. Eisenbud [6, Propositions 21.2 and 21.5]

$$\begin{aligned} \epsilon : M/N &\longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M/N, R), R) \\ x + N &\longmapsto (\epsilon_x : \theta \mapsto \theta(x + N)) \end{aligned}$$

is an R-module isomorphism. Since $a + N \neq 0 \in M/N$, it follows that $\epsilon_a \neq 0$ whence there is some $\theta \in \mathrm{Hom}_R(M/N, R)$ with $\theta(a + N) \neq 0$. If $\pi : M \to M/N$ denotes the canonical projection, then $\theta \circ \pi : M \to R$ satisfies $\theta(N) = \{0\}$ and $\theta(a + N) \neq 0$. □

In the following lemma we summarize some basic properties of the $\mathsf{s}(\cdot, M)$-invariant.

Lemma 3.4. *Let R be a commutative ring, M an R-module and $A, B \subset M$.*

(1) $\mathsf{s}(A, M) \le |A|$.
(2) *Let C be an R-module and $\theta \in \mathrm{Hom}_R(M, C)$ such that $0 \notin \theta(A\setminus\{0\})$. Then* $\mathsf{s}(A, M) \le |\theta(A)|$.
(3) $\mathsf{s}(A \cup B, M) \le \mathsf{s}(A, M) + \mathsf{s}(B, M)$.
(4) *If $B \subset A$ with $\mathsf{s}(B, M) < \mathsf{s}(A, M)$, then $A\setminus B \ne \emptyset$.*
(5) *Suppose that $A = A_1 \cup \cdots \cup A_t$ where $\mathsf{s}(A_i, M) = 1$ for all $i \in [1, t]$ and $t = \mathsf{s}(A, M)$. Then for every non-empty set $I \subset [1, t]$ we have $\mathsf{s}(\bigcup_{i\in I} A_i, M) = |I|$.*

Proof. Without restriction we may suppose that $0 \notin A \cup B$.

1. Since $A = \bigcup_{a\in A}(a + \{0\})$, the assertion follows.
2. Since
$$A = \bigcup_{b\in\theta(A)} (A \cap \theta^{-1}(b))$$
and since by Proposition 3.3(1) $\mathsf{s}(A \cap \theta^{-1}(b), M) = 1$, it follows that $\mathsf{s}(A, M) \le |\theta(A)|$.
3. If $A = \bigcup_{i=1}^{\mathsf{s}(A,M)} A_i$ and $B = \bigcup_{j=1}^{\mathsf{s}(B,M)} B_j$ with proper cosets A_i, B_j, then $A \cup B$ is the union of the $A_i's$ and $B_j's$ whence $\mathsf{s}(A \cup B, M) \le \mathsf{s}(A, M) + \mathsf{s}(B, M)$.
4. Suppose that $B \subset A$ and $\mathsf{s}(B, M) < \mathsf{s}(A, M)$. Then $A = B \cup (A\setminus B)$ and
$$\mathsf{s}(B, M) < \mathsf{s}(A, M) \le \mathsf{s}(B, M) + \mathsf{s}(A\setminus B, M)$$
whence $\mathsf{s}(A\setminus B, M) \ne 0$ and $A\setminus B \ne \emptyset$.
5. Obvious. □

Definition 3.5. Let R be a commutative ring and M a free R-module with basis $X_1, \dots, X_l$ for some $l \in \mathbb{N}$.

(1) For every $\mathbf{0} \ne \boldsymbol{k} \in \mathbb{N}_0^l$ we set
$$A_R^l(\boldsymbol{k}) = A^l(\boldsymbol{k}) = A(\boldsymbol{k}) = \left\{ \sum_{i=1}^{l} a_i X_i \mid 0 \le a_i \le k_i \text{ for every } i \in [1, l] \right\} \subset M.$$
(2) Let G be an Abelian group and $S = \prod_{i=1}^{l} g_i \in \mathcal{F}(G)$ a sequence in G. We set
$$A_R^l(S) = A(S) = \left\{ \sum_{i\in I} X_i \mid \emptyset \ne I \subset [1, l], \sum_{i\in I} g_i = 0 \right\} \subset A_R^l(\mathbf{1}).$$

In particular, we write $A_R^l(\mathbf{1}) = A_R^l((1, \dots, 1))$ and we may interpret $A_R^l(\mathbf{1})$ as the set of vertices of the cube in M. Clearly, $A_R^l(\boldsymbol{k})$ depends on the choice of a basis in M but $\mathsf{s}(A_R^l(\boldsymbol{k}), M)$ is independent of the basis whence we simply write $\mathsf{s}(A_R^l(\boldsymbol{k}), R^l)$.

Whenever for a sequence S one has $\mathsf{s}(A_R^l(\mathbf{1})\setminus A_R^l(S), R^l) < \mathsf{s}(A_R^l(\mathbf{1}), R^l)$, then $A_R^l(S) \ne \emptyset$ whence S is not zero-sumfree. In this way we shall give a new proof that the Davenport constant of C_p^r equals $r(p-1)+1$ (see the discussion after Theorem 6.6).

Lemma 3.6. *Let R be a commutative ring, M an R-module, $\{X_1, \ldots, X_l\} \subset M$ an independent subset and $1 \ne S = \prod_{\nu=1}^{l} X_\nu^{m_\nu} \in \mathcal{F}(M)$.*

(1) $\Sigma(S) = A_R^l(\boldsymbol{m})\setminus\{0\} \subset \langle X_1, \ldots, X_l\rangle_R$ *and S is zero-sumfree if and only if either* $\operatorname{char}(R) = 0$ *or* $\boldsymbol{m} \in [0, \operatorname{char}(R) - 1]^l$.

(2) *Suppose that S is zero-sumfree and let $\boldsymbol{0} \ne \boldsymbol{k} \le \boldsymbol{m}$ and $I \subset [1, l]$. Then*

$$1 \le \mathsf{s}\left(\Sigma\left(\prod_{\nu=1}^{l} X_\nu^{k_\nu}\right), M\right) \le \mathsf{s}(\Sigma(S), M)$$

$$\le \mathsf{s}\left(\Sigma\left(\prod_{i\in[1,l]\setminus I} X_i^{m_i}\right), M\right) + \sum_{i\in I} m_i \le \sum_{i=1}^{l} m_i = |S|.$$

(3) *If* $\operatorname{char}(R) = n$ *and p is a prime divisor of n with $p < n$ and $p < l$, then* $\mathsf{s}(A_R^l(\boldsymbol{1}), R^l) \le l - 1$.

Proof. 1. By definition we have $\Sigma(S) = A_R^l(\boldsymbol{m})\setminus\{0\}$. S is zero-sumfree if and only if $0 \notin \Sigma(S)$ if and only if for every $\boldsymbol{k} \le \boldsymbol{m}$ the equation $\sum_{\nu=1}^{l} k_\nu X_\nu = 0$ implies that $\boldsymbol{k} = 0$. Since $X_1, \ldots, X_l$ are independent elements, the assertion follows.

2. Since $\Sigma(X_i)\setminus\{0\} = \{X_i\} = X_i + \{0\} \subset M$ is a proper coset, it follows that $\mathsf{s}(\Sigma(X_i), M) = 1$ for every $i \in [1, l]$. Since $\boldsymbol{0} \ne \boldsymbol{k} \le \boldsymbol{m}$, we have $\Sigma\left(\prod_{\nu=1}^{l} X_\nu^{k_\nu}\right) \subset \Sigma(S)$ and Lemma 3.4(3) implies that $\mathsf{s}\left(\Sigma\left(\prod_{\nu=1}^{l} X_\nu^{k_\nu}\right), M\right) \le \mathsf{s}(\Sigma(S), M)$.

Next we show that

$$\mathsf{s}(\Sigma(S), M) \le \mathsf{s}\left(\Sigma\left(\prod_{i=1}^{l-1} X_i^{m_i}\right), M\right) + m_l$$

which implies the remaining inequalities by an inductive argument.

