Tone rows and tropes

Harald Fripertinger
Karl-Franzens-Universität Graz

September 3, 2013, General Mathematics Seminar
Mathematics Research Unit, University of Luxembourg

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1. Introduction: Pitch classes, tone rows, similarity operations.

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4. The database.

Pitch and Scale

A tone in music is described by its fundamental frequency $f > 0$, which we call its pitch. It is usually given in Hertz (Hz), which is defined as the number of periodic cycles of a sine wave within a second. Two tones with frequencies f and $2f$ form the interval of an octave. In well tempered music (or equal temperament) an octave is divided into 12 equal parts. We speak of a 12-scale. Therefore the frequencies f_i, $1 \leq i \leq 11$, of the 11 tones between f and $2f$ would be $f \cdot 2^{i/12}$. (The factor $2^{1/12} = \sqrt[12]{2}$ describes the frequency ratio of a semi tone interval.)

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Disregarding the fact that human beings can hear tones only in the range from 20 Hz to 20000 Hz, in general the set of all tones in a 12-scale (which contains a tone with frequency f) is countably infinite and is given by $\{f \cdot 2^{k/12} \mid k \in \mathbb{Z}\}$. Since we are not interested in the particular frequencies we omit the factor f and each tone is represented by an integer k. Consequently, $\mathbb{Z}$ is a model of a 12-scale.

Pitch classes

From the musical perception we deduce that tones which are an integer multiple of an octave apart have a similar quality. We speak of octave equivalence. In music analysis and 12-tone composition usually it is not important which octave a certain tone belongs to, therefore, tones being a whole number of octaves apart are considered to be equivalent and are collected to a pitch class. E. g., if a' is the tone with frequency $f = 440$Hz, then the pitch class of a' consists of all tones $\ldots$, A, a, a', a'', a''', $\ldots$ These are the tones with frequencies $f \cdot 2^k \mathrm{Hz} = f \cdot 2^{12k/12} \mathrm{Hz}$, $k \in \mathbb{Z}$. Let n be an integer, then by $\overline{n}$ we denote the subset $\{12k + n \mid k \in \mathbb{Z}\}$. It is the residue class of n modulo 12. Of course $\overline{n} = \overline{n + 12k}$ for any $k \in \mathbb{Z}$.

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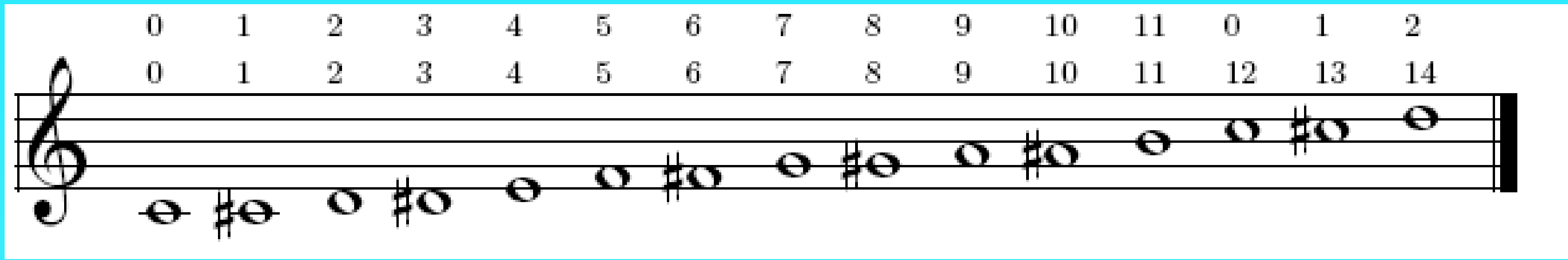
Using $\mathbb{Z}$ as the model of a 12-scale the twelve pitch classes are the subsets $\bar{i}$ for $0 \leq i < 12$. It is now clear that the pitch classes in the 12-scale $\mathbb{Z}$ coincide with the residue classes in $\mathbb{Z}_{12} := \mathbb{Z} \bmod 12\mathbb{Z}$.

The chromatic scale

Here is a short part of the chromatic scale together with the labelling of the tones in $\mathbb{Z}$ and pitch classes in $\mathbb{Z}_{12}$.



Tone rows

A ***tone row*** is a sequence of 12 tones so that tones in different positions belong to different pitch classes. Therefore, we describe a tone row by a mapping

$$f\colon \{1,\dots,12\} \to \mathbb{Z}, \qquad \overline{f(i)} \neq \overline{f(j)},\ i \neq j,$$

where $\overline{f(i)}$ is the residue class of $f(i)$, $i \in \{1,\dots,12\}$. The set $\{1,\dots,12\}$ is the set of all order numbers or time positions. The value $f(i)$, $i \in \{1,\dots,12\}$ is the tone in i-th position of the tone row f.

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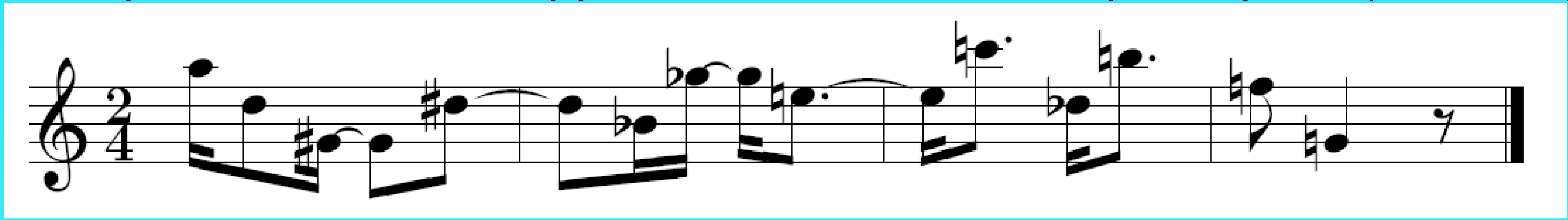
This leads to a total of $12! = 12\cdot 11\cdots 2\cdot 1 = 479\,001\,600$ tone rows.

O. Messiaen: Le Merle Noir

A piece for flute and piano composed in 1951. It is not a standard example. The tone row appears in the coda of the piano part. (Min 5.06)



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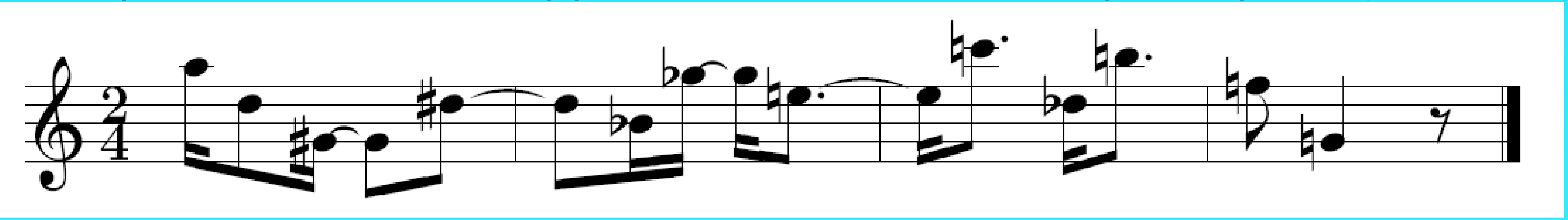

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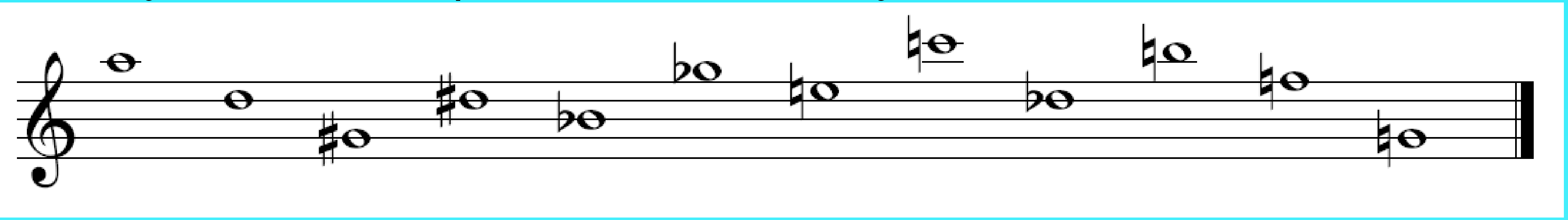

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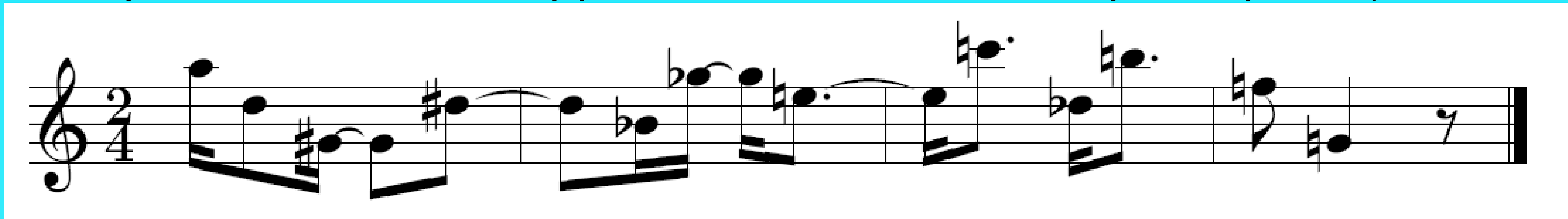

