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SETS OF LENGTHS IN ATOMIC UNIT-CANCELLATIVE FINITELY PRESENTED MONOIDS

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Abstract. For an element a of a monoid H, its set of lengths $\mathsf{L}(a) \subset \mathbb{N}$ is the set of all positive integers k for which there is a factorization $a = u_1 \cdot \ldots \cdot u_k$ into k atoms. We study the system $\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$ with a focus on the unions $\mathcal{U}_k(H) \subset \mathbb{N}$ of all sets of lengths containing a given $k \in \mathbb{N}$. The Structure Theorem for Unions—stating that for all sufficiently large k, the sets $\mathcal{U}_k(H)$ are almost arithmetical progressions with the same difference and global bound—has attracted much attention for commutative monoids and domains. We show that it holds true for the not necessarily commutative monoids in the title satisfying suitable algebraic finiteness conditions. Furthermore, we give an explicit description of the system of sets of lengths of the monoids $B_n = \langle a, b \mid ba = b^n \rangle$ for $n \in \mathbb{N}_{\ge 2}$. Based on this description, we show that the monoids B_n are not transfer Krull.

1. Introduction. By an *atomic unit-cancellative monoid*, we mean an associative semigroup with unit element where every non-unit can be written as a finite product of atoms (irreducible elements) and where equations of the form $au = a$ or $ua = a$ imply that u is a unit. Let H be an atomic unit-cancellative monoid. If $a = u_1 \cdot \ldots \cdot u_k$ with $a \in H$ and atoms $u_1, \ldots, u_k \in H$, then k is called a *factorization length* of a, and the set $\mathsf{L}(a) \subset \mathbb{N}$ of all possible factorization lengths is called the *set of lengths* of a.

The system $\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$ of all sets of lengths (for convenience, one defines $\mathsf{L}(a) = \{0\}$ for units $a \in H$) is a well-studied means of describing the non-uniqueness of factorizations of H. If there is an $a \in H$ with $|\mathsf{L}(a)| > 1$, say $k, \ell \in \mathsf{L}(a)$, then for all $N \in \mathbb{N}$, $\mathsf{L}(a^N) \supset \{(N-i)k + i\ell \mid i \in [0, N]\}$ and hence $|\mathsf{L}(a^N)| > N$. Thus $\mathcal{L}(H)$ either consists of singletons only or it contains arbitrarily large sets.

For every $k \in \mathbb{N}$, let $\mathcal{U}_k(H)$ denote the set of all $\ell \in \mathbb{N}$ for which there is an equation of the form $u_1 \cdot \ldots \cdot u_k = v_1 \cdot \ldots \cdot v_\ell$ with atoms $u_1, \ldots, u_k, v_1, \ldots, v_\ell \in H$. Thus $\mathcal{U}_k(H)$ is the union of all sets of lengths containing k.

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Systems of sets of lengths, together with all invariants derived from them, such as unions, sets of distances, and more, are well-studied invariants in factorization theory. For a long time the focus was on commutative and cancellative monoids which mainly stem from ring theory, such as monoids of non-zero elements of integral domains or monoids of invertible ideals [2, 20, 17, 12]. Recently, first steps were made to study the arithmetic of commutative but not necessarily cancellative monoids [13, 11, 15].

Although various concepts of unique factorization in non-commutative rings have been studied for decades (see [34] for a survey), a systematic investigation of arithmetic phenomena in non-commutative rings has only been started in recent years [33, 5, 8, 6, 9, 7]. In many of these papers their authors construct so-called (weak) transfer homomorphisms from the non-commutative ring R under consideration to a commutative monoid H such that many arithmetical phenomena of R and H coincide and, in particular, $\mathcal{L}(R) = \mathcal{L}(H)$. To mention one of these deep results explicitly, consider a bounded hereditary noetherian prime ring R: if every stably free left R-ideal is free, then there are a commutative Krull monoid H and a weak transfer homomorphism $\theta\colon R \to H$ implying that $\mathcal{L}(R) = \mathcal{L}(H)$ [32, Theorem 4.4].

In the present note we study atomic unit-cancellative monoids which are finitely presented. Under mild algebraic finiteness conditions, we show that all unions $\mathcal{U}_k(H)$ are finite and that, apart from globally bounded initial and end parts, they are arithmetical progressions. Thus $\mathcal{L}(H)$ satisfies the Structure Theorem for Unions (Lemma 3.1 and Theorem 3.6).

Section 4 is devoted to the monoids $B_n = \langle a, b \mid ba = b^n \rangle$ for $n \in \mathbb{N}_{\ge 2}$. They are Möbius monoids anti-isomorphic to the semigroups $S_{2,n}$ from [31] with a unit element adjoined, playing an important role in Möbius inversion and in the study of monoids with one defining relation generating varieties of finite axiomatic rank [31, 30, 28, 29]. It is easy to see that the monoids B_n are atomic, unit-cancellative, and right-cancellative, but not left-cancellative. We provide an explicit description of the system $\mathcal{L}(B_n)$ and of the unions $\mathcal{U}_k(B_n)$ for all $k \ge 2$ (Theorem 4.2 and Corollary 4.3). Such explicit descriptions are very rare in the literature ([11, Corollary 16], [22]). They allow us to show that there is no weak transfer homomorphism $\theta\colon B_n \to H$ where H is any commutative Krull monoid (Corollary 4.4). Furthermore, we prove that the system $\mathcal{L}(B_n)$ is closed under set addition, and hence $\mathcal{L}(B_n)$ is a reduced atomic unit-cancellative monoid with set addition as operation (Theorem 4.5).

2. Preliminaries. We denote by $\mathbb{N}$ the set of positive integers. For $x \in \mathbb{R}_{\ge 0}$, we denote by $\lfloor x \rfloor \in \mathbb{N}_0$ the largest integer which is smaller than or equal to x. For $a, b \in \mathbb{Z}$, $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$ denotes the discrete interval. Let $L, L' \subset \mathbb{Z}$ be subsets of the integers. Then $L + L' = \{\ell + \ell' \mid$

$\ell \in L$, $\ell' \in L'\}$ is their sumset, with $m + L = \{m\} + L$ for every $m \in \mathbb{Z}$. For $d \in \mathbb{N}$, $d \cdot L = \{da \mid a \in L\}$ denotes the *dilation* of L by d. Thus, for $d \in \mathbb{N}$, $q \in \mathbb{N}_0$, and $m \in \mathbb{Z}$, $m + d \cdot [0, q] = \{m, m+d, \ldots, m+qd\}$ is an arithmetical progression with difference d.