Let $\nu \in [1, m_l]$. Since S is zero-sumfree and $\{X_1, \ldots, X_l\}$ is independent, we obtain that

$$0 \ne -\nu X_l \notin \Sigma\left(\prod_{i=1}^{l-1} X_i^{m_i}\right) \subset \langle X_1, \ldots, X_{l-1}\rangle_R$$

and that $\nu X_l \notin \langle X_1, \ldots, X_{l-1}\rangle_R$. Thus for

$$B_\nu = \{\nu X_l\} + \left(\Sigma\left(\prod_{i=1}^{l-1} X_i^{m_i}\right) \cup \{0\}\right) \subset \{\nu X_l\} + \langle X_1, \ldots, X_{l-1}\rangle_R$$

we obtain that $\mathsf{s}(B_\nu, R^l) = 1$. Since $\Sigma(S) = \Sigma\left(\prod_{i=1}^{l-1} X_i^{m_i}\right) \cup \bigcup_{\nu=1}^{m_l} B_\nu$, Lemma 3.4 implies the assertion.

3. Suppose $\{X_1, \ldots, X_l\}$ is a basis of R^l, $\operatorname{char}(R) = n$ and p a prime divisor of n with $p < \min\{n, l\}$. For $i \in \mathbb{N}$ we set

$$A_i = \left\{\sum_{j\in I} X_j \mid I \subset [1, l] \text{ with } |I| = i\right\}$$

and obtain that

$$\mathrm{ev}_{\boldsymbol{c}}(A_i) = i + n\mathbb{Z} \qquad \text{where } \boldsymbol{c} = (1 + n\mathbb{Z}, \dots, 1 + n\mathbb{Z})$$

whence $\mathsf{s}(A_i, R^l) = 1$. Furthermore, $\mathsf{s}(A_1 \cup A_{p+1}, R^l) = 1$, because

$$\mathrm{ev}_{\boldsymbol{c}}(A_1) = \frac{n}{p} + n\mathbb{Z} = \mathrm{ev}_{\boldsymbol{c}}(A_{p+1}) \qquad \text{for } \boldsymbol{c} = \left(\frac{n}{p} + n\mathbb{Z}, \dots, \frac{n}{p} + n\mathbb{Z}\right).$$

Thus we infer that

$$A_R^l(\mathbf{1}) = \bigcup_{i \in [1,l]} A_i = (A_1 \cup A_{p+1}) \cup \bigcup_{i \in [1,l] \setminus \{1, p+1\}} A_i$$

which implies that

$$\mathsf{s}(A_R^l(\mathbf{1}), R^l) \le l - 1. \quad \square$$

Lemma 3.7. *Let R be a commutative ring, $l \in \mathbb{N}$ and $k \in [1, l-1]$. In $R[X, Y_{i,j} \mid i \in [1,k],\ j \in [1,l]\,]$ we have the following polynomial identity:*

$$\sum_{\emptyset \ne J \subset [1,l]} (-1)^{|J|} \prod_{i=1}^{k} \left(X - \sum_{j \in J} Y_{i,j} \right) = -X^k.$$

Proof. see Lemma 9.3 in [16]. $\square$

Proposition 3.8. *Let R be a commutative ring, M an R-module, C an R-algebra and S a zero-sumfree sequence in M. Suppose $\Sigma(S) = A_1 \cup \cdots \cup A_k$ where $k < |S|$ and $\theta_i(A_i) = \{b_i\}$ with $\theta_i \in \mathrm{Hom}_R(M, C)$ and $b_i \in C \setminus \{0\}$ for all $i \in [1,k]$. Then $\prod_{i=1}^k b_i^k = 0$.*

Proof. Let $b = \prod_{i=1}^k b_i$ and for $i \in [1,k]$ we set $\theta_i' = \frac{b}{b_i} \cdot \theta_i \in \mathrm{Hom}_R(M, C)$ whence $\theta_i'(A_i) = \{b\}$.

Suppose that $S = \prod_{\nu=1}^{|S|} f_\nu$ and let $\emptyset \ne J \subset [1, |S|]$. Since $\Sigma(S) = A_1 \cup \cdots \cup A_k$, there exists some $\lambda \in [1,k]$ such that $\sum_{j \in J} f_j \in A_\lambda$. This implies that

$$\sum_{j \in J} \theta_\lambda'(f_j) = \theta_\lambda' \left(\sum_{j \in J} f_j \right) = b$$

whence

$$\prod_{i=1}^{k} \left(b - \sum_{j \in J} \theta_i'(f_j) \right) = 0.$$

Using Lemma 3.8 with $X = b$ and $Y_{i,j} = \theta_i'(f_j)$ we infer that $0 = -b^k$. $\square$

Theorem 3.9. *Let R be a field, $l \in \mathbb{N}$ and S be a zero-sumfree sequence in R^l.*

(1) $\mathsf{s}(\Sigma(S), R^l) \ge |S|$.
(2) *If* $\mathrm{supp}(S) \subset R^l$ *is independent, then* $\mathsf{s}(\Sigma(S), R^l) = |S|$.

(3) *Let* $\boldsymbol{k} \in \mathbb{N}^l$ *if* $\mathrm{char}(R) = 0$, *and* $\boldsymbol{k} \in [0, \mathrm{char}(R)-1]^l$ *otherwise. Then* $\mathsf{s}(A^l_R(\boldsymbol{k}), R^l) = \sum_{i=1}^l k_i$.

Proof. (1) Assume to the contrary that $\mathsf{s}(\Sigma(S), R^l) < |S|$. Then $\Sigma(S) = A_1 \cup \cdots \cup A_k$ with $k < |S|$ and $\mathsf{s}(A_1, R^l) = \cdots = \mathsf{s}(A_k, R^l) = 1$. By Proposition 3.3 there exist $\theta_i \in \mathrm{Hom}_R(R^l, R)$ and $b_i \in R \setminus \{0\}$ such that $\theta_i(A_i) = \{b_i\}$ for every $i \in [1, k]$. Thus Proposition 3.8 implies that $\prod_{\nu=1}^k b_\nu^k = 0$ whence $b_\nu = 0$ for some $\nu \in [1, k]$, a contradiction.

(2) Lemma 3.6(2) implies that $\mathsf{s}(\Sigma(S), R^l) \le |S|$ whence the assertion follows from (1).

(3) If $\{X_1, \ldots, X_l\}$ is a basis of R^l, then by Lemma 3.6(1) $T = \prod_{i=1}^l X_i^{k_i}$ is zero-sumfree and $\Sigma(T) = A^l_R(\boldsymbol{k}) \setminus \{0\}$. Hence the assertion follows from (2). □

Remark 3.10. For $\boldsymbol{k} = \mathbf{1}$ and $R = \mathbb{Z}/p\mathbb{Z}$ the result on $\mathsf{s}(A^l_R(\boldsymbol{k}), R^l)$ was proved by Gao in [11] and for $\boldsymbol{k} = \mathbf{1}$ and R the real numbers a first proof was given by Alon and Füredi in [1]. For further results of this type see also [4] and [16].

4. On $\mathsf{s}(M, M)$ for finite Abelian groups M

Let M be a finite Abelian group with exponent $\exp(M) = n$. Then M may be considered as an R-module with $R = \mathbb{Z}/n\mathbb{Z}$ and the R-submodules coincide with the subgroups of M. Since M is finite, Lemma 3.4 shows that

$$\mathsf{s}(M, M) \le |M| < \infty.$$

In this section we study $\mathsf{s}(M, M)$ and for simplicity we set $\mathsf{s}(M) = \mathsf{s}(M, M)$.