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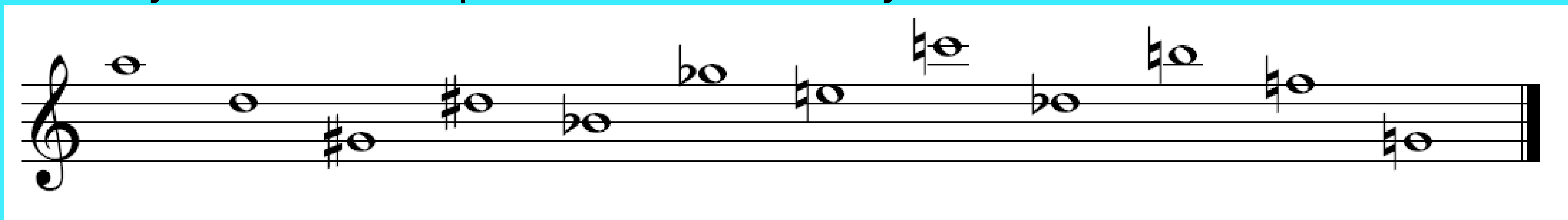
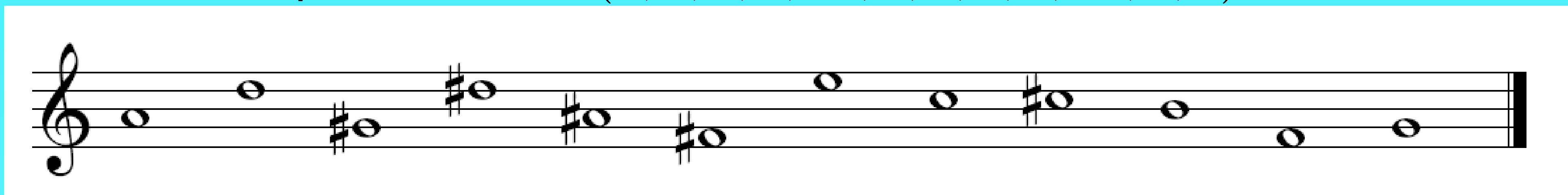


Reduction to pitch classes $(9, 2, 8, 3, 10, 6, 4, 0, 1, 11, 5, 7)$.



Circular representation of a tone row

In order to present a tone row as a graph

Example Tone row of Le Merle Noir

$$f := (f(1), \ldots, f(12)) = (9, 2, 8, 3, 10, 6, 4, 0, 1, 11, 5, 7):$$

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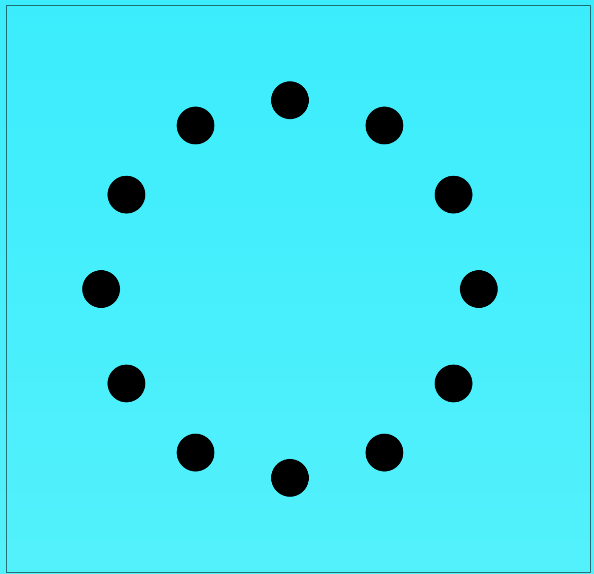

Circular representation of a tone row

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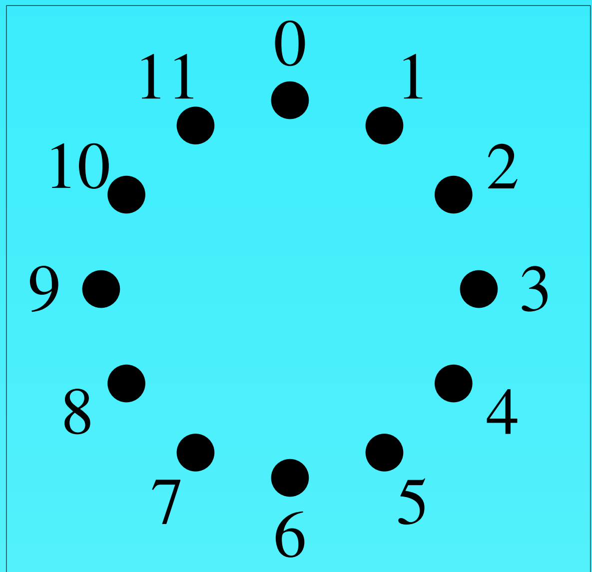

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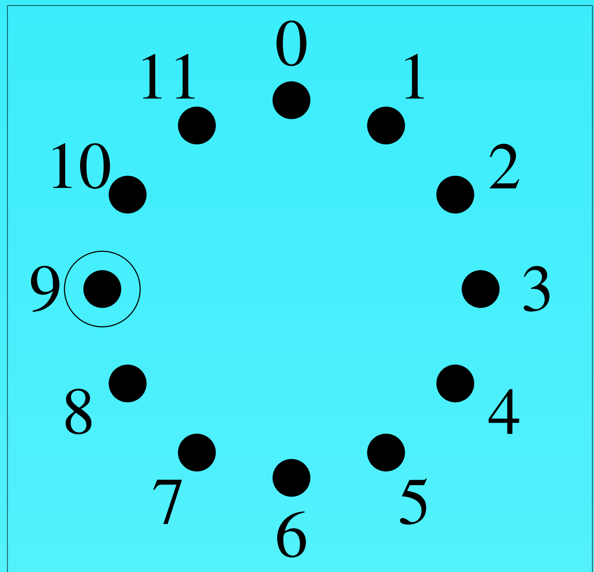

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In order to present a tone row as a graph we draw the 12 pitch classes as a regular 12-gon, we label them, we indicate the first tone, and we connect pitch classes which occur in consecutive position in the tone row.

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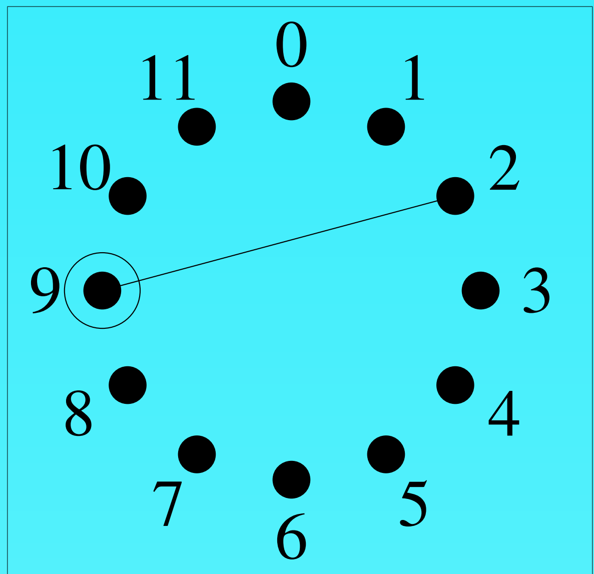

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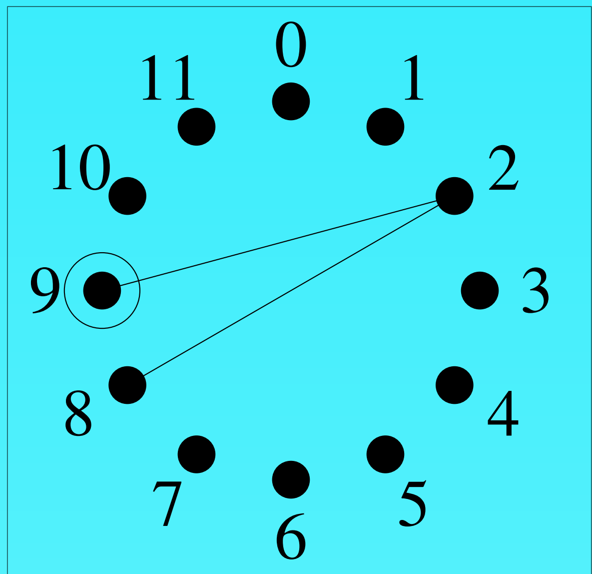

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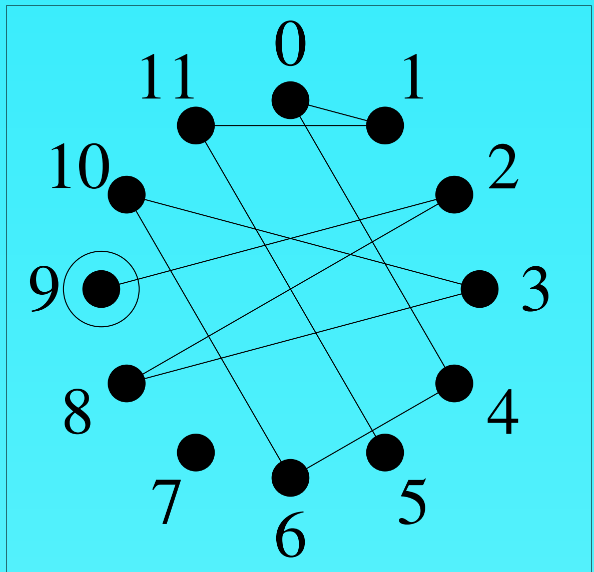

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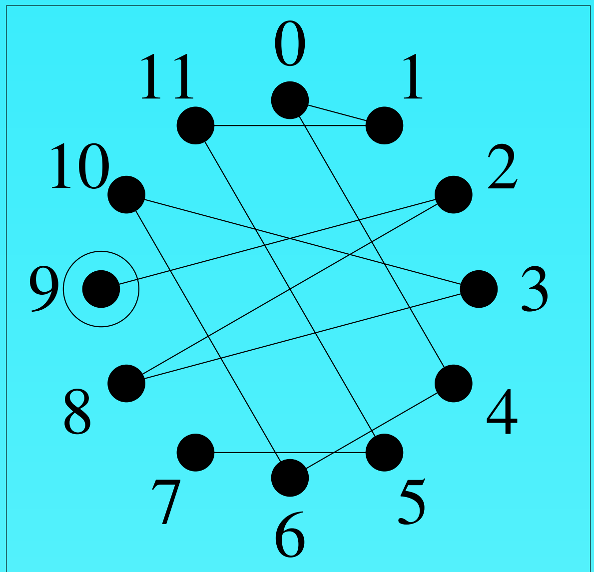

Transposing a tone row

Transposing a tone row by a semi tone is moving each tone one semi tone up. This operation is also possible for pitch classes. Define

$$T: \mathbb{Z}_{12} \to \mathbb{Z}_{12} : i \mapsto i + 1$$

then the transposed of a tone row f by k semitones, $k \in \mathbb{Z}$, is $T^k \circ f$.

