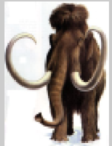

Musical Enumeration Theory Applied to the Classification of Canons

Harald Fripertinger
Karl-Franzens-Universität Graz

Colloquium on Mathematical Music Theory
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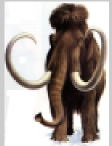

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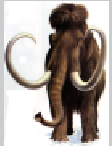

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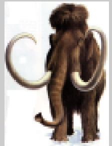

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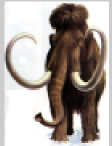

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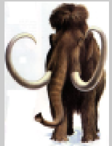

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We will discuss the following topics:
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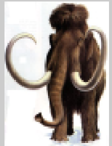
— examples of discrete structures in music theory,
— some results on the enumeration of canons.

Discrete Structures

Discrete structures are objects which can be constructed as:



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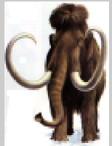
– subsets, unions, products of finite sets,



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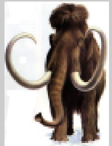

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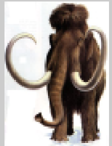

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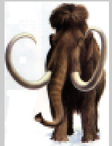

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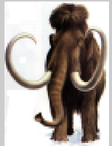

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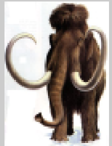
Examples: graphs, necklaces, designs, codes, matroids, switching functions, molecules in chemistry, spin-configurations in physics, objects of local music theory.

Classification of Discrete Structures I

The process of ***classification*** provides more detailed information about the objects in a discrete structure. We distinguish different steps in the process of classification:



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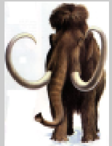

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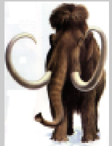

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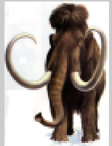

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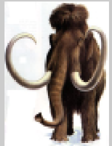

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step 1: Determine the number of different objects.

step 2: Determine the number of objects with certain properties.

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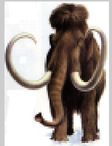
step 4: Generate the objects of a discrete structure uniformly at random.

Labelled Graphs

A simple labelled graph is described by a finite set of labelled vertices V and by a set of edges connecting two vertices. An edge is usually described as a 2-set of vertices, indicating which two vertices are connected by the corresponding edge.



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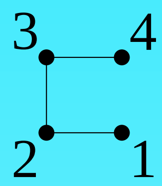
For instance $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$ describes the graph 

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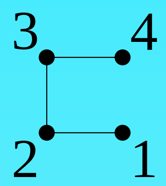
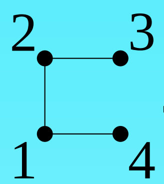
For instance $V = \{1,2,3,4\}$ and $E = \{\{1,2\},\{2,3\},\{3,4\}\}$ describes the graph 

on 4 vertices. Usually the labelling of the vertices is not important, whence this graph is considered to be essentially the same as 

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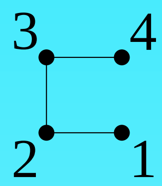
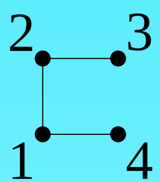
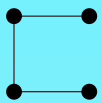
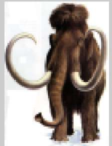
For instance $V = \{1,2,3,4\}$ and $E = \{\{1,2\},\{2,3\},\{3,4\}\}$ describes the graph 

on 4 vertices. Usually the labelling of the vertices is not important, whence this graph is considered to be essentially the same as 

. All graphs obtained by relabelling of one graph are collected to an unlabelled graph, an equivalence class of labelled graphs. In our example we get  .

Unlabelled Graphs on 4 Vertices

As we have just seen, often the elements of a discrete structure are themselves classes of objects which are considered to be equivalent. In one class all those elements are collected which are assumed to be not essentially different. These classes are caused by relabelling of labelled structures or by otherwise naturally motivated equivalence relations.



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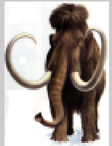

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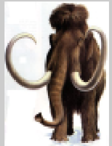

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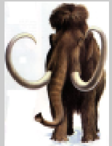
Example: Classification of unlabelled simple graphs:
step 1: There are 11 graphs on 4 vertices.
step 2: There exists exactly one graph with 0, 1, 5 or 6 edges; two graphs with 2 or 4 edges; three graphs with 3 edges.