Definition 4.1. We define a homomorphism $\mathsf{L} : (\mathbb{N}, \cdot) \to (\mathbb{N}_0, +)$ by

$$\mathsf{L} : \mathbb{N} \to \mathbb{N}_0$$
$$n \mapsto \sum_{p \in \mathbb{P}} \mathsf{v}_p(n)(p-1).$$

Lemma 4.2. *Let M be a finite Abelian group.*

(1) *If $N < M$ is a subgroup, then*

$$\mathsf{s}(N) = \mathsf{s}(N, M) \le \mathsf{s}(M) \le \mathsf{s}(N) + \mathsf{s}(M/N).$$

(2) $\mathsf{s}(M) \le \mathsf{L}(|M|)$.

(3) *If* $\mathsf{s}(M) = \mathsf{L}(|M|)$, *then* $\mathsf{s}(N) = \mathsf{L}(|N|)$ *for all subgroups* $N < M$.

Proof. (1) Let $N < M$, $N \setminus \{0\} = \bigcup_{i=1}^{\mathsf{s}(N)} (g_i + N_i)$, with all $N_i < M$ and all $g_i \in M \setminus N_i$, and let $M/N \setminus \{N\} = \bigcup_{i=1}^{\mathsf{s}(M/N)} ((a_i + N) + H_i/N)$ with all $N < H_i < M$ and all $a_i \in M \setminus H_i$. If $x \in M \setminus N$, then there is some $i \in [1, \mathsf{s}(M/N)]$ such that

$x + N \in (a_i + N) + H_i/N$ whence $(x - a_i) + N \in H_i/N$, $x - a_i \in H_i$ and $x \in a_i + H_i$. Thus

$$M \setminus \{0\} = \bigcup_{i=1}^{\mathsf{s}(N)} (g_i + N_i) \cup \bigcup_{i=1}^{\mathsf{s}(M/N)} (a_i + H_i)$$

whence $\mathsf{s}(M) \le \mathsf{s}(N) + \mathsf{s}(M/N)$.

By definition we have $\mathsf{s}(N, M) \le \mathsf{s}(N, N)$ and $\mathsf{s}(N, M) \le \mathsf{s}(M, M)$. Hence it remains to verify that $\mathsf{s}(N) \le \mathsf{s}(N, M)$. Let $N \setminus \{0\} = \bigcup_{i=1}^{t} (g_i + N_i)$ with $t = \mathsf{s}(N, M)$, $N_i < M$ and $g_i \in M \setminus N_i$ for every $i \in [1, t]$. By the minimality of t we infer that there is some $0 \ne t_i \in N \cap (g_i + N_i)$ whence $t_i + N_i = g_i + N_i$ for every $i \in [1, t]$. Therefore it follows that

$$N \setminus \{0\} = \bigcup_{i=1}^{t} ((t_i + N_i) \cap N) = \bigcup_{i=1}^{t} (t_i + (N_i \cap N))$$

with $t_i \in N \setminus (N_i \cap N)$ for every $i \in [1, t]$. This implies that $\mathsf{s}(N) = \mathsf{s}(N, N) \le t = \mathsf{s}(N, M)$.

(2) We proceed by induction on $|M|$. If $|M| = 1$, then $\mathsf{s}(M) = 0 = \mathsf{L}(1)$. Suppose that $|M| > 1$ and let $N < M$ be a subgroup of index $(M : N) = p$ for some prime $p \in \mathbb{P}$. Then (1) and induction hypothesis imply that

$$\mathsf{s}(M) \le \mathsf{s}(N) + \mathsf{s}(M/N) \le \mathsf{s}(N) + (p - 1) = \mathsf{L}(|N|) + (p - 1) = \mathsf{L}(|M|).$$

(3) Suppose that $\mathsf{s}(M) = \mathsf{L}(|M|)$. It suffices to show that for all subgroups $N < M$ with $(M : N) \in \mathbb{P}$ we have $\mathsf{s}(N) = \mathsf{L}(|N|)$. Then the assertion follows by induction. Let $p \in \mathbb{P}$ and $N < M$ a subgroup with $(M : N) = p$. Using (1) and (2) we infer that

$$\mathsf{L}(|M|) = \mathsf{s}(M) \le \mathsf{s}(N) + (p - 1) \le \mathsf{L}(|N|) + (p - 1) = \mathsf{L}(|M|)$$

whence $\mathsf{s}(N) = \mathsf{L}(|N|)$. □

Lemma 4.3. *Let M be a finite Abelian group and $\theta < M$ a subgroup.*

(1) *Let $M \setminus \{0\} = \bigcup_{i=1}^{\mathsf{s}(M)} (g_i + N_i)$ where, for every $i \in [1, \mathsf{s}(M)]$, $N_i < M$ is a subgroup and $g_i \in M \setminus N_i$, and let I consist of those $i \in [1, \mathsf{s}(M)]$ such that $(g_i + N_i) \cap \theta \ne \emptyset$. If $x + \theta \not\subset \bigcup_{i \in I} (g_i + N_i)$ for every $x \in M \setminus \theta$, then $\mathsf{s}(M) \ge \mathsf{s}(\theta) + \mathsf{s}(M/\theta)$.*

(2) *If $\mathsf{s}(M) \ge \mathsf{s}(\theta) + \mathsf{s}(M/\theta)$, $\mathsf{s}(\theta) = \mathsf{L}(|\theta|)$ and $\mathsf{s}(M/\theta) = \mathsf{L}(|M/\theta|)$, then $\mathsf{s}(M) = \mathsf{L}(|M|)$.*

Proof. 1. We set $J = [1, \mathsf{s}(M)] \setminus I$. For every $i \in I$ there are $h_i \in N_i$ and $t_i \in \theta$ such that $g_i + h_i = t_i$ whence $g_i + N_i = t_i + N_i$, and since $N_i \ne g_i + N_i$, it follows that $t_i \notin N_i$. Thus we obtain that

$$(*) \qquad \theta \setminus \{0\} = \bigcup_{i \in I} ((t_i + N_i) \cap \theta) = \bigcup_{i \in I} (t_i + (N_i \cap \theta)).$$

We assert that

$$(**) \qquad M/\theta \setminus \{\theta\} = \bigcup_{j \in J} ((g_j + \theta) + (N_j + \theta)/\theta)$$

is a covering by proper cosets. Then $(*)$ and $(**)$ imply that

$$\mathsf{s}(M) = |I| + |J| \geq \mathsf{s}(\theta) + \mathsf{s}(M/\theta).$$

Let $i \in [1, \mathsf{s}(M)]$. If $g_i + \theta \in (N_i + \theta)/\theta$, then there is some $h_i \in N_i$ with $g_i + \theta = h_i + \theta$ whence $g_i \in N_i + \theta$ and $(g_i + N_i) \cap \theta \neq \emptyset$. Thus it follows that $(g_j + \theta) + (N_j + \theta)/\theta$ is a proper coset of M/θ for every $j \in J$.

To verify equality, let $x \in M \setminus \theta$. Since $x + \theta \not\subset \bigcup_{i \in I}(g_i + N_i)$, there exists some $y \in x + \theta$ and some $j \in J$ such that $y \in g_j + N_j$. Then $x + \theta = y + \theta \subset g_j + N_j + \theta$ whence $x + \theta \in (g_j + \theta) + (N_j + \theta)/\theta$.

