Tone rows and tropes

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September 3, 2013, General Mathematics Seminar
Mathematics Research Unit, University of Luxembourg
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1. Introduction: Pitch classes, tone rows, similarity operations.
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3. Orbits of tone rows, stabilizers of tone rows, interval structure, trope structure.
4. The database.
Pitch and Scale

A tone in music is described by its fundamental frequency $f > 0$, which we call its pitch. It is usually given in Hertz (Hz), which is defined as the number of periodic cycles of a sine wave within a second. Two tones with frequencies $f$ and $2f$ form the interval of an octave. In well tempered music (or equal temperament) an octave is divided into 12 equal parts. We speak of a 12-scale. Therefore the frequencies $f_i$, $1 \leq i \leq 11$, of the 11 tones between $f$ and $2f$ would be $f \cdot 2^{i/12}$. (The factor $2^{1/12} = \sqrt[12]{2}$ describes the frequency ratio of a semi tone interval.)
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Disregarding the fact that human beings can hear tones only in the range from 20 Hz to 20000 Hz, in general the set of all tones in a 12-scale (which contains a tone with frequency $f$) is countably infinite and is given by $\{f \cdot 2^{k/12} \mid k \in \mathbb{Z}\}$. Since we are not interested in the particular frequencies we omit the factor $f$ and each tone is represented by an integer $k$. Consequently, $\mathbb{Z}$ is a model of a 12-scale.
Pitch classes

From the musical perception we deduce that tones which are an integer multiple of an octave apart have a similar quality. We speak of octave equivalence. In music analysis and 12-tone composition usually it is not important which octave a certain tone belongs to, therefore, tones being a whole number of octaves apart are considered to be equivalent and are collected to a pitch class. E.g., if $a'$ is the tone with frequency $f = 440\text{Hz}$, then the pitch class of $a'$ consists of all tones . . . , A, a, a', a'', a''', . . . These are the tones with frequencies $f \cdot 2^k\text{Hz} = f \cdot 2^{12k/12}\text{Hz}, k \in \mathbb{Z}$. Let $n$ be an integer, then by $\bar{n}$ we denote the subset \( \{12k + n \mid k \in \mathbb{Z}\} \). It is the residue class of $n$ modulo 12. Of course $\bar{n} = n + 12k$ for any $k \in \mathbb{Z}$. 
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Using \( \mathbb{Z} \) as the model of a 12-scale the twelve pitch classes are the subsets \( \bar{i} \) for \( 0 \leq i < 12 \). It is now clear that the pitch classes in the 12-scale \( \mathbb{Z} \) coincide with the residue classes in \( \mathbb{Z}_{12} := \mathbb{Z} \mod 12\mathbb{Z} \).
The chromatic scale

Here is a short part of the chromatic scale together with the labelling of the tones in $\mathbb{Z}$ and pitch classes in $\mathbb{Z}_{12}$. 

![Chromatic Scale Diagram]
A **tone row** is a sequence of 12 tones so that tones in different positions belong to different pitch classes. Therefore, we describe a tone row by a mapping

$$f: \{1, \ldots, 12\} \rightarrow \mathbb{Z}, \quad f(i) \neq f(j), \quad i \neq j,$$

where $\overline{f(i)}$ is the residue class of $f(i), i \in \{1, \ldots, 12\}$. The set $\{1, \ldots, 12\}$ is the set of all order numbers or time positions. The value $f(i), i \in \{1, \ldots, 12\}$ is the tone in $i$-th position of the tone row $f$. 
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where \( f(i) \) is the residue class of \( f(i), \ i \in \{1,\ldots,12\} \). The set \( \{1,\ldots,12\} \) is the set of all order numbers or time positions. The value \( f(i), \ i \in \{1,\ldots,12\} \) is the tone in \( i \)-th position of the tone row \( f \).

Since different tones in \( f \) must belong to different pitch classes and since the actual choice of \( f(i) \) in its pitch class is not important, we consider tone rows as functions \( f: \{1,\ldots,12\} \rightarrow \mathbb{Z}_{12} \). A function \( f: \{1,\ldots,12\} \rightarrow \mathbb{Z}_{12} \) is a tone row, if and only if \( f \) is bijective. Therefore, the set of all tone rows coincides with the set of all bijective functions from \( \{1,\ldots,12\} \) to \( \mathbb{Z}_{12} \).