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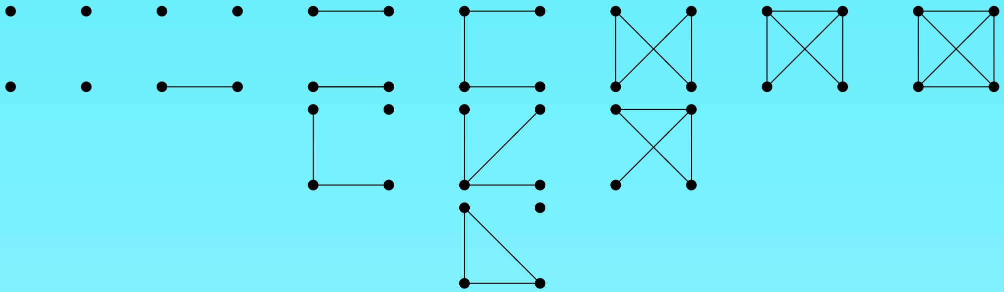

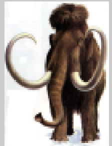

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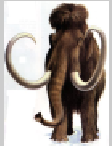

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Classification of Discrete Structures II

1. Determine the finite set X on which the discrete structure is defined.

2. Describe how the discrete structure is constructed over X.

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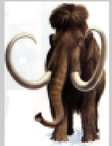
3. If the objects of a discrete structure are equivalence classes, introduce a suitable group action on X to describe the equivalence classes as orbits.

4. Proceed with the four steps for classification (enumeration, construction, random generation) as described above.



Group Actions

A multiplicative group G with neutral element 1 acts on a set X if there exists a mapping

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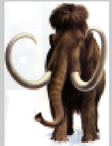

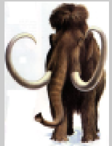

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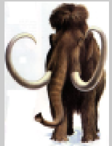

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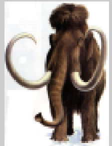

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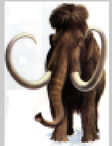

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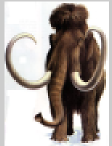
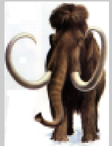
If G and X are finite sets, then we speak of a ***finite group action***.



Orbits under Group Actions

A group action ${}_G X$ defines the following equivalence relation on X.
$x_1 \sim x_2$ if and only if there is some $g \in G$ such that $x_2 = g x_1$.



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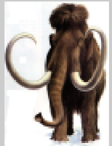
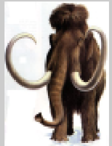
The set of orbits of G on X is indicated as

$$G \backslash\backslash X := \{G(x) \mid x \in X\}.$$





Orbits under Group Actions

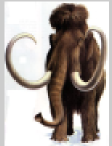
A group action ${}_GX$ defines the following equivalence relation on X. $x_1 \sim x_2$ if and only if there is some $g \in G$ such that $x_2 = gx_1$. The equivalence classes $G(x)$ with respect to $\sim$ are the ***orbits*** of G on X. Hence, the orbit of x under the action of G is

$$G(x) = \{gx \mid g \in G\}.$$

The set of orbits of G on X is indicated as

$$G\backslash\backslash X := \{G(x) \mid x \in X\}.$$

Theorem. *The equivalence classes of any equivalence relation can be represented as orbits under a suitable group action.*



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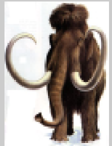
Stabilizers and Fixed Points

Let ${}_G X$ be a group action. For each $x \in X$ the ***stabilizer*** G_x of x is the set of all group elements which do not change x, in other words

$$G_x := \{ g \in G \mid gx = x \} .$$

It is a subgroup of G.



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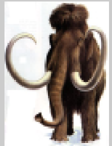
Lagrange Theorem. *If ${}_G X$ is a finite group action then the size of the orbit of $x \in X$ equals*

$$|G(x)| = \frac{|G|}{|G_x|}.$$



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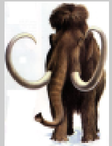
Finally, the ***set of all fixed points*** of $g \in G$ is denoted by

$$X_g := \{x \in X \mid gx = x\}\,.$$

Enumeration under Group Actions

Let ${}_G X$ be finite group action. The main tool for determining the number of different orbits is the

Cauchy Frobenius Lemma.