2. Lemma 4.2 implies that

$$\mathsf{L}(|M|) \geq s(M) \geq \mathsf{s}(\theta) + \mathsf{s}(M/\theta) = \mathsf{L}(|\theta|) + \mathsf{L}(|M/\theta|) = \mathsf{L}(|M|)$$

whence the assertion follows. □

Proposition 4.4. *Let M be a finite Abelian group.*

(1) *If $M = M_1 \oplus M_2$ with $\gcd\{|M_1|, |M_2|\} = 1$, then $\mathsf{s}(M) \geq \mathsf{s}(M_1) + \mathsf{s}(M_2)$.*
(2) *If $M = \oplus_{i=1}^{k} M_i$ a direct decomposition into subgroups with $\gcd\{|M_i|, |M_j|\} = 1$ for all $1 \leq i < j \leq k$ and $\mathsf{s}(M_i) = \mathsf{L}(|M_i|)$ for every $i \in [1, k]$, then $\mathsf{s}(M) = \mathsf{L}(|M|)$.*
(3) *If $\exp(M) = \prod_{i=1}^{s} p_i^{n_i}$ and $\mathsf{s}((C_{p_i^{n_i}})^{r_i}) = \mathsf{L}(|(C_{p_i^{n_i}})^{r_i}|)$ where r_i is the p_i-rank of M, then $\mathsf{s}(M) = \mathsf{L}(|M|)$.*

Proof. (1) Suppose that $M = M_1 \oplus M_2$. We verify the assumption of Lemma 4.3 with $\theta = M_1$. Then the assertion follows. With all notations as in Lemma 4.3, let $x \in M \setminus M_1$ whence $x + M_1 = b + M_1$ for some $b \in M_2 \setminus \{0\}$. Then there exists some $\lambda \in [1, \mathsf{s}(M)]$ such that $b \in g_\lambda + N_\lambda$. It suffices to verify that

$$(g_\lambda + N_\lambda) \cap M_1 = \emptyset$$

(whence $\lambda \notin I$ and $b \notin \bigcup_{i \in I}(g_i + N_i)$).

We set $N_\lambda = H$ and since $\gcd\{|M_1|, |M_2|\} = 1$, it follows that $H = H_1 \oplus H_2$ with $H_i < M_i$. Assume to the contrary that

$$(b + H) \cap M_1 = (g_\lambda + H) \cap M_1 \neq \emptyset.$$

Then there are $h_1 \in H_1, h_2 \in H_2$ and $m_1 \in M_1$ such that $b + h_1 + h_2 = m_1$ whence $b + h_2 = m_1 - h_1 \in M_1 \cap M_2 = \{0\}$. Therefore $b = -h_2 \in H_2 < H$ and $g_\lambda + H = b + H = H$, a contradiction.

(2) Lemmas 4.2 and 4.3 imply that

$$\mathsf{L}(|M|) = \sum_{i=1}^{k} \mathsf{L}(|M_i|) = \sum_{i=1}^{k} \mathsf{s}(M_i) \leq \mathsf{s}(M) \leq \mathsf{L}(|M|).$$

(3) If for $i \in [1, s]$ M_i denotes the p_i-subgroup of M, then $M_i < (C_{p_i^{n_i}})^{r_i}$ and Lemma 4.2 implies that $\mathsf{s}(M_i) = \mathsf{L}(|M_i|)$. Thus the assertion follows from (1). □

Let M be a finite Abelian p-group. A subset $\{e_1, \dots, e_t\} \subset M$ is called independent, if $\sum_{i=1}^{t} m_i e_i = 0$, with $m_1, \dots, m_t \in \mathbb{Z}$, implies that $m_1 e_1 = \cdots = m_t e_t = 0$. Every independent subset is contained in a maximal independent subset, and each two maximal independent subsets have the same number of elements, which is denoted by $\mathsf{r}(M)$ and is called the *rank of* M. Let $\mathrm{soc}(M) = \{x \in M \mid px = 0\}$ denote the *socle of* M. Then $\mathrm{soc}(M)$ is an $\mathbb{F}_p$-vector space with $\dim_{\mathbb{F}_p}(\mathrm{soc}(M)) = \mathsf{r}(\mathrm{soc}(M)) = \mathsf{r}(M)$.

Lemma 4.5. *Let M be a finite Abelian p-group, $N < M$ a subgroup, $g \in M \setminus N$ and $\theta = \mathrm{soc}(M)$.*

(1) *If $\mathsf{r}(N) = \mathsf{r}(M)$, then $\theta < N$.*

(2) *If $g \in M \setminus N$, then there exists some $N^* < M$ such that $N < N^*$, $pg \in N^*$, $g \notin N^*$ and $\mathsf{r}(N^* + \langle g \rangle) = \mathsf{r}(M)$.*

(3) *If $\mathsf{r}(N + \langle g \rangle) = \mathsf{r}(M)$, $pg \in N$ and $(g + N) \cap \theta = \emptyset$, then $\mathsf{r}(N) = \mathsf{r}(M)$.*

(4) *If $M = (C_{p^n})^r$ with $r, n \in \mathbb{N}$ and $(g + N) \cap \theta \neq \emptyset$, then there are $e^* \in M$ and $N^* < M$ such that $M = \langle e^* \rangle \oplus N^*$, $N < N^*$ and $p^{n-1} e^* + N = g + N$.*

Proof. 1. Clearly, we have $\mathrm{soc}(N) < \mathrm{soc}(M)$. If $\mathsf{r}(N) = \mathsf{r}(M)$, then $\mathrm{soc}(N)$ and $\mathrm{soc}(M)$ are $\mathbb{F}_p$-vector spaces with the same dimension whence $\mathrm{soc}(M) = \mathrm{soc}(N) < N$.

2. Let $g \in M \setminus N$ and $N_1 = \langle N, pg \rangle$. Assume to the contrary, that $g \in N_1$. Then there are $a \in \mathbb{Z}$ and $h \in N$ such that $g = -a(pg) + h$ whence $(1 + ap)g = h$. If $x, y \in \mathbb{Z}$ with $x\, \mathrm{ord}(g) + y(1 + ap) = 1$, then $g = (1 - x\, \mathrm{ord}(g))g = yh \in N$, a contradiction.

If $\mathsf{r}(M) = \mathsf{r}(N_1 + \langle g \rangle)$, we set $N^* = N_1$. Suppose that $\mathsf{r}(M) > \mathsf{r}(N_1 + \langle g \rangle)$ and set $N_1 + \langle g \rangle = \oplus_{i=1}^{t} \langle e_i \rangle$ with $t = \mathsf{r}(N_1 + \langle g \rangle)$. Then $\{e_1, \dots, e_t\} \subset M$ is contained in a maximal independent subset $E \subset M$ whence $|E| = \mathsf{r}(M)$. We set $Q = \langle E \setminus \{e_1, \dots, e_t\} \rangle$ and $N^* = N_1 + Q$. Then $N < N_1 < N^*$, $pg \in N^*$ and $\mathsf{r}(N^* + \langle g \rangle) = \mathsf{r}(M)$. Assume to the contrary that $g \in N^*$. Then there are $n \in N_1$ and $q \in Q$ such that $g = n + q$ whence $q = g - n \in N_1 + \langle g \rangle \cap Q = \{0\}$ and $g = n \in N_1$, a contradiction.

3. Suppose that $(g + N) \cap \theta = \emptyset$ and $\mathsf{r}(N + \langle g \rangle) = \mathsf{r}(M)$. We assert that

$$\mathrm{soc}(N) = \mathrm{soc}(N + \langle g \rangle)$$

which implies that $\mathsf{r}(N) = \mathsf{r}(M)$. Obviously, $\mathrm{soc}(N) < \mathrm{soc}(N + \langle g \rangle)$, and we choose some $x \in \mathrm{soc}(N + \langle g \rangle)$. Since $pg \in N$, we have $x = ag + n$ with $n \in N$ and $a \in [0, p-1]$. Assume to the contrary that $a > 0$. Then there is some $a' \in [1, p-1]$ and some $k \in \mathbb{Z}$ such that $aa' = 1 + kp$. Then $a'x = g + n'$ with $n' = kpg + a'n \in N$. Then $0 = a'px$, but $g + n' \in g + N$ implies that $p(g + n') \neq 0$, a contradiction.

