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RESEARCH ARTICLE

Finitary Monoids

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1. Introduction

In this paper we introduce finitary monoids as a new class of monoids (by a monoid we always mean a commutative cancellative semigroup with identity element). The main examples of finitary monoids are v-noetherian G-monoids (see [6]) and in particular finitely generated monoids, strongly primary monoids (see [8]) and abstract congruence monoids (see [9]). Finitary monoids satisfy the ACCP (ascending chain condition on principal ideals), finite direct products of finitary monoids are finitary, and saturated submonoids with torsion class group of finitary monoids are finitary again.

The main motivation for introducing finitary monoids stems from the theory of non-unique factorizations in integral domains. In general, the multiplicative monoids even of noetherian integral domains are highly complicated. It turned out to be a standard method in factorization theory to construct suitable auxiliary monoids of a relatively simple structure which reflect the arithmetical properties of the integral domains in question. Up to now the auxiliary monoids in the centre of interest in factorization theory are finitely generated and finitely primary monoids (see [16]) and saturated submonoids of finite products of strongly primary monoids with finite class group (see [8]). All these monoids are finitary, and it is our feeling that, to a large extent, finitary monoids should take over their role in factorization theory (see [19] and Example 4.16).

In Section 2 we fix our notations and recall some fundamentals from the ideal theory of monoids. In Section 3 we define finitary monoids, prove some structural properties (Theorems 3.5 and 3.8), discuss examples and establish the central arithmetical result on finitary monoids (Theorem 3.10). Section 4 deals with v-noetherian G-monoids which turn out to be finitary and have only finitely many prime s-ideals (Theorem 4.6). In particular, we shall prove that the multiplicative monoid of a noetherian domain is finitary (resp. a v-noetherian G-monoid) if and only if the domain is one-dimensional and semilocal (Proposition 4.14).

2. Preliminaries

Let $\mathbb{N}$ denote the positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and for $a, b \in \mathbb{Z}$ we set $[a, b] = \{\nu \in \mathbb{Z} \mid a \le \nu \le b\}$.

Throughout, a monoid is a commutative cancellative semigroup with identity element. If not denoted otherwise, we use multiplicative notation. Let H be a monoid with unit element $1_H = 1 \in H$. We denote by $H^\times$ the group of invertible elements of H, and we call H reduced if $H^\times = \{1\}$. We denote by $H_{\rm red} = \{aH^\times \mid a \in H\}$ the associated reduced monoid of H and by $\mathcal{Q}(H)$ a (fixed) quotient group of H whose elements we write as fractions $\frac{a}{b}$ (where $a, b \in H$).

An element $u \in H \setminus H^\times$ is called

- irreducible (or an atom) if, for all $a, b \in H$, $u = ab$ implies $a \in H^\times$ or $b \in H^\times$.
- prime (or a prime element) if, for all $a, b \in H$, $u \mid ab$ implies $u \mid a$ or $u \mid b$.

Every prime element is irreducible. The monoid H is called atomic (resp. factorial), if every $a \in H \setminus H^\times$ has a factorization into a product of irreducible (resp. prime) elements. It is well known that H is factorial if and only if H is atomic and every irreducible element is prime (see [17], Theorem 10.7).

For a submonoid $S \subset H$ we always assume that $\mathcal{Q}(S) \subset \mathcal{Q}(H)$, and we denote by

$$S^{-1}H = \left\{ \frac{a}{s} \mid a \in H, s \in S \right\} \subset \mathcal{Q}(H)$$

the quotient monoid of H with respect to S. In particular, $H^{-1}H = \mathcal{Q}(H)$.

A submonoid $S \subset H$ is called

- saturated, if $a, b \in S$, $c \in H$ and $a = bc$ implies that $c \in S$ (equivalently, $S = H \cap \mathcal{Q}(S)$).
- divisor-closed, if $a \in H$, $b \in S$ and $a \mid b$ implies that $a \in S$.
- cofinal, if for every $c \in H$ there is some $a \in S$ such that $c \mid a$.

By definition, every divisor-closed submonoid is saturated.

For a subset $U \subset \mathcal{Q}(H)$, let $[U] \subset H$ denote the submonoid and $\langle U \rangle$ the subgroup of $\mathcal{Q}(H)$ generated by U. For a subset $U \subset H$, let

$$[[U]]_H = [[U]] = \{x \in H \mid x \text{ divides some } s \in [U]\}$$

denote the smallest divisor-closed submonoid of H containing U. We say that H is finitely generated, if $H = [U]$ for some finite subset $U \subset H$. If $U = \{u_1, \ldots, u_m\}$, we set (as usual) $[U] = [u_1, \ldots, u_m]$ and $\langle U \rangle = \langle u_1, \ldots, u_m \rangle$.

For subsets $X, Y \subset \mathcal{Q}(H)$, we set $(X : Y) = \{z \in \mathcal{Q}(H) \mid zY \subset X\}$, and $X^{-1} = (H : X)$. We denote by

$$\tilde{H} = \{x \in \mathcal{Q}(H) \mid x^n \in H \text{ for some } n \in \mathbb{N}\}$$

the root closure of H and by

$$\hat{H} = \{x \in \mathcal{Q}(H) \mid \text{ there exists some } c \in H \text{ such that } cx^n \in H \text{ for all } n \in \mathbb{N}\}$$

the complete integral closure of H. We have

$$H \subset \tilde{H} \subset \hat{H} \subset \mathcal{Q}(H).$$

Our main reference for ideal theory is [17]. Note however that a monoid in [17] contains a zero element. Thus in order to apply results of [17] we either have to adjoin a zero element to our monoids and ideals or have to remove the zero element from the monoids and ideals in [17]. For the readers convenience, we recall some central notions.

By an ideal system r on H we mean a map

$$r\colon \begin{cases} \mathbb{P}(H) & \to \mathbb{P}(H) \\ X & \mapsto \quad X_r \end{cases}$$

such that the following conditions are fulfilled for all $X, Y \subset H$ and all $c \in H$.

(Id 1): $X \subset X_r$

(Id 2): $X \subset Y_r$ implies $X_r \subset Y_r$.

(Id 3): $\{c\}_r = cH$.

(Id 4): $cX_r = (cX)_r$.

For an ideal system r on H we denote by

$$\mathcal{I}_r(H) = \{X_r \mid X \subset H\} = \{J \subset H \mid J = J_r\}$$

the set of all r-ideals of H. Note that $\emptyset = \emptyset_r \in \mathcal{I}_r(H)$. We define the ideal system r_s on H by

$$X_{r_s} = \bigcup_{\substack{E \subset X \\ E \text{ is finite}}} E_r \quad \text{for} \quad X \subset H,$$

and we call r finitary if $r = r_s$. The ideal system r_s is always finitary. We say that H is r-noetherian, if the ascending chain condition for r-ideals is satisfied. Note that H is r-noetherian if and only if H is r_s-noetherian, and then $r = r_s$.

The most important ideal systems are the s-system, the v-system and the t-system. For $X \subset H$, they are defined by

$$X_s = XH, \qquad X_v = (X^{-1})^{-1} \qquad \text{and} \qquad t = v_s.$$

Then $X_s \subset X_t \subset X_v$ and $\mathcal{I}_v(H) \subset \mathcal{I}_t(H) \subset \mathcal{I}_s(H)$. Note that H is v-noetherian if and only if H is t-noetherian, and then $t = v$. For every ideal system r on H, we have $\mathcal{I}_v(H) \subset \mathcal{I}_r(H) \subset I_s(H)$ and $X_s \subset X_r \subset X_v$ for every subset $X \subset H$. If r is finitary, then $X_r \subset X_t$ and $\mathcal{I}_t(H) \subset \mathcal{I}_r(H)$.