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Let $_GX$ be finite group action. The main tool for determining the number of different orbits is the

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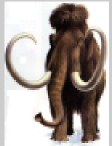

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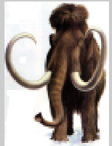

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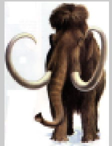

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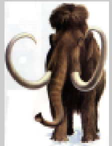
$$\sum_{g\in G}|X_g| = \sum_{g\in G}\sum_{x:gx=x} 1 =$$



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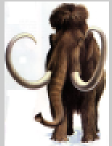

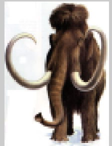

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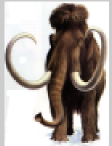

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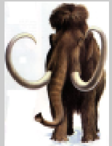

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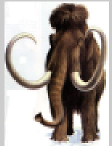
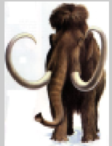


Symmetry types of mappings

The most important applications of classification under group actions can be described as mappings between two sets. Group actions on the domain X or range Y induce group actions on Y^X.



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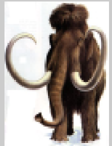

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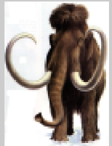

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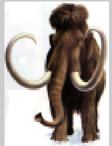
— Then the direct product $H \times G$ acts on Y^X by

$$(H \times G) \times Y^X \to Y^X, \qquad ((h,g), f) \mapsto \overline{h} \circ f \circ \overline{g}^{-1}.$$



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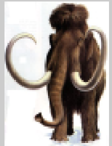
Discrete structures in Music Theory

Reduction of all pitches modulo one octave yields the notion of ***pitch-classes***. In an n***-scale*** there are exactly n tones in one octave, whence there are n pitch classes which are described as elements of $Z_n := \mathbb{Z}/n\mathbb{Z}$, the ***residue class ring modulo*** n.



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Musical transposing or inversion can be described as ***transposing by one pitch-class***, T, or ***inversion at pitch-class 0***, I, which are permutations on Z_n. They motivate cyclic groups, $\langle T \rangle$, dihedral groups, $\langle T, I \rangle$, or affine groups as permutation groups acting on Z_n.

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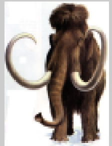

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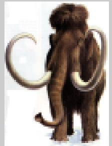

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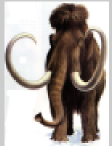
Mosaics are partitions of Z_n. These are collections π of subsets of Z_n, such that the empty set is not an element of π and such that for each $i \in Z_n$ there is exactly one $P \in \pi$ with $i \in P$.



Discrete structures in Music Theory (cont.)

Next to the pitch component we also introduce the time as a second parameter.

For $n \geq 3$ a ***tone-row*** in Z_n is a bijective mapping $f\colon \{1, \ldots, n\} \to Z_n$ where $f(i)$ is the pitch class of the tone which occurs in i-th position in the tone-row. Usually two tone-rows f_1, f_2 are considered to be equivalent if f_1 can be constructed by transposing, inversion and retrograde inversion R of f_2. Thus the similarity classes of tone-rows are the $D_n \times \langle R \rangle$ orbits on the set of these bijections.

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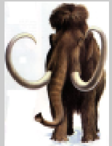
In the most simple model of a motive, each note is described by its onset and by its pitch. In other words, we put a coordinate system over a line of notes. If there are exactly m possible onsets and n pitch classes, a k***-motive*** is considered to be a k-subset of $Z_m \times Z_n$. In the case $m = n$, Mazzola showed that the group of all affine mappings on Z_n^2 is a musically motivated group.



Rhythms

When investigating a rhythm again we are only interested in the onsets, not in the duration of the different notes. Moreover, we want to forget all information about the pitch.

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We also assume that we found a subdivision of the rhythm, i.e. a regular pulsation, into equidistant beats such that all rhythmical events coincide with some of these beats. If the rhythm is covered by a pulsation of n beats, then it can be described as a $0,1$-vector of length n, whence as a function $f\colon\{0,\dots,n-1\}\to\{0,1\}$.