4. Let $M = (C_{p^n})^r$ with $r, n \in \mathbb{N}$ and $(g + N) \cap \theta \neq \emptyset$. Then there is some $e \in \theta$ and some $n \in N$ such that $e = g - n$ whence $g + N = e + N$. Since $e \notin N$ and $pe = 0$, it follows that $\langle e \rangle \cap N = \{0\}$. There is some $e^* \in M$ with $p^{n-1} e^* = e$ and obviously $\langle e^* \rangle \cap N = \{0\}$. Then N is contained in a maximal subset N^* such that $\langle e^* \rangle \cap N^* = \{0\}$. Thus we obtain that $M = \langle e^* \rangle \oplus N^*$ (cf. [22], 4.2.7). □

Proposition 4.6. *Let M be a finite Abelian p-group.*

(1) *If M is elementary, then $\mathsf{s}(M) = \mathsf{L}(|M|)$.*
(2) *If M is cyclic, then $\mathsf{s}(M) = \mathsf{L}(|M|)$.* □

Proof. (1) If M is elementary with basis $X_1, \dots, X_l$, then $M = A^l_{\mathbb{Z}/p\mathbb{Z}}((p-1, \dots, p-1))$ whence the assertion follows from Theorem 3.9.

(2) Let M be a cyclic group. We proceed by induction on $|M|$. If $|M| = p$, then M is elementary and the assertion follows from (1). To do the induction step, we set $\theta = \operatorname{soc}(M)$. If we can verify the assumption of Lemma 4.3, then the assertion follows.

Let $M\backslash\{0\} = \bigcup_{i=1}^{\mathsf{s}(M)}(g_i + N_i)$ where, for every $i \in [1, \mathsf{s}(M)]$, $N_i < M$ is a subgroup and $g_i \in M\backslash N_i$. Let I consist of those $i \in [1, \mathsf{s}(M)]$ such that $(g_i + N_i) \cap \theta \neq \emptyset$. By Lemma 4.5 we may suppose that $pg_i \in N_i$ for every $i \in [1, \mathsf{s}(M)]$.

Let $x \in M\backslash\theta$. We have to verify that

$$x + \theta \not\subset \bigcup_{i \in I}(g_i + N_i).$$

If $\lambda \in [1, \mathsf{s}(M)]$ with $x \in g_\lambda + N_\lambda$, then $0 \neq px \in N_\lambda$ whence $1 = \mathsf{r}(N_\lambda) = \mathsf{r}(M)$. Thus $\theta < N_\lambda$, $(g_\lambda + N_\lambda) \cap \theta \subset (g_\lambda + N_\lambda) \cap N_\lambda = \emptyset$ whence $\lambda \notin I$ and $x \notin \bigcup_{i \in I}(g_i + N_i)$. □

Theorem 4.7. *Let M be a finite Abelian group. If $M = M_1 \oplus M_2$, where M_1 is cyclic, $\exp(M_2)$ squarefree and $\gcd\{|M_1|, |M_2|\} = 1$, then $\mathsf{s}(M) = \mathsf{L}(|M|)$.*

Proof. If M_1 is cyclic, then M_1 is a direct sum of cyclic groups of prime power order. If $\exp(M_2)$ is squarefree, then M_2 is a direct sum of elementary p-groups. Thus the assertion follows from Propositions 4.4 and 4.6. □

5. Zero sets

Zero sets play a crucial part in establishing the connection between covering problems and zero-sum problems.

Definition 5.1. Let R be a commutative ring and C an R-module. A subset A of a finitely generated free R-module M is called a *zero set over C*, if $0 \in \theta(A)$ for every $\theta \in \operatorname{Hom}_R(M, C)$.

We continue with a characterization of zero sets in the case where C is a direct sum of submodules. If R is a field and C an R-vector space, then zero sets allow a very simple characterization.

Proposition 5.2. *Let R be a commutative ring, $k \in \mathbb{N}$ and $C = \oplus_{i=1}^k C_i$ an R-module.*

(1) *For a subset A of some finitely generated free R-module M the following conditions are equivalent:*

  (a) *A is a zero set over C.*

(b) *For every partition (resp. for every decomposition)* $A = A_1 \cup \cdots \cup A_k$ *there is some* $i \in [1, k]$ *such that* A_i *is a zero set over* C_i.

(2) *Suppose that* R *is a field and* $C_1 = \cdots = C_k = R$. *For a subset* $A \subset R^l$ *the following conditions are equivalent:*

(a) A *is a zero set over* C.
(b) $A \cap H \neq \emptyset$ *for all submodules* $H < R^l$ *with* $\mathsf{r}(H) \geq l - k$.

(3) *Suppose that* $R = \mathbb{Z}/p\mathbb{Z}$ *for some prime* $p \in \mathbb{P}$ *and let* $C = \mathbb{F}_q$ *be the field with* $q = p^k$ *elements. For a subset* $A \subset \langle X_1, \ldots, X_l \rangle_R \subset R[X_1, \ldots, X_l]$ *the following conditions are equivalent:*

(a) A *is a zero set over* C.
(b) $\prod_{f \in A} f \in \langle X_i^q - X_i \mid i \in [1, l] \rangle_R$.

Proof. 1. $(a) \Longrightarrow (b)$ Assume to the contrary that $A = A_1 \cup \cdots \cup A_k$ and no A_i is a zero set over C_i. Hence for every $i \in [1, k]$ there is some $\theta_i \in \mathrm{Hom}_R(M, C_i)$ such that $\theta_i(A_i) \subset C_i \setminus \{0\}$. Therefore $\theta = (\theta_1, \ldots, \theta_k) \in \mathrm{Hom}_R(M, C)$ and $\theta(A) \subset C \setminus \{0\}$, a contradiction.

$(b) \Longrightarrow (a)$ Assume to the contrary that A is not a zero set over C. Then there is some $\theta = (\theta_1, \ldots, \theta_k) \in \mathrm{Hom}_R(M, C)$ such that $\theta(A) \subset C \setminus \{0\}$. For $i \in [1, k]$ we set $A_i = \{a \in A \mid \theta_i(a) \neq 0\}$ and obtain that $A = A_1 \cup \cdots \cup A_k$ and no A_i is a zero set over C_i, a contradiction.

2. Every submodule $H < R^l$ with $\mathsf{r}(H) \geq l - k$ is the intersection of k (not necessarily different) hyperplanes, say $H = H_1 \cap \cdots \cap H_k$, and for every H_i there is some $\theta_i \in \mathrm{Hom}_R(R^l, R)$ such that $H_i = \ker(\theta_i)$. Thus $A \cap H \neq \emptyset$ if and only if there is some $a \in A$ such that for $\theta = (\theta_1, \ldots, \theta_k) \in \mathrm{Hom}_R(R^l, R^k)$ we have $\theta(a) = 0$ whence the assertion follows.

3. A is a zero set over $\mathbb{F}_q$ if and only if for all $\theta \in \mathrm{Hom}_R(R[X_1, \ldots, X_l], \mathbb{F}_q)$ we have $0 \in \theta(A)$, which holds if and only if for all $\boldsymbol{c} \in \mathbb{F}_q^l$ there is some $f \in A$ such that $f(\boldsymbol{c}) = 0$. This is equivalent to the fact that for all $\boldsymbol{c} \in \mathbb{F}_q^l$ we have $\prod_{f \in A} f(\boldsymbol{c}) = 0$ and the assertion follows. □

In the following we want to point out that many classical problems in zero-sum theory allow a straightforward formulation in terms of zero sets.

Let G be a finite Abelian group with exponent n. A first problem, which is still unsolved for general G, is to determine the *Davenport constant* $\mathsf{D}(G)$ *of* G which is defined as the maximal length of a minimal zero-sum sequence in G (equivalently, $\mathsf{D}(G)$ is the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ with $|S| \geq l$ contains a zero-sum subsequence). The paper of Erdös–Ginzburg–Ziv [10] was a starting point for investigations of subsequences of given sequences which have sum zero and satisfy certain additional properties. For a subset $\Lambda \subset \mathbb{N}$ let $\eta_\Lambda(G)$ denote the smallest integer $l \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ has a zero-sum subsequence T with $|T| \in \Lambda$.

