A REALIZATION THEOREM FOR SETS OF LENGTHS IN NUMERICAL MONOIDS

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Abstract. We show that for every finite nonempty subset L of $\mathbb{N}_{\geq 2}$ there are a numerical monoid H and a squarefree element $a \in H$ whose set of lengths $\mathsf{L}(a)$ is equal to L.

1. Introduction

In the last decade the arithmetic of numerical monoids has found wide interest in the literature. Since numerical monoids are finitely generated, every element of a given monoid can be written as a sum of atoms and all arithmetical invariants describing the non-uniqueness of factorizations are finite. The focus of research was on obtaining precise values for the arithmetical invariants (e.g., [1, 21, 2]), on their interplay with minimal relations of a given presentation (e.g., [7]), and also on computational aspects (e.g., [13, 14] and [10] for a software package in GAP). A further direction of research was to establish realization results for arithmetical parameters. This means to show that there are numerical monoids whose arithmetical parameters have prescribed values. So for example, it was proved only recently that every finite set (with some obvious restrictions) can be realized as the set of catenary degrees of a numerical monoid ([22]). The goal of the present note is to show a realization theorem for sets of lengths.

Let H be a numerical monoid. If $a \in H$ and $a = u_1 + \ldots + u_k$, where $u_1, \ldots, u_k$ are atoms of H, then k is called a factorization length of a and the set $\mathsf{L}(a) \subset \mathbb{N}$ of all factorization lengths is called the set of lengths of a. Further, $\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$ denotes the system of sets of lengths of H. It is easy to see that all sets of lengths are finite nonempty and can get arbitrarily large, and it is well-known that they have a well-defined structure (see the beginning of Section 3). As a converse, we show in the present paper that for every finite nonempty set $L \subset \mathbb{N}_{\geq 2}$ there is a numerical monoid H and a squarefree element $a \in H$ such that $\mathsf{L}(a) = L$ (Theorem 3.3). In fact, we show more precisely that the number of factorizations of each length can be prescribed. Several types of realization results for sets of lengths are known in the literature, most of them in the setting of Krull monoids (see [20, 16, 23, 11, 12], [15, Theorem 7.4.1]). However, we know that if H is a numerical monoid, then $\mathcal{L}(H) \neq \mathcal{L}(H')$ for every Krull monoid H' (see [18, Theorem 5.5] and note that every numerical monoid is strongly primary).

It is an open problem which finite sets of positive integers can occur as sets of distances of numerical monoids. Based on our main result we can show that every finite set is contained in a set of distances of a numerical monoid (Corollary 3.4). There is a vibrant interplay between numerical monoids, and more generally affine monoids, and the associated semigroup algebras ([4, 3, 6]). In Corollary 3.5 we shift our realization result from numerical monoids to numerical semigroup algebras.

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2. Background on the arithmetic of numerical monoids

We denote by $\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$ the set of prime numbers, positive integers, integers, and rational numbers respectively. For $a, b \in \mathbb{Q}$, let $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$ be the discrete interval of integers lying between a and b. If $A, B \subset \mathbb{Z}$, then $A + B = \{a + b \mid a \in A, b \in B\}$ denotes the sumset and $kA = A + \ldots + A$ is the k-fold sumset for every $k \in \mathbb{N}$. If $A = \{m_1, \ldots, m_k\} \subset \mathbb{Z}$ with $m_{i-1} < m_i$ for each $i \in [2, k]$, then $\Delta(A) = \{m_i - m_{i-1} \mid i \in [2, k]\} \subset \mathbb{N}$ is the set of distances of L. Note that $\Delta(A) = \emptyset$ if and only if $|A| \le 1$.

By a *monoid*, we mean a commutative cancellative semigroup with identity element. Let H be a monoid. Then $H^\times$ denotes the group of invertible elements, $\mathsf{q}(H)$ the quotient group of H, and $\mathcal{A}(H)$ the set of atoms (irreducible elements) of H. We say that H is reduced if the identity element is the only invertible element. We call $H_{\text{red}} = H/H^\times$ the reduced monoid associated to H. A *numerical monoid* is a submonoid of $(\mathbb{N}_0, +)$ whose complement in $\mathbb{N}_0$ is finite. Every numerical monoid is finitely generated, reduced, and its quotient group is $\mathbb{Z}$. For any set P, let $\mathcal{F}(P)$ denote the free abelian monoid with basis P. Then, using additive notation, every element $a \in \mathsf{q}(\mathcal{F}(P))$ can be written uniquely in the form

$$a = \sum_{p \in P} l_p p \,,$$

where $l_p \in \mathbb{Z}$ for each $p \in P$, and all but finitely many l_p are equal to 0. For $a = \sum_{p \in P} l_p p \in \mathcal{F}(P)$, we set $|a| = \sum_{p \in P} l_p \in \mathbb{N}_0$ and call it the length of a.