For example, the function $f=(f(0),\dots,f(7))=(1,0,0,1,1,0,1,0)$ describes the following rhythm of length 8, 

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or

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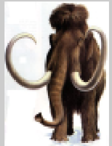
0,1-Vectors

The cyclic group C_n generated by $\pi_n := (0,1,\ldots,n-1)$ acts on the set of all mappings from $\underline{n} := \{0,1,\ldots,n-1\}$ to $\{0,1\}$ according to

$$C_n \times \{0,1\}^{\underline{n}} \to \{0,1\}^{\underline{n}}, \quad (\sigma, f) \mapsto f \circ \sigma^{-1}.$$

If we write f as a vector $f = (f(0),\ldots,f(n-1))$, then $f \circ \pi_n^j = (f(n+j),\ldots,f(n-1),f(0),\ldots,f(n+j-1))$. Hence, the orbit $C_n(f)$ consists of all cyclic shifts of f.



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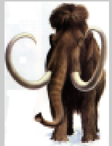
Using the natural order $0 < 1$, the set $\{0,1\}^{\underline{n}}$ is totally ordered by the lexicographical order. For $f,g \in \{0,1\}^{\underline{n}}$ we say

$$f < g :\Leftrightarrow \exists i \in \underline{n} : f(j) = g(j) \text{ for } j < i \text{ and } f(i) < g(i).$$

We choose the smallest element of an orbit as its canonical representative.



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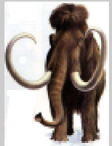
0,1-Vectors (cont.)

For example the orbit of $f = (10011010)$ under C_8 contains the vectors (10011010), (00110101), (01101010), (11010100), (10101001), (01010011), (10100110), and (01001101).



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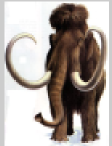
The stabilizer of $f \in \{0,1\}^{\underline{n}}$ is a subgroup of C_n, whence again a cyclic group. We call f ***acyclic*** if its stabilizer consists of the identity only. If f is acyclic, then the canonical representative of $C_n(f)$ is called a ***Lyndon word***.

The function of the last example is acyclic, and (00110101) is a Lyndon word.



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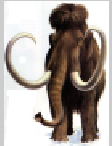
The stabilizer of $f \in \{0,1\}^{\underline{n}}$ is a subgroup of C_n, whence again a cyclic group. We call f ***acyclic*** if its stabilizer consists of the identity only. If f is acyclic, then the canonical representative of $C_n(f)$ is called a ***Lyndon word***.

The function of the last example is acyclic, and (00110101) is a Lyndon word.

The vector (01100110) has a nontrivial stabilizer, since $\pi_8^4 \neq \mathrm{id}$ is also contained in it.



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We identify a 0,1-vector f with the set $f^{-1}(\{1\})$ of pre-images of 1. Then f is the characteristic function of $f^{-1}(\{1\})$. In this case we also assume that $f^{-1}(\{1\}) \subseteq Z_n$.

Canons

A ***canon*** of length n consisting of $t \geq 1$ voices V_i is a set $\{V_1, \ldots, V_t\}$ of 0,1-vectors $V_i \neq 0$ of length n, such that

1. $V_i \in C_n(V_1)$ for $1 \leq i \leq t$,
2. V_1 is acyclic,
3. the set of differences in $K := \bigcup_{i=1}^{t} V_i$ generates Z_n, i.e.

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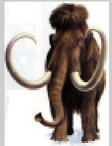
Two canons $\{V_1, \dots, V_t\}$ and $\{W_1, \dots, W_s\}$ are called ***isomorphic*** if $s = t$ and if there exists some $\sigma \in C_n$ and a permutation τ in the symmetric group S_t such that $\sigma(V_i) = W_{\tau(i)}$ for $1 \leq i \leq t$.

Inner and Outer Rhythm of a Canon

The canon $\{V_1, \ldots, V_t\}$ can be described as a pair (V_1, f), where V_1 is the ***inner*** and f the ***outer rhythm*** of the canon. The inner rhythm describes the rhythm of any voice. The outer rhythm determines how the different voices are distributed over the n beats of a canon.



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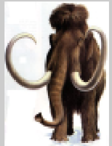
For example consider $V_1 = (10011010)$, $V_2 = (01010011)$, and $V_3 = (11010100)$. We get a score of the form

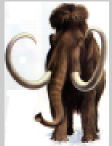
$$\begin{array}{c} 10011010 \\ 01010011 \\ 11010100 \end{array}$$

Hence the outer rhythm of this canon is $f = (10010100)$.