Clearly, $\eta_{\mathbb{N}}(G)$ is just the Davenport constant $\mathsf{D}(G)$ and the invariants $\eta_\Lambda(G)$ for $\Lambda = \{|G|\}$, $\Lambda = \{n\}$ and $\Lambda = [1, n]$ have found considerable attention in the literature (cf. [8, 12, 19, 24] and the references given there). If $\Lambda = \{\lambda\}$, then we set $\eta_\lambda(G) = \eta_\Lambda(G)$.

Main Lemma 5.3. *Let G be a finite Abelian group with exponent n, $R = \mathbb{Z}/n\mathbb{Z}$ and $\Lambda \subset \mathbb{N}$ a subset. Then $\eta_\Lambda(G)$ is the smallest integer $l \in \mathbb{N}$ such that the subset*

$$A = \left\{ \sum_{i \in I} X_i \mid I \subset [1, l], |I| \in \Lambda \right\} \subset A_R^l(\mathbf{1}) \subset R^l = \langle X_1, \ldots, X_l \rangle_R$$

is a zero set over the R-module G.

Proof. Recall that for every $\theta \in \mathrm{Hom}_R(R^l, G)$ there is some $\boldsymbol{c} \in G^l$ such that $\theta = \mathrm{ev}_{\boldsymbol{c}} : R^l \to G$. A sequence $S = \prod_{i=1}^l c_i \in \mathcal{F}(G)$ has a zero-sum subsequence $T = \prod_{i \in I} c_i$ with $|T| = |I| \in \Lambda$ if and only if there exists some $f \in A$ such that $\mathrm{ev}_{\boldsymbol{c}}(f) = f((c_1, \ldots, c_l)) = 0$. This implies the assertion. □

Hence in this interpretation of zero-sum problems we fix the R-module C (here $C = \mathbb{Z}/n\mathbb{Z}$) and vary over the ranks of the free R-modules M. This motivates the following definition.

Definition 5.4. Let R be a commutative ring. For an R-module C we set

$$\mathsf{s}^*(C) = \sup\{\mathsf{s}(A, M) \mid A \text{ is a subset of a free } R\text{-module } M \text{ with finite rank and } A \text{ is not a zero set over } C\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Proposition 5.5. *Let R be a commutative ring and C an R-module.*

(1) $\mathsf{s}^*(C) \le |C| - 1$.
(2) *A subset A of some free R-module M with finite rank, which satisfies $\mathsf{s}(A, M) \ge k\mathsf{s}^*(C) + 1$, is a zero set over C^k.*

Proof. (1) If a subset A of some free R-module M with finite rank is not a zero set over C and $\theta \in \mathrm{Hom}_R(M, C)$ such that $\theta(A) \subset C \setminus \{0\}$, then Lemma 3.4(2) implies that

$$\mathsf{s}(A, M) \le |\theta(A)| \le |C| - 1.$$

Thus we obtain that $\mathsf{s}^*(C) \le |C| - 1$.

(2) Let A be a set having the above properties and assume to the contrary, that A is not a zero set over C^k. Then by Proposition 5.2(1) there exists a partition $A = A_1 \cup \cdots \cup A_k$ such that no A_i is a zero set over C. This implies that

$$\mathsf{s}(A, M) \le \sum_{i=1}^{k} s(A_i, M) \le k\mathsf{s}^*(C),$$

a contradiction. □

At the end of this section we want to show in an explicit example how zero-sum problems can be attacked via zero sets (see also Theorem 6.6).

Let $n \in \mathbb{N}$ be a positive integer with $n \ge 2$. An old conjecture, going back to Kemnitz, states that

$$\eta_n(C_n \oplus C_n) = 4n - 3.$$

It is easy to see that $\eta_n(C_n \oplus C_n) \ge 4n - 3$ and quite recently it was proved that for prime powers we have $\eta_n(C_n \oplus C_n) \le 4n - 2$ (see [14, 21, 23]).

Main Lemma 5.6. *Suppose that for every prime $p \in \mathbb{P}$ and for*

$$A = \left\{ \sum_{i \in I} X_i \mid I \subset [1, l], |I| = p \right\} \subset R^l = \langle X_1, \ldots, X_l \rangle_R,$$

where $R = \mathbb{Z}/p\mathbb{Z}$ and $l = 4p-3$, we have $\mathsf{s}(A, R^l) \geq 2p-1$. Then $\eta_n(C_n \oplus C_n) = 4n-3$ for every $n \in \mathbb{N}$.

Proof. It suffices to verify that

$$(*) \quad \eta_n(C_n \oplus C_n) \leq 4n - 3.$$

Since $\eta_n(\cdot)$ is multiplicative, it suffices to show $(*)$ for prime numbers.

Let $p \in \mathbb{P}$ be a prime number and $R = \mathbb{Z}/p\mathbb{Z}$. Using Proposition 5.5 we infer that $\mathsf{s}^*(R) \leq p - 1$ and

$$\mathsf{s}(A, R^{4p-3}) \geq 2p - 1 \geq 2(\mathsf{s}^*(R)) + 1$$

whence A is a zero set over $R \oplus R$. Thus Main Lemma 5.3 implies that $\eta_p(R \oplus R) \leq 4p - 3$. □

6. The case $G = C_n^r$

In this final section we concentrate on groups G of the form $G = C_n^r$ and study the maximal possible length of minimal zero-sum sequences in G and consider the structure of such sequences. To begin with, let $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r$. Then it is easy to see that $\mathsf{D}(G) \geq 1 + \sum_{i=1}^r (n_i - 1)$. Equality holds for p-groups and for groups with rank $r \leq 2$, but for every $r \geq 4$ there are infinitely many groups for which the above inequality is strict (see [13], Theorem 3.3 in [16, 18] and the references cited there). Although for p-groups the precise value of the Davenport constant is known, we have almost no information about the structure of minimal zero-sum sequences S with $|S| = \mathsf{D}(G)$. We start with a structural result for such sequences in elementary p-groups (Theorem 6.2 and Corollary 6.3). Then we consider the Davenport constant for groups $G = C_n^r$ where n is not a prime power.

Lemma 6.1. *Let $G = C_n^r$ with $n, r \in \mathbb{N}$, $n \geq 2$ and $S = \prod_{i=1}^l g_i \in \mathcal{F}(G)$. Then $\mathsf{s}(A_R^l(\mathbf{1}) \setminus A_R^l(S), R^l) \leq r(n-1)$ where $R = \mathbb{Z}/n\mathbb{Z}$.*

Proof. Let $\{e_1, \ldots, e_r\}$ be a basis of G. For every $i \in [1, l]$ we set $g_i = \sum_{\nu=1}^r c_{\nu,i} e_\nu$ with $c_{\nu,i} \in \mathbb{Z}$. For $\nu \in [1, r]$ and $m \in [1, n-1]$ let

$$\begin{aligned} A_{\nu,m} &= \left\{ \sum_{i \in I} X_i \in A_R^l(\mathbf{1}) \mid I \subset [1, l], \sum_{i \in I} (c_{\nu,i} + n\mathbb{Z}) = m + n\mathbb{Z} \right\} \\ &\subset \langle X_1, \ldots, X_l \rangle_R = R^l. \end{aligned}$$

Then $\mathsf{s}(A_{\nu,m}, R^l) = 1$, since for $\boldsymbol{c}_\nu = (c_{\nu,1} + n\mathbb{Z}, \ldots, c_{\nu,l} + n\mathbb{Z}) \in R^l$ we have

$$\mathrm{ev}_{\boldsymbol{c}_\nu}(A_{\nu,m}) = \sum_{i \in I} (c_{\nu,i} + n\mathbb{Z}) = m + n\mathbb{Z} \in R \setminus \{0\}.$$