We recall some arithmetical concept of monoids. Since our focus is on numerical monoids we use additive notation. Let H be an additively written monoid. The (additively written) free abelian monoid $\mathsf{Z}(H) = \mathcal{F}(\mathcal{A}(H_{\text{red}}))$ is called the *factorization monoid* of H and the canonical epimorphism $\pi \colon \mathsf{Z}(H) \to H_{\text{red}}$ is the factorization homomorphism. For $a \in H$ and $k \in \mathbb{N}$,

$$\begin{aligned} \mathsf{Z}_H(a) = \mathsf{Z}(a) = \pi^{-1}(a + H^\times) \subset \mathsf{Z}(H) \quad & \text{is the } \textit{set of factorizations} \text{ of } a\,, \\ \mathsf{Z}_{H,k}(a) = \mathsf{Z}_k(a) = \{z \in \mathsf{Z}(a) \mid |z| = k\} \quad & \text{is the } \textit{set of factorizations} \text{ of } a \text{ of length } k, \quad \text{and} \\ \mathsf{L}_H(a) = \mathsf{L}(a) = \big\{|z| \;\big|\; z \in \mathsf{Z}(a)\big\} \subset \mathbb{N}_0 \quad & \text{is the } \textit{set of lengths} \text{ of } a\,. \end{aligned}$$

Thus, by definition, $\mathsf{L}(a) = \{0\}$ if and only if $a \in H^\times$ and $\mathsf{L}(a) = \{1\}$ if and only if $a \in \mathcal{A}(H)$. The monoid H is said to be atomic if $\mathsf{Z}(a) \ne \emptyset$ for all $a \in H$ (equivalently, every non invertible element is a finite sum of atoms). We call

- $\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$ the *system of sets of lengths* of H, and
- $\Delta(H) = \bigcup_{L \in \mathcal{L}(H)} \Delta(L)$ the *set of distances* (also called *delta set*) of H.

Every numerical monoid H is atomic with finite set of distances $\Delta(H)$, and $\Delta(H) = \emptyset$ if and only if $H = \mathbb{N}_0$.

3. A realization theorem for sets of lengths

The goal of this section is to prove our main realization theorem, namely that for every finite nonempty subset $L \subset \mathbb{N}_{\ge 2}$ there exists a numerical monoid H such that L is a set of lengths of H (Theorem 3.3). We show the existence of this monoid by an explicit recursive construction over the size of L. Instead of working with numerical monoids directly, we work in the setting of finitely generated additive submonoids of the nonnegative rationals. Additive submonoids of $(\mathbb{Q}_{\ge 0}, +)$ are called Puiseux monoids and have recently been studied in a series of papers by F. Gotti et al. (e.g., [19]). In the setting of Puiseux monoids all arithmetical concepts refer to addition and not to multiplication of rationals. In particular, an element a of a Puiseux monoid H is said to be squarefree if there are no nonzero elements $b, c \in H$ such that $a = b + b + c$.

Clearly, the constructed numerical monoid heavily depends on the given set L. This is inevitable because for every fixed numerical monoid H, sets of lengths have a well-defined structure. Indeed, there is a constant $M \in \mathbb{N}_0$ (just depending on H) such that every $L \in \mathcal{L}(H)$ has the form

$$L = y + \big(L' \cup \{\nu d \mid \nu \in [0,l]\} \cup L''\big) \ \subset \ y + d\mathbb{Z}\,, \tag{3.1}$$

where $d = \min \Delta(H)$, $y \in \mathbb{Z}$, $L' \subset [-M,-1]$, and $L'' \subset ld + [1,M]$ ([15, Theorem 4.3.6]).

We start with a technical lemma.