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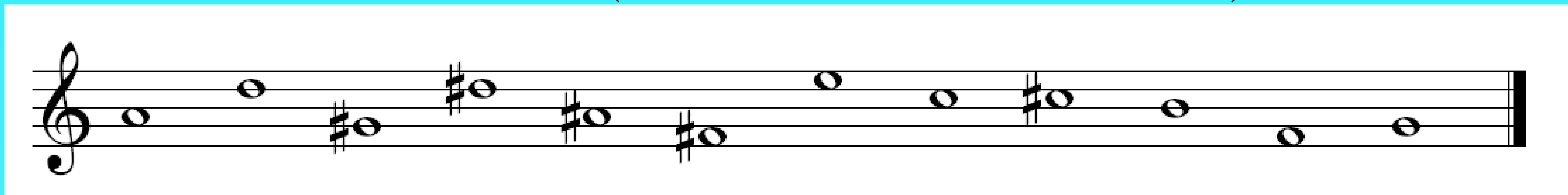
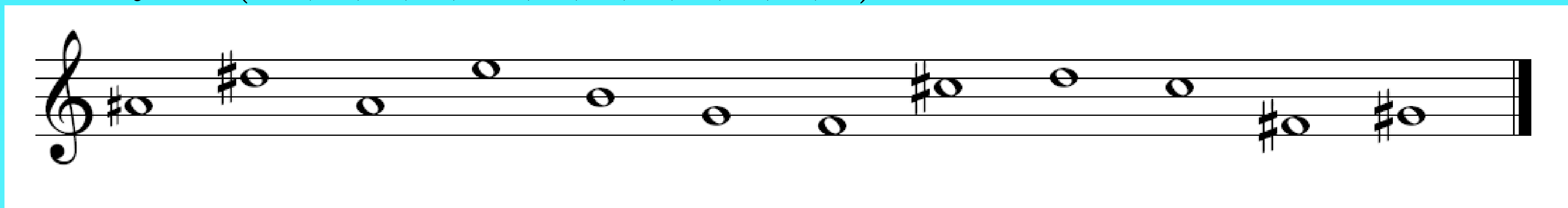
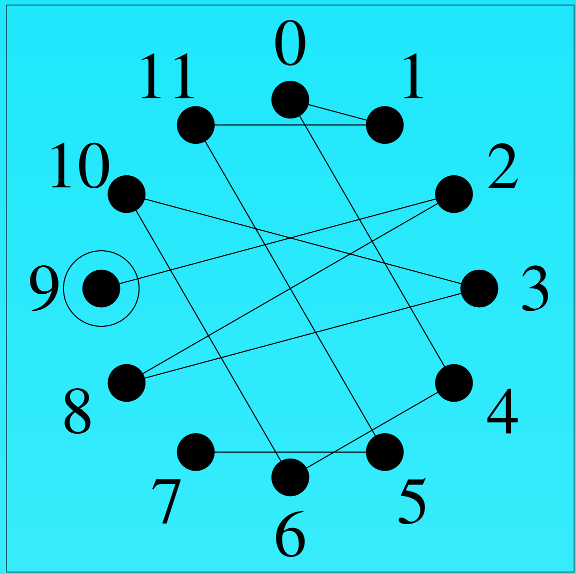
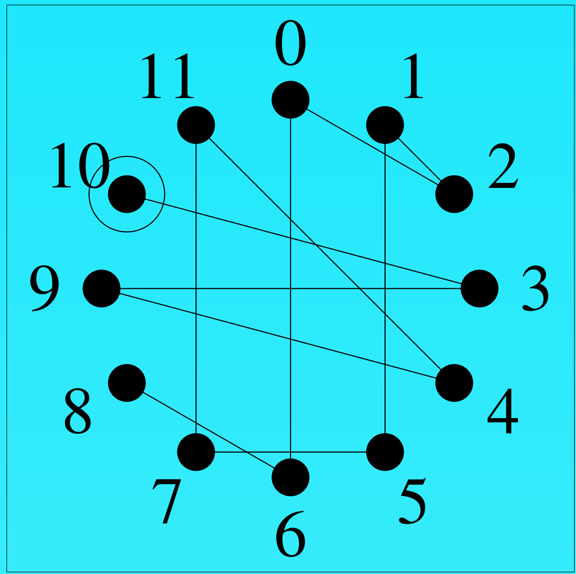


is $T \circ f = (10, 3, 9, 4, 11, 7, 5, 1, 2, 0, 6, 8)$



Transposing means a rotation of the circular representation.





Inversion of a tone row

There exist two inversion operators. Operators of the first kind fix a single tone t_0. For any tone $t_1 \neq t_0$ there exists a positive integer r so that t_1 is either r semitones higher or r semitones lower than t_0. By inversion t_1 is mapped onto the tone t_2 which is exactly r semitones lower, or higher than t_0.

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For operators of the second kind there exist two tones which are exactly one semitone apart. They will be interchanged by this operator. We call them t_0 and t_0+1. For any tone $t_1 \notin \{t_0, t_0+1\}$ there exists a positive integer r so that t_1 is either r semitones higher than t_0+1 or r semitones lower than t_0. By inversion t_1 is mapped onto the tone t_2 which is exactly r semitones lower than t_0, or exactly r semitones higher than t_0+1.

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Define inversion at pitch class 0 by

$$I\colon \mathbb{Z}_{12} \to \mathbb{Z}_{12} : i \mapsto -i.$$

Moreover, $I \circ T^k = T^{-k} \circ I$, $k \in \{0, \ldots, 11\}$. All inversion operators on $\mathbb{Z}_{12}$ can be written as compositions $T^r \circ I$ for $r \in \{0, \ldots, 11\}$. If r is even, then $T^r \circ I$ consists of exactly two fixed points and five cycles of length two, otherwise it consists of six cycles of length two.

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Inversion of a tone row f at 0 is defined as $I \circ f$.

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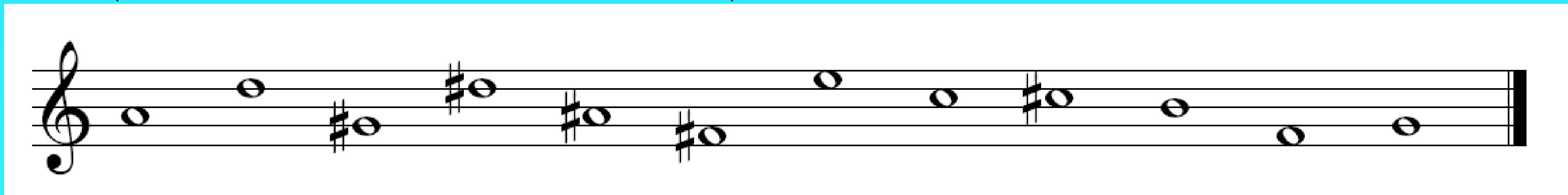
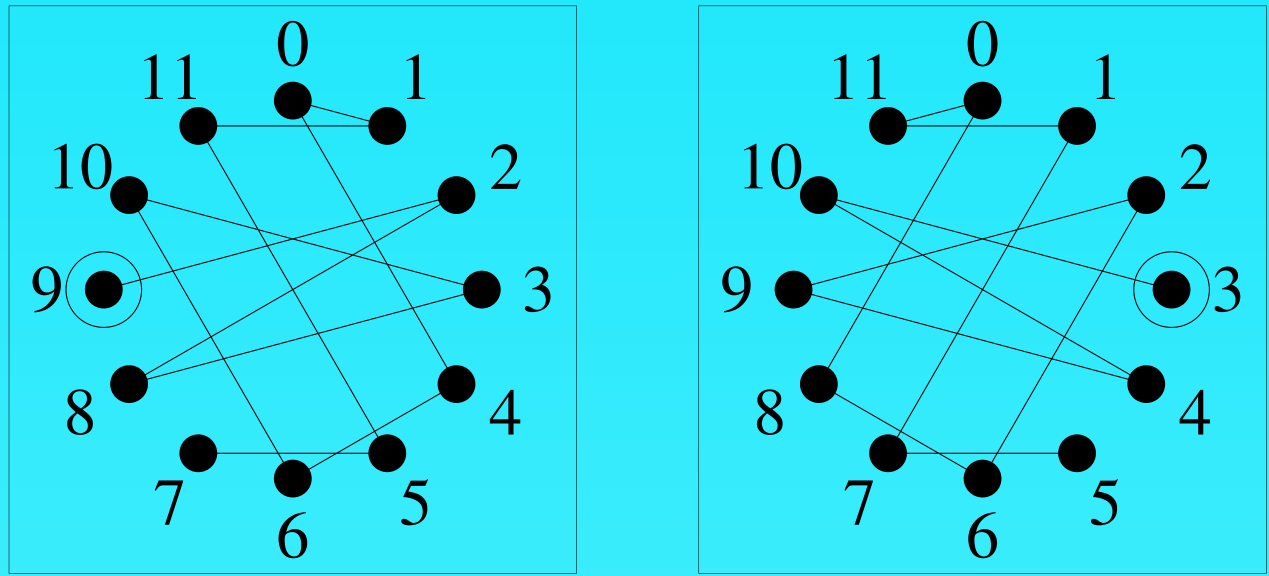
Inversion of a tone row f at 0 is defined as $I \circ f$. E.g. the transposed of $f = (9, 2, 8, 3, 10, 6, 4, 0, 1, 11, 5, 7)$



is $I \circ f = (3, 10, 4, 9, 2, 6, 8, 0, 11, 1, 7, 5)$

The circular representation of the inversion of f is the mirror image of the circular representation of f.