This leads to a total of \( 12! = 12 \cdot 11 \cdot \ldots \cdot 2 \cdot 1 = 479001600 \) tone rows.
O. Messiaen: Le Merle Noir

A piece for flute and piano composed in 1951. It is not a standard example. The tone row appears in the coda of the piano part. (Min 5.06)
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Reduction to pitch classes \((9, 2, 8, 3, 10, 6, 4, 0, 1, 11, 5, 7)\).
Circular representation of a tone row

In order to present a tone row as a graph

Example Tone row of Le Merle Noir

\[ f := (f(1), \ldots, f(12)) = (9, 2, 8, 3, 10, 6, 4, 0, 1, 11, 5, 7) : \]
Circular representation of a tone row

In order to present a tone row as a graph we draw the 12 pitch classes as a regular 12-gon,

**Example** Tone row of Le Merle Noir

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Circular representation of a tone row

In order to present a tone row as a graph we draw the 12 pitch classes as a regular 12-gon, we label them,

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In order to present a tone row as a graph, we draw the 12 pitch classes as a regular 12-gon, we label them, we indicate the first tone, and we connect pitch classes which occur in consecutive position in the tone row.

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![Diagram](image)
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\[ f := (f(1), \ldots, f(12)) = (9, 2, 8, 3, 10, 6, 4, 0, 1, 11, 5, 7) : \]
Transposing a tone row

Transposing a tone row by a semi tone is moving each tone one semi tone up. This operation is also possible for pitch classes. Define

\[ T: \mathbb{Z}_{12} \to \mathbb{Z}_{12} : i \mapsto i + 1 \]

then the transposed of a tone row \( f \) by \( k \) semitones, \( k \in \mathbb{Z} \), is \( T^k \circ f \).
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then the transposed of a tone row \( f \) by \( k \) semitones, \( k \in \mathbb{Z} \), is \( T^k \circ f \). E.g. the transposed of \( f = (9, 2, 8, 3, 10, 6, 4, 0, 1, 11, 5, 7) \) is \( T \circ f = (10, 3, 9, 4, 11, 7, 5, 1, 2, 0, 6, 8) \).
Transposing means a rotation of the circular representation.
Inversion of a tone row

There exist two inversion operators. Operators of the first kind fix a single tone $t_0$. For any tone $t_1 \neq t_0$ there exists a positive integer $r$ so that $t_1$ is either $r$ semitones higher or $r$ semitones lower than $t_0$. By inversion $t_1$ is mapped onto the tone $t_2$ which is exactly $r$ semitones lower, or higher than $t_0$. 
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For operators of the second kind there exist two tones which are exactly one semitone apart. They will be interchanged by this operator. We call them $t_0$ and $t_0 + 1$. For any tone $t_1 \notin \{t_0, t_0 + 1\}$ there exists a positive integer $r$ so that $t_1$ is either $r$ semitones higher than $t_0 + 1$ or $r$ semitones lower than $t_0$. By inversion $t_1$ is mapped onto the tone $t_2$ which is exactly $r$ semitones lower than $t_0$, or exactly $r$ semitones higher than $t_0 + 1$. 
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Define inversion at pitch class 0 by

\[
I: \mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12} : i \mapsto -i.
\]
Moreover, $I \circ T^k = T^{-k} \circ I$, $k \in \{0, \ldots, 11\}$. All inversion operators on $\mathbb{Z}_{12}$ can be written as compositions $T^r \circ I$ for $r \in \{0, \ldots, 11\}$. If $r$ is even, then $T^r \circ I$ consists of exactly two fixed points and five cycles of length two, otherwise it consists of six cycles of length two.
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Inversion of a tone row $f$ at 0 is defined as $I \circ f$. 
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Inversion of a tone row $f$ at 0 is defined as $I \circ f$. E.g. the transposed of $f = (9, 2, 8, 3, 10, 6, 4, 0, 1, 11, 5, 7)$

is $I \circ f = (3, 10, 4, 9, 2, 6, 8, 0, 11, 1, 7, 5)$
The circular representation of the inversion of $f$ is the mirror image of the circular representation of $f$. 