Lemma 3.1. *Let $k \in \mathbb{N}_{\ge 2}$. Then there exist pairwise distinct nonzero $c_1, \ldots, c_k \in [-k^{k-1}, k^{k-1}]$ with $c_1 + \ldots + c_k = 0$ such that for all primes $p > (k+1)k^{k-1}$ the following property holds: if $l_1, \ldots, l_k \in \mathbb{N}_0$ such that $\sum_{i=1}^k l_i c_i \equiv 0 \mod p$, then*

$$l_1 = \ldots = l_k = 0 \quad \text{or} \quad l_1 = \ldots = l_k = 1 \quad \text{or} \quad l_1 + \ldots + l_k > k.$$

Proof. For $i \in [1, k-1]$ we define $c_i = k^{i-1}$, and we set $c_k = -\sum_{i=1}^{k-1} c_i$. Then clearly,

$$c_k = -\sum_{i=1}^{k-1} c_i = -\sum_{i=1}^{k-1} k^{i-1} = -\frac{k^{k-1}-1}{k-1}\,.$$

Now we choose a prime $p > (k+1)k^{k-1}$ and $l_1, \ldots, l_k \in \mathbb{N}_0$ such that $\sum_{i=1}^k l_i c_i \equiv 0 \mod p$ and $\sum_{i=1}^k l_i > 0$. We may distinguish the following two cases.

CASE 1: $\sum_{i=1}^{k-1} l_i c_i \ge p$ or $l_k c_k \le -p$.

If $p \le \sum_{i=1}^{k-1} l_i c_i \le \big(\sum_{i=1}^{k-1} l_i\big) c_{k-1}$, then

$$\sum_{i=1}^{k} l_i \ge \sum_{i=1}^{k-1} l_i \ge \frac{p}{c_{k-1}} > \frac{(k+1)k^{k-1}}{c_{k-1}} \ge k+1\,.$$

If $p \le l_k |c_k|$, then

$$\sum_{i=1}^{k} l_i \ge l_k \ge \frac{p}{|c_k|} > \frac{(k+1)k^{k-1}}{|c_k|} \ge k+1\,.$$

CASE 2: $\sum_{i=1}^{k-1} l_i c_i < p$ and $l_k c_k > -p$.

Since $\sum_{i=1}^k l_i c_i \equiv 0 \mod p$, we infer that $\sum_{i=1}^k l_i c_i = 0$. Suppose that there is a $j \in [1,k]$ with $l_j \ge k$. Since at least two elements of $l_1, \ldots, l_k$ are positive, it follows that $\sum_{i=1}^k l_i > k$. Suppose that $l_i \in [0, k-1]$ for all $i \in [1,k]$. Since $0 = \sum_{i=1}^k l_i c_i$, the definition of $c_1, \ldots, c_k$ implies that

$$\sum_{i=1}^{k-1} l_i k^{i-1} = \sum_{i=1}^{k-1} l_k k^{i-1}\,.$$

By the uniqueness of the k-adic digit expansion, we infer that $l_i = l_k$ for all $i \in [1, k-1]$. If $l_1 = 1$, then $l_1 = \ldots = l_k = 1$. If $l_1 > 1$, then $l_1 + \ldots + l_k = kl_1 > k$. □

The following proposition will be our key tool to do the recursive construction step in Theorem 3.3. For every prime $p \in \mathbb{P}$, we denote by v_p the usual p-adic valuation of the rationals, that is, for $q \in \mathbb{Q} \setminus \{0\}$, $\mathsf{v}_p(q)$ the integer j such that $q = p^j \frac{a}{b}$ with integers a, b such that $p \nmid ab$. Moreover, we set $\mathsf{v}_p(0) = \infty$.

Proposition 3.2. *Let $k \in \mathbb{N}_{\ge 2}$ and $H \subset (\mathbb{Q}_{\ge 0}, +)$ be a finitely generated monoid with $\mathbb{N}_0 \subset H$ and $\mathcal{A}(H) \subset \mathbb{Q}_{<1}$. Then there exists a finitely generated monoid H' with $H \subset H'$ and $\mathcal{A}(H) \subset \mathcal{A}(H') \subset \mathbb{Q}_{<1}$ such that the following properties are satisfied:*

(a) *For all $u \in H$ with $u < 1$ we have $\mathsf{Z}_H(u) = \mathsf{Z}_{H'}(u)$.*
(b) *$\mathsf{Z}_{H'}(1) = \mathsf{Z}_H(1) \uplus \{q_1 + \ldots + q_k\}$, where $q_1, \ldots, q_k$ are pairwise distinct and $\mathcal{A}(H') = \mathcal{A}(H) \uplus \{q_1, \ldots, q_k\}$.*