An r-ideal $\mathfrak{p} \subset H$ is called prime if $H \setminus \mathfrak{p}$ is a submonoid of H. We denote by r-spec(H) the set of prime r-ideals, and we set $r\text{-spec}(H)^\bullet = r\text{-spec}(H) \setminus \{\emptyset\}$. Note that a subset $\mathfrak{p} \subset H$ is a prime s-ideal if and only if $H \setminus \mathfrak{p}$ is a divisor-closed submonoid of H. For an s-ideal $\mathfrak{a} \subset H$, we define its radical by

$$\sqrt{\mathfrak{a}} = \{a \in H \mid a^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}\}.$$

If r is a finitary ideal system and $\mathfrak{a} \in \mathcal{I}_r(H)$, then

$$\sqrt{\mathfrak{a}} = \bigcap_{\substack{\mathfrak{p} \in r\text{-spec}(H) \\ \mathfrak{p} \supset \mathfrak{a}}} \mathfrak{p}.$$

Let $\pi\colon H \to H_{\text{red}}$ denote the canonical epimorphism. Then the ideal system r induces an ideal system r_{red} on H_{red} by means of $\pi(X)_{r_{\text{red}}} = \pi(X_r)$. A subset $\mathfrak{a} \subset H$ is an r-ideal if and only if $\pi(H)$ is an r_{red}-ideal. In particular, $\mathcal{I}_r(H)$ and $\mathcal{I}_{r_{\text{red}}}(H_{\text{red}})$ are isomorphic lattices, and thus the r-ideal theory of H "coincides" with the r_{red}-ideal theory of H_{red}. The ideal systems s_{red}, v_{red} and t_{red} on H_{red} are just the systems s, v and t on H_{red}, respectively.

3. Finitary monoids

We start with a simple technical lemma.

Lemma 3.1. *Let H be a monoid and $\mathfrak{b} \subset H$ an s-ideal.*

1. *Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_r \subset H$ be s-ideals and $n_1, \ldots, n_r \in \mathbb{N}_0$ such that $\mathfrak{a}_i^{n_i} \subset \mathfrak{b}$ for every $i \in [1, r]$. If $n = n_1 + \cdots + n_r$, then $(\mathfrak{a}_1 \cup \cdots \cup \mathfrak{a}_r)^n \subset \mathfrak{b}$.*

2. *Let r be an ideal system on H, $\mathfrak{b}$ an r-ideal and $\mathfrak{a} \subset H$ an r-finitely generated r-ideal such that $\mathfrak{a} \subset \sqrt{\mathfrak{b}}$. Then there exists some $n \in \mathbb{N}$ such that $\mathfrak{a}^n \subset \mathfrak{b}$.*

Proof. 1. If $a \in (\mathfrak{a}_1 \cup \cdots \cup \mathfrak{a}_r)^n$, then $a = a_1^{m_1} \cdot \ldots \cdot a_r^{m_r}$, where $a_i \in \mathfrak{a}_i$, $m_i \in \mathbb{N}_0$ and $n = m_1 + \cdots + m_r$. Since $n = n_1 + \cdots + n_r$, there exists some $i \in [1, r]$ such that $m_i \geq n_i$, which implies $a_i^{m_i} \in \mathfrak{b}$ and hence $a \in \mathfrak{b}$.

2. By [17], Theorem 6.7.(v). ∎

Lemma 3.2. *Let H be a monoid and $U \subset H$ a subset.*

1. *If $a, b \in H$, then $b \in \sqrt{aH}$ if and only if $a \in [[b]]$. In particular, $\sqrt{aH} = \sqrt{bH}$ if and only if $[[a]] = [[b]]$.*

2. *$\{\sqrt{uH} \mid u \in U\} = \{\sqrt{aH} \mid a \in H\}$ if and only if $\{[[u]] \mid u \in U\} = \{[[a]] \mid a \in H\}$.*

3. *If $U \subset H \setminus H^\times$ and $U \cap [[b]] \neq \emptyset$ for every $b \in H \setminus H^\times$, then $H \setminus H^\times = \sqrt{UH}$.*

4. *If s-spec(H) is finite, then $\{[[a]] \mid a \in H\}$ is the set of all divisor-closed submonoids of H.*

Proof. 1. If $a, b \in H$, then $a \in [[b]]$ if and only if $a \mid b^n$ for some $n \in \mathbb{N}$, and the latter condition it equivalent to $b \in \sqrt{aH}$.

2. follows from 1.

3. Clearly, $U \subset H \setminus H^\times$ implies $\sqrt{UH} \subset H \setminus H^\times$. If $b \in H \setminus H^\times$, then there exists some $u \in U \cap [[b]]$, and by 1. we obtain $b \in \sqrt{uH} \subset \sqrt{UH}$.

4. Let $S \subset H$ be a divisor-closed submonoid. Since s-spec(H) is finite, the set of divisor-closed submonoids of H is also finite. Let $[[a]]$ be maximal in $\{[[b]] \mid b \in S\}$. We assert that $[[a]] = S$. Indeed, if $b \in S \setminus [[a]]$, then $[[a]] \subsetneq [[ab]] \subset S$ contradicts the maximal choice of $[[a]]$. ■

Definition 3.3. Let H be a monoid. A subset $U \subset H \setminus H^\times$ is called an *almost generating set of H*, if there exists some $n \in \mathbb{N}$ such that

$$(H \setminus H^\times)^n \subset UH.$$

We denote by $\mathcal{M}(U)$ the smallest possible $n \in \mathbb{N}$ for which the above inclusion holds.

Lemma 3.4. *Let H be a monoid and $U = \{u_1, \ldots, u_r\} \subset H \setminus H^\times$ an almost generating set of H.*

1. *For any $k_1, \ldots, k_r \in \mathbb{N}$, the set $\{u_1^{k_1}, \ldots, u_r^{k_r}\}$ is an almost generating set of H.*

2. *Let $v_1, \ldots, v_r \in H \setminus H^\times$ be such that $v_i \mid u_i$ for all $i \in [1, r]$. Then $\{v_1, \ldots, v_r\}$ is an almost generating set of H. In particular, if H is atomic, then H possesses an almost generating set consisting of atoms.*

3. *Let $V \subset U$ be such that $\{[[v]] \mid v \in V\}$ is the set of all minimal elements of $\{[[u]] \mid u \in U\}$ (with respect to set-theoretical inclusion).*

   (a) *V is an almost generating set of H.*

(b) $\{[[v]] \mid v \in V\}$ *is the set of all minimal elements of* $\{[[a]] \mid a \in H\backslash H^\times\}$.

(c) *If* W *is any almost generating set of* H, *then* $\{[[v]] \mid v \in V\} \subset \{[[w]] \mid w \in W\}$.

Proof. 1. We set $\mathfrak{a} = UH = u_1H \cup \cdots \cup u_rH$, $\mathfrak{b} = \{u_1^{k_1}, \ldots, u_r^{k_r}\}H$ and $k = \max\{k_1, \ldots, k_r\}$. Then $(u_iH)^k = u_i^kH \subset \mathfrak{b}$ for all $k \in [1, r]$, and by Lemma 3.1.1 there exists some $N \in \mathbb{N}$ such that $\mathfrak{a}^N \subset \mathfrak{b}$. If $n \in \mathbb{N}$ is such that $(H\backslash H^\times)^n \subset \mathfrak{a}$, then $(H\backslash H^\times)^{Nn} \subset \mathfrak{b}$, whence $\{u_1^{k_1}, \ldots, u_r^{k_r}\}$ is an almost generating set of H.

2. Obvious, since $v_i \mid u_i$ implies $u_iH \subset v_iH$.

3. (a) It is sufficient to prove that if $u, v \in U$ and $[[v]] \subset [[u]]$, then $U\backslash\{u\}$ is an almost generating set of H. If $u, v \in U$ and $[[v]] \subset [[u]]$, then $v \mid u^k$ for some $k \in \mathbb{N}$. By 1., $(U\backslash\{u\}) \cup \{u^k\}$ is an almost generating set, and by 2. the set $(U\backslash\{u\}) \cup \{v\} = U\backslash\{u\}$ is also an almost generating set of H.

(b) Let $a \in H\backslash H^\times$. We have to show that there exists some $v \in V$ such that $[[v]] \subset [[a]]$. Obviously, 3.(a) implies the existence of some $n \in \mathbb{N}$ and $v \in V$ such that $a^n \in vH$. By Lemma 3.2.1 we obtain $v \in [[a]]$ and hence $[[v]] \subset [[a]]$.