![Circular representation of f and its inversion](image.png)
Retrograde and cyclic shift of a tone row

Consider the permutations $R = (1, 12)(2, 11) \cdots (6, 7)$ and $S = (1, 2, 3, \ldots, 12)$, then the retrograde of the tone row $f$ is $f \circ R$ and the cyclic shift of $f$ is $f \circ S$. 
Retrograde and cyclic shift of a tone row

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\[
\begin{align*}
f &= (9, 2, 8, 3, 10, 6, 4, 0, 1, 11, 5, 7), \\
f \circ S &= (2, 8, 3, 10, 6, 4, 0, 1, 11, 5, 7, 9) \\
\text{and} \\
f \circ R &= (7, 5, 11, 1, 0, 4, 6, 10, 3, 8, 2, 9).
\end{align*}
\]
Permutation groups

The groups $\langle T, I \rangle$ and $\langle S, R \rangle$ are permutation groups, both isomorphic to the dihedral group $D_{12}$ consisting of 24 elements.
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Theorem. Let $\pi$ be a permutation of $\mathbb{Z}_{12}$, then $\pi(i + 1) = \pi(i) + 1$ for all $i \in \mathbb{Z}_{12}$, or $\pi(i + 1) = \pi(i) - 1$ for all $i \in \mathbb{Z}_{12}$, if and only if $\pi \in D_{12}$. 
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We consider also the quart-circle $Q$ defined by $Q: \mathbb{Z}_{12} \to \mathbb{Z}_{12}, i \mapsto 5i$.

It replaces a chromatic scale by a sequence of quarts, the quint-circle $I \circ Q$ by a sequence of quints, since $(I \circ Q)(i) = 7i, i \in \mathbb{Z}_{12}$.

The group $\langle T, I, Q \rangle$ is the group of all affine mappings on $\mathbb{Z}_{12}$ which we abbreviate by $\text{Aff}_1(\mathbb{Z}_{12})$. It is the set of all mappings $f(i) = ai + b$, where $a, b \in \mathbb{Z}$, $\gcd(a, 12) = 1$. Thus $a \in \{1, 5, 7, 11\}$ and $b \in \{0, \ldots, 11\}$.

The same operation on the order numbers is the fife-step $F$. 
Equivalent tone rows

A tone row $f'$ is considered to be **equivalent** to a tone row $f$ if $f'$ can be constructed from $f$ by any combination of transposing, inversion, cyclic shift and retrograde.
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Let $\mathcal{R}$ be the set of all tone rows, i.e. bijective mappings from $\{1, \ldots, 12\}$ to $\mathbb{Z}_{12}$. We consider the following mapping

$$(\langle T, I \rangle \times \langle S, R \rangle) \times \mathcal{R} \rightarrow \mathcal{R}$$

$$(\varphi, \pi), f) \mapsto \varphi \circ f \circ \pi^{-1}. \quad (*)$$

This mapping defines a group action.
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A tone row $f'$ is equivalent to $f$ if and only if $f' = \varphi \circ f \circ \pi^{-1}$ for some $(\varphi, \pi)$ in $\langle T, I \rangle \times \langle S, R \rangle$. 

E.g. The retrograde of the inversion of the tone row in Le Merle Noir is \( I \circ f \circ R = (5, 7, 1, 11, 0, 8, 6, 2, 9, 4, 10, 3) \).
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Circular representation of all equivalent rows: We start with $f$. 

![Circular representation of all equivalent rows](image)
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Circular representation of all equivalent rows: We start with $f$. Because of transposing we delete the labels.
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Circular representation of all equivalent rows: We start with $f$. Because of transposing we delete the labels. Inversion of $f$ is the mirror of the given graph. Because of retrograde we don’t show the first tone. Because of cyclic shifts we insert the missing edge.