A positive integer $d \in \mathbb{N}$ is said to be a *distance* of L if there is an $a \in L$ such that $[a, a+d] \cap L = \{a, a+d\}$, and $\Delta(L) \subset \mathbb{N}$ is the *set of distances* of L. If $L \subset \mathbb{N}_0$, then the *elasticity* $\rho(L)$ is defined as $\rho(L) = \sup(L \cap \mathbb{N}) / \min(L \cap \mathbb{N}) \in \mathbb{Q}_{\ge 1} \cup \{\infty\}$ if $L \cap \mathbb{N} \ne \emptyset$ and $\rho(L) = 1$ if $L \cap \mathbb{N} = \emptyset$.

Let $\mathcal{L}$ be a family of subsets of $\mathbb{N}_0$. For each $k \in \mathbb{N}$, we set

$$\mathcal{U}_k(\mathcal{L}) = \bigcup_{k \in L,\, L \in \mathcal{L}} L \subset \mathbb{N} \quad \text{and} \quad \rho_k(\mathcal{L}) = \sup \mathcal{U}_k(\mathcal{L}).$$

Furthermore, we call

- $\Delta(\mathcal{L}) = \bigcup_{L \in \mathcal{L}} \Delta(L) \subset \mathbb{N}$ the *set of distances* of $\mathcal{L}$,
- $\rho(\mathcal{L}) = \sup\{\rho(L) \mid L \in \mathcal{L}\} \in \mathbb{R}_{\ge 1} \cup \{\infty\}$ the *elasticity* of $\mathcal{L}$,

and we say that $\mathcal{L}$ has *accepted elasticity* if there is some $L \in \mathcal{L}$ such that $\rho(L) = \rho(\mathcal{L}) < \infty$.

By a *monoid* we mean an associative semigroup with a unit element. If not stated otherwise, we use multiplicative notation. Let H be a monoid with unit element $1_H = 1$. We denote by $H^\times$ the group of invertible elements of H, and say that H is *reduced* if $H^\times = \{1\}$. We say that H is (left and right) *unit-cancellative* if the following two properties are satisfied:

- if $a, u \in H$ and $a = au$, then $u \in H^\times$,
- if $a, u \in H$ and $a = ua$, then $u \in H^\times$.

Clearly, every cancellative monoid is unit-cancellative. Unit-cancellativity is a frequently studied property, by many authors and under many different names (the corresponding concept in ring theory is called présimplifiable; it was introduced by Bouvier and further studied by D. D. Anderson et al. [4, 3]). An element $u \in H$ is said to be *irreducible* (or an *atom*) if $u \notin H^\times$ and an equation $u = ab$ with $a, b \in H$ implies that $a \in H^\times$ or $b \in H^\times$. We denote by $\mathcal{A}(H)$ the set of atoms of H, and we say that H is *atomic* if every non-unit can be written as a finite product of atoms of H.

For a set P, let $\mathcal{F}^*(P)$ be the free monoid with basis P and let $\mathcal{F}(P)$ be the free abelian monoid with basis P. We denote by $|\cdot| \colon \mathcal{F}^*(P) \to \mathbb{N}_0$ the homomorphism mapping each word to its length. Similarly, if

$$a = \prod_{p \in P} p^{\mathsf{v}_p(a)} \in \mathcal{F}(P), \quad \text{where} \quad \mathsf{v}_p \colon \mathcal{F}(P) \to \mathbb{N}_0 \text{ is the } p\text{-adic exponent},$$

then $|a| = \sum_{p \in P} \mathsf{v}_p(a) \in \mathbb{N}_0$ is the length of a. Let D be a monoid. A submonoid $H \subset D$ is said to be *saturated* if the conditions $a \in D$, $b \in H$, and ($ab \in H$ or $ba \in H$) imply that $a \in H$ (instead of saturated also the terms *full* or *grouplike* are used [25, p. 64]). A commutative monoid H is *Krull* if its associated reduced monoid is a saturated submonoid of a free abelian monoid [20, Theorem 2.4.8].

Let H be an atomic unit-cancellative monoid. If $a = u_1 \cdot \ldots \cdot u_k \in H$, where $k \in \mathbb{N}$ and $u_1, \ldots, u_k \in \mathcal{A}(H)$, then k is called a *factorization length*, and

$$\mathsf{L}_H(a) = \mathsf{L}(a) = \{k \in \mathbb{N} \mid a \text{ has a factorization of length } k\} \subset \mathbb{N}$$

is called the *set of lengths* of a. For a unit $\epsilon \in H^\times$ we set $\mathsf{L}(\epsilon) = \{0\}$. Then

$$\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$$

denotes the *system of sets of lengths* of H. We say that H is

- a BF-*monoid* (a bounded factorization monoid) if $\mathsf{L}(a)$ is finite and non-empty for all $a \in H$,
- *half-factorial* if $|\mathsf{L}(a)| = 1$ for all $a \in H$.

If H is not half-factorial, then there is an $a \in H$ with $|\mathsf{L}(a)| > 1$, whence $\mathsf{L}(a^n) \supset \mathsf{L}(a) + \cdots + \mathsf{L}(a)$ has more than n elements for every $n \in \mathbb{N}$. We set

- $\mathcal{U}_k(H) = \mathcal{U}_k(\mathcal{L}(H))$ for every $k \in \mathbb{N}$, the *union of sets of lengths* (containing k),
- $\Delta(H) = \Delta(\mathcal{L}(H))$, the *set of distances* of H,
- $\rho(H) = \rho(\mathcal{L}(H))$, the *elasticity* of H.

3. Reduced atomic unit-cancellative monoids. In this section we study sets of lengths of reduced atomic unit-cancellative monoids. Under suitable additional algebraic finiteness conditions we show that their systems of sets of lengths satisfy the Structure Theorem for Unions (Theorem 3.6). To start with, we recall some concepts needed to formulate that theorem.

A non-empty subset $L \subset \mathbb{N}_0$ is called an *almost arithmetical progression* (AAP for short) if $L \subset \min L + d\mathbb{Z}$ and

$$L \cap [M + \min L, -M + \sup L]$$

is an arithmetical progression with difference d, where $d \in \mathbb{N}$, $M \in \mathbb{N}_0$, and with the conventions that arithmetical progressions are non-empty and that $[M + \min L, -M + \sup L] = \mathbb{N}_{\geq M + \min L}$ if L is infinite. Let $\mathcal{L}$ be a family of subsets of $\mathbb{N}_0$

- $\mathcal{L}$ is *directed* if $1 \in L$ for some $L \in \mathcal{L}$ and, for all $L_1, L_2 \in \mathcal{L}$, there is $L' \in \mathcal{L}$ with $L_1 + L_2 \subset L'$;
- $\mathcal{L}$ satisfies the *Structure Theorem for Unions* if there are $d \in \mathbb{N}$ and $M \in \mathbb{N}_0$ such that $\mathcal{U}_k(\mathcal{L})$ is an AAP with difference d and bound M for all sufficiently large $k \in \mathbb{N}$.