Inner and Outer Rhythm of a Canon (cont.)

Given a canon (V_1, f) of length n, there exists an isomorphic canon (L, f') where L is a Lyndon word, the canonical representative of $C_n(V_1)$, and f' is the canonical representative of $C_n(f)$.



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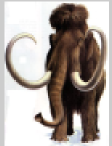
The different voices of the canon given by (L, f') are $j + L$ for $j \in f'$.



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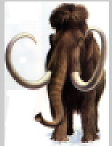
Conversely, not each pair (L, f) where L is a Lyndon word and f a 0,1-vector determines a canon of length n.

Lemma. *Let $L \neq 0$ be a Lyndon word, and let f be a 0,1-vector both of length n. The pair (L, f) does not describe a canon if and only if there exists an integer $d > 1$ such that $d \mid n$, $d \mid k - l$ for all $k, l \in L$, and $d \mid k - l$ for all $k, l \in f$.*



Enumeration of Canons

Theorem. *The number of isomorphism classes of canons of length n is*

$$K_n := \sum_{d|n} \mu(d)\lambda(n/d)\alpha(n/d),$$

where μ *is the* **Moebius function***,* $\lambda(1) = 1$*,*

$$\lambda(r) = \frac{1}{r}\sum_{s|r} \mu(s) 2^{r/s} \text{ for } r > 1,$$

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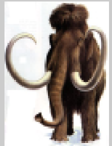

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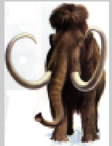
The ideas of this proof and of the previous Lemma can be used in order to construct all rhythmical canons of length n.

Rhythmic Tiling Canons

In our last example of a canon, $((00110101), (01001010))$

$$10011010$$
$$01010011$$
$$11010100$$

we saw that there are beats where no, one, two, or three voices are playing at the same time.



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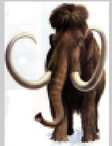
A canon is called a ***rhythmic tiling canon*** if $Z_n = \dot{\bigcup}_{i=1}^{t} V_i$. In other words, the voices are pairwise disjoint and cover entirely Z_n. The canon (L, f) is a tiling canon if and only if $L + f = Z_n$ and $|L|\,|f| = n$, thus Z_n is the direct sum of L and f.



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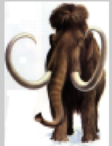
For example $((00000101),(00110011))$ is the canon $\begin{array}{c}01000001\\10100000\\00010100\\00001010\end{array}$.



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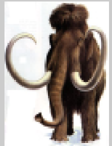
Enumeration of Tiling Canons

Classification of rhythmic tiling canons by computing complete lists of their representatives. We have such lists for $n \leq 40$.

Comparison of T_n the number of nonisomorphic tiling canons and K_n the numbers of canons of length n.

n	T_n	K_n
2	1	1
3	1	5
4	2	13
5	1	41
6	3	110
7	1	341
8	6	1035
9	4	3298



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12	23	117455
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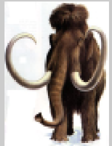
14	13	1.370798
15	25	4.780715
16	49	16788150
17	1	59451809



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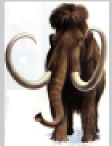
$n = 40$: $T_n = 64989$, $K_n = 755.578639.350274.265100$.



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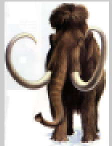
Regular Complementary Canons of Maximal Category

A rhythmic tiling canon described by (L, f) is a ***regular complementary canon of maximal category*** (RCMC-canon) if both L and f are acyclic.



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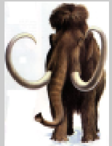

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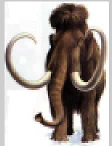

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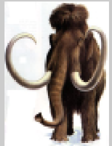
Z_n is not a Hajós group if and only if n can be expressed in the form $p_1 p_2 n_1 n_2 n_3$ with p_1, p_2 primes, $n_i \geq 2$ for $1 \leq i \leq 3$, and $\gcd(n_1 p_1, n_2 p_2) = 1$.

How does such a canon sound?

$[14, 8, 1, 5, 4, 4, 9, 9, 4, 6, 4, 9, 9, 4, 4, 5, 1, 8]$, $[57, 12, 12, 3, 12, 12]$



Vuza's Algorithm

If Z_n is not a Hajós group, Vuza presents an algorithm for constructing two acyclic vectors L and f of length n, such that $|L| = n_1 n_2$, $|f| = p_1 p_2 n_3$, and $L + f = Z_n$.