Database: The equivalence class of $(9, 2, 8, 3, 10, 6, 4, 0, 1, 11, 5, 7)$
A multiplicative group $G$ with neutral element 1 acts on a set $X$ if there exists a mapping

$$*: G \times X \rightarrow X, \quad *(g, x) \mapsto g \ast x (= gx),$$
A multiplicative group $G$ with neutral element $1$ acts on a set $X$ if there exists a mapping

$$\star : G \times X \rightarrow X, \quad \star(g, x) \rightarrow g \star x (= gx),$$

such that

$$(g_1 g_2) \star x = g_1 \star (g_2 \star x), \quad g_1, g_2 \in G, \ x \in X,$$
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and

$$1 \ast x = x, \quad x \in X.$$
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Notation: We usually write $gx$ instead of $g \ast x$, and $\bar{g}: X \rightarrow X, \bar{g}(x) = gx.$
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If $G$ and $X$ are finite sets, then we speak of a **finite group action**.
Orbits under Group Actions

A group action $\gamma X$ defines the following equivalence relation on $X$. $x_1 \sim x_2$ if and only if there is some $g \in G$ such that $x_2 = gx_1$. 
A group action $G X$ defines the following equivalence relation on $X$. $x_1 \sim x_2$ if and only if there is some $g \in G$ such that $x_2 = gx_1$. The equivalence classes $G(x)$ with respect to $\sim$ are the **orbits** of $G$ on $X$. Hence, the orbit of $x$ under the action of $G$ is

$$G(x) = \{gx \mid g \in G\}.$$
Orbits under Group Actions

A group action $G \times X$ defines the following equivalence relation on $X$. $x_1 \sim x_2$ if and only if there is some $g \in G$ such that $x_2 = gx_1$. The equivalence classes $G(x)$ with respect to $\sim$ are the **orbits** of $G$ on $X$. Hence, the orbit of $x$ under the action of $G$ is

$$G(x) = \{ gx \mid g \in G \}.$$ 

The set of orbits of $G$ on $X$ is indicated as

$$G \backslash X := \{ G(x) \mid x \in X \}.$$
Orbits under Group Actions

A group action $G X$ defines the following equivalence relation on $X$. $x_1 \sim x_2$ if and only if there is some $g \in G$ such that $x_2 = gx_1$. The equivalence classes $G(x)$ with respect to $\sim$ are the orbits of $G$ on $X$. Hence, the orbit of $x$ under the action of $G$ is

$$G(x) = \{gx \mid g \in G\}.$$ 

The set of orbits of $G$ on $X$ is indicated as

$$G \backslash X := \{G(x) \mid x \in X\}.$$ 

**Theorem.** The equivalence classes of any equivalence relation (or the blocks of any partition) can be represented as orbits under a suitable group action.
Let $G^X$ be a group action. For each $x \in X$ the stabilizer $G_x$ of $x$ is the set of all group elements which do not change $x$, in other words

$$G_x := \{ g \in G \mid gx = x \}.$$ 

It is a subgroup of $G$. 
Let \( G \times X \) be a group action. For each \( x \in X \) the **stabilizer** \( G_x \) of \( x \) is the set of all group elements which do not change \( x \), in other words

\[
G_x := \{ g \in G \mid gx = x \}.
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It is a subgroup of \( G \).

**Lagrange Theorem.** If \( G \times X \) is a finite group action then the size of the orbit of \( x \in X \) equals

\[
|G(x)| = \frac{|G|}{|G_x|}.
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**Lagrange Theorem.** If \( G \times X \) is a finite group action then the size of the orbit of \( x \in X \) equals

\[
|G(x)| = \frac{|G|}{|G_x|}.
\]

Finally, the **set of all fixed points** of \( g \in G \) is denoted by

\[
X_g := \{ x \in X \mid gx = x \}.
\]
Let $G \times X$ be finite group action. The main tool for determining the number of different orbits is the

**Lemma of Cauchy–Frobenius.**
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Enumeration under Group Actions

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**Proof.**

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\[
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\]

**Proof.**

\[
\sum_{g \in G} |X_g| = \sum_{g \in G} \sum_{x : gx = x} 1
\]
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\]

\[
= |G| \sum_{x \in X} \frac{1}{|G(x)|} = |G| \sum_{\omega \in G \setminus X} \sum_{x \in \omega} \frac{1}{|\omega|} =
\]


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**Proof.**

$$\sum_{g \in G} |X_g| = \sum_{g \in G} \sum_{x \in X : gx = x} 1 = \sum_{x \in X