The next result provides a characterization of when the Structure Theorem for Unions holds in the setting of directed families of subsets of the non-negative integers.

LEMMA 3.1. *Let $\mathcal{L}$ be a directed family of subsets of $\mathbb{N}_0$ such that $\Delta(\mathcal{L})$ is finite non-empty, and set $d = \min \Delta(\mathcal{L})$. Let $\ell \in \mathbb{N}$ be such that $\{\ell, \ell+d\} \subset L$ for some $L \in \mathcal{L}$. Then $q = d^{-1} \max \Delta(\mathcal{L})$ is a non-negative integer and the following statements are equivalent:*

(a) *$\mathcal{L}$ satisfies the Structure Theorem for Unions.*
(b) *There exists $M \in \mathbb{N}$ such that $\mathcal{U}_k(\mathcal{L}) \cap [\rho_{k-\ell q}(\mathcal{L})+\ell q, \rho_k(\mathcal{L})-M]$ is either empty or an arithmetical progression with difference d for all sufficiently large k.*

A proof of Lemma 3.1 together with a variety of properties of directed families and of consequences of the Structure Theorem for Unions can be found in [15, Section 2]. In particular, we recall that if the Structure Theorem holds and $\Delta(\mathcal{L})$ is non-empty, then the unions are AAPs with difference $\min \Delta(\mathcal{L})$.

We now analyze condition (b) in Lemma 3.1. If there is some $\ell \in \mathbb{N}$ such that $\rho_\ell(\mathcal{L}) = \infty$, then $\rho_k(\mathcal{L}) = \infty$ for all $k \geq \ell$, whence the intersection in (b) is empty. Suppose that $\rho_k(\mathcal{L}) < \infty$ for all $k \in \mathbb{N}$. If there is an $M \in \mathbb{N}$ such that $\rho_k(\mathcal{L}) - \rho_{k-1}(\mathcal{L}) \leq M$ for all $k \in \mathbb{N}_{\geq 2}$, then (b) holds (for details see [15, Section 2]).

Let H be an atomic unit-cancellative monoid. If $u \in \mathcal{A}(H)$, then $\mathsf{L}(u) = \{1\}$. If $a, b \in H$, then $\mathsf{L}(a) + \mathsf{L}(b) \subset \mathsf{L}(ab)$. Thus, if $H \neq H^\times$, then $\mathcal{L}(H)$ is a directed family and the characterization in Lemma 3.1 applies to it.

Two elements $a, b \in H$ are said to be *associated* (we write $a \simeq b$) if $a \in H^\times b H^\times$. Suppose that $aH^\times = H^\times a$ for all $a \in H$. Then being associated is a congruence relation on H, and $H_{\mathrm{red}} = H/{\simeq}$ is a reduced atomic monoid. For every $a \in H$, we have $[a]_\simeq = aH^\times$ and $\mathsf{L}_H(a) = \mathsf{L}_{H_{\mathrm{red}}}(aH^\times)$, whence $\mathcal{L}(H) = \mathcal{L}(H_{\mathrm{red}})$. We will formulate all our results for reduced atomic monoids but they also apply to non-reduced monoids (see, for example, Remark 3.7(2)).

We now suppose that H is reduced atomic and unit-cancellative. We define its factorization monoid as the monoid of formal products of atoms and distinguish between the commutative and the non-commutative case. Thus we call

$$\mathsf{Z}(H) = \begin{cases} \mathcal{F}^*(\mathcal{A}(H)) & \text{if } H \text{ is non-commutative,} \\ \mathcal{F}(\mathcal{A}(H)) & \text{if } H \text{ is commutative} \end{cases}$$

the *factorization monoid* of H. Then $\pi\colon \mathsf{Z}(H) \to H$ denotes the canonical epimorphism. If $a \in H$, then $\mathsf{Z}(a) = \pi^{-1}(a) \subset \mathsf{Z}(H)$ is the *set of factorizations* of a, and $\mathsf{L}(a) = \{|z| \mid z \in \mathsf{Z}(a)\} \subset \mathbb{N}_0$ is the *set of lengths* of a, as introduced above. We say that H is an *FF-monoid* (a finite factorization monoid) if $\mathsf{Z}(a)$ is finite and non-empty for every $a \in H$. Note that every reduced atomic unit-cancellative FF-monoid is a Möbius monoid [28, Section 2.1]. The *monoid of relations* is defined as

$$\sim_H = \{(x,y) \in \mathsf{Z}(H) \times \mathsf{Z}(H) \mid \pi(x) = \pi(y)\},$$

and a *distance* on H is a map $\mathsf{d} \colon \sim_H \to \mathbb{N}_0$ satisfying the following conditions for all $z, z', z'' \in \sim_H$:

(D1) $\mathsf{d}(z,z) = 0$,
(D2) $\mathsf{d}(z,z') = \mathsf{d}(z',z)$,
(D3) $\mathsf{d}(z,z') \le \mathsf{d}(z,z'') + \mathsf{d}(z'',z')$,
(D4) $\mathsf{d}(xz,xz') = \mathsf{d}(zy,z'y) = \mathsf{d}(z,z')$ for all x,y, and
(D5) $\big||z| - |z'|\big| \le \mathsf{d}(z,z') \le \max\{|z|,|z'|,1\}$.

Having distance functions at our disposal, we can introduce the concept of catenary degrees. For an element $a \in H$, the *catenary degree* $\mathsf{c}_{\mathsf{d}}(a) \in \mathbb{N}_0 \cup \{\infty\}$ (of a with respect to the distance function d) is the minimal $N \in \mathbb{N}_0 \cup \{\infty\}$ such that for any two factorizations z, z' of a there are factorizations $z = z_0, z_1, \ldots, z_n = z'$ of a such that $\mathsf{d}(z_{i-1}, z_i) \le N$ for all $i \in [1,n]$. The *catenary degree* (in distance d) of H is

$$\mathsf{c}_{\mathsf{d}}(H) = \sup\{\mathsf{c}_{\mathsf{d}}(a) \mid a \in H\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Property **(D5)** easily implies that $\sup \Delta(\mathsf{L}(a)) \le \mathsf{c}_{\mathsf{d}}(a)$ for every $a \in H$, and hence (by [8, Lemma 4.2])

$$\sup \Delta(H) \le \mathsf{c}_{\mathsf{d}}(H). \tag{3.1}$$

First, suppose that H is commutative (and still reduced atomic unit-cancellative). To recall the usual distance function, consider two factorizations $z, z' \in \mathsf{Z}(H)$. Then there exist $\ell, m, n \in \mathbb{N}_0$ and $u_1, \ldots, u_\ell, v_1, \ldots, v_m, w_1, \ldots, w_n \in \mathcal{A}(H)$ with $\{v_1, \ldots, v_m\} \cap \{w_1, \ldots, w_n\} = \emptyset$ such that

$$z = u_1 \cdot \ldots \cdot u_\ell v_1 \cdot \ldots \cdot v_m \quad \text{and} \quad z' = u_1 \cdot \ldots \cdot u_\ell w_1 \cdot \ldots \cdot w_n.$$

Then the map $\mathsf{d} \colon \sim_H \to \mathbb{N}_0$ defined by $\mathsf{d}(z,z') = \max\{m,n\}$ is a distance function satisfying **(D1)**–(**D5**). If H is commutative and cancellative but not half-factorial, then the stronger inequality $2 + \sup \Delta(H) \le \mathsf{c}_{\mathsf{d}}(H)$ holds [20, Chapter 1].