Hence it suffices to prove that

$$A_R^l(\mathbf{1})\backslash A_R^l(S) \subset \bigcup_{\nu=1}^{r}\bigcup_{m=1}^{n-1} A_{\nu,m}.$$

To verify the inclusion, let $f = \sum_{i\in I} X_i \in A_R^l(\mathbf{1})\backslash A_R^l(S) \subset R^l$. Then $\sum_{i\in I} g_i \neq 0$ whence there exists some $\nu \in [1, r]$ with $\sum_{i\in I} c_{\nu,i} e_\nu \neq 0$. Therefore, $\sum_{i\in I} c_{\nu,i} e_\nu = m + n\mathbb{Z}$ for some $m \in [1, n-1]$ i.e. $f \in A_{\nu,m}$. □

Theorem 6.2. *Let G be an elementary p-group and $S \in \mathcal{F}(G)$ a zero-sumfree sequence with maximal length. Then for every subsequence T of S and every cyclic subgroup H of G we have $|\Sigma(T) \cap H| \le |T|$.*

Proof. Let $R = \mathbb{Z}/p\mathbb{Z}$, $r \in \mathbb{N}$ and $H < G = (\mathbb{Z}/p\mathbb{Z})^r$ a cyclic subgroup. For $H = \{0\}$ the assertion is obvious whence we suppose that $|H| = p$ and set $G = H' \oplus H$. Suppose that

$$S = \prod_{\nu=1}^{r(p-1)} a_\nu \qquad \text{and} \qquad T = \prod_{\nu=r(p-1)+1-t}^{r(p-1)} a_\nu$$

with $t = |T| \in \mathbb{N}$. If $t \ge p$, the assertion is obvious. So we suppose that $t \le p-1$.

For every $i \in [1, |S|]$ we write $a_i = b_i + c_i$ with $b_i \in H'$ and $c_i \in H$ and we set

$$U' = \prod_{\nu=1}^{l} b_\nu \in \mathcal{F}(H') \qquad \text{where } l = r(p-1) - t.$$

Theorem 3.9 implies that $\mathsf{s}(A_R^l(\mathbf{1}), R^l) = l$ and Lemma 6.1 yields that $\mathsf{s}(A_R^l(\mathbf{1})\backslash A_R^l(U'), R^l) \le (r-1)(p-1)$. We have

$$\begin{aligned} A_R^l(U') &= \left\{ \sum_{i\in I} X_i \mid I \subset [1,l], \sum_{i\in I} b_i = 0 \right\} \\ &= \left\{ \sum_{i\in I} X_i \mid I \subset [1,l], \sum_{i\in I} a_i = \sum_{i\in I} c_i \in H \right\} \subset R^l = \langle X_1, \ldots, X_l \rangle_R. \end{aligned}$$

Since $0 \notin \Sigma(S)$, it follows that

$$0 \notin \left\{ \sum_{i\in I} c_i = \sum_{i\in I} a_i \mid I \subset [1,l], \sum_{i\in I} b_i = 0 \right\} = \mathrm{ev}_{\mathbf{c}}(A_R^l(U')).$$

Using Lemma 3.4 we infer that

$$\begin{aligned} |\mathrm{ev}_{\mathbf{c}}(A_R^l(U'))| &\ge \mathsf{s}(A_R^l(U'), R^l) \\ &\ge \mathsf{s}(A_R^l(\mathbf{1}), R^l) - \mathsf{s}(A_R^l(\mathbf{1})\backslash A_R^l(U'), R^l) \\ &\ge r(p-1) - t - (r-1)(p-1) \\ &= p - 1 - t. \end{aligned}$$

We set

$$A = \Sigma(T) \cap H,\ A' = A \cup \{0\},\ B = \mathrm{ev}_{\boldsymbol{c}}(A_R^l(U')) \qquad \text{and} \qquad B' = B \cup \{0\}.$$

Then $A' + B' = A \cup B \cup (A + B) \cup \{0\}$, $A \cup B \cup (A + B) \subset \Sigma(S) \cap H \subset H \setminus \{0\}$ and $|A \cup B \cup (A + B)| = |A' + B'| - 1$. Since 0 has exactly one representation of the form $0 = a + b$ for some $a \in A'$ and some $b \in B'$, a theorem of Kemperman ([20], Theorem 3.2) implies that

$$|A' + B'| \geq |A'| + |B'| - 1.$$

Therefore we obtain that

$$\begin{aligned} p - 1 &= |H \setminus \{0\}| \geq |A \cup B \cup (A + B)| \geq |A'| + |B'| - 2 \\ &= |A| + |B| \geq p - 1 + (|\Sigma(T) \cap H| - |T|) \end{aligned}$$

whence $|\Sigma(T) \cap H| \leq |T|$. □

Corollary 6.3. *Let G be an elementary p-group and $S \in \mathcal{F}(G)$ a zero-sumfree sequence with maximal length. Then each two distinct elements of S are independent.*

Proof. Let g_1, g_2 be two elements occurring in the sequence S and suppose that they are dependent. We have to show that $g_1 = g_2$. Clearly $H = \langle g_1 \rangle = \langle g_2 \rangle$ is a cyclic subgroup of G and $T = g_1 \cdot g_2$ is a subsequence of S with $\Sigma(T) = \{g_1, g_2, g_1 + g_2\} \subset H$. Then Theorem 6.2 implies that $|\Sigma(T) \cap H| \leq |T| = 2$ whence $g_1 = g_2$. □

Remark 6.4. Let G be an elementary p group with rank r.

(1) Let $S \in \mathcal{F}(G)$ be a zero-sumfree sequence with length $r(p - 1)$ and $g \in G$ with $\mathsf{v}_g(S) = i \in [1, p - 1]$. If $H = \langle g \rangle$ and $T = g^i$, then $\Sigma(T) \cap H = \{\nu g \mid \nu \in [1, i]\}$ whence $|\Sigma(T) \cap H| = i = |T|$. Thus Theorem 6.2 is sharp in this case.

(2) We briefly discuss what is known about the structure of a minimal zero-sum sequence $S \in \mathcal{F}(G)$ with maximal length i.e. with $|S| = \mathsf{D}(G) = r(p - 1) + 1$.

   (a) If $r = 1$, then it is obvious that S has the form $S = g^p$ for some $0 \neq g \in G$.
   (b) If $r = 2$, it is conjectured that there exists some $g \in G$ which occurs $p - 1$ times in S (i.e. with $\mathsf{v}_g(S) = p - 1$; cf. Section 4 [13], [16] and [15]).
   (c) If $r \geq 2p - 1$, then there exists some minimal zero-sum sequence $T \in \mathcal{F}(G)$ with $|T| = \mathsf{D}(G)$, which is squarefree (i.e. $\mathsf{v}_g(S) \leq 1$ for all $g \in G$; cf. Theorem 7.3 in [16]).

Finally we study the Davenport constant for groups $G = C_n^r$ where n is not necessarily a prime power. It is still conjectured that for every $n \geq 2$ and every $r \geq 1$ we have

(∗) $\qquad \mathsf{D}(C_n^r) = r(n - 1) + 1$

(see [2]) but up to now there is no strong evidence why this should be true (cf. [7], page 462).

We conjecture that for $R = \mathbb{Z}/n\mathbb{Z}$ and all $r \in \mathbb{N}$

(∗∗) $\qquad \mathsf{s}(A_R^{r(n-1)+1}(\mathbf{1}), R^{r(n-1)+1}) = r\mathsf{L}(n) + 1.$

After a further lemma we show in our final result that (∗∗) implies (∗).