Proof. We set

$$\mathcal{A}(H) = \Big\{ \frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s} \Big\}$$

where $a_i, b_i \in \mathbb{N}$ with $\gcd(a_i, b_i) = 1$ for all $i \in [1, s]$. Let $c_1, \ldots, c_k \in [-k^{k-1}, k^{k-1}]$ such that all properties of Lemma 3.1 are satisfied. We choose a prime number $p \in \mathbb{N}$ such that

$$p \nmid \operatorname{lcm}(b_1, \ldots, b_s) \quad \text{and} \quad p > (k+1)k^{k-1} \,,$$

and we define

$$q_i = \frac{p + c_i}{kp} \quad \text{for every} \quad i \in [1, k] \,.$$

By construction, we have $q_1 + \ldots + q_k = 1$ and $\mathsf{v}_p(q_i) = -1$ whence $q_i \notin H$ for all $i \in [1, k]$. We define

$$H' = \big[H, q_1, \ldots, q_k \big] \subset (\mathbb{Q}_{\ge 0}, +)$$

to be the additive submonoid of nonnegative rationals generated by the elements of H and by $q_1, \ldots, q_k$. Thus H' is generated by $\mathcal{A}(H) \cup \{q_1, \ldots, q_k\}$ whence finitely generated. Since H' is reduced, [15, Proposition 1.1.7] implies that H' is atomic and

$$\mathcal{A}(H') \subset \mathcal{A}(H) \cup \{q_1, \ldots, q_k\} \,. \tag{3.2}$$

We continue with the following assertions.

A1. $\{q_1, \ldots, q_k\} \subset \mathcal{A}(H')$.

A2. Let $u \in H$ and suppose that u has a factorization $z \in \mathsf{Z}_{H'}(u)$ which is divisible by some element from $\{q_1, \ldots, q_k\}$. Then either $u > 1$ or $z = q_1 + \ldots + q_k \in \mathsf{Z}_{H'}(u)$ (whence in particular $u = 1$).

Proof of **A1**. Assume to the contrary that there is an $i \in [1, k]$ such that $q_i \notin \mathcal{A}(H')$. Since $q_i \notin H$, it is divisible by an atom from $\mathcal{A}(H') \setminus \mathcal{A}(H) \subset \{q_1, \ldots, q_k\}$, say $q_i = q_j + b$ with $j \in [1, k] \setminus \{i\}$ and $b \in H' \setminus \{0\}$. We claim that $b \notin H$. Since $0 \ne b = q_i - q_j$ and $0 \ne |c_i - c_j| \le 2k^{k-1} < p$,

$$\mathsf{v}_p(q_i - q_j) = \mathsf{v}_p\Big(\frac{c_i - c_j}{kp} \Big) = -1$$

which implies that $b \notin H$. Thus there is an $l \in [1, k]$ such that $b = q_l + d$ with $d \in H' \subset \mathbb{Q}_{\ge 0}$. Since $p > (k+1)k^{k-1} \ge 3k^{k-1} \ge |c_i| + |c_j| + |c_l|$, it follows that

$$q_i = q_j + q_l + d \ge q_j + q_l = \frac{2p + c_j + c_l}{kp} > \frac{p + c_i}{kp} = q_i \,,$$

a contradiction. □[Proof of **A1**]

Proof of **A2**. Since u has a factorization which is divisible by some element from $\{q_1, \ldots, q_k\}$, there are $l_1, \ldots, l_k \in \mathbb{N}_0$ and $v \in H$ such that

$$u = v + \sum_{i=1}^{k} l_i q_i \quad \text{and} \quad \sum_{i=1}^{k} l_i > 0 \,.$$