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Retrograde and cyclic shift of a tone row

Consider the permutations $R = (1,12)(2,11)\cdots(6,7)$ and $S = (1,2,3,\ldots,12)$, then the retrograde of the tone row f is $f \circ R$ and the cyclic shift of f is $f \circ S$.

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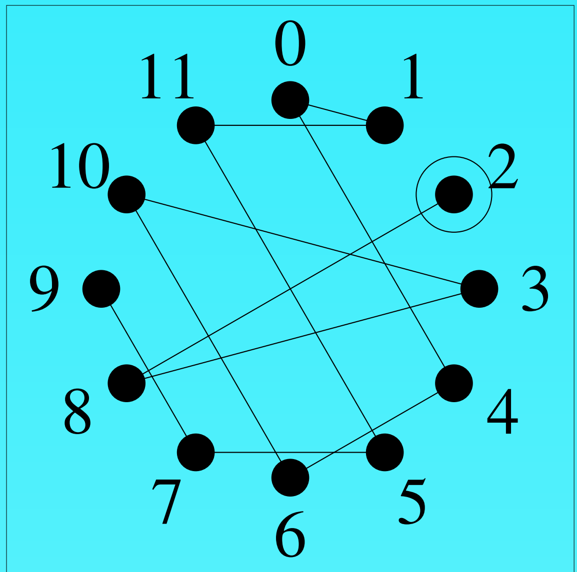
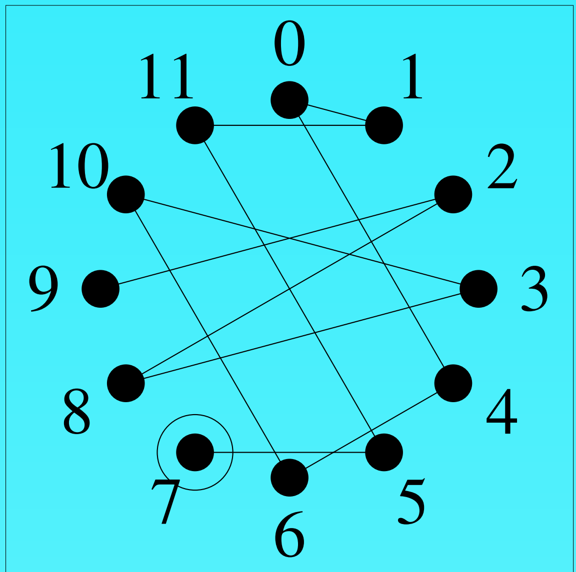
$f = (9,2,8,3,10,6,4,0,1,11,5,7)$,

$f \circ S = (2,8,3,10,6,4,0,1,11,5,7,9)$ and

$f \circ R = (7,5,11,1,0,4,6,10,3,8,2,9)$.





Permutation groups

The groups $\langle T, I\rangle$ and $\langle S, R\rangle$ are permutation groups, both isomorphic to the ***dihedral group*** D_{12} consisting of 24 elements.

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Theorem. *Let π be a permutation of $\mathbb{Z}_{12}$, then $\pi(i+1) = \pi(i) + 1$ for all $i \in \mathbb{Z}_{12}$, or $\pi(i+1) = \pi(i) - 1$ for all $i \in \mathbb{Z}_{12}$, if and only if $\pi \in D_{12}$.*

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$C_{12}=\langle T\rangle$ is a ***cyclic group*** of order 12. It is a subgroup of D_{12}.

We consider also the ***quart-circle*** Q defined by $Q\colon\mathbb{Z}_{12}\to\mathbb{Z}_{12}$, $i\mapsto 5i$.

It replaces a chromatic scale by a sequence of quarts, the quint-circle $I\circ Q$ by a sequence of quints, since $(I\circ Q)(i)=7i$, $i\in\mathbb{Z}_{12}$.
The group $\langle T,I,Q\rangle$ is the group of all ***affine mappings*** on $\mathbb{Z}_{12}$ which we abbreviate by $\mathrm{Aff}_1(\mathbb{Z}_{12})$. It is the set of all mappings $f(i)=ai+b$, where $a,b\in\mathbb{Z}$, $\gcd(a,12)=1$. Thus $a\in\{1,5,7,11\}$ and $b\in\{0,\dots,11\}$.

The same operation on the order numbers is the ***fife-step*** F.

Equivalent tone rows

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Let $\mathscr{R}$ be the set of all tone rows, i.e. bijective mappings from $\{1,\ldots,12\}$ to $\mathbb{Z}_{12}$. We consider the following mapping

$$(\langle T,I\rangle \times \langle S,R\rangle) \times \mathscr{R} \to \mathscr{R}$$

$$((\varphi,\pi),f) \mapsto \varphi \circ f \circ \pi^{-1}. \qquad (*)$$

This mapping defines a ***group action***.

Equivalent tone rows

A tone row f' is considered to be ***equivalent*** to a tone row f if f' can be constructed from f by any combination of transposing, inversion, cyclic shift and retrograde.

Let $\mathscr{R}$ be the set of all tone rows, i.e. bijective mappings from $\{1,\ldots,12\}$ to $\mathbb{Z}_{12}$. We consider the following mapping

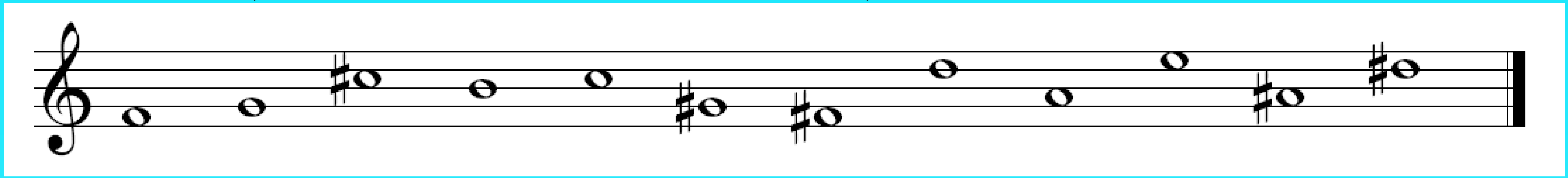
$$(\langle T,I\rangle \times \langle S,R\rangle) \times \mathscr{R} \to \mathscr{R}$$

$$((\varphi,\pi),f) \mapsto \varphi \circ f \circ \pi^{-1}. \qquad (*)$$

This mapping defines a ***group action***.

A tone row f' is equivalent to f if and only if $f' = \varphi \circ f \circ \pi^{-1}$ for some (φ,π) in $\langle T,I\rangle \times \langle S,R\rangle$.

E.g. The retrograde of the inversion of the tone row in Le Merle Noir is $I \circ f \circ R = (5, 7, 1, 11, 0, 8, 6, 2, 9, 4, 10, 3)$.



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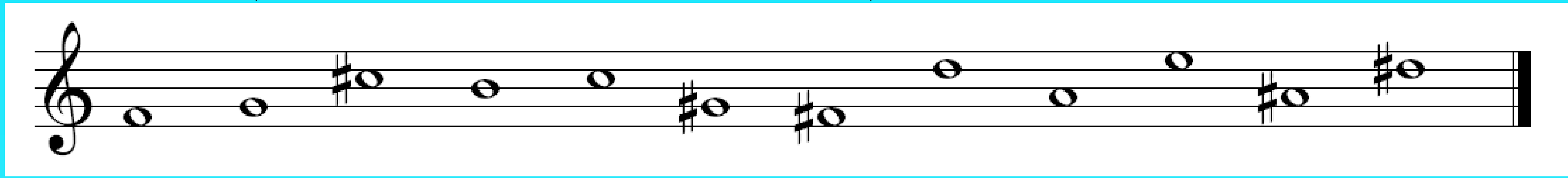
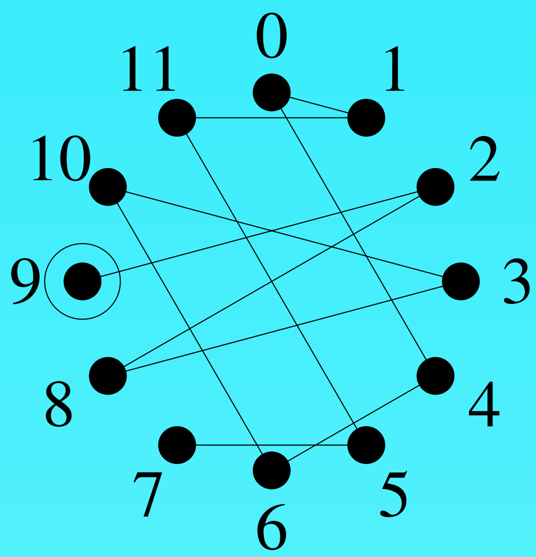


Circular representation of all equivalent rows: We start with f.