(c) If $v \in V$, then there exists some $n \in \mathbb{N}$ and $w \in W$ such that $v^n \in wH$ and hence $[[w]] \subset [[v]]$ by Lemma 3.2.1. By (b), there exists some $v_0 \in V$ such that $[[v_0]] \subset [[w]] \subset [[v]]$, and by the choice of V we obtain $[[v]] = [[v_0]] = [[w]]$. ∎

Let H be a monoid. An element $a \in H$ is called *archimedean*, if

$$\bigcap_{n \geq 0} a^n H = \emptyset.$$

The monoid H is called *archimedean*, if every $a \in H\backslash H^\times$ is archimedean. Every monoid satisfying the ACCP (ascending chain condition on principal ideals) is archimedean and atomic (see [17], Ch. 3, Ex. 6, and Proposition 10.3).

Let H be an atomic monoid, $a \in H\backslash H^\times$ and $a = u_1 \cdot \ldots \cdot u_k$ a factorization of a into irreducible elements $u_1, \ldots, u_k$. Then k is called the length of that factorization, and

$$\mathsf{L}(a) = \{k \in \mathbb{N} \mid a \text{ has a factorization of length } k\} \subset \mathbb{N}$$

is called the set of lengths of a. For $a \in H^\times$ we set $\mathsf{L}(a) = \{0\}$. The monoid H is called a BF-monoid (bounded factorization monoid), if it is atomic and $\mathsf{L}(a)$ is finite for every $a \in H$. Every BF-monoid satisfies the ACCP (see [15], Corollary 1).

Theorem 3.5. *Let H be a monoid with $H \neq H^\times$. Then the following statements are equivalent:*

1. *H has a finite almost generating set consisting of archimedean elements.*
2. *H is a BF-monoid and possesses a finite almost generating set.*
3. *H satisfies the ACCP and has a finite subset $U \subset H \backslash H^\times$ such that for some $n \in \mathbb{N}$ we have*

$$H = UH \cup \{a \in H \mid \sup \mathsf{L}(a) < n\}.$$

Proof. 1. $\Longrightarrow$ 2. Let $U \subset H \backslash H^\times$ be a finite almost generating set of H consisting of archimedean elements, and $a \in H \backslash H^\times$. Then there exists some $N \in \mathbb{N}$ such that $a \notin u^N H$ for every $u \in U$, and we assert that $\max \mathsf{L}(a) < \mathcal{M}(U)N|U|$ (then $\mathsf{L}(a)$ is finite). Assume, to the contrary, that there exists a factorization $a = a_1 \cdot \ldots \cdot a_t$, where $a_1, \ldots, a_t \in H \backslash H^\times$ and $t \geq \mathcal{M}(U)N|U|$. By the very definition of $\mathcal{M}(U)$ and Lemma 3.1.1, we obtain

$$a \in (H \backslash H^\times)^{\mathcal{M}(U)N|U|} \subset (UH)^{N|U|} \subset \bigcup_{u \in U} u^N H,$$

a contradiction.

2. $\Longrightarrow$ 3. Every BF-monoid satisfies the ACCP, and if U is an almost generating set of H and $a \in H$ satisfies $\max \mathsf{L}(a) = t \geq \mathcal{M}(U)$, then

$$a \in (H \backslash H^\times)^t \subset (H \backslash H^\times)^{\mathcal{M}(U)} \subset UH.$$

3. $\Longrightarrow$ 1. Since H satisfies the ACCP, H is archimedean. If $a \in (H \backslash H^\times)^n$, then $\sup \mathsf{L}(a) \geq n$ whence $a \in UH$. Thus U is an almost generating set of H. ■

Definition 3.6. A monoid H is called *finitary*, if $H \neq H^\times$ and H satisfies the equivalent conditions of Theorem 3.5.

Example 3.7. The following four classes of monoids are finitary: 1. Finitely generated monoids; 2. Strongly primary monoids; 3. v-noetherian G-monoids; 4. Abstract congruence monoids. We discuss here finitely generated and strongly primary monoids. v-noetherian G-monoids will be investigated in detail in Section 4. For the definition of congruence monoids we refer the reader to [11], §3. A thorough investigation of congruence monoids will be presented in [9] where we shall also prove that they are finitary.

Finitely generated monoids. Let H be a monoid such that H_{red} is finitely generated and $H \neq H^\times$. Then there exists a finite subset $U \subset H \backslash H^\times$ such that $H = [U \cup H^\times]$, and consequently $H \backslash H^\times \subset UH$. It follows from [15], Theorem 2, that H_{red} (and hence H) is a BF-monoid. Therefore H is finitary, U is an almost generating set of H and $\mathcal{M}(U) = 1$.

Strongly primary monoids: A monoid H is called strongly primary if $H \ne H^\times$, and there exists a function $\mathcal{M}\colon H \to \mathbb{N}_0$ such that $(H \setminus H^\times)^{\mathcal{M}(a)} \subset aH$ for every $a \in H \setminus H^\times$ (see [8]).

Let H be strongly primary. If $a, a_1, \ldots, a_r \in H \setminus H^\times$ and $a = a_1 \cdot \ldots \cdot a_r$, then $a_1 \cdot \ldots \cdot a_{r-1} \in (H \setminus H^\times)^{r-1} \setminus aH$, whence $r \le \mathcal{M}(a)$. Hence we obtain $\max \mathsf{L}(a) \le \mathcal{M}(a)$ for all $a \in H \setminus H^\times$, and consequently H is a BF-monoid. Therefore H is finitary, and for every $a \in H \setminus H^\times$ the singleton $\{a\}$ is an almost generating set of H satisfying $\mathcal{M}(\{a\}) = \mathcal{M}(a)$.

A monoid H is called primary if $H \ne H^\times$, and if $\emptyset$ and $H \setminus H^\times$ are the only prime s-ideals of H (see [17], Ch. 15.5 for various equivalent characterizations). Obviously, every strongly primary monoid is primary. We assert that, conversely, every primary finitary monoid is strongly primary.

Let H be a primary finitary monoid, $U \subset H \setminus H^\times$ an almost generating set of H and $a \in H \setminus H^\times$. Since H is primary, we have $[[u]] = H$ for every $u \in H \setminus H^\times$. Since U is an almost generating set, the same is true for $U \cup \{a\}$ and, by Lemma 3.4.3, $\{a\}$ is also an almost generating set of H. Hence there exists some $n \in \mathbb{N}$ such that $(H \setminus H^\times)^n \subset aH$, whence H is strongly primary.

Theorem 3.8. *Let H be a monoid.*

1. *H is finitary if and only if H_{red} is finitary.*
2. *Let H be finitary and $S \subset H$ a saturated submonoid such that $S \ne S^\times$.*
   (a) *If H possesses a finite almost generating set U such that $U \subset S$, then U is an almost generating set of S and S is finitary.*
   (b) *If $\mathcal{Q}(H)/\mathcal{Q}(S)H^\times$ is a torsion group, then H has an almost generating set U with $U \subset S$.*
   (c) *If H is a finite direct product of strongly primary monoids, then S is finitary.*
   (d) *If S is divisor-closed, then S is finitary.*
3. *Let $k \in \mathbb{N}$ and $H_1, \ldots, H_k$ be submonoids of H such that $H = H_1 \times \cdots \times H_k$, and $H_i \ne H_i^\times$ for all $i \in [1, k]$. Then H is finitary if and only if H_i is finitary for every $i \in [1, k]$.*

Proof. 1. Let $\pi\colon H \to H_{\mathrm{red}}$ be the canonical epimorphism. Then a subset $U \subset H \setminus H^\times$ is an almost generating set of H if and only if $\pi(U)$ is an almost generating set of H_{red}. Since H is a BF-monoid if and only if H_{red} is a BF-monoid, the assertion follows by Theorem 3.5.

2. Since $S \subset H$ is saturated, we have $S^\times = H^\times \cap S$ and $uH \cap S = uS$ for every $u \in S$. By [15], Theorem 3, S is a BF-monoid. We shall use Theorem 3.5 to prove that S is finitary.

(a) Let $U \subset S$ be a finite almost generating set of H. Then

$$(S \setminus S^\times)^{\mathcal{M}(U)} = S \cap (H \setminus H^\times)^{\mathcal{M}(U)} \subset UH \cap S = US.$$

Hence U is an almost generating set of S, and therefore S is finitary.