Since $\mathsf{v}_p(u) \ge 0$ and $\mathsf{v}_p(v) \ge 0$, it follows that

$$0 \le \mathsf{v}_p(u - v) = \mathsf{v}_p\Big(\sum_{i=1}^{k} l_i q_i \Big) = \mathsf{v}_p\Big(\frac{\sum_{i=1}^{k} l_i p + \sum_{i=1}^{k} l_i c_i}{kp} \Big)$$

whence $\sum_{i=1}^{k} l_i c_i \equiv 0 \mod p$. Therefore Lemma 3.1 implies that

$$l_1 = \ldots = l_k = 1 \quad \text{or} \quad \sum_{i=1}^{k} l_i > k \,.$$

If $\sum_{i=1}^k l_i > k$ and $j \in [1,k]$ with $q_j = \min\{q_1, \ldots, q_k\}$, then

$$u = v + \sum_{i=1}^{l} l_i q_i \ge (k+1) q_j = (k+1) \frac{p + c_j}{kp} > 1 \,, \tag{3.3}$$

where the last inequality uses that $p > (k+1)k^{k-1} \ge (k+1)|c_j|$. If $l_1 = \ldots = l_k = 1$, then

$$u = \sum_{i=1}^{l} l_i q_i + v = q_1 + \ldots + q_k + v = 1 + v \,.$$

Thus $v > 0$ implies $u > 1$ and $v = 0$ implies $u = 1$ and $z = q_1 + \ldots + q_k$. □[Proof of **A2**]

If $u \in \mathcal{A}(H)$, then $u < 1$ by assumption and **A2** implies that u is not divisible by any element from $\{q_1, \ldots, q_k\}$ and therefore $u \in \mathcal{A}(H')$. Thus we obtain that $\mathcal{A}(H) \subset \mathcal{A}(H')$ and together with **A1** and (3.2), it follows that

$$\mathcal{A}(H') = \mathcal{A}(H) \uplus \{q_1, \ldots, q_k\} \,. \tag{3.4}$$

Thus, we have that

$$\mathsf{Z}(H) = \mathcal{F}(\mathcal{A}(H)) \subset \mathcal{F}(\mathcal{A}(H')) = \mathsf{Z}(H') \quad \text{and} \quad \mathsf{Z}_H(u) \subset \mathsf{Z}_{H'}(u) \tag{3.5}$$

for every $u \in H$. If $u < 1$, then **A2** implies that $\mathsf{Z}_H(u) = \mathsf{Z}_{H'}(u)$.

It remains to show Property (b) given in Proposition 3.2, namely that

$$\mathsf{Z}_H(1) \uplus \{q_1 + \ldots + q_k\} = \mathsf{Z}_{H'}(1) \,.$$

We see from Equation (3.5) that $\mathsf{Z}_H(1) \uplus \{q_1 + \ldots + q_k\} \subset \mathsf{Z}_{H'}(1)$. Conversely, let z be a factorization of 1 in H'. Then either $z \in \mathsf{Z}_H(1)$ or z is divisible (in $\mathsf{Z}(H')$) by some element from $\{q_1, \ldots, q_k\}$. In the latter case **A2** implies that $z = q_1 + \ldots + q_k \in \mathsf{Z}(H')$. □

Theorem 3.3. *Let $L \subset \mathbb{N}_{\ge 2}$ be a finite nonempty set and $f \colon L \to \mathbb{N}$ a map. Then there exist a numerical monoid H and a squarefree element $a \in H$ such that*

$$\mathsf{L}(a) = L \quad \textit{and} \quad |\mathsf{Z}_k(a)| = f(k) \quad \textit{for every } k \in L \,. \tag{3.6}$$

Proof. Every finitely generated submonoid of $(\mathbb{Q}_{\ge 0}, +)$ is isomorphic to a numerical monoid (cf. [19, Proposition 3.2]) and the isomorphism maps squarefree elements onto squarefree elements. Thus it is sufficient to show that, for every set L and every map f as in the statement of the theorem, there is a finitely generated submonoid H of the nonnegative rationals with $\mathbb{N}_0 \subset H$ and $\mathcal{A}(H) \subset \mathbb{Q}_{<1}$ such that the element $a = 1 \in H$ is squarefree in H and has the properties given in (3.6).

Clearly, it is equivalent to consider nonzero maps $f \colon \mathbb{N}_{\ge 2} \to \mathbb{N}_0$ with finite support and to find a monoid H as above such that $|\mathsf{Z}_k(1)| = f(k)$ for every $k \in \mathbb{N}_{\ge 2}$ and 1 is squarefree in H. For every nonzero map $f \colon \mathbb{N}_{\ge 2} \to \mathbb{N}_0$ with finite support $\sum_{k \ge 2} f(k)$ is a positive integer and we proceed by induction on this sum.