Now suppose that H is not commutative. Then the above definition of factorizations and of the associated factorization monoid coincides with the concept of rigid factorizations studied by Baeth and Smertnig who also introduced the concept of abstract distance functions as given above ([33, Remark 3.3], [8], [34, Section 5]). A well-studied example is the *Levenshtein distance*, $z, z' \in \mathsf{Z}(H)$ as the minimum number of operations needed to transform z into z', where an operation is a substitution, deletion, or insertion of an atom $a \in \mathcal{A}(H)$ (see [8, Section 3] for this and other distance functions).

Recall that even monoids with a single defining relation need not be atomic (e.g, $H = \langle a, b \mid a = bab \rangle$), and there are also finitely generated commutative monoids which are not atomic (e.g., [27, Theorem 3 and Example 6]). However, every unit-cancellative monoid satisfying the ACC on principal right ideals and on principal left ideals is atomic ([33, Proposition 3.1] for the cancellative case, [15, Lemma 3.1] for the commutative unit-cancellative case, and [16, Theorem 2.6] for the general case). Clearly, the ACC on all right ideals is a much stronger condition, as the next lemma shows.

LEMMA 3.2. *Let H be a reduced submonoid of a group satisfying the* ACC *on right ideals.*

(1) *H satisfies the* ACC *on left ideals and H is atomic with finite set of atoms.*
(2) *If $S \subset H$ is a saturated submonoid, then S satisfies the* ACC *on right ideals.*

Proof. See [25, Lemmas 4.1.1 and 4.2.5]. ∎

LEMMA 3.3. *Let H be a reduced atomic cancellative monoid.*

(1) *The monoid of relations $\sim_H \subset \mathsf{Z}(H) \times \mathsf{Z}(H)$ is a saturated submonoid and a reduced cancellative BF-monoid.*
(2) *If $\sim_H$ satisfies the* ACC *on right ideals, then it is finitely generated.*
(3) *If H is commutative and finitely generated, then $\sim_H$ satisfies the ACC on ideals and is finitely generated.*

Proof. (1) To show that $\sim_H$ is saturated, let $(x_1, y_1), (x_2, y_2) \in \mathsf{Z}(H) \times \mathsf{Z}(H)$. Suppose that $(x_1, y_1) \in \sim_H$, and either $(x_1, y_1)(x_2, y_2) \in \sim_H$ or $(x_2, y_2)(x_1, y_1) \in \sim_H$, say the former. Then $\pi(x_1) = \pi(y_1) \in H$ and $\pi(x_1)\pi(x_2) = \pi(y_1)\pi(y_2) \in H$. Thus the cancellativity of H implies that $\pi(x_2) = \pi(y_2) \in H$, whence $(x_2, y_2) \in \sim_H$. Since $\mathsf{Z}(H) \times \mathsf{Z}(H)$ is a reduced cancellative monoid, so is $\sim_H$ and it is a BF-monoid by [19, Lemma 2].

(2) By Lemma 3.2(1), the set $\mathcal{A}(\sim_H)$ of atoms is finite, whence $\sim_H$ is finitely generated.

(3) Suppose that H is commutative and finitely generated. Then $\mathsf{Z}(H)$ is finitely generated by definition and it satisfies the ACC on ideals by [24, Theorem 3.6]. Thus (1) and Lemma 3.2(2) imply that $\sim_H$ satisfies the ACC on ideals, and hence $\sim_H$ is finitely generated by (2). ∎

A reduced atomic monoid is said to be *finitely presented* if $A = \mathcal{A}(H)$ is finite and there is a finite set of relations

$$R \subset \sim_H$$

which generates $\sim_H$ as a congruence relation (this means that one first defines a reflexive, symmetric relation $E \subset \mathsf{Z}(H) \times \mathsf{Z}(H)$ by defining $x \sim_E y$

if and only if $x = sut$ and $y = svt$ with $s, t \in \mathsf{Z}(H)$ and $(u, v) \in R \cup R^{-1} \cup \{(x, x) \mid x \in \mathsf{Z}(H)\}$, and then one takes the transitive closure of E; for details see [14, Section 1.5]). Thus, if $\sim_H$ is finitely generated as a monoid, then H is finitely presented. As usual we write $H = \langle A \mid R \rangle = \langle A \mid x_1 = y_1, \ldots, x_m = y_m \rangle$ if $R = \{(x_1, y_1), \ldots, (x_m, y_m)\} \subset \sim_H$.

If an atomic monoid H is presented by homogenous relations (that is, for all $(x, y) \in R$, we have $|x| = |y|$), then H is half-factorial (that is, $\Delta(H) = \emptyset$), and conversely (monoids presented by homogenous relations are of much interest in the study of finitely presented algebras, e.g., [10, 26]). Our first result states that finitely presented monoids have finite sets of distances.

Proposition 3.4. *Let H be a reduced atomic unit-cancellative and finitely presented monoid with $H \neq \{1\}$, say $H = \langle \mathcal{A}(H) \mid x_1 = y_1, \ldots, x_m = y_m \rangle$, and let $\mathsf{d} \colon \sim_H \to \mathbb{N}_0$ be a distance on H. Then*

$$\sup \Delta(H) \le \mathsf{c}_{\mathsf{d}}(H) \le \max\{\mathsf{d}(x_1, y_1), \ldots, \mathsf{d}(x_m, y_m)\} < \infty.$$

Moreover, if $\mathsf{c}_{\mathsf{d}}(\pi(x_i)) \ge \mathsf{d}(x_i, y_i)$ for every $i \in [1, m]$, then

$$\mathsf{c}_{\mathsf{d}}(H) = \max\{\mathsf{d}(x_1, y_1), \ldots, \mathsf{d}(x_m, y_m)\}.$$

Proof. It is sufficient to prove the upper bound for $\mathsf{c}_{\mathsf{d}}(H)$; then the upper bound for $\sup \Delta(H)$ follows from (3.1).