(b) Suppose that $\mathcal{Q}(H)/\mathcal{Q}(S)H^\times$ is a torsion group. Let $\{v_1, \ldots, v_r\} \subset H \setminus H^\times$ be an almost generating set of H, and let $k \in \mathbb{N}$ be such that $v_i^k \in \mathcal{Q}(S)H^\times$, say $v_i^k = u_i \varepsilon_i$, where $u_i \in \mathcal{Q}(S)$ and $\varepsilon_i \in H^\times$. By Lemma 3.4, $\{u_1, \ldots, u_r\}$ is an almost generating set of H, and since $u_i = \varepsilon_i^{-1} v_i^k \in H \cap \mathcal{Q}(S) = S$ for all $i \in [1, r]$, it follows that $\{u_1, \ldots, u_r\} \subset S$.

(c) Suppose that $H = \prod_{i=1}^n H_i$ where $H_1, \ldots, H_n$ are strongly primary monoids. Then every $a \in H$ may be written uniquely in the form $a = a_1 \cdot \ldots \cdot a_n$ where $a_i \in H_i$ for every $i \in [1, n]$. Let $T \colon H \to \{0, 1\}^n$ be defined by $T(a) = (\alpha_1, \ldots, \alpha_n)$ where $a \in H$ and $\alpha_i = 0$ if and only if $a_i \in H_i^\times$ for $i \in [1, n]$. Let $U \subset S \setminus S^\times$ be a finite set with $T(U) = T(S \setminus S^\times)$ and let

$$m = 2^n M \qquad \text{where} \qquad M = \max\{\mathcal{M}(u_i) \mid u \in U, i \in [1, n]\}.$$

We show that

$$(S \setminus S^\times)^m \subset US.$$

Let $a \in (S \setminus S^\times)^m$, say $a = \prod_{\nu=1}^m b^{(\nu)}$ with $b^{(1)}, \ldots, b^{(m)} \in S \setminus S^\times$. Since $|T(S)| \le 2^n$, there exists some $\Lambda \subset [1, m]$ with $|\Lambda| = M$, say $\Lambda = [1, M]$, such that $T(b^{(1)}) = \cdots = T(b^{(M)}) = (\beta_1, \ldots, \beta_n) \in \{0, 1\}^n$. We set $\prod_{\nu=1}^M b^{(\nu)} = b = b_1 \cdot \ldots \cdot b_n$ with $b_i \in H_i$ for every $i \in [1, n]$. Let $u \in U$ with $T(u) = (\beta_1, \ldots, \beta_n)$. If for some $i \in [1, n]$ we have $\beta_i = 1$, then $b_i \in (H_i \setminus H_i^\times)^M$ whence $u_i \mid_H b_i$. If for some $i \in [1, n]$ we have $\beta_i = 0$, then $u_i \in H_i^\times, b_i \in H_i^\times$ and thus $u_i \mid_H b_i$. Therefore we obtain that $u \mid_H b \mid_H a$ whence $u \mid_S a$ since $S \subset H$ is saturated.

(d) Let $U \subset H \setminus H^\times$ be a finite almost generating set of H, and let $S \subset H$ be divisor-closed. Then

$$(S \setminus S^\times)^{\mathcal{M}(U)} \subset UH \cap S = (U \cap S)S,$$

and therefore $U \cap S$ is a finite almost generating set of S. Hence S is finitary.

3. It suffices to consider the case $k = 2$. If H is finitary, then H_1 and H_2 are divisor-closed submonoids of H, and therefore they are finitary by 2.

Suppose that, for $i \in \{1, 2\}$, H_i is finitary and $U_i \subset H_i \setminus H_i^\times$ is a finite almost generating set of H_i. We shall prove that $\mathsf{L}(a)$ is finite for every $a \in H \setminus H^\times$, and $U = U_1 \cup U_2 \cup U_1 U_2$ is an almost generating set of H. Set $M = \max\{\mathcal{M}(U_1), \mathcal{M}(U_2)\}$.

If $a \in H \setminus H^\times$, then $a = a_1 a_2$, where $a_i \in H_i$ and $a_1 \notin H^\times$ or $a_2 \notin H^\times$. If $a_2 \in H^\times$, then $\mathsf{L}(a) = \mathsf{L}(a_1)$ and $a^M \in U_1 H_1 H_2^\times \subset UH$. If $a_1 \in H^\times$, then $\mathsf{L}(a) = \mathsf{L}(a_2)$ and $a^M \in H_1^\times U_2 H_2 \subset UH$. If $a_1 \notin H_1^\times$ and $a_2 \notin H_2^\times$, then factorizations of a into irreducibles are performed component-wise, and therefore $\mathsf{L}(a) = \mathsf{L}(a_1) + \mathsf{L}(a_2)$ and $a^M \in U_1 H_1 U_2 H_2 = UH$. Hence the assertion follows. ∎

Example 3.9. A monoid H possessing a finite almost generating set which is not finitary. We consider the additive monoid

$$H = (\mathbb{Z} \times \mathbb{R}_{>0}) \cup (\mathbb{N}_0 \times \{0\}) \subset (\mathbb{R}^2, +).$$

We assert that H is a reduced valuation monoid with $|s\text{-spec}(H)^\bullet| = 2$ (for the theory of valuation monoids see [17], Ch. 15 and 16). Indeed, if $(m,x),(n,y) \in H$, then $(m,x) \mid (n,y)$ if and only if $(n-m, y-x) \in H$, and this is equivalent to either $y > x$ or $[y = x$ and $n \ge m]$. Hence the principal s-ideals of H form a chain, and therefore H is a valuation monoid. Obviously, H is reduced. If $y \in \mathbb{R}_{>0}$ and $n \in \mathbb{Z}$, then $[[(n,y)]] = H$, and if $n \in \mathbb{N}$, then $[[(n,0)]] = \mathbb{N}_0 \times \{0\}$. Therefore $H \setminus H^\times = H \setminus \{(0,0)\} = (1,1) + H$ and $H \setminus (\mathbb{N}_0 \times \{0\})$ are the only non-empty prime s-ideals of H. Since H is not a discrete valuation monoid, it is not atomic (see [17], Theorem 16.4) and thus not finitary.

Since $H \setminus H^\times = (1,0) + H$, the singleton $\{(1,0)\}$ is an almost generating set of H.

Next we study the arithmetic of finitary monoids. We show that locally tame finitary monoids have finite catenary degree. This is already known for various subclasses of finitary monoids. The proof given here for the general case is so simple that it encourages us to believe that finitary monoids might serve as an appropriate conceptual tool in factorization theory.

We recall the notions of local tameness and of the catenary degree. For the relevance of these two fundamental concepts in factorization theory we refer the reader to the survey article [7].

Let H be an atomic monoid, $A = \mathcal{A}(H_{\rm red})$, $\mathsf{Z}(H) = \mathcal{F}(A)$ the free abelian monoid with basis A and $\pi\colon \mathsf{Z}(H) \to H_{\rm red}$ the canonical epimorphism. For $z = a_1 \cdot \ldots \cdot a_r \in \mathsf{Z}(H)$ (where $r \in \mathbb{N}_0$ and $a_1, \ldots, a_r \in A$) we call $|z| = r$ the length of z. For $z = y a_1 \cdot \ldots \cdot a_k$ and $z' = y b_1 \cdot \ldots \cdot b_l \in \mathsf{Z}(H)$ (where $y \in \mathsf{Z}(H)$, $k, l \in \mathbb{N}_0$, $a_1, \ldots, a_k, b_1, \ldots, b_l \in A$ and $\{a_1, \ldots, a_k\} \cap \{b_1, \ldots, b_l\} = \emptyset$) we call $\mathsf{d}(z, z') = \max\{k, l\} \in \mathbb{N}_0$ the distance between z and z'.

For $a \in H$, the elements of $\mathsf{Z}(a) = \pi^{-1}(aH^\times)$ are called the factorizations of a. Let $z, z' \in \mathsf{Z}(a)$ and $N \in \mathbb{N}_0$. By an N-chain of factorizations from z to z' we mean a finite sequence $(z_0, z_1, \ldots, z_k)$ in $\mathsf{Z}(a)$ such that $z_0 = z$, $z_k = z'$ and $\mathsf{d}(z_{i-1}, z_i) \le N$ for every $i \in [1, k]$. We denote by $\mathsf{c}(a) \in \mathbb{N}_0 \cup \{\infty\}$ the minimal $N \in \mathbb{N}_0 \cup \{\infty\}$ such that for each two factorizations $z, z' \in \mathsf{Z}(a)$ there is an N-chain of factorizations from z to z'. Obviously, $\mathsf{c}(a) \le \sup \mathsf{L}(a)$. We call

$$\mathsf{c}(H) = \sup\{\mathsf{c}(a) \mid a \in H\} \in \mathbb{N}_0 \cup \{\infty\}$$

the catenary degree of H.