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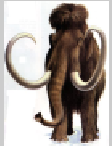

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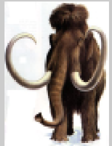

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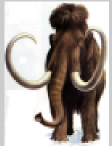
He also proves that when (L, f) describes an RCMC-canon, then also (kL, f), (kL, kf) have this property for all $k \in Z_n^*$. (Here $kL = \{k\ell \mid \ell \in L\}$.)

Research by: G. Hajós, L. Rédei, N.G. de Bruijn, A.D. Sands.

Enumeration of Vuza Constructible Canons

An RCMC-canon which can be constructed by Vuza's algorithm and the previous remarks is called ***Vuza constructible canon***.



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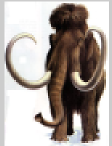

An RCMC-canon which can be constructed by Vuza's algorithm and the previous remarks is called ***Vuza constructible canon***.

Enumeration of nonisomorphic Vuza constructible canons by construction: This means that complete lists of these canons are available!

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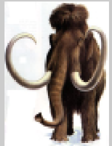
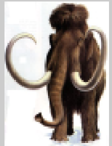
n	p_1	p_2	n_1	n_2	n_3	$\#L$	$\#f$	$\#(L,f)$
72	2	3	2	3	2	3	6	18
108	2	3	2	3	3	3	180	540
120	2	3	2	5	2	16	20	320
120	2	5	2	3	2	8	6	48
144	2	3	4	3	2	6	36	216
144	2	3	2	3	4	3	2808	8424
168	2	3	2	7	2	104	42	4368
168	2	7	2	3	2	16	6	96
180	2	3	2	3	5	3	45360	136080
180	2	3	2	5	3	16	1000	16000
180	2	5	2	3	3	8	252	2016
180	2	3	5	3	2	9	60	540
180	3	5	3	2	2	6	12	72
200	2	5	2	5	2	125	20	2500
240	2	3	4	5	2	32	120	3840
240	2	5	4	3	2	16	36	576
240	2	3	2	5	4	16	26000	416000
240	2	5	2	3	4	8	6264	50112



Do there exist RCMC-canons which are not Vuza constructible?



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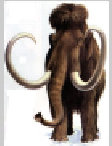

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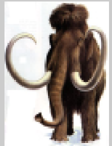

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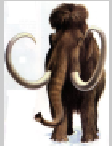

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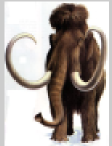
Idea: If you have an RCMC-canon, then subdivide each beat into 2 beats. Replace each quarter note or rest by 2 eighth notes or rests in the inner rhythm and enlarge the outer rhythm in a suitable way.



Do there exist RCMC-canons which are not Vuza constructible?

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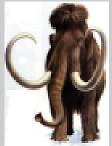

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Among the 216 RCMC-canons of length $2 \cdot 72 = 144$ with $|L| = 12$ we did not find a canon which was constructed in this way from the 18 canons of length 72.

From L equal to

[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1]

From L equal to

[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1]

and f equal to

[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,1,0,0,0,0,1,0,0,0,0,0,1,0,
0,0,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,0,0,1,1]

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0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1]

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0,0,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,0,0,1,1]

we get the canon with L' equal to

[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,
0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1]

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0,0,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,0,0,1,1]

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0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
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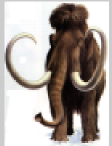

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We can do much better

Define a composition of two 0,1-vectors $f \in \{0,1\}^{\underline{n}}$, $g \in \{0,1\}^{\underline{m}}$. Then $f(g)$ is the vector in $\{0,1\}^{\underline{nm}}$ where each 1 in f is replaced by g and each 0 in f is replaced by $0^m = 0\dots0$. Each $i \in \underline{nm}$ can uniquely be written as $i = qm + r$ with $q \in \underline{n}$ and $r \in \underline{m}$. Then

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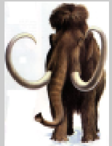

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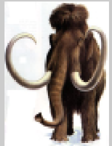

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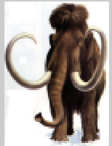

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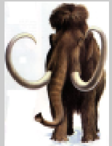

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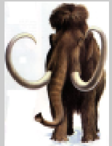

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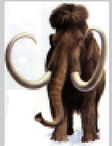

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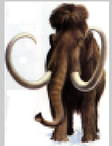

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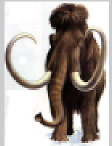
7. If f is a Lyndon word and g is the canonical representative of $C_m(g)$, then $f(g)$ is a Lyndon word.