Proposition 6.5. *Let $R = \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 2$ and $A \subset R^l$ for some $l \in \mathbb{N}$.*

(1) *If A is not a zero set over R, then $\mathsf{s}(A, R^l) \leq \mathsf{s}(R, R) = \mathsf{L}(n)$.*
(2) *If n is a product of distinct primes, then $\mathsf{s}(A^l_R((n-1, \ldots, n-1)), R^l) = l\mathsf{L}(n)$.*
(3) *If n is prime and $l = r(n-1)+1$ for some $r \in \mathbb{N}_0$, then $\mathsf{s}(A^l_R(\mathbf{1}), R^l) = r\mathsf{L}(n) + 1$.*

Proof. 1. Theorem 4.7 implies that $\mathsf{s}(R, R) = \mathsf{L}(|R|) = \mathsf{L}(n)$.

Let $X_1, \ldots, X_l$ be a basis of R^l and for $\boldsymbol{c} \in R^l$ and $f \in A$ let $f(\boldsymbol{c}) = \mathrm{ev}_{\boldsymbol{c}}(f)$. Suppose that A is not a zero set over R. Then there exists some $\boldsymbol{c} \in R^l$ such that

$$\{f(\boldsymbol{c}) \mid f \in A\} \subset R \setminus \{0\}.$$

Suppose that

$$R \setminus \{0\} \subset \bigcup_{i=1}^{t} (a_i + H_i)$$

where $t = \mathsf{s}(R, R)$ and for all $i \in [1, t]$ let $H_i = \langle m_i + n\mathbb{Z} \rangle$ with $1 < m_i \mid n$ and $a_i \in R \setminus H_i$. For $i \in [1, t]$ let

$$A_i = \{f \in A \mid f(\boldsymbol{c}) \in a_i + H_i\}$$

and since for every $f \in A_i$ we have $f(\frac{n}{m_i}\boldsymbol{c}) = \frac{n}{m_i} a_i \neq 0 \in \mathbb{Z}/n\mathbb{Z}$, it follows that $\mathsf{s}(A_i, R^l) = 1$. Since $A = A_1 \cup \cdots \cup A_t$, we finally infer that $\mathsf{s}(A, R^l) \leq t$.

2. By definition we have $A^l_R((n-1, \ldots, n-1)) = R^l$ whence Theorem 4.7 implies that $\mathsf{s}(R^l, R^l) = \mathsf{L}(|R^l|) = \mathsf{L}(n^l) = l\mathsf{L}(n)$.

3. This follows from Theorem 3.9 and the definition of $\mathsf{L}(\cdot)$. □

Theorem 6.6. *Let $G = C_n^r$ with $n, r \in \mathbb{N}$, $n \geq 2$ and suppose that $\mathsf{s}(A_R^{r(n-1)+1}(\mathbf{1}), R^{r(n-1)+1}) = r\mathsf{L}(n) + 1$ where $R = \mathbb{Z}/n\mathbb{Z}$. Then $D(G) = r(n-1)+1$.*

Proof. Assume to the contrary that $D(G) > r(n-1)+1$. Then by Lemma 5.3 $A_R^{r(n-1)+1}(\mathbf{1})$ is not a zero set over G. By Proposition 5.2 there exists a partition $A_R^{r(n-1)+1}(\mathbf{1}) = A_1 \cup \cdots \cup A_r$ such that no A_i is a zero set over R. Then Lemma 3.4 and Proposition 6.5 imply that

$$\mathsf{s}(A_R^{r(n-1)+1}(\mathbf{1}), R^{r(n-1)+1}) \leq \sum_{i=1}^{r} \mathsf{s}(A_i, R^{r(n-1)+1}) \leq r\mathsf{L}(n),$$

a contradiction. □

Finally we point out that our methods give two new proofs of the well-known fact that $\mathsf{D}(C_p^r) = r(p-1)+1$ (for a discussion of further proofs see [3], Section 6). Firstly, the result follows from Theorem 6.6 and Proposition 6.5(3). For a second proof, let $S \in \mathcal{F}(C_p^r)$ be a sequence with $|S| = l = r(p-1)+1$. We have to show that S is not zero-sumfree. Lemma 6.1 and Proposition 6.5(3) imply that

$$\mathsf{s}(A^l_R(\mathbf{1}) \setminus A^l_R(S), R^l) \leq r(p-1) < r(p-1)+1 = \mathsf{s}(A^l_R(\mathbf{1}), R^l)$$

whence $A^l_R(S), R^l) \neq \emptyset$ and S is not zero-sumfree.

Acknowledgements

We would like to thank Florian Kainrath for valuable discussions on module theory. We also would like to thank the referee for his careful reading and all his suggestions. This work has been supported by NSFC with grant number 19971058

References

[1] N. Alon, F. Füredi, Covering the cube by affine hyperplanes, European. J. Combin. 14 (1993) 79–83.

[2] N. Alon, S. Friedland, G. Kalai, Regular subgraphs of almost regular graphs, J. Combin. Theory Ser. B 37 (1984) 79–91.

[3] N. Alon, Tools from higher algebra, in: Handbook of Combinatorics, vol. 2, North-Holland, 1995, pp. 1749–1783.

[4] N. Alon, Combinatorial nullstellensatz, Combin. Probab. Comput. 8 (1999) 7–29.

[5] D.D. Anderson, Factorization in Integral Domains, Pure and Applied Mathematics, vol. 189, Marcel Dekker, 1997.

[6] D. Eisenbud, Commutative Algebra With a View Toward Algebraic Geometry, Graduate Texts in Mathematics, vol. 150, Springer, 1995.

[7] R.C. Baker, W. Schmidt, Diophantine problems in variables restricted to the values of 0 and 1, J. Number Theory 12 (1980) 460–486.

[8] Y. Caro, Zero-sum problems—a survey, Discrete Math. 152 (1996) 93–113.

[9] S. Chapman, On the Davenports constant, the cross number, and their application in factorization theory, in: Zero-dimensional Commutative Rings, Lecture Notes in Pure Appl. Math., vol. 171, Marcel Dekker, 1995, pp. 167–190.

[10] P. Erdös, A. Ginzburg, A. Ziv, Theorem in the additive number theory, Bull. Research Council Israel 10 (1961) 41–43.

[11] W. Gao, On the additivity of integers, Adv. Math. 19 (1990) 488–492.

[12] W. Gao, A combinatorial problem on finite Abelian groups, J. Number Theory 58 (1995) 100–103.

[13] W. Gao, On Davenport's constant of finite Abelian groups with rank three, Discrete Math. 222 (2000) 111–124.

[14] W. Gao, Note on a zero-sum problem, J. Combin. Theory Ser. A 95 (2001) 387–389.

[15] W. Gao, A. Geroldinger, On zero-sum sequences in $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ (submitted).

[16] W. Gao, A. Geroldinger, On long minimal zero sequences in finite Abelian groups, Period. Math. Hungar. 38 (1999) 179–211.

[17] A. Geroldinger, Y.O. Hamidoune, Zero-sumfree sequences in cyclic groups and some arithmetical application, J. Théor. des Nombres Bordeaux 14 (2002) 221–239.

[18] A. Geroldinger, R. Schneider, On Davenport's constant, J. Combin. Theory Ser. A 61 (1992) 147–152.

[19] Y.O. Hamidoune, O. Ordaz, A. Ortunio, On a combinatorial theorem of Erdös, Ginzburg and Ziv, Combin. Probab. Comput. 7 (1998) 403–412.

[20] J.H.B. Kemperman, On small sumsets in an Abelian group, Acta Math. 103 (1960) 63–88.

[21] A. Kemnitz, On a lattice point problem, Ars Combin. 16 (1983) 151–160.

[22] D. Robinson, A Course in the Theory of Groups, Graduate Texts in Mathematics, vol. 80, Springer, 1996.

[23] L. Ronyai, On a conjecture of Kemnitz, Combinatorica 20 (2000) 569–573.

[24] W. Schmid, On zero-sum subsequences in finite Abelian groups, Integers 1 (2001) A1.