For $u \in A$, let $\mathsf{t}(H, u)$ denote the smallest $t \in \mathbb{N}_0 \cup \{\infty\}$ such that for every $a \in H$ the following holds: if $\mathsf{Z}(a) \cap u\mathsf{Z}(H) \ne \emptyset$ and $z \in \mathsf{Z}(a)$, then there exists some $z' \in \mathsf{Z}(a) \cap u\mathsf{Z}(H)$ such that $\mathsf{d}(z, z') \le N$. We say that H is locally tame, if $\mathsf{t}(H, u) < \infty$ for every $u \in A$.

Theorem 3.10. *Every locally tame finitary monoid has finite catenary degree.*

Proof. Let H be a locally tame finitary monoid. Then H is a BF-monoid, and by Lemma 3.4.2, H possesses a finite almost generating set U such that

$U \subset \mathcal{A}(H)$. For $u \in U$, we set $\bar{u} = uH^\times \in \mathcal{A}(H_{\rm red})$. We shall prove that

$$\mathsf{c}(a) \le M = \max\{\mathcal{M}(U), \mathsf{t}(H, \bar{u}) \mid u \in U\} \quad \text{for all} \quad a \in H.$$

Let $a \in H$ be given. Since H is a BF-monoid, we may proceed by induction on $\max \mathsf{L}(a)$. If $\max \mathsf{L}(a) < \mathcal{M}(U)$, there is nothing to do. Thus suppose that $\max \mathsf{L}(a) \ge \mathcal{M}(U)$, and let $z, z' \in \mathsf{Z}(a)$ be factorizations of a. By Theorem 3.5 we have $a \in UH$, whence $a = ub$ for some $u \in U$ and $b \in H$. There exist factorizations $y, y' \in \mathsf{Z}(b)$ (whence $\bar{u}y, \bar{u}y' \in \mathsf{Z}(a)$)such that $d(z, \bar{u}y) \le \mathsf{t}(H, \bar{u})$ and $d(z', \bar{u}y') \le \mathsf{t}(H, \bar{u})$. Since $\max \mathsf{L}(b) < \max \mathsf{L}(a)$, there exists an M-chain of factorizations $(y_0, \ldots, y_k)$ from y to y'. Then $(z, \bar{u}y_0, \ldots, \bar{u}y_k, z')$ an M-chain of factorizations from z to z'. ■

4. v-noetherian G-monoids

Lemma 4.1. *Let H be a monoid.*

1. *If $\mathfrak{p} \in s$-spec(H), then*
$$\mathfrak{p} = \bigcup_{\substack{\mathfrak{q} \subset \mathfrak{p} \\ \mathfrak{q} \in t\text{-spec}(H)}} \mathfrak{q}.$$
2. *s-spec(H) is finite if and only if t-spec(H) is finite.*

Proof. 1. If $a \in \mathfrak{p}$, then aH is a t-ideal satisfying $aH \cap (H \setminus \mathfrak{p}) = \emptyset$, and by [17], Corollary 6.3, there exists some $\mathfrak{q} \in t$-spec(H) such that $aH \subset \mathfrak{q} \subset \mathfrak{p}$. This implies one inclusion and the other one is obvious.

2. If t-spec(H) is finite, then s-spec(H) is finite by 1., and since s-spec$(H) \supset t$-spec(H), the assertion follows. ■

Lemma 4.2. *Let H be a monoid and $a \in H$. Then the following conditions are equivalent:*

1. *$a \in \mathfrak{p}$ for every $\mathfrak{p} \in s$-spec$(H)^\bullet$.*
2. *$a \in \mathfrak{p}$ for every $\mathfrak{p} \in t$-spec$(H)^\bullet$.*
3. *Every non-empty s-ideal contains a power of a.*
4. *$\mathcal{Q}(H) = [H \cup \{a^{-1}\}]$.*
5. *$H = [[a]]$.*

Proof. The equivalence of 1., 3. and 4. was proved in [6], Lemma 4, and the equivalence of 1. and 2. follows by Lemma 4.1.1.

1. $\Longrightarrow$ 5. If $[[a]] \ne H$, then $H \setminus [[a]] \in s$-spec$(H)^\bullet$ and $a \notin H \setminus [[a]]$.

5. $\Longrightarrow$ 1. If $\mathfrak{p} \in s$-spec$(H)^\bullet$ and $a \notin \mathfrak{p}$, then $[[a]] \cap \mathfrak{p} = \emptyset$ and therefore $[[a]] \ne H$. ■

Definition 4.3. A monoid H is said to be a *G-monoid*, if there exists some $a \in H$ satisfying the equivalent conditions of Lemma 4.2.

Proposition 4.4. *Let H be a monoid.*

1. *H is a G-monoid if and only if $H_{\rm red}$ is a G-monoid.*
2. *If H is a G-monoid and $S \subset H$ a cofinal saturated submonoid, then S is a G-monoid.*
3. *If H is a G-monoid and D is a monoid such that $H \subset D \subset \mathcal{Q}(H)$, then D is a G-monoid.*
4. *Let $k \in \mathbb{N}$ and let $H_1, \ldots, H_k$ be submonoids of H such that $H = H_1 \times \cdots \times H_k$. Then H is a G-monoid if and only if H_i is a G-monoid for every $i \in [1, k]$.*

Proof. 1. If $a \in H$, then $H = [[a]]$ if and only if $H_{\rm red} = [[aH^\times]]$. Thus the assertion follows from Lemma 4.2.5.

2. Let $a \in H$ be such that $H = [[a]]_H$, and let $S \subset H$ be a cofinal saturated submonoid. Then there exists some $u \in S$ such that $a \,|_H\, u$, and we assert that $S = [[u]]_S$. Indeed, if $c \in S$, then $c \,|_H\, a^n$ for some $n \in \mathbb{N}$, hence $c \,|_H\, u^n$, and since $S \subset H$ is saturated, we obtain $c \,|_S\, u^n$ whence $c \in [[u]]_S$.

3. This follows by Lemma 4.2.4.

4. It suffices to consider the case $k = 2$. If $a_1 \in H_1$ and $a_2 \in H_2$, then it follows from the very definition that $H = [[a_1 a_2]]$ if and only if $H_1 = [[a_1]]$ and $H_2 = [[a_2]]$. Hence, by Lemma 4.2.5, H is a G-monoid if and only if H_1 and H_2 are both G-monoids. ∎

Next we investigate monoids H for which s-spec(H) is finite in more detail.

Proposition 4.5. *Let H be a monoid.*

1. *If s-spec(H) is finite, then H is a G-monoid. In particular, every primary monoid is a G-monoid.*
2. *s-spec(H) is finite if and only if s-spec$(H_{\rm red})$ is finite.*
3. *If s-spec(H) is finite and $S \subset H$ is a saturated submonoid, then s-spec(S) is finite.*
4. *If s-spec(H) is finite and $S \subset H$ is a submonoid, then s-spec$(S^{-1}H)$ is finite.*
5. *Let $k \in \mathbb{N}$ and let $H_1, \ldots, H_k$ be submonoids of H such that $H = H_1 \times \cdots \times H_k$. Then s-spec(H) is finite if and only if s-spec(H_i) is finite for every $i \in [1, k]$.*

Proof. 1. If $s\text{-spec}(H)^\bullet = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ and $a_i \in \mathfrak{p}_i$ for all $i \in [1, r]$, then $a_1 \cdot \ldots \cdot a_r \in \mathfrak{p}_i$ for all $i \in [1, r]$. Therefore H is a G-monoid by Lemma 4.2.1.

2. If $\pi\colon H \to H_{\text{red}}$ denotes the canonical epimorphism, then the assignment $I \mapsto \pi(I)$ defines a bijective map $s\text{-spec}(H) \to s\text{-spec}(H_{\text{red}})$.