We choose $a \in H$ and two factorizations $z, z' \in \mathsf{Z}(H)$ with $\pi(z) = \pi(z') = a$. Since $R = \{(x_1, y_1), \ldots, (x_m, y_m)\}$ generates the congruence relation defining H, there are $z = z_0, \ldots, z_n = z' \in \pi^{-1}(a)$ where z_j arises from z_{j-1} by replacing some x_i by some y_i (or conversely) for some $i \in [1, m]$ and all $j \in [1, n]$. Thus $\mathsf{d}(z_{j-1}, z_j) \le \max\{\mathsf{d}(x_1, y_1), \ldots, \mathsf{d}(x_m, y_m)\}$ and $\mathsf{c}_{\mathsf{d}}(a) \le \max\{\mathsf{d}(x_1, y_1), \ldots, \mathsf{d}(x_m, y_m)\}$, as asserted.

To verify the "moreover" statement, let $\pi \colon \mathsf{Z}(H) \to H$ denote the factorization homomorphism, and for every $i \in [1, m]$ set $a_i = \pi(x_i)$. Then x_i, y_i are factorizations of a_i, and the assumption implies that

$$\mathsf{d}(x_i, y_i) \le \mathsf{c}_{\mathsf{d}}(a_i) \le \max\{\mathsf{d}(x_1, y_1), \ldots, \mathsf{d}(x_m, y_m)\},$$

hence $\mathsf{c}_{\mathsf{d}}(H) = \max\{\mathsf{d}(x_1, y_1), \ldots, \mathsf{d}(x_m, y_m)\}$. ∎

Proposition 3.5. *Let H be a reduced atomic unit-cancellative monoid with $H \neq \{1\}$ and suppose that any of the following conditions holds:*

(a) *There is an $L \in \mathcal{L}(H)$ such that $\rho(L) = \rho(H) < \infty$.*
(b) *H is cancellative and its monoid of relations is finitely generated.*
(c) *H is commutative and finitely generated.*

Then there is an $M \in \mathbb{N}$ such that $\rho_k(H) - \rho_{k-1}(H) \le M$ for all $k \in \mathbb{N}_{\ge 2}$.

Remark. Since $\rho_1(H) = 1$, the existence of M as above implies in particular that $\rho_k(H) < \infty$ for all $k \in \mathbb{N}_{\ge 2}$.

Proof of Proposition 3.5. (a) Since H is a unit-cancellative monoid, the system $\mathcal{L}(H)$ is a directed family. Thus if the elasticity is accepted, then the assertion on the growth behavior of the $\rho_k(H)$ follows from an associated statement in the setting of directed families [15, Proposition 2.8].

(b) It is sufficient to verify that (a) holds. By definition,

$$\begin{aligned}\rho(H) &= \{\rho(L) \mid L \in \mathcal{L}(H)\} = \{\sup \mathsf{L}(a)/\min \mathsf{L}(a) \mid a \in H\} \\ &= \{|x|/|y| \mid (x,y) \in \sim_H\}.\end{aligned}$$

Suppose that $A \subset \sim_H$ is a finite minimal generating set (note that A is symmetric, whence $(x,y) \in A$ implies $(y,x) \in A$). We assert that

$$\rho(H) = \max\{|x'|/|y'| \mid (x',y') \in A\}. \tag{3.2}$$

We show that $|x|/|y| \le \max\{|x'|/|y'| \mid (x',y') \in A\}$ for all $(x,y) \in \sim_H$ and proceed by induction on $|x| + |y|$. If $(x,y) \in A$, then the assertion holds. Suppose that $(x,y) \notin A$. Then (x,y) is a product of two elements from $\sim_H$, both distinct from the identity element, say $(x,y) = (x_1x_2, y_1y_2)$ where $(x_1,y_1),(x_2,y_2) \in \sim_H$ and $|x_\nu| + |y_\nu| < |x| + |y|$ for $\nu \in [1,2]$. Then the induction hypothesis implies that

$$\frac{|x|}{|y|} = \frac{|x_1| + |x_2|}{|y_1| + |y_2|} \le \max\left\{\frac{|x_1|}{|y_1|}, \frac{|x_2|}{|y_2|}\right\} \le \max\{|x'|/|y'| \mid (x',y') \in A\}.$$

Thus (3.2) holds and hence there is an $L \in \mathcal{L}(H)$ such that $\rho(L) = \rho(H)$.
(c) This follows from [15, Proposition 3.4]. ∎

Theorem 3.6. *Let H be a reduced atomic unit-cancellative monoid with $H \ne \{1\}$. If any of the following conditions holds, then there is an $M \in \mathbb{N}$ such that $\rho_k(H) - \rho_{k-1}(H) \le M$ for all $k \in \mathbb{N}_{\ge 2}$, and $\mathcal{L}(H)$ satisfies the Structure Theorem for Unions.*

(a) *H is finitely presented and there is an $L \in \mathcal{L}(H)$ such that $\rho(L) = \rho(H) < \infty$.*
(b) *H is cancellative and its monoid of relations is finitely generated.*
(c) *H is commutative and finitely generated.*

Proof. If (b) holds, then H is finitely presented as observed above, and if (c) holds, then H is finitely presented by Rédei's Theorem [23, Section VI.1]. In all three cases the set of distances $\Delta(H)$ is finite by Proposition 3.4. Thus condition (b) in Lemma 3.1 holds by Proposition 3.5 and hence $\mathcal{L}(H)$ satisfies the Structure Theorem for Unions. ∎

An overview of commutative monoids and rings satisfying the Structure Theorem for Unions can be found in [15, Section 3]. Based on these results it follows that the Structure Theorem for Unions holds for (not necessarily commutative) transfer Krull monoids of finite type [19, Theorem 13] (the systems of sets of lengths of such monoids coincide with the corresponding systems of commutative Krull monoids). Crucial examples of transfer Krull monoids and domains are due to Baeth and Smertnig [33, 5, 8, 6, 32], and for an overview we refer to [19, Section 4].

REMARK 3.7. (1) There is a commutative Krull monoid H with finite set of distances and finite kth elasticities $\rho_k(H)$ for all $k \in \mathbb{N}$ whose system $\mathcal{L}(H)$ does not satisfy the Structure Theorem for Unions. In this case there is no $M \in \mathbb{N}$ such that $\rho_k(H) - \rho_{k-1}(H) \le M$ for all $k \in \mathbb{N}_{\ge 2}$ [15, Theorem 4.2].

(2) A monoid H is called *almost commutative* [8, Section 6] if being associated is a congruence relation on H and the associated reduced monoid is commutative. Thus, if H is almost commutative and H_{red} is finitely generated, then $\mathcal{L}(H) = \mathcal{L}(H_{\mathrm{red}})$ satisfies the Structure Theorem for Unions by condition (c) of Theorem 3.6.