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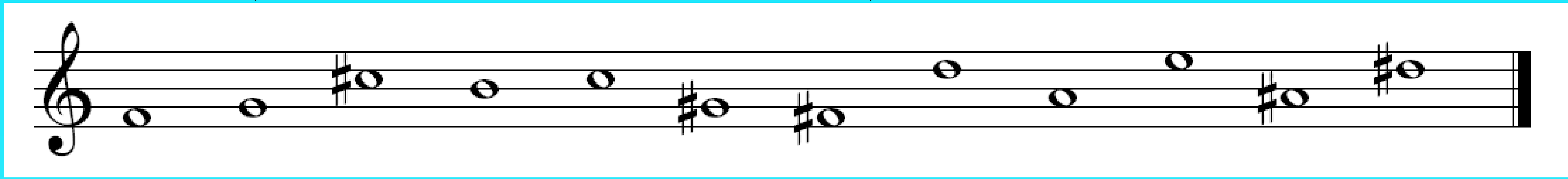

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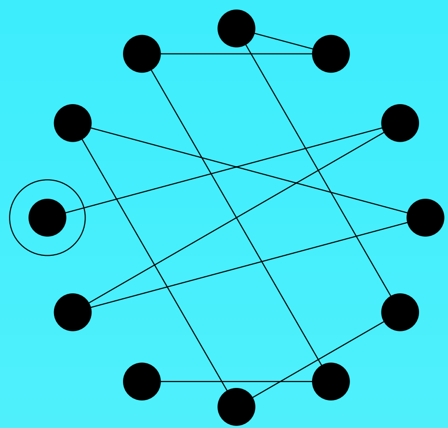

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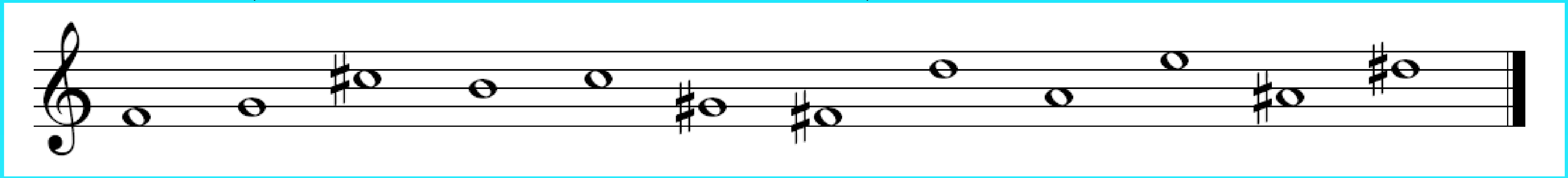

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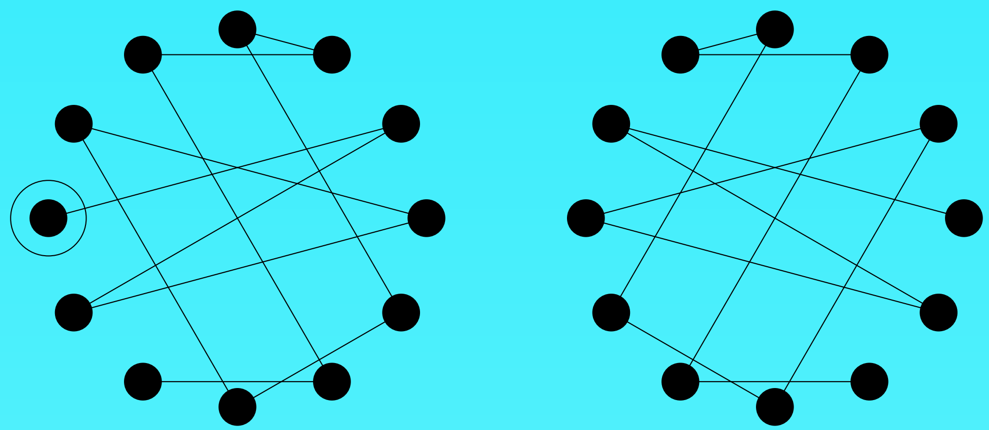
Inversion of f is the mirror of the given graph.



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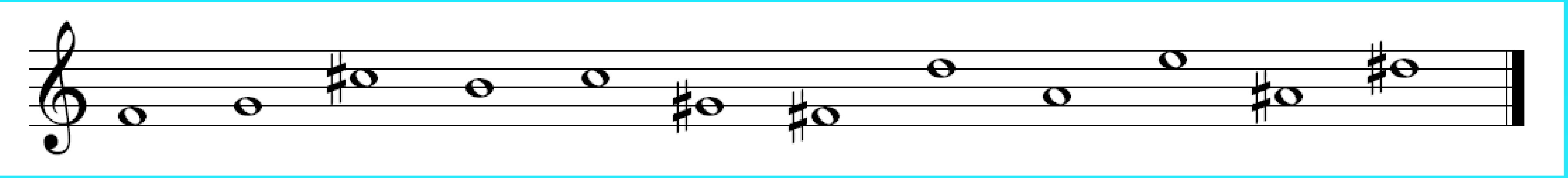

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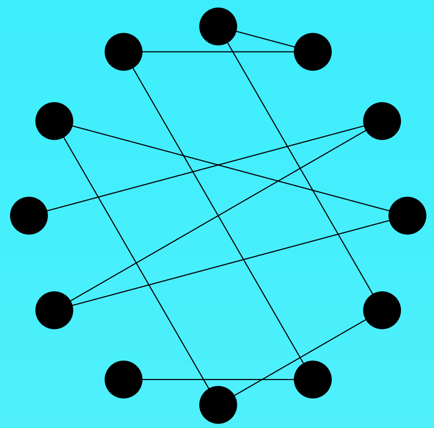

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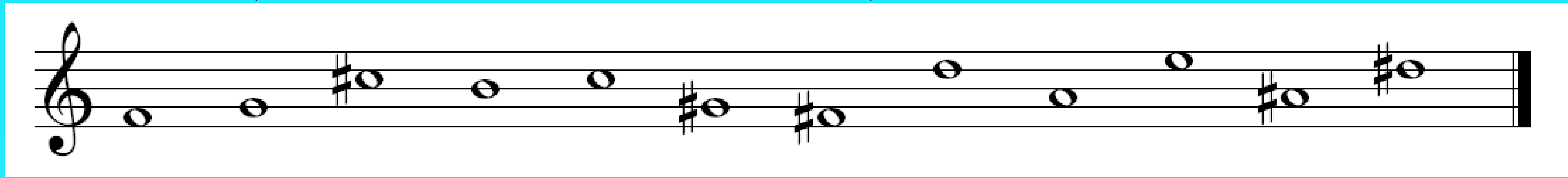

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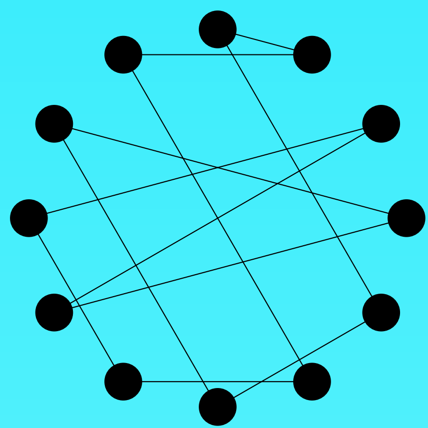

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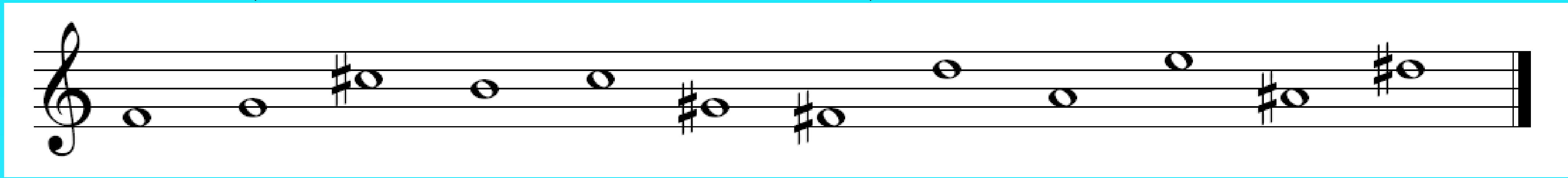

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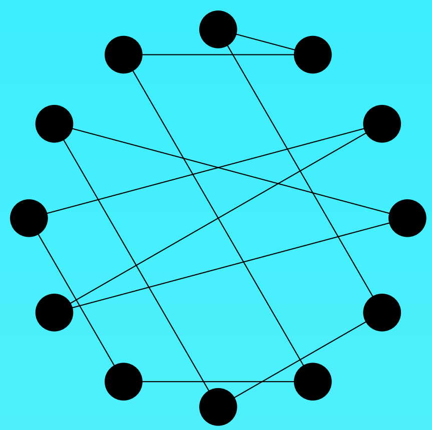

Database: The equivalence class of (9,2,8,3,10,6,4,0,1,11,5,7)

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If G and X are finite sets, then we speak of a ***finite group action***.

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Theorem. *The equivalence classes of any equivalence relation (or the blocks of any partition) can be represented as orbits under a suitable group action.*

Stabilizers and Fixed Points

Let ${}_G X$ be a group action. For each $x \in X$ the ***stabilizer*** G_x of x is the set of all group elements which do not change x, in other words

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— Then the direct product $H \times G$ acts on Y^X by

$$(H \times G) \times Y^X \to Y^X, \qquad ((h,g),f) \mapsto \bar{h} \circ f \circ \bar{g}^{-1}.$$

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Classification of tone rows

Equivalent tone rows are collected in one orbit of tone rows under the group action of $(*)$. If we know all the orbits then we know all non-equivalent tone rows.

How many orbits are there?