3. Let $S \subset H$ be a saturated submonoid. It $T \subset S$ is any submonoid, then $[[T]]_S = [[T]]_H \cap S$. Now, if $\mathfrak{p} \in s\text{-spec}(H)$, then $\bar{\mathfrak{p}} = H \setminus [[S \setminus \mathfrak{p}]]_H \in s\text{-spec}(H)$, and

$$\bar{\mathfrak{p}} \cap S = S \setminus ([[S \setminus \mathfrak{p}]]_H \cap S) = S \setminus [[S \setminus \mathfrak{p}]]_S = \mathfrak{p}.$$

Hence the finiteness of $s\text{-spec}(H)$ implies the finiteness of $s\text{-spec}(S)$.

4. By [17], Theorem 4.4 **viii)** and Theorem 7.2.

5. It suffices to consider the case $k = 2$. We shall prove that the divisor-closed submonoids of $S \subset H_1 \times H_2$ are exactly the monoids $S = S_1 \times S_2$, where $S_1 \subset H_1$ and $S_2 \subset H_2$ are divisor-closed submonoids.

Clearly, if $S_1 \subset H_1$ and $S_2 \subset H_2$ are divisor-closed submonoids, the S_1S_2 is a divisor-closed submonoid of H. Conversely, let $S \subset H$ be a divisor-closed submonoid. For $i \in \{1, 2\}$, let $p_i\colon H \to H_i$ be the canonical projection. Then $p_1(S) \subset H_1$ and $p_2(S) \subset H_2$ are divisor-closed submonoids, and $S \subset p_1(S)p_2(S)$. We assert that equality holds. Indeed, suppose that $s = p_1(s')p_2(s'') \in p_1(S)p_2(S)$, where $s', s'' \in S$. Since $p_1(s') \mid_S s'$ and $p_2(s'') \mid_S s''$, it follows that $p_1(s') \in S$, $p_2(s'') \in S$ and hence also $s \in S$. ∎

Example 4.6. A G-monoid possessing infinitely many prime s-ideals. Let H be the additive monoid of all bounded sequences $(n_i)_{i\in\mathbb{N}} \in \mathbb{N}_0^{\mathbb{N}}$. Then $\mathcal{Q}(H)$ consists of all bounded sequences $(n_i)_{i\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$, and if $\mathbf{e}$ denotes the constant sequence with value -1, then $\mathcal{Q}(H) = [H \cup \{\mathbf{e}\}]$. Hence H is a G-monoid. For every $\nu \in \mathbb{N}$, the sequence $\mathbf{p}_\nu = (\delta_{\nu,i})_{i\in\mathbb{N}}$ is a prime element of H. Hence H possesses infinitely many prime s-ideals.

Theorem 4.7. *Let H be a v-noetherian G-monoid and $H \ne H^\times$.*

1. *H is finitary and $s\text{-spec}(H)$ is finite.*
2. *If $(H\colon \hat{H}) \ne \emptyset$, then $\hat{H}$ is a Krull monoid, and $s\text{-spec}(\hat{H})$ is finite.*

Proof. 1. By Lemma 4.2.2, there exists some $a \in H$ lying in all $\mathfrak{p} \in t\text{-spec}(H)^\bullet$. Since H is v-noetherian, we have $v = t$, and therefore $t\text{-spec}(H)$ is finite by [17], Theorem 24.2. Hence $s\text{-spec}(H)$ is finite by Lemma 4.1.2.

Every v-noetherian monoid satisfies the ACCP and therefore it is archimedean. Hence it remains to prove that H has a finite almost generating set.

For every $\mathfrak{q} \in t\text{-spec}(H)$, we consider the prime s-ideal

$$\mathfrak{q}^* = \bigcup_{\substack{\mathfrak{p} \in t\text{-spec}(H) \\ \mathfrak{p} \ne \mathfrak{q}}} \mathfrak{p} \in s\text{-spec}(H).$$

By Lemma 3.2.4, there exists some $a_{\mathfrak q} \in H\setminus H^\times$ such that $H\setminus\mathfrak q^* = [[a_{\mathfrak q}]]$. In particular, $a_{\mathfrak q} \notin \mathfrak p$ for all $\mathfrak p \in t\text{-spec}(H)\setminus\{\mathfrak q\}$, and therefore, by [17], Proposition 6.6 and Theorem 6.7,

$$\sqrt{a_{\mathfrak q}H} = \bigcup_{\substack{\mathfrak p\in t\text{-spec}(H)\\ a_{\mathfrak q}\in\mathfrak p}} \mathfrak p = \mathfrak q.$$

Since H is v-noetherian, every $\mathfrak q \in t\text{-spec}(H)$ is v-finitely generated. Hence there exists some $n \in \mathbb N$ such that $\mathfrak q^n \subset a_{\mathfrak q}H$ for all $\mathfrak q \in t\text{-spec}(H)$.

We assert that

$$(H\setminus H^\times)^n \subset \{a_{\mathfrak q} \mid \mathfrak q \in t\text{-spec}(H)^\bullet\}H,$$

whence $\{a_{\mathfrak q} \mid \mathfrak q \in t\text{-spec}(H)^\bullet\}$ is an almost generating set of H. Indeed, if $a \in H\setminus H^\times$, then there exists some $\mathfrak q \in t\text{-spec}(H)^\bullet$ such that $a \in \mathfrak q$ and hence $a^n \in a_{\mathfrak q}H$.

2. By [17], Theorem 24.8.(i), $\hat H$ is a Krull monoid, and hence it is v-noetherian. By Proposition 4.4.3, $\hat H$ is a G-monoid, and by 1. $s\text{-spec}(\hat H)$ is finite. ∎

We do not know whether the assumption $(H:\hat H)\neq\emptyset$ in Theorem 4.7.2 is necessary. If H is a G-monoid, then $\hat{\hat H}$ is completely integrally closed by [6], Theorem 4, but $\hat H$ need not be completely integrally closed (see [12], Theorem 3). However, we do not know of an example of a v-noetherian G-monoid for which $\hat H$ is not a Krull monoid.

Recall that a monoid H is said to be seminormal, if the following condition is satisfied:

If $x \in \mathcal Q(H)$ and $x^n \in H$ for all sufficiently large $n \in \mathbb N$, then $x \in H$.

The condition of seminormality plays a central role in order to obtain strong results on the structure of the complete integral closure (see [21], [4] and [5]). In particular, the complete integral closure of a seminormal v-noetherian domain is a Krull domain (see [2], Theorem 2.9).

Here we show that for a seminormal G-monoid H we always have $(H:\hat H)\neq\emptyset$ (a corresponding result for integral domains was proved in [4], Lemma 3.8).

Proposition 4.8. *Let H be a G-monoid and*

$$\mathfrak j = \bigcap_{\mathfrak p\in s\text{-spec}(H)^\bullet} \mathfrak p.$$

1. *If $a \in \mathfrak j$ and $x \in \hat H$, then there exists some $k \in \mathbb N$ such that $a^k x^n \in H$ for all $n \in \mathbb N$.*
2. *If H is seminormal, then $\hat H = (\mathfrak j:\mathfrak j)$ and $\mathfrak j \subset (H:\hat H)$.*

Proof. 1. Suppose that $a \in \mathfrak{j}$, $x \in \hat{H}$ and let $c \in H$ such that $cx^n \in H$ for all $n \in \mathbb{N}$. Since by Lemma 4.2 $[[a]] = H$, there exist $b \in H$ and $k \in \mathbb{N}$ such that $a^k = bc$ and therefore $a^k x^n = bcx^n \in H$ for all $n \in \mathbb{N}$.

2. It is sufficient to prove that $\mathfrak{j}\hat{H} \subset \mathfrak{j}$. Indeed, then $\hat{H} \subset (\mathfrak{j} : \mathfrak{j})$ and the other inclusion follows by [17], Theorem 14.1. In particular, we obtain that $\mathfrak{j}\hat{H} \subset H$ and hence $\mathfrak{j} \subset (H : \hat{H})$.