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New Canons from Old Canons

Theorem.

1. If (L_1, f_1) is a canon of length n_1 and (L_2, f_2) is a canon of length n_2, then $(L_1(L_2), f_1(f_2))$ is a canon of length $n_1 n_2$.

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2. If (L_1, f_1) is a tiling canon and (L_2, f_2) is a tiling canon, then $(L_1(L_2), f_1(f_2))$ is a tiling canon.

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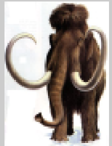

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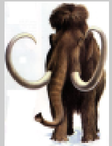
2. If (L_1, f_1) is a tiling canon and (L_2, f_2) is a tiling canon, then $(L_1(L_2), f_1(f_2))$ is a tiling canon.

3. If (L_1, f_1) is an RCMC-canon and (L_2, f_2) is a tiling canon, then $(L_1(L_2), f_1(f_2))$ is an RCMC-canon.



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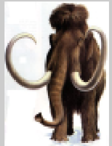
Are these all RCMC-Canons?



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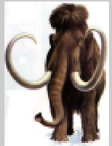
The answer is:



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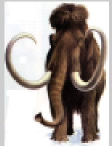
The answer is: **NO!**



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Construction: Consider an RCMC-canon (L, f) of length n. For fixed L try to find all Lyndon words f', such that $L + f' = Z_n$.

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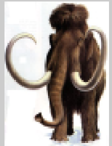
Inspired by discussions with E. Amiot, I developed a backtracking algorithm for this problem. For given inner rhythm L it finds all outer rhythms f so that (L, f) is an RCMC-canon.



Complete lists of RCMC-canons

If (L, f) is a tiling canon of length n, then the weights of L and f are divisors of n.

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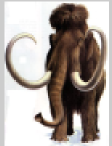
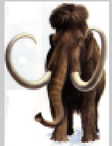
Theorem. (Sands) If $\mathrm{wt}(L)$ or $\mathrm{wt}(f)$ is a prime power, then f has cyclic symmetries. Thus (L, f) is not an RCMC-canon.





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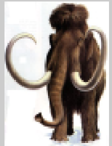
Construction: Find all suitable decompositions of n into two integers, $n = rs$. If r and s are not powers of a prime assume that $r \leq s$ and continue with the following construction. Determine all Lyndon words of length n and weight r over $\{0, 1\}$. In order to decrease their number, determine the number of orbits of these Lyndon words under the action of the affine group. For each of these representatives L determine all acyclic outer rhythms f, such that $L + f = Z_n$.



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All RCMC-canons of length 72 and 108

We have to consider the two compositions $72 = 6 * 12$ and $108 = 6 * 18$.

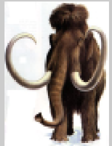
There are 2.169882 Lyndon words of length 72 and weight 6 over $\{0, 1\}$. There remain just 3 Lyndon words which can be extended to an RCMC-canon. For each Lyndon word there exist (the same 6) outer rhythms which can be used to determine an RCMC-canon. Thus all RCMC-canons of length 72 are Vuza constructible.

All RCMC-canons of length 72 and 108

We have to consider the two compositions $72 = 6 * 12$ and $108 = 6 * 18$.

There are 2.169882 Lyndon words of length 72 and weight 6 over $\{0, 1\}$. There remain just 3 Lyndon words which can be extended to an RCMC-canon. For each Lyndon word there exist (the same 6) outer rhythms which can be used to determine an RCMC-canon. Thus all RCMC-canons of length 72 are Vuza constructible.

There are 17.717859 Lyndon words of length 108 and weight 6 over $\{0, 1\}$. They are collected into 514754 orbits. There remains only one orbit representative which can be extended to an RCMC-canon. (This orbit contains 3 different Lyndon words.) For each of these Lyndon words there exist (the same 252) outer rhythms which can be used to determine an RCMC-canon. Thus some RCMC-canons of length 108 area not Vuza constructible.



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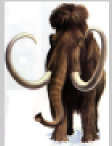

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