A monoid H is said to be *normalizing* if $aH = Ha$ for all $a \in H$. Normalizing monoids play an important role in the study of semigroup algebras [25, 1, 26], and normalizing Krull monoids are almost commutative [18].

(3) Let H be as in Theorem 3.6, namely a reduced atomic unit-cancellative monoid. We compare conditions (a), (b), and (c). As shown in Proposition 3.5, condition (b) implies (a), and if H is cancellative, then (c) implies (b) by Lemma 3.3(3). However, none of the reverse implications is true in general. A commutative finitely generated monoid without accepted elasticity can be found in [15, Remarks 3.11].

The monoids B_n discussed in Section 4 satisfy the conclusions of Theorem 3.6 but none of the conditions (a)–(c). On the other hand, it is easy to see that there are cancellative monoids with a single defining relation, whose sets of distances are finite by Proposition 3.4 but whose kth elasticities are infinite for all $k \ge 2$. For example, consider the monoid $H = \langle a, b \mid a^2 = ba^2b \rangle$. Then H is reduced atomic with $\mathcal{A}(H) = \{a, b\}$, and H is cancellative because it is an Adyan monoid. Clearly, $\rho_2(H) = \infty$ and hence $\rho_k(H) = \infty$ for all $k \ge 2$.

(4) Baeth and Smertnig [7] verify the Structure Theorem for Unions for local quaternion orders. Their result and the present Theorem 3.6 yield the first non-commutative monoids H for which it can be shown that $\mathcal{L}(H)$ satisfies the Structure Theorem for Unions without showing that $\mathcal{L}(H) = \mathcal{L}(B)$ for some commutative monoid B (see also Corollary 4.4 below).

4. Sets of lengths of the monoid $B_n = \langle a, b \mid ba = b^n \rangle$. For $n \in \mathbb{N}_0$, consider the monoid

$$B_n = \langle a, b \mid ba = b^n \rangle .$$

Then B_0 is the bicyclic monoid, and B_1 is the submonoid of right units of Warne's 2-dimensional bicyclic monoid. If $a^k b^m \in B_0$, then $a^k b^m =$

$a^kb^mba = a^kb^mbaba = \cdots$, and if $a^kb^m \in B_1$ with $m \neq 0$, then $a^kb^m = a^kb^ma = a^kb^maa = \cdots$. Therefore, B_0 and B_1 are not BF-monoids.

Suppose that $n \geq 2$. In this case B_n is a Möbius monoid [28, Secton 2.1]. Observe that multiplication in B_n is given by

$$a^kb^m \cdot a^rb^s = \begin{cases} a^{k+r}b^s & \text{if } m = 0, \\ a^kb^{m+(n-1)r+s} & \text{if } m > 0. \end{cases}$$

Clearly, B_n is reduced, unit-cancellative, right-cancellative, but not left-cancellative, and atomic with $\mathcal{A}(B_n) = \{a, b\}$.

THEOREM 4.1. *Let $n \in \mathbb{N}_{\geq 2}$.*

(1) *If d denotes the Levenshtein distance, then $\mathsf{c_d}(B_n) = n - 1$, and B_n is half-factorial if and only if $n = 2$.*
(2) *For every $k, m \in \mathbb{N}_0$ we have*

$$\mathsf{L}(a^kb^m) = k + m - q_{m,n}(n-2) + (n-2) \cdot [0, q_{m,n}],$$

where

$$q_{m,n} = \begin{cases} \left\lfloor \frac{m}{n-1} \right\rfloor & \text{if } (n-1) \nmid m, \\ \frac{m}{n-1} - 1 & \text{if } (n-1) \mid m \text{ and } m \neq 0, \\ 0 & \text{if } m = 0. \end{cases}$$

Proof. Since $\mathsf{d}(ab, b^n) = n - 1$, Proposition 3.4 implies that $\mathsf{c_d}(B_n) = n - 1$.

Let $k, m \in \mathbb{N}_0$. If $m = 0$, then $\mathsf{L}(a^k) = \{k\}$, and hence the claim holds. Suppose that $m > 0$. Pick $u, v \in \mathcal{F}^*(\{a, b\})$. If v is directly derivable from u (that is, $v = w_1b^nw_2$ and $u = w_1baw_2$, or $v = w_1baw_2$ and $u = w_1b^nw_2$), then $|v| = |u| \pm (n-2)$. This shows that $\Delta(\mathsf{L}(a^kb^m)) \subset \{n-2\}$, and hence B_n is half-factorial if and only if $n = 2$.

For integers m and n ($m \geq 0$, $n \geq 2$), we define $q_0, q_1, \ldots$ and $r_0, r_1, \ldots$ as quotients and remainders in the following sequence of divisions (with q_i being the first quotient which vanishes):

$$\begin{aligned} m &= nq_0 + r_0 \quad (r_0 < n;\ q_0 \neq 0), \\ q_0 + r_0 &= nq_1 + r_1 \quad (r_1 < n;\ q_1 \neq 0), \\ q_1 + r_1 &= nq_2 + r_2 \quad (r_2 < n;\ q_2 \neq 0), \\ &\cdots \\ q_{i-1} + r_{i-1} &= n \cdot 0 + r_i \quad (r_i < n;\ 0 = q_i), \end{aligned}$$

and we set $q_{m,n} = q_0 + q_1 + \cdots + q_i$. If $m \geq n$, then

$$\begin{aligned} a^kb^m &= a^kb^{m-(n-1)}a = a^kb^{m-2(n-1)}a^2 = \cdots = a^kb^{m-q_0(n-1)}a^{q_0} = \cdots \\ &= a^kb^{m-(q_0+q_1)(n-1)}a^{q_0+q_1} = \cdots = a^kb^{m-q_{m,n}(n-1)}a^{q_{m,n}}. \end{aligned}$$

This shows that $k+m-q_{m,n}(n-2)+(n-2)\cdot[0,q_{m,n}] \subset \mathsf{L}(a^k b^m)$. Conversely, by construction and the fact that $\Delta(\mathsf{L}(a^k b^m)) \subset \{n-2\}$, any other factorization length of $a^k b^m$ is contained in

$$k+m-q_{m,n}(n-2)+(n-2)\cdot[0,q_{m,n}].$$

It remains to show that $q_{m,n}$ has the asserted value. By adding terms in the columns in the above sequence of divisions (i.e., in the defining table of $q_0, q_1, \ldots$ and $r_0, r_1, \ldots$) we see that

$$q_{m,n} = q_0 + \cdots + q_i = \frac{m-r_i}{n-1}.$$

Note that $m>0$ implies $r_i>0$. If $n-1 \mid m$ then $r_i = n-1$, whence

$$q_{m,n} = \frac{m}{n-1} - 1.$$

If $(n-1) \nmid m$, then $0 < r_i/(n-1) < 1$ and

$$q_{m,n} = \frac{m}{n-1} - \frac{r_i}{n-1} = \left\lfloor \frac{m}{n-1} \right\rfloor. \qquad \blacksquare$$

If $n=2$, then B_n is half-factorial, whence $\mathcal{L}(B_n) = \{\{k\} \mid k \in \mathbb{N}_0\}$. Thus in our further study of $\mathcal{L}(B_n)$ we always suppose that $n \ge 3$.