Suppose that $x \in \hat{H}$ and $a \in \mathfrak{j}$. By 1.there exists some $k \in \mathbb{N}$ such that $a^k x^n \in H$ for all $n \in \mathbb{N}$. For $n > k$ we obtain

$$(ax)^n = a^{n-k}(a^k x^n) \in aH \subset H$$

whence $ax \in H$ by seminormality, and even $ax \in \sqrt{aH} \subset \sqrt{\mathfrak{j}} = \mathfrak{j}$. ■

Theorem 4.9. *Let H be a monoid.*

1. *H is a v-noetherian G-monoid if and only if H_{red} is a v-noetherian G-monoid.*
2. *Every saturated submonoid of a v-noetherian G-monoid is a v-noetherian G-monoid.*
3. *If H is a v-noetherian G-monoid and $S \subset H$ a submonoid, then $S^{-1}H$ is a v-noetherian G-monoid.*
4. *Let $k \in \mathbb{N}$ and let $H_1, \ldots, H_k$ be submonoids of H such that $H = H_1 \times \cdots \times H_k$. Then H is v-noetherian G-monoid if and only if H_i is a v-noetherian G-monoid for every $i \in [1, k]$.*

Proof. 1. If $\pi\colon H \to H_{\mathrm{red}}$ denotes the canonical epimorphism, then the assignment $I \mapsto \pi(I)$ defines a bijective map $\mathcal{I}_v(H) \to \mathcal{I}_v(H_{\mathrm{red}})$. Hence H is v-noetherian if and only if H_{red} is v-noetherian. By Proposition 4.4.1, H is a G-monoid if and only if H_{red} is a G-monoid.

2. Let H be a v-noetherian G-monoid and $S \subset H$ a saturated submonoid. By Theorem 4.6.1, s-spec(H) is finite. Hence s-spec(S) is finite by Proposition 4.7.3, and therefore S is a G-monoid by Proposition 4.7.1. By [10], Proposition 6.6, S is v-noetherian.

3. By Proposition 4.4.3., $S^{-1}H$ is a G-monoid, and by [17], Corollary 24.1, $S^{-1}H$ is v-noetherian.

4. If H is a v-noetherian G-monoid and $i \in [1, k]$, then $H_i \subset H$ is a saturated submonoid and hence a v-noetherian G-monoid by 2. If $H_1, \ldots, H_k$ are v-noetherian G-monoids, then H is a G-monoid by Proposition 4.4.3, and H is v-noetherian by [10], Proposition 6.7. ■

Corollary 4.10. *Let H be a monoid such that H_{red} is finitely generated. Then H is a v-noetherian G-monoid.*

Proof. By Theorem 4.9.1 we may suppose that H is finitely generated, say $H = [u_1, \ldots, u_r]$. Then $H = [[u_1 \cdot \cdots \cdot u_r]]$ and, by Lemma 4.2.5, H

is a G-monoid. By [17], Theorem 3.6, H is s-noetherian and hence v-noetherian. ∎

Remark 4.11. **1.** There exists a primary monoid H (whence $|s\text{-spec}(H)^\bullet| = 1$ and thus H is a G-monoid) for which $\hat{H}$ is factorial (whence $\hat{H}$ a Krull monoid) with finitely many non-associated prime elements and $(H : \hat{H}) \neq \emptyset$, but H is not v-noetherian (see [9]). This example is in contrast to the Eakin-Nagata theorem in ring theory (see [20], Theorem 3.7).

2. We consider the (multiplicative) monoid $H = \{1\} \cup 2\mathbb{N} \subset \mathbb{N}$. We assert that H is finitary and v-noetherian, but not a G-monoid. Obviously, H is a reduced BF-monoid, which is finitary since $(H \setminus \{1\})^2 \subset 2H$. If p is a prime, then $I_p = 2p\mathbb{N}$ is a prime s-ideal of H, but there is no $a \in H$ lying in all I_p. Hence H is not a G-monoid. In order to prove that H is v-noetherian, we show that $\mathcal{I}_v(H) = \{2a\mathbb{N} \mid a \in \mathbb{N}\} \cup \{\emptyset\}$.

Clearly, $\mathcal{Q}(H) = \mathbb{Q}_{>0}$. If $\emptyset \neq X \subset 2\mathbb{N} = H \setminus \{1\}$, then there exists some $a \in \mathbb{N}$ and a subset $X_0 \subset \mathbb{N}$ such that $\gcd(X_0) = 1$ and $X = 2aX_0$. Then $X^{-1} = a^{-1}\mathbb{N}$ and $X_v = (a^{-1}\mathbb{N})^{-1} = 2a\mathbb{N}$. Therefore X is a v-ideal if and only if $X = 2a\mathbb{N}$ for some $a \in \mathbb{N}$.

The monoid $\{1\} \cup 2\mathbb{N}$ is the simplest example of a non-trivial abstract congruence monoid (see [9]).

Proposition 4.12. *Let H be a Krull monoid and $U \subset H \setminus H^\times$ an almost generating set. Then we have $\mathfrak{p} \cap U \neq \emptyset$ for all $\mathfrak{p} \in v\text{-spec}(H)^\bullet$.*

Proof. Assume to the contrary that

$$\Omega = \{\mathfrak{p} \in v\text{-spec}(H)^\bullet \mid \mathfrak{p} \cap U \neq \emptyset\} \subsetneq v\text{-spec}(H)^\bullet.$$

Then there exists an element $a \in H$ such that $a \in \mathfrak{q}$ for some $\mathfrak{q} \in v\text{-spec}(H)^\bullet \setminus \Omega$, and we choose a in such a way that

$$d = |\{\mathfrak{p} \in \Omega \mid a \in \mathfrak{p}\}|$$

becomes minimal. Since U is an almost generating set of H, there exists some $u \in H$ and $n \in \mathbb{N}$ such that $a^n \in uH$. Thus it follows that $a \in \mathfrak{p}$ for some $\mathfrak{p} \in \Omega$, hence $d \geq 1$, and we set

$$\{\mathfrak{p} \in \Omega \mid a \in \mathfrak{p}\} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_d\}.$$

For $\mathfrak{p} \in v\text{-spec}(H)^\bullet$, we denote by $\mathsf{v}_\mathfrak{p}$ the $\mathfrak{p}$-adic valuation of $\mathcal{Q}(H)$ (see [17], 26.4). For $i \in [1, d]$, we set $v_i = \mathsf{v}_{\mathfrak{p}_i}(u) \in \mathbb{N}_0$ and $w_i = \mathsf{v}_{\mathfrak{p}_i}(a) \in \mathbb{N}$. By construction, $\mathsf{v}_\mathfrak{p}(u) = 0$ for all $\mathfrak{p} \in v\text{-spec}(H)^\bullet \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_d\}$, and $\mathsf{v}_\mathfrak{q}(a) > 0$. Let $j \in [1, d]$ be such that

$$\frac{v_j}{w_j} = \max\left\{\frac{v_1}{w_1}, \ldots, \frac{v_d}{w_d}\right\},$$

and consider the element $a' = u^{-w_j} a^{v_j} \in \mathcal{Q}(H)$. We assert that $a' \in H$ contradicts the minimal choice of a. Indeed, if $i \in [1,d]$, then $\mathsf{v}_{\mathfrak{p}_i}(a') = -w_j v_i + v_j w_i \ge 0$, $\mathsf{v}_{\mathfrak{p}_j}(a') = 0$, and if $\mathfrak{p} \in v\text{-spec}(H)^\bullet \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_d\}$, then $\mathsf{v}_{\mathfrak{p}}(u) = 0$ and hence $\mathsf{v}_{\mathfrak{p}}(a') = \mathsf{v}_{\mathfrak{p}}(a) \ge 0$. In particular, $a' \in H$, $\mathsf{v}_{\mathfrak{q}}(a') > 0$, and

$$\{\mathfrak{p} \in \Omega \mid a' \in \mathfrak{p}\} \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_d\} \setminus \{\mathfrak{p}_j\},$$

a contradiction. ∎

Corollary 4.13. *Let H be a Krull monoid and $H \ne H^\times$. Then the following conditions are equivalent:*

1. *H is a G-monoid.*
2. *H is finitary.*
3. *s-spec(H) is finite.*
4. *H_{red} is finitely generated.*

Proof. Since H is v-noetherian, Theorem 4.7 shows that 1. $\Longrightarrow$ 2. and 1. $\Longrightarrow$ 3., and Proposition 4.5.1 shows that 3. $\Longrightarrow$ 1.

