Enumeration of non-isomorphic canons

The cyclic group C_n acts on the set of all functions from Z_n to $\{0,1\}$ by

$$C_n \times \{0,1\}^{Z_n} \to \{0,1\}^{Z_n} \qquad (\sigma, f) \mapsto f \circ \sigma^{-1}.$$

As the *canonical representative* of the *orbit* $C_n(f) = \{f \circ \sigma \mid \sigma \in C_n\}$ we choose the function $f_0 \in C_n(f)$, such that $f_0 \leq h$ for all $h \in C_n(f)$.

A function $f \in \{0,1\}^{Z_n}$ (or the corresponding vector $(f(0), f(1), \ldots, f(n-1))$) is called *acyclic* if $C_n(f)$ consists of n different objects. The canonical representative of the orbit of an acyclic function is called a *Lyndon word*.

Lemma 1. *The pair* (L, A) *does not describe a canon in* Z_n *if and only if there exists a divisor* $d > 1$ *of* n *such that* $L(i) = 1$ *implies* $i \equiv d - 1 \bmod d$ *and* $\chi_{A_0}(i) = 1$ *implies* $i \equiv d - 1 \bmod d$*, where* χ_{A_0} *is the canonical representative of* $C_n(\chi_A)$*.*

Theorem 2. *The number of isomorphism classes of canons in* Z_n *is*

$$K_n := \sum_{d|n} \mu(d)\lambda(n/d)\alpha(n/d),$$

where μ *is the* Moebius function, $\lambda(1) = 1$,

$$\lambda(r) = \frac{1}{r}\sum_{s|r} \mu(s)2^{r/s} \text{ for } r > 1,$$

and

$$\alpha(r) = \frac{1}{r}\sum_{s|r} \varphi(s)2^{r/s} - 1 \text{ for } r \geq 1,$$

where φ *is the* Euler totient function*.*

Enumeration of rhythmic tiling canons

There exist more complicated definitions of canons. A canon described by the pair (R, A) of inner and outer rhythm defines a *rhythmic tiling canon* in Z_n with voices V_a for $a \in A$ if and only if

1. the voices V_a cover entirely the cyclic group Z_n,

2. the voices V_a are pairwise disjoint.

Rhythmic tiling canons with the additional property

3. both R and A are aperiodic,

are called *regular complementary canons of maximal category*.

Hence rhythmic tiling canons are canons which are also mosaics. More precisely, if $|A| = t$, then they are mosaics consisting of t blocks of size n/t, whence they are of block-type λ where

$$\lambda_i = \begin{cases} t & \text{if } i = n/t \\ 0 & \text{otherwise.} \end{cases}$$

However, the description of the isomorphism classes of canons as pairs $(L, C_n(A))$ consisting of Lyndon words L and C_n-orbits of subsets A of Z_n with some additional properties can also be applied for the determination of complete sets of representatives of non-isomorphic canons in Z_n.

There exist fast algorithms for computing all Lyndon words of length n over $\{0, 1\}$ and all C_n-orbit representatives of subsets of Z_n.

Vuza showed that regular complementary canons of maximal category occur only for certain values of n, actually only for *non-Hajós-groups* Z_n. The smallest n for which Z_n is not a Hajós-group is $n = 72$ which is still much further than the scope of our computations. Hence, we deduce that for all n such that Z_n is a Hajós-group the following is true:

Lemma 3. *If a pair* (L, A_0) *describes a regular tiling canon in a Hajós-group* Z_n*, then* A_0 *is not aperiodic.*

This reduces dramatically the number of pairs which must be tested.

The group Z_n is a Hajós group if the decomposition of n is not too complicated. If n is of the form

$$p^k \text{ for } k \geq 0,\ p^k q \text{ for } k \geq 1,\ p^2 q^2,$$

$$p^k qr \text{ for } k \in \{1, 2\},\ pqrs$$

for distinct primes p, q, r and s, then Z_n is a Haós group and Vuza proved that for these n there do not exist regular complementary canons of maximal category. Moreover, he described a method how to construct such canons for all Z_n which are not Haós groups. Then n can be expressed in the form $p_1p_2n_1n_2n_3$ with p_1, p_2 primes, $n_i \geq 2$ for $1 \leq i \leq 3$, and $\gcd(n_1p_1, n_2p_2) = 1$. Vuza presents an algorithm for constructing two aperiodic subsets L and A of Z_n, such that $|L| = n_1n_2$, $|A| = p_1p_2$, and $L + A = Z_n$. Hence, L or A can serve as the inner rhythm and the other set as the outer rhythm of such a canon. Moreover, it is important to mention that there is some freedom for constructing these two sets, and each of these two sets can be constructed independently from the other one. He also proves that when L and A satisfy $L + A = Z_n$, then also (kL, A), (kL, kA) have this property for all $k \in Z_n^*$.

Some interesting open problems

1. In his papers, Vuza did not prove that each regular complementary canons of maximal category can be constructed with his method. Is it possible to find regular complementary canons of maximal category which cannot be produced by Vuza's approach?

2. Is there a more elegant method for enumerating regular tiling canons?

3. When enumerating isomorphism classes of mosaics in Z_n we could apply groups different from the cyclic group C_n. How to do this for canons? For the group C_n, a canon was given as a pair (L, A) with certain properties. L was an acyclic vector, so probably in all generalizations we must assume that L does not have cyclic symmetries. A was considered to be a subset of Z_n, but actually A describes the onset distribution of the different voices, whence it is actually a subset of

the acting group C_n. When considering the group $\mathrm{Aff}_1(Z_n)$ consisting of all affine mappings $\pi_{a,b} : Z_n \to Z_n$, $i \mapsto ai + b$, for $a \in Z_n^*$ and $b \in Z_n$, acting on Z_n, then A must be considered as a subset of this group. If L has just the trivial symmetry, then each $\pi_{a,b}(L)$ describes another voice of the canon. If the stabilizer U of L is non-trivial, but it does not contain symmetries of the form $\pi_{1,b}$ for $b \neq 0$, then A must be considered as a subset of $U\backslash\backslash \mathrm{Aff}_1(Z_n)$. When computing the number of non-isomorphic canons in this setting, we get a much bigger number of different canons, since usually many different voices start at the same onset. So maybe in this situation we should restrict to canons, such that different voices have different onsets in Z_n. But when speaking of onsets of voices we can get some problems with symmetries $\pi_{a,b}$ of L for $a \neq 1$ and $b \neq 0$. So maybe we should not allow any symmetries of L. But then we will not get a complete overview over all canons in Z_n. Still the property that $K - K$ generates Z_n was not considered for these generalizations.