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	acting group	# of orbits
1.	$\langle T\rangle \times \langle R\rangle$	19 960 320
2.	$\langle T, I\rangle \times \langle R\rangle$	9 985 920
3.	$\langle T\rangle \times \langle S\rangle$	3 326 788
4.	$\langle T, I\rangle \times \langle S\rangle$	1 664 354
5.	$\langle T, I\rangle \times \langle S, R\rangle$	836 017
6.	$\langle T, I, Q\rangle \times \langle S, R\rangle$	419 413
7.	$\langle T, I\rangle \times \langle S, R, F\rangle$	419 413
8.	$\langle T, I, Q\rangle \times \langle S, R, F\rangle$	211 012

Arnold Schönberg considered tone rows as equivalent if they can be determined by transposing, inversion and/or retrograde from a single tone row. Thus, his model corresponds to the settings in 2.

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According to Theorem the dihedral group is the biggest group which preserves the neighbor relations in $\mathbb{Z}_{12}$. Therefore, we consider the settings of **5. as the standard settings for our classification**. There is also big evidence that ***Josef Matthias Hauer*** was considering this equivalence relation. Also ***Ron C. Read*** considers in [7, page 546] this notion as the natural equivalence relation on the set of all tone rows. He also determines 836 017 as the number of pairwise non-equivalent tone-rows. Previously, this number was already determined in a geometric problem by ***Solomon W. Golomb*** and ***Lloyd R. Welch*** in [4]. In their manuscript [5] ***David J. Hunter*** and ***Paul T. von Hippel*** also consider the cyclic shift as a symmetry operation for tone rows.

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- Given two tone rows f_1 and f_2 belonging to the same orbit, determine an element $g \in G$ so that $f_2 = g f_1$.

The normal form

We define a ***total order*** on $\mathbb{Z}_{12}$ by assuming that $\overline{0} < \overline{1} < \ldots < \overline{11}$. We represent the tone row f as a vector of length 12 of the form $(f(1), \ldots, f(12))$. Using this total order, the set of tone rows written as vectors is totally ordered by the ***lexicographical order***. We say the tone row f_1 is smaller than the tone row f_2, and we write $f_1 < f_2$, if there exists an integer $i \in \{1, \ldots, 12\}$ so that $f_1(i) < f_2(i)$ and $f_1(j) = f_2(j)$ for all $1 \leq j < i$.

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Given a group G which describes the equivalence of tone rows and a tone row f, we compute the orbit $G(f)$ of f by applying all elements of G to f. By doing this we obtain the set $\{gf \mid g \in G\}$ which contains at most $|G|$ tone rows. As the ***standard representative*** of this orbit, or as the ***normal form*** of f, we choose the smallest element in $G(f)$ with respect to the lexicographical order.

Database: The normal form (9,2,8,3,10,6,4,0,1,11,5,7)

The stabilizer type of a tone row

Let f be a tone row and let G be a group describing the equivalence classes of tone rows. From Theorem we already know that the size of the orbit of f depends on the size of its stabilizer G_f. The stabilizer G_f is a subgroup of G.

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The ***stabilizer type*** of the orbit $G(f)$ is the conjugacy class $\tilde{G}_f = \{gG_fg^{-1} \mid g \in G\}$, since the stabilizer of gf is gG_fg^{-1}, $g \in G$.

The stabilizer type of a tone row

Let f be a tone row and let G be a group describing the equivalence classes of tone rows. From Theorem we already know that the size of the orbit of f depends on the size of its stabilizer G_f. The stabilizer G_f is a subgroup of G.

The ***stabilizer type*** of the orbit $G(f)$ is the conjugacy class $\tilde{G}_f = \{gG_fg^{-1} \mid g \in G\}$, since the stabilizer of gf is gG_fg^{-1}, $g \in G$.

In our standard situation we have ***17 different stabilizer types***. We list all these stabilizer types $\tilde{U}_i$ and the number of $D_{12} \times D_{12}$-orbits of tone rows which have stabilizer type $\tilde{U}_i$. These numbers were computed by applying Burnside's Lemma. For doing this, we used the computer algebra system GAP.

Examples:

- Consider $f = (0, 1, 10, 8, 9, 11, 5, 3, 2, 4, 7, 6)$, then $T^6 f = (6, 7, 4, 2, 3, 5, 11, 9, 8, 10, 1, 0)$ and $fR = (6, 7, 4, 2, 3, 5, 11, 9, 8, 10, 1, 0)$, thus $(T^6, R) * f = f$.

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  and $fR = (6, 7, 4, 2, 3, 5, 11, 9, 8, 10, 1, 0)$,
  thus $(T^6, R) * f = f$.

- Consider $f = (5, 4, 0, 7, 3, 2, 8, 9, 1, 6, 10, 11)$,
  then $TIf = (8, 9, 1, 6, 10, 11, 5, 4, 0, 7, 3, 2) = fS^6$
  and $T^6 f = (11, 10, 6, 1, 9, 8, 2, 3, 7, 0, 4, 5) = fR$.
  The stabilizer of f is $\{\mathrm{id}, (TI, S^6), (T^6, R), (T^7 I, S^6 R)\}$.

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- Consider $f = (0,1,10,8,9,11,5,3,2,4,7,6)$,
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  and $fR = (6,7,4,2,3,5,11,9,8,10,1,0)$,
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- The chromatic scale has a stabilizer of order 24.

  Database: The stabilizer of the chromatic scale

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- The chromatic scale has a stabilizer of order 24.

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- 99.93% of all orbits of tone rows have the trivial stabilizer type.

name	generators	$\lvert U \rvert$	$\lvert \tilde{U} \rvert$	$\lvert \tilde{U}_i \backslash\backslash \mathscr{R} \rvert$
$\tilde{U}_1$	identity	1	1	827 282
$\tilde{U}_2$	(TI, S^6)	2	6	912
$\tilde{U}_3$	(T^6, R)	2	3	912
$\tilde{U}_4$	(T^6, S^6)	2	1	130
$\tilde{U}_5$	(I, SR)	2	36	942
$\tilde{U}_6$	(TI, R)	2	36	5 649
$\tilde{U}_7$	(T^4, S^4)	3	2	11
$\tilde{U}_8$	(T^3, S^3)	4	2	2
$\tilde{U}_9$	$(TI, S^6), (T^6, R)$	4	36	96
$\tilde{U}_{10}$	$(I, SR), (T^6, S^6)$	4	18	12
$\tilde{U}_{11}$	$(TI, R), (T^6, S^6)$	4	18	42
$\tilde{U}_{12}$	$(I, SR), (T^4, S^4)$	6	24	2
$\tilde{U}_{13}$	$(TI, R), (T^4, S^4)$	6	24	15
$\tilde{U}_{14}$	$(I, SR), (T^3, S^3)$	8	36	6
$\tilde{U}_{15}$	$(TI, R), (T^2, S^2)$	12	12	2
$\tilde{U}_{16}$	$(I, SR), (T, S)$	24	12	1
$\tilde{U}_{17}$	$(I, SR), (T, S^5)$	24	12	1

Database: Search for tone rows of given stabilizer type

The interval structure

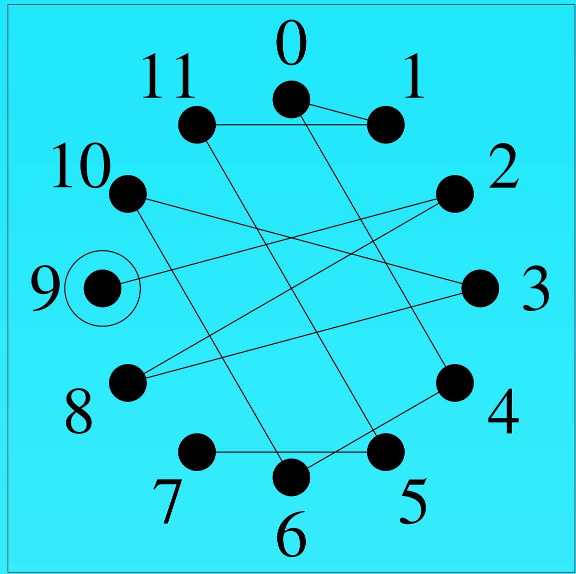
The ***interval*** from a to b in $\mathbb{Z}_{12}$ is the difference $b-a \in \mathbb{Z}_{12}$. This is the minimum number of steps in clockwise direction from a to b in the regular 12-gon. The tone row f determines the following sequence of eleven intervals

$$(f(2)-f(1), f(3)-f(2), \ldots, f(12)-f(11)). \qquad (**)$$

Since we consider a tone-row as a closed polygon we also have to add the closing interval $f(1)-f(12)$. Consequently, the ***interval structure*** of the tone-row $f\colon \{1,\ldots,12\} \to \mathbb{Z}_{12}$ is the function $g\colon \{1,\ldots,12\} \to \mathbb{Z}_{12} \setminus \{0\}$, defined by

$$g(i) := \begin{cases} f(i+1)-f(i) & \text{for } 1 \le i \le 11 \\ f(1)-f(12) & \text{for } i = 12. \end{cases}$$

The interval structure of $f = (9, 2, 8, 3, 10, 6, 4, 0, 1, 11, 5, 7)$



is $g = (5, 6, 7, 7, 8, 10, 8, 1, 10, 6, 2, 2)$.

To the $D_{12} \times D_{12}$-orbit of the tone row f corresponds to the $\langle I \rangle \times D_{12}$-orbit of the interval structure g. It is given by its standard representative $(1, 8, 10, 8, 7, 7, 6, 5, 2, 2, 6, 10)$.