3. $\Longleftrightarrow$ 4. By Lemma 4.1.2, s-spec(H) is finite if and only if t-spec$(H) = v$-spec(H) is finite. By [17], Theorem 23.4, H possesses a divisor theory. If $\partial\colon H \to D$ is a divisor theory, then $D \cong \mathcal{I}_t(H)$, and $\mathcal{I}_t(H)$ is free with basis v-spec$(H)^\bullet$ (see [17], Theorem 20.5 and Corollary 23.3). Hence s-spec(H) is finite if and only if H has a divisor theory with only finitely many prime divisors, and by [14] Satz 1, this is true if and only if H_{red} is finitely generated.

2. $\Longrightarrow$ 3. Let $U \subset H \setminus H^\bullet$ be a finite almost generating set of H. By Proposition 4.12, every $\mathfrak{p} \in v$-spec$(H)^\bullet$ contains some $u \in U$. Since every $u \in U$ lies in only finitely many v-prime ideals of H it follows that v-spec(H) and hence also s-spec(H) is finite. ∎

We close with a few remarks concerning integral domains. If R is an integral domain, then we denote by $R^\bullet = R \setminus \{0\}$ its multiplicative monoid and by $R^\# = R^\bullet / R^\times$ the associated reduced monoid. The d-system on $R^\bullet$ is defined by $X_d = {}_R\langle X \rangle \setminus \{0\}$. Thus a subset $\mathfrak{a} \subset R^\bullet$ is a d-ideal if and only if $\mathfrak{a} \cup \{0\} \subset R$ is a usual ring ideal. The domain R is called a G-domain, if $\bigcap_{\mathfrak{p} \in d\text{-spec}(R^\bullet)^\bullet} \mathfrak{p} \ne \emptyset$ (see [13], §31 for the theory of G-domains). Since d is a finitary ideal system on $R^\bullet$, Proposition 11.6 in [17] implies that

$$\bigcap_{\mathfrak{p} \in d\text{-spec}(R^\bullet)^\bullet} \mathfrak{p} = \bigcap_{\mathfrak{p} \in t\text{-spec}(R^\bullet)^\bullet} \mathfrak{p}.$$

Hence $R^\bullet$ is a G-monoid if and only if R is a G-domain. If R is a one-dimensional semilocal domain, then d-spec$(R^\bullet)^\bullet = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ is finite whence $\{0\} \subsetneq \prod_{i=1}^r \mathfrak{p}_i \subset \bigcap_{i=1}^r \mathfrak{p}_i$ and R is a G-domain.

An integral domain is called a Mori domain if it satisfies the ascending chain condition on divisorial ideals. For a survey article concerning recent results on Mori domains we refer the reader to [3]. Note that an integral domain R is a Mori domain if and only if $R^\bullet$ is v-noetherian. Hence if R is a one-dimensional semilocal Mori domain, then $R^\bullet$ is a v-noetherian G-monoid. We summarize our discussion in the following result.

Proposition 4.14. *For a noetherian domain R the following conditions are equivalent:*

1. *R is a one-dimensional semilocal domain.*
2. *$R^\bullet$ is a G-monoid.*
3. *$R^\bullet$ is finitary.*

Proof. 1. $\Longrightarrow$ 2. follows from above and 2. $\Longrightarrow$ 3. from Theorem 4.7.1. To verify 3. $\Longrightarrow$ 1., let $U \subset R^\bullet \backslash R^\times$ be a finite almost generating set of $R^\bullet$. The set $\mathcal{P}$ of minimal prime d-ideals lying over some $u \in U$ is finite and by Krull's principal ideal theorem every ideal in $\mathcal{P}$ has height one. If $x \in R^\bullet \backslash R^\times$, then $x^{\mathcal{M}(U)} \in \mathfrak{p}$ for some $\mathfrak{p} \in \mathcal{P}$ whence $x \in \mathfrak{p}$. Thus if $\mathfrak{m} \subset R$ is a maximal d-ideal, then $\mathfrak{m} \subset \bigcup_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}$ whence $\mathfrak{m} \subset \mathfrak{p}$. Therefore R is one-dimensional and semilocal. ∎

Proposition 4.14 does not stay valid any longer for Mori domains, as we point out in the following remark.

Remark 4.15. **1.** There are Mori domains in which Krull's principal ideal theorem does not hold: let k be a field and $A = k + Xk[X, Y]$ where X and Y are indeterminates. Then A is a Mori domain, the ideal M generated by $\{XY^n \mid n \in \mathbb{N}_0\}$ is a prime ideal which is minimal over X but has height two (see [1], Example 3.6 (c)).

2. Let R be a Mori domain, $a \in R$ and $\mathfrak{p} \subset R$ a prime d-ideal which is minimal over a and has height greater than one. Then $R_\mathfrak{p}$ is a Mori domain, not one-dimensional and $\mathfrak{p}_\mathfrak{p} = R_\mathfrak{p} \backslash R_\mathfrak{p}^\times$ is the only prime d-ideal lying over $aR_\mathfrak{p}$ whence it is a v-ideal by [17], Proposition 6.6. Therefore we have $\sqrt{aR_\mathfrak{p}} = \mathfrak{p}_\mathfrak{p}$ and by Lemma 3.1.2 there is some $n \in \mathbb{N}$ such that $(R_\mathfrak{p} \backslash R_\mathfrak{p}^\times)^n \subset aR_\mathfrak{p}$. Thus $R_\mathfrak{p}^\bullet$ is a finitary monoid and $\{a\}$ is an almost generating set. Note that $R_\mathfrak{p}^\bullet$ is not a primary monoid, since $R_\mathfrak{p}$ is not one-dimensional.

We close with a further example of v-noetherian G-monoids arising in the theory of non-unique factorizations.

Example 4.16. *Local monoids in noetherian weakly Krull domains are v-noetherian G-monoids.* Let R be a noetherian weakly Krull domain (cf. [16]), i.e. a noetherian domain such that

$$R = \bigcap_{\mathfrak{p} \in \mathfrak{X}(R)} R_\mathfrak{p}$$

where $\mathfrak{X}(R)$ denotes the set of height one prime ideals of R. Then the canonical map

$$\Phi\colon\ R^{\#} \to \coprod_{\mathfrak{p}\in\mathfrak{X}(R)} R_{\mathfrak{p}}^{\#}$$

is a divisor homomorphism (note that for every $r \in R^{\bullet}$ we have $r \in R_{\mathfrak{p}}^{\times}$ for all but finitely many $\mathfrak{p} \in \mathfrak{X}(R)$). Let $x \in R^{\#}$ and $V(x) = \{\mathfrak{p} \in \operatorname{spec}(R) \mid x \in \mathfrak{p}\}$. Then

$$[[x]]_{R^{\#}} \subset R^{\#}$$

is called the *local monoid belonging to* x. These monoids are studied in detail in [19], section three. It turned out that

$$[[x]]_{R^{\#}} = \{y \in R^{\#} \mid V(y) \subset V(x)\} = \{y \in R^{\#} \mid V(y) \cap \mathfrak{X}(R) \subset V(x) \cap \mathfrak{X}(R)\}$$

and that

$$\Phi\,|_{[[x]]_{R^{\#}}}\colon\ [[x]]_{R^{\#}} \to \prod_{\mathfrak{p}\in V(x)\cap\mathfrak{X}(R)} R_{\mathfrak{p}}^{\#} \qquad (*)$$

is a cofinal divisor homomorphism whose class group is a subgroup of the class group of R ([19], Proposition 3.2 and Theorem 3.4). Since each $R_{\mathfrak{p}}$ is a one-dimensional, local noetherian domain, $R_{\mathfrak{p}}^{\#}$ is a v-noetherian G-monoid by Proposition 4.14. Since $V(x) \cap \mathfrak{X}(R)$ is finite, $(*)$ and Theorem 4.9.2 imply that the local monoid $[[x]]_{R^{\#}}$ is a v-noetherian G-monoid.

Local monoids allow to obtain finiteness results for arithmetical invariants (such as the catenary degree or local tame degrees) under very moderate finiteness assumptions on the class group of R (see [18], Section 5 and [19], Section 4).

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