To do the base case, let $f \colon \mathbb{N}_{\ge 2} \to \mathbb{N}_0$ be a map with $\sum_{k \ge 2} f(k) = 1$. Let $k \in \mathbb{N}_{\ge 2}$ with $f(k) = 1$. We have to find a finitely generated monoid $H \subset (\mathbb{Q}_{\ge 0}, +)$ with $\mathcal{A}(H) \subset \mathbb{Q}_{<1}$ and pairwise distinct atoms $q_1, \ldots, q_k \in H$ such that $\mathsf{Z}_H(1) = \{q_1 + \ldots + q_k\}$.

We proceed along the lines of the proof of **A2** in Proposition 3.2. Indeed, we choose $c_1, \ldots, c_k \in [-k^{k-1}, k^{k-1}]$ such that all properties of Lemma 3.1 are satisfied and pick a prime number $p \in \mathbb{N}$ with $p > (k+1)k^{k-1}$. We set

$$q_i = \frac{p + c_i}{kp} \quad \text{for every} \quad i \in [1,k]$$

and define $H = [q_1, \ldots, q_k] \subset \mathbb{Q}_{\ge 0}$. By [15, Proposition 1.1.7], $\mathcal{A}(H) \subset \{q_1, \ldots, q_k\}$. Since for all (not necessarily distinct) $r, s, t \in [1,k]$ we have $q_r < q_s + q_t$, it follows that $q_r \in \mathcal{A}(H)$. Thus we obtain that $\mathcal{A}(H) = \{q_1, \ldots, q_k\}$. Since $q_1 + \ldots + q_k = 1$, it follows that $\{q_1 + \ldots + q_k\} \subset \mathsf{Z}_H(1)$. To show equality, let $l_1, \ldots, l_k \in \mathbb N_0$ such that $1 = \sum_{i=1}^k l_i q_i$. It follows that $\sum_{i=1}^k l_i c_i \equiv 0 \mod p$. Therefore Lemma 3.1 implies that

$$l_1 = \ldots = l_k = 1 \quad \text{or} \quad \sum_{i=1}^k l_i > k \,.$$

If $\sum_{i=1}^k l_i > k$ and $j \in [1,k]$ with $q_j = \min\{q_1, \ldots, q_k\}$, then

$$1 = \sum_{i=1}^l l_i q_i \ge (k+1) q_j = (k+1)\frac{p+c_j}{kp} > 1 \,, \tag{3.7}$$

a contradiction. Thus $l_1 = \ldots = l_k = 1$ and the claim follows.

Now let $N \in \mathbb N_{\ge 2}$ and suppose that the assertion holds all nonzero maps $f \colon \mathbb N_{\ge 2} \to \mathbb N_0$ with finite support and with $\sum_{k \ge 2} f(k) < N$. Let $f_0 \colon \mathbb N_{\ge 2} \to \mathbb N_0$ with $\sum_{k \ge 2} f_0(k) = N$. We choose an element $k_0 \in \mathbb N_{\ge 2}$ with $f(k_0) \ne 0$ and define a map $f_1 \colon \mathbb N_{\ge 2} \to \mathbb N_0$ as $f_1(k_0) = f_0(k_0) - 1$ and $f_1(k) = f_0(k)$ for all $k \in \mathbb N_{\ge 2} \setminus \{k_0\}$. By the induction hypothesis, there exists a finitely generated monoid $H_1 \subset (\mathbb Q_{\ge 0}, +)$ with $\mathbb N_0 \subset H_1$ and $\mathcal A(H_1) \subset \mathbb Q_{<1}$ such that $|\mathsf Z_{H_1,k}(1)| = f_1(k)$ for every $k \in \mathbb N_{\ge 2}$ and 1 is squarefree in H_1. By Proposition 3.2 there exist a finitely generated monoid $H_0 \subset (\mathbb Q_{\ge 0}, +)$ such that

$$\mathsf Z_{H_0}(1) = \mathsf Z_{H_1}(1) \uplus \{q_1 + \ldots + q_{k_0}\},$$

where $q_1, \ldots, q_{k_0}$ are pairwise distinct and $\mathcal A(H_0) = \mathcal A(H_1) \uplus \{q_1, \ldots, q_{k_0}\} \subset \mathbb Q_{<1}$. Since $q_1, \ldots, q_{k_0}$ are pairwise distinct and since 1 was squarefree in H_1, it follows that 1 is squarefree in H_0. Moreover, $\mathsf Z_{H_0,k}(1) = \mathsf Z_{H_1,k}(1)$ for all $k \in \mathbb N_{\ge 2} \setminus \{k_0\}$ and $\mathsf Z_{H_0,k_0}(1) = \mathsf Z_{H_1,k_0}(1) \uplus \{q_1 + \ldots + q_{k_0}\}$. In particular, we have $|\mathsf Z_{H_0,k}(1)| = f_0(k)$ for every $k \in \mathbb N_{\ge 2}$. □

We continue with a corollary on sets of distances. Let H be an atomic monoid with nonempty set of distances $\Delta(H)$. Then it is easy to verify that $\min \Delta(H) = \gcd \Delta(H)$, and the question is which finite sets D with $\min D = \gcd D$ can be realized as a set of distances in a given class of monoids or domains. The question has an affirmative answer in the class of finitely generated Krull monoids ([17]). If H is a numerical monoid generated by two atoms, say $\mathcal A(H) = \{n_1, n_2\}$, then $\Delta(H) = \{|n_2 - n_1|\}$ whence every singleton occurs as a set of distances of a numerical monoid. There are periodicity results on individual sets $\Delta(\mathsf L(a))$ for elements in a numerical monoid ([8]), but the only realization result beyond the simple observation above is due to Colton and Kaplan ([9]). They show that every two-element set D with $\min D = \gcd D$ can be realized as the set of distances of a numerical monoid. As a consequence of Theorem 3.3 we obtain that every finite set is contained in the set of distances of a numerical monoid (this was achieved first by explicit constructions in [5, Corollary 4.8]).

Corollary 3.4. *For every finite nonempty subset $D \subset \mathbb N$ there is a numerical monoid H such that $D \subset \Delta(H)$.*

Proof. Let $D = \{d_1, \ldots, d_k\} \subset \mathbb N$ be a finite nonempty subset. By Theorem 3.3 there is a numerical monoid H such that $L = \{2, 2+d_1, 2+d_1+d_2, \ldots, 2+d_1+\ldots+d_k\} \in \mathcal L(H)$ whence $D = \Delta(L) \subset \Delta(H)$. □

Let K be a field and H a numerical monoid. The semigroup algebra

$$K[H] = \Big\{ \sum_{h \in H} a_h X^h \mid a_h \in H \text{ for all } h \in H \text{ and almost all } a_h \text{ are zero} \Big\} \subset K[X]$$

is a one-dimensional noetherian domain and its integral closures $K[X]$ is a finitely generated module over $K[H]$. Thus $K[H]$ is weakly Krull, $\operatorname{Pic}(K[H])$ is finite if K is finite whence it satisfies all arithmetical finiteness results established for weakly Krull Mori domains with finite class group (see [15] for basic information). However, all results on $\mathcal L\big(K[H]\big)$ so far depend on detailed information on the Picard group and the distribution of height one prime ideals not containing the conductor in the Picard group.

Corollary 3.5. *Let K be a field, $L \subset \mathbb{N}_{\ge 2}$ a finite nonempty set, and $f \colon L \to \mathbb{N}$ a map. Then there is a numerical monoid H and a squarefree element $g \in K[H]$ such that*

$$\mathsf{L}_{K[H]}(g) = L \quad \text{and} \quad |\mathsf{Z}_{K[H],k}(g)| = f(k) \quad \text{for every } k \in L\,.$$

Proof. By Theorem 3.3 there is a numerical monoid H and a squarefree element $c \in H$ having the required properties. Clearly, the additive monoid H is isomorphic to the multiplicative monoid of monomials

$$H' = \{X^h \mid h \in H\} \subset K[H]\,.$$

Since $K[H]^\times = K^\times$, the monoid $H'' = \{cX^h \mid h \in H, c \in K^\times\} \subset K[H]$ is a divisor-closed submonoid and $H''_{\mathrm{red}} \cong H'$. Thus for every $k \in L$ and the element $g = X^c \in K[H]$ we obtain that

$$|\mathsf{Z}_{K[H],k}(X^c)| = |\mathsf{Z}_{H'',k}(X^c)| = |\mathsf{Z}_{H',k}(X^c)| = |\mathsf{Z}_{H,k}(c)| = f(k)$$

whence the assertion follows. □

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