THEOREM 4.2. *Let $n \in \mathbb{N}_{\ge 3}$. Then*

$$\mathcal{L}(B_n) = \{x+(n-2)\cdot[0,q] \mid x,q \in \mathbb{N}_0 \text{ with } x=q=0 \text{ or } x>q\}.$$

Proof. By definition, $\mathsf{L}(1)=\{0\}$, and this is the only $L \in \mathcal{L}(B_n)$ with $0 \in L$. First we show that $\mathcal{L}(B_n)$ is contained in the right hand side. Let $k,m \in \mathbb{N}_0$ with $k+m>0$. By Theorem 4.1, we have

$$\mathsf{L}(a^k b^m) = k+m-q_{m,n}(n-2)+(n-2)\cdot[0,q_{m,n}],$$

and so we need to verify that

$$(*) \qquad k+m-q_{m,n}(n-2) > q_{m,n}.$$

If $q_{m,n}=0$, then $(*)$ holds. Otherwise,

$$\frac{m-n}{n-1} < q_{m,n} < \frac{m}{n-1},$$

which implies that $0 < m - q_{m,n}(n-1) < n$, and thus $(*)$ holds.

Conversely, let $x,q \in \mathbb{N}_0$ with $x>q$. We choose $k \in \mathbb{N}_0$ such that $0 < x-q-k < n$ and set $m = q(n-1)+x-q-k$. If $n-1 \mid m$, then $x-q-k = n-1$ and $q_{m,n}=q$, whence

$$\mathsf{L}(a^k b^m) = k+m-q_{m,n}(n-2)+(n-2)\cdot[0,q_{m,n}] = x+(n-2)\cdot[0,q].$$

If $n-1 \nmid m$, then again $q_{m,n} = \left\lfloor \frac{m}{n-1} \right\rfloor = q$ and hence

$$\mathsf{L}(a^k b^m) = k+m-q_{m,n}(n-2)+(n-2)\cdot[0,q_{m,n}] = x+(n-2)\cdot[0,q]. \qquad \blacksquare$$

Corollary 4.3. *Let* $n \in \mathbb{N}_{\geq 3}$.

(1) $\rho(B_n) = n - 1 > \rho(L)$ *for every* $L \in \mathcal{L}(B_n)$.
(2) *For every* $\ell \geq 2$ *we have*

$$\mathcal{U}_\ell(B_n) = \ell - q_{\ell,n}(n-2) + (n-2) \cdot [0, q_{\ell,n} + \ell - 1].$$

Proof. (1) The explicit description of $\mathcal{L}(B_n)$ in Theorem 4.2 implies that

$$\begin{aligned}\rho(B_n) &= \sup\{\max L / \min L \mid \{0\} \neq L \in \mathcal{L}(B_n)\} \\ &= \sup\left\{ \frac{x + (n-2)q}{x} \;\middle|\; x, q \in \mathbb{N}_0,\, x > q \right\} = n - 1 > \frac{\max L}{\min L}\end{aligned}$$

for every $L \in \mathcal{L}(B_n)$.

(2) Let $\ell \geq 2$. By Theorem 4.2, $\mathcal{U}_\ell(B_n)$ is a union of arithmetical progressions having difference $n-2$ and containing ℓ. Thus $\mathcal{U}_\ell(B_n)$ is an arithmetical progression with difference $n-2$ which contains ℓ. The maximum of $\mathcal{U}_\ell(B_n)$ comes from a set of the form $x + (n-2) \cdot [0, q]$ with $x > q$ and $\ell \in x + (n-2) \cdot [0, q]$, so $\max \mathcal{U}_\ell(B_n) = \ell + (n-2)(\ell - 1)$. To determine $\min \mathcal{U}_\ell(B_n)$ we have to find the maximal $y \in \mathbb{N}$ such that

$$\ell - y(n-2) + (n-2) \cdot [0, y] = \ell \quad \text{and} \quad y < \ell - y(n-2),$$

which implies $y < \ell/(n-1)$, and thus $y = q_{\ell,n}$. ∎

Let G be an additive abelian group and $G_0 \subset G$ a subset. We introduce a commutative Krull monoid having a combinatorial flavor. Because of connections with additive combinatorics, the elements $S \in \mathcal{F}(G_0)$ are called sequences over G_0. Let $\sigma \colon \mathcal{F}(G_0) \to G$ be the homomorphism defined by $\sigma(g) = g$ for every $g \in G_0$. Thus, if $S = g_1 \cdot \ldots \cdot g_\ell \in \mathcal{F}(G_0)$, then $\sigma(S) = g_1 + \cdots + g_\ell \in G$ denotes its sum. Then

$$\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) \mid \sigma(S) = 0\} \subset \mathcal{F}(G_0)$$

is a submonoid, called the *monoid of zero-sum sequences* over G_0. Clearly, $\mathcal{B}(G_0) \subset \mathcal{F}(G_0)$ is a saturated submonoid, whence $\mathcal{B}(G_0)$ is a commutative Krull monoid (see the definition of Krull monoids in Section 2).

We recall the concept of weak transfer homomorphisms ([5, Definition 3.1] and [8, Definition 2.1]). Let H and B be atomic unit-cancellative monoids. A monoid homomorphism $\theta \colon H \to B$ is called a *weak transfer homomorphism* if the following two properties are satisfied:

(T1) $B = B^\times \theta(H) B^\times$ and $\theta^{-1}(B^\times) = H^\times$.
(WT2) If $a \in H$, $n \in \mathbb{N}$, $v_1, \ldots, v_n \in \mathcal{A}(B)$ and $\theta(a) = v_1 \cdot \ldots \cdot v_n$, then there exist $u_1, \ldots, u_n \in \mathcal{A}(H)$ and a permutation $\tau \in S_n$ such that $a = u_1 \cdot \ldots \cdot u_n$ and $\theta(u_i) \in B^\times v_{\tau(i)} B^\times$ for each $i \in [1, n]$.

If $\theta \colon H \to B$ is a weak transfer homomorphism, then it is easy to check that $\theta(\mathcal{A}(H)) = \mathcal{A}(B)$ and $\mathcal{L}(H) = \mathcal{L}(B)$. An atomic unit-cancellative monoid H

is said to be a *transfer Krull monoid* if one of the following two equivalent properties is satisfied:

(a) There is a commutative Krull monoid B and a weak transfer homomorphism $\theta\colon H \to B$.
(b) There is an abelian group G, a subset $G_0 \subset G$, and a weak transfer homomorphism $\theta\colon H \to \mathcal{B}(G_0)$.

In case (b) we say that H is a transfer Krull monoid G_0. Thus by definition, every commutative Krull monoid is transfer Krull, but there is an impressive list of transfer Krull, monoids which are not commutative Krull (for a survey see [19, Section 4]). Since the class of commutative Krull monoids is huge and since for most classes of rings and monoids only qualitative finiteness or infiniteness results for arithmetical invariants are known but no precise values or explicit descriptions (such as the one in Theorem 4.2), only for small classes of monoids do we know that they are not transfer Krull, and all of them are commutative [22].

Corollary 4.4. *Let* $n \in \mathbb{N}_{\ge 3}$.

(1) *For every reduced atomic cancellative monoid* H *whose monoid of relations is finitely generated, we have* $\mathcal{L}(H) \ne \mathcal{L}(B_n)$.
(2) *The monoid* B_n *is not a transfer Krull monoid.*

Proof. (1) Let H be a reduced atomic cancellative monoid whose monoid of relations is finitely generated. Then H has accepted elasticity by Proposition 3.5 (indeed, in its proof we showed that condition (b) there implies condition (a)). Since B_n does not have accepted elasticity by Corollary 4.3.1, the claim follows.

(2) Assume to the contrary that there is an abelian group G, a subset $G_0 \subset G$, and a weak transfer homomorphism $\theta\colon B_n \to \mathcal{B}(G_0)$. Since the set of atoms $\mathcal{A}(B_n)$ is finite, the set of atoms

$$\mathcal{A}(\mathcal{B}(G_0)) = \theta(\mathcal{A}(B_n))$$

is finite. Thus the set

$$G_1 := \bigcup_{U \in \mathcal{A}(\mathcal{B}(G_0))} \{g \in G_0 \mid \mathsf{v}_g(U) > 0\} \ \subset G_0$$

is finite and $\theta(B_n) \subset \mathcal{B}(G_1)$. Consequently, we have a weak transfer homomorphism $\theta\colon B_n \to \mathcal{B}(G_1)$ and hence $\mathcal{L}(B_n) = \mathcal{L}(\mathcal{B}(G_1))$. Since G_1 is finite, $\mathcal{B}(G_1)$ is finitely generated [20, Theorem 3.4.2]. Clearly, $\mathcal{B}(G_1)$ is reduced atomic cancellative, and since it is finitely generated, its monoid of relations is finitely generated by Lemma 3.3(3), contradicting (1). ∎

Let H be an atomic unit-cancellative monoid with $H \ne H^\times$. Then, as already mentioned, for every $L_1, L_2 \in \mathcal{L}(H)$ there is an $L \in \mathcal{L}(H)$ such that

$L_1+L_2 \subset L$ (clearly, if $L_i = \mathsf{L}(a_i)$ for $i=1,2$, then $\mathsf{L}(a_1)+\mathsf{L}(a_2) \subset \mathsf{L}(a_1a_2)$). We say that $\mathcal{L}(H)$ is *closed under set addition* if for every $L_1, L_2 \in \mathcal{L}(H)$ we have $L_1 + L_2 \in \mathcal{L}(H)$. Whereas the first property holds for all atomic unit-cancellative monoids (indeed, all $\mathcal{L}(H)$ are directed families), the property of being closed under set addition is extremely restrictive. Clearly, if H is a BF-monoid and $\mathcal{L}(H)$ is closed under set addition, then $\mathcal{L}(H)$ itself is a reduced atomic unit-cancellative monoid with set addition as operation and with zero element $\mathsf{L}(1) = \{0\}$.

The system $\mathcal{L}(H)$ is closed under set addition in both extremal cases, namely when either H is half-factorial (in this case $\mathcal{L}(H) = \{\{k\} \mid k \in \mathbb{N}_0\}$) or when every finite subset $L \subset \mathbb{N}_{\ge 2}$ is a set of lengths (this holds true e.g., for transfer Krull monoids over infinite abelian groups). If H is a transfer Krull monoid over a finite abelian group G, then $\mathcal{L}(H)$ is closed under set addition if and only if $\exp(G)+\mathsf{r}(G) \le 5$, where $\exp(G)$ denotes the exponent and $\mathsf{r}(G)$ the rank of G [21, Theorem 1.1].

In our final show we show that $\mathcal{L}(B_n)$ is closed under set addition, and we determine its monoid-theoretical structure.

THEOREM 4.5. *Let $n \in \mathbb{N}_{\ge 3}$.*

(1) *$\mathcal{L}(B_n)$ is closed under set addition.*
(2) *There is a monoid isomorphism*

$$\Phi\colon \mathcal{L}(B_n) \to H, \qquad x + (n-2)\cdot[0,q] \mapsto (x,q),$$

where $H = \{(k,i) \mid k,i \in \mathbb{N}_0,\ k=i=0 \text{ or } k>i\} \subset (\mathbb{N}_0^2,+)$. Clearly, H is a reduced atomic commutative cancellative half-factorial monoid, and we have $\mathcal{A}(H) = \{(k,k-1) \mid k \in \mathbb{N}\}$.

Proof. By Theorem 4.2, we have $\mathcal{L}(B_n) = \{x + (n-2)\cdot[0,q] \mid x,q \in \mathbb{N}_0,\ x=q=0 \text{ or } x>q\}$.

(1) Clearly, $\{0\}$ is the zero element of $\mathcal{L}(B_n)$, and if $x_1,x_2,q_1,q_2 \in \mathbb{N}_0$ with $x_i > q_i$ for $i \in [1,2]$, then

$$\begin{aligned}(x_1 + (n-2)\cdot[0,q_1]) &+ (x_2 + (n-2)\cdot[0,q_2]) \\ &= (x_1+x_2) + (n-2)\cdot[0,q_1+q_2] \in \mathcal{L}(B_n).\end{aligned}$$

Thus $\mathcal{L}(B_n)$ is closed under set addition and hence a monoid with set addition as operation.

(2) Obviously, Φ is an isomorphism. As a submonoid of the free abelian monoid $(\mathbb{N}_0^2,+)$, H is reduced atomic commutative cancellative, and clearly $\mathcal{A}(H) = \{(k,k-1) \mid k \in \mathbb{N}\}$. To show that H is half-factorial, consider an equation of the form

$$(k,i) = (k_1,k_1-1) + \cdots + (k_r,k_r-1) = (k_1',k_1'-1) + \cdots + (k_s',k_s'-1),$$

which implies that $r = k-i = s$. ∎

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