Database: Search for tone rows of given interval structure

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All-interval rows

A tone row f is called an ***all-interval row*** if all elements of $\mathbb{Z}_{12} \setminus \{0\}$ occur in the sequence $(**)$. Then each element of $\mathbb{Z}_{12} \setminus \{0\}$ occurs exactly once in $(**)$. Hence, $\{g(j) \mid 1 \leq j \leq 11\} = \{1, \ldots, 11\}$ and, therefore,

$$g(12) = -\sum_{j=1}^{11} g(j) = -\sum_{i \in \mathbb{Z}_{12} \setminus \{0\}} i = 6.$$

Generalizing this notion, the $D_{12} \times D_{12}$-orbit of f contains all-interval rows if and only if each possible value occurs in the interval structure of f. In this situation the interval 6 occurs exactly twice and all other intervals exactly once in the interval structure of f. Thus the interval type of f looks like $(1,1,1,1,1,2,1,1,1,1,1)$. In this situation we call the orbit $D_{12} \times D_{12}(f)$ an all-interval orbit.

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Example:

Consider the vector $(1, -2, 3, -4, 5, -6, 7, -8, 9, -10, 11, 6)$ in $\mathbb{Z}_{12}^{12}$ which is the interval structure (1,10,3,8,5,6,7,4,9,2,11,6) of the tone row (0,1,11,2,10,3,9,4,8,5,7,6). This is an all interval-row.

On enumeration and listing of all-interval rows see [2, 6, 1, 3].

Database: Search for all-interval rows

Orbits of hexachords

A ***hexachord*** in the 12-scale $\mathbb{Z}_{12}$ is a 6-subset of $\mathbb{Z}_{12}$. There exist $\binom{12}{6} = 924$ different hexachords in the set $\mathscr{H} = \{A \subset \mathbb{Z}_{12} \mid |A| = 6\}$.

If a group G acts on $\mathbb{Z}_{12}$, then the induced action of $g \in G$ on a k-subset A of $\mathbb{Z}_{12}$, $k \in \{1,\dots,12\}$, is given by $g * A := \{g * a \mid a \in A\}$.

The number of orbits of hexachords under the action of a group is easily computed by an application of Pólya's Theorem. We get

G	$\lvert G \backslash\backslash \mathscr{H} \rvert$
C_{12}	80
D_{12}	50
$\mathrm{Aff}_1(\mathbb{Z}_{12})$	34

Database: List all k-chords

Tropes

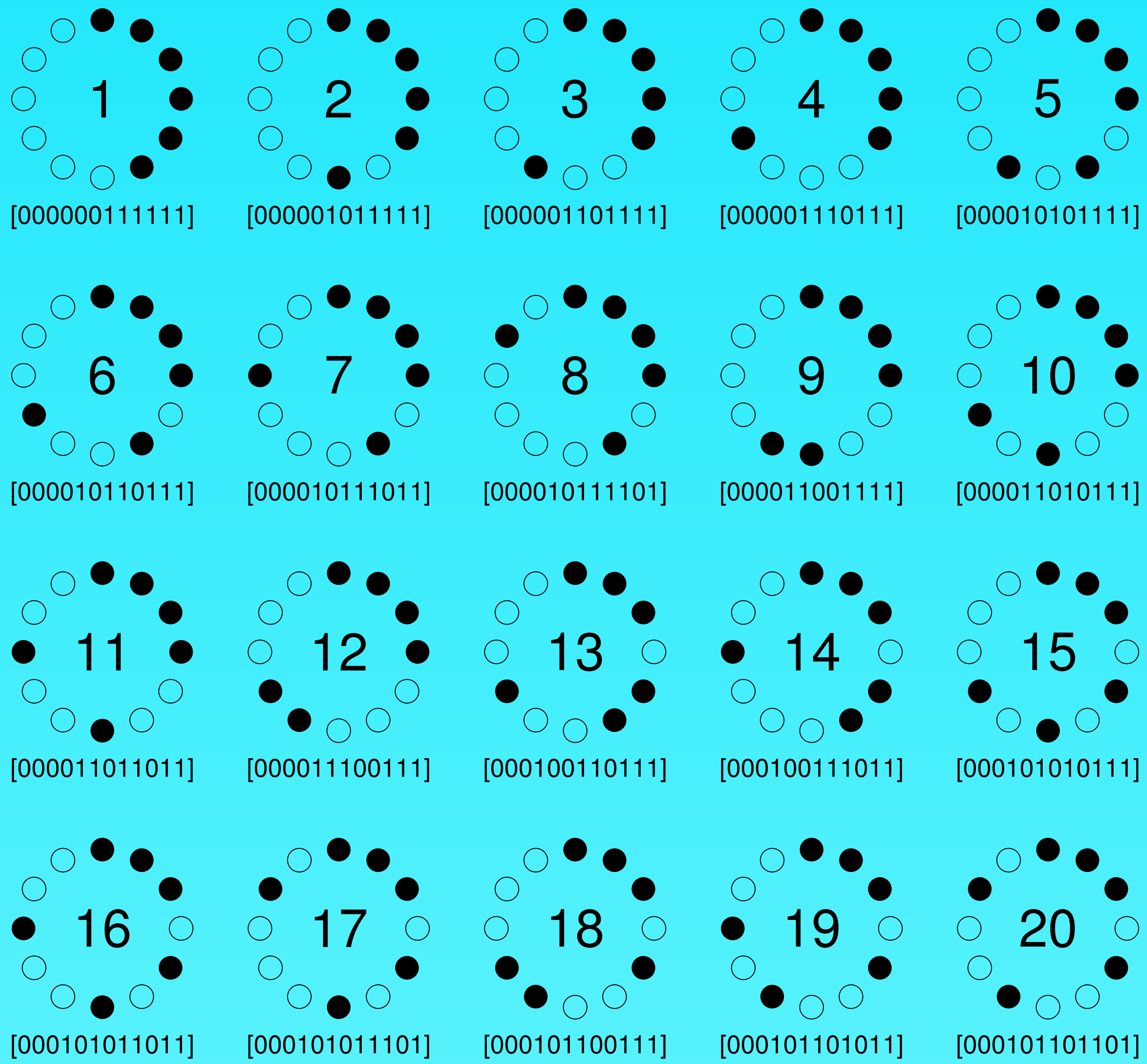
Let A be a hexachord, then its complement $A' := \mathbb{Z}_{12} \setminus A$ is also a hexachord. Now we consider "pairs" $\{A, A'\}$ of hexachords which we call ***tropes***. (We use quotation marks around the word pair, since $\{A, A'\}$ is actually not a pair, but a 2-set of hexachords!)

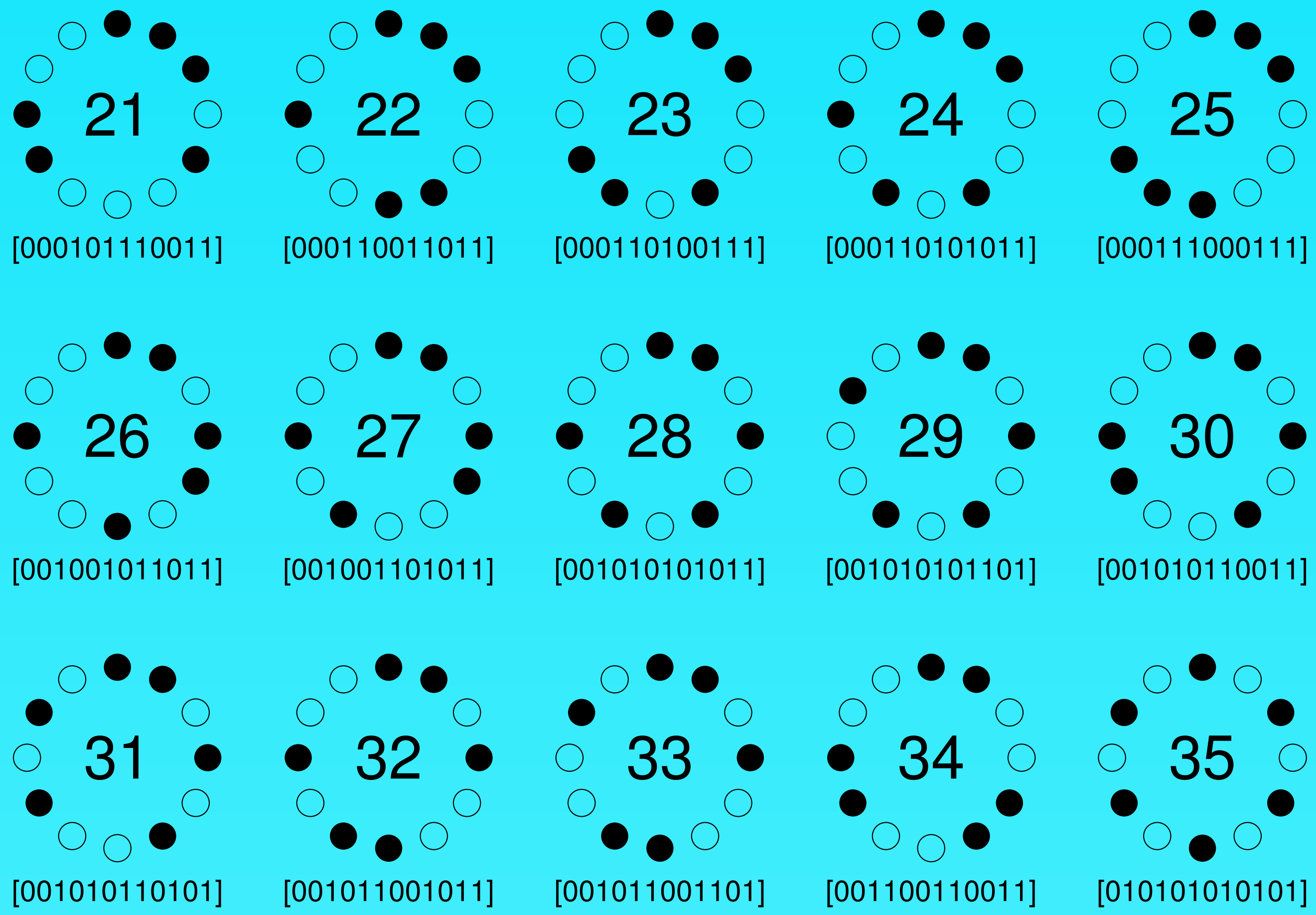
The set of all tropes will be indicated by $\mathscr{T} := \{\{A, \mathbb{Z}_{12} \setminus A\} \mid A \in \mathscr{H}\}$. In total there exist $924/2 = 462$ tropes in the 12-scale. If a group G acts on $\mathbb{Z}_{12}$, then the induced action of $g \in G$ on a trope $\{A, A'\}$ is given by $g * \{A, A'\} := \{g * A, g * A'\}$.
Number of orbits of tropes:

G	$\lvert G \backslash\backslash \mathscr{T} \rvert$
C_{12}	44
D_{12}	35
$\mathrm{Aff}_1(\mathbb{Z}_{12})$	26

Database: List all tropes

List of all 35 orbits of tropes

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[000110100111]
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[000110101011]
25
[000111000111]
26
[001001011011]
27
[001001101011]
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[001010101011]
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[001010101101]
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[001010110011]
31
[001010110101]
32
[001011001011]
33
[001011001101]
34
[001100110011]
35
[010101010101]

Trope structure of a tone row

Consider a tone row f. We obtain in a natural way six "pairs" of hexachords defined by f:

$$
\begin{aligned}
\tau_1 &:= \{\{f(1),f(2),f(3),f(4),f(5),f(6)\},\{f(7),f(8),f(9),f(10),f(11),f(12)\}\}\\
\tau_2 &:= \{\{f(2),f(3),f(4),f(5),f(6),f(7)\},\{f(8),f(9),f(10),f(11),f(12),f(1)\}\}\\
\tau_3 &:= \{\{f(3),f(4),f(5),f(6),f(7),f(8)\},\{f(9),f(10),f(11),f(12),f(1),f(2)\}\}\\
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From τ_i to τ_{i+1} exactly 2 pitch classes are changing their hexachord. They form the ***moving pair*** $\{f(i),f(i+6)\}$. The two tropes τ_i and τ_{i+1} are called ***connectable***.

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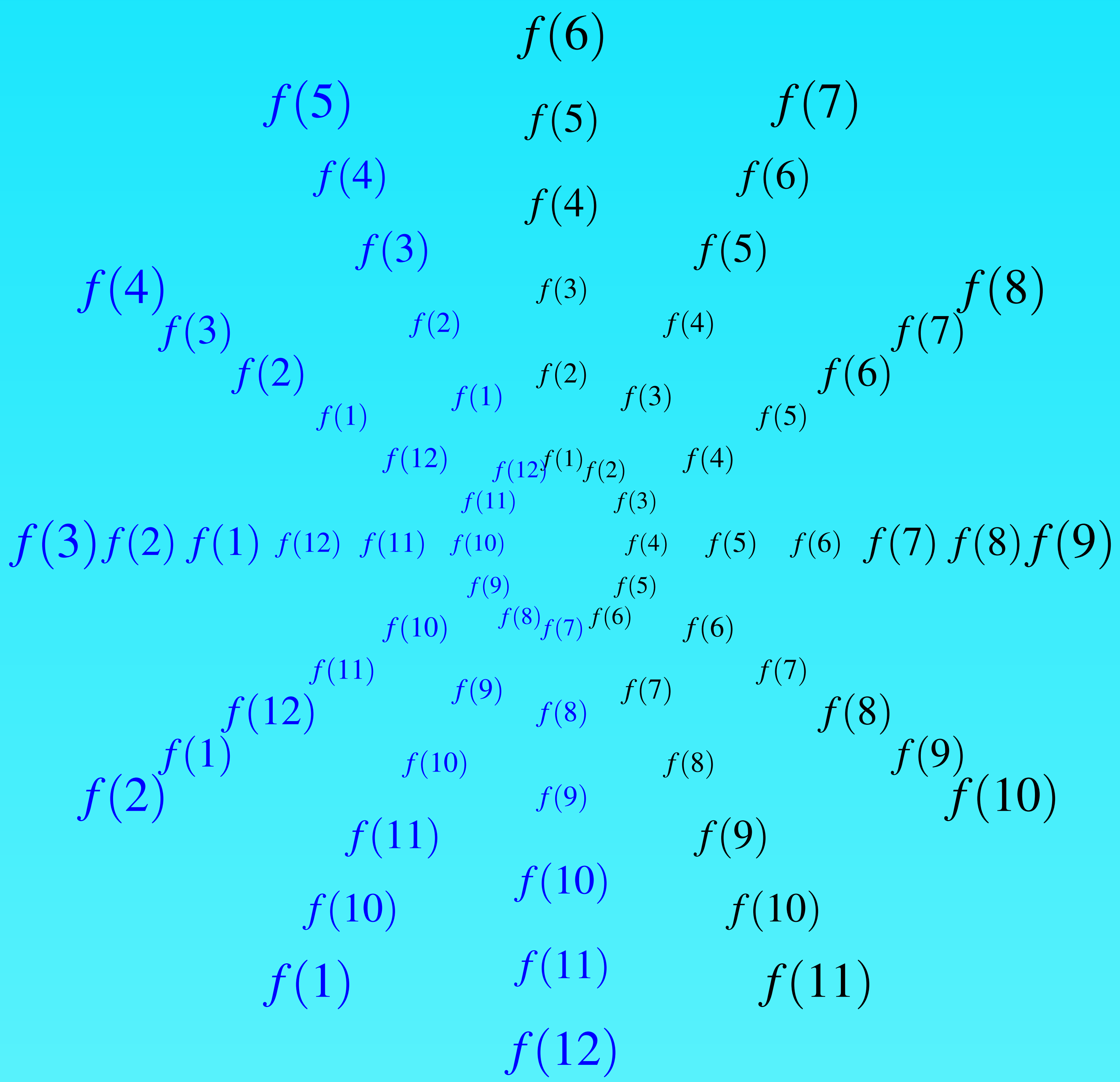
Theorem. *There exists a tone row f so that $\sigma\colon \{1,\ldots,6\} \to \{1,\ldots,35\}$ is the trope number sequence of f, if and only if for $1 \leq r \leq 6$ there exists a representative τ_r of the $\sigma(r)$-th D_{12}-orbit of tropes, so that*

- *τ_r and τ_{r+1} are connectable with the moving pair $\{i_r, j_r\}$, $1 \leq r \leq 5$, and*
- *τ_6 and τ_1 are connectable with the moving pair $\{i_6, j_6\}$, and*
- *each element of $\mathbb{Z}_{12}$ is moving exactly once, i.e.*

$$\bigcup_{r=1}^{6} \{i_r, j_r\} = \mathbb{Z}_{12}.$$


f(6)
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f(4) f(6)
f(4)
f(3) f(5)
f(4) f(3) f(8)
f(2) f(4)
f(3) f(7)
f(2) f(2) f(6)
f(1) f(3)
f(1) f(5)
f(12) f(4)
f(12) f(1) f(2)
f(11) f(3)
f(3) f(2) f(1) f(12) f(11) f(10) f(4) f(5) f(6) f(7) f(8) f(9)
f(9) f(5)
f(8) f(6)
f(10) f(7) f(6)
f(11) f(7)
f(9) f(7)
f(12) f(8) f(8)
f(1) f(9)
f(10) f(8)
f(2) f(9) f(10)
f(11) f(9)
f(10)
f(10) f(10)
f(1) f(11) f(11)
f(12)

Let f be a tone row.

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Trope number sequence: Replace the tropes by the numbers of their D_{12}-orbits, $s_f\colon \{1,\ldots,6\} \to \{1,\ldots,35\}$, $s_f(i)$ is the number of the orbit $D_{12}(\tau_i)$, $1 \leq i \leq 6$.

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We associate the $D_{12} \times D_{12}$-orbit of the tone row f with the D_{12}-orbit of s_f where the dihedral group D_{12} acts on the domain of s_f. We call this orbit of s_f the ***trope structure*** of the orbit $(D_{12} \times D_{12})(f)$.

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From the database it is possible to deduce that there are 538 139 different trope structures. There are trope structures, e.g. (1,1,1,1,1,1), which determine a unique $D_{12} \times D_{12}$-orbit of tone rows. But there exist also two trope structures namely (10,18,22,14,22,18) and (10,18,22,14,22,27) which belong to 48 $D_{12} \times D_{12}$-orbits of tone rows.

Database: Compute the trope structure or Search for the trope structure

Tropes and stabilizers of tone rows

There is a close connection between the stabilizer type of a tone row and its trope structure. E.g.

Theorem. *Let f be a tone row. The pair (T^6, R) belongs to the stabilizer of f, if and only if the following assertions hold true.*

- *f has exactly four different trope numbers.*
- *The trope number sequence of f is of the form $(t_1, t_2, t_3, t_4, t_3, t_2)$, where t_1 belongs to $\{1, 8, 14, 31, 34\}$, which are the numbers of those tropes $\{A, A'\}$ so that $T^6(A) = A'$, and t_4 is an element of $\{25, 32, 35\}$, which are the numbers of those tropes so that $T^6(A) = A$.*
- *There exists a trope sequence $(\tau_1, \ldots, \tau_6)$ where τ_r belongs to the t_r-th D_{12}-orbit of tropes, $1 \leq r \leq 6$, which satisfies the properties of Theorem and $\tau_6 = T^6(\tau_2)$ and $\tau_5 = T^6(\tau_3)$.*

The database

Data were computed with SYMMETRICA [9] and GAP [8].

Search routines using perl. Graphics uses javascript.

Database: Database on tone rows and tropes

1. We will check for Le Merle Noir.

2. We determine all information on this tone row.

3. Search for all-interval rows and retrieve musical information on those used by Alban Berg. Who else used similar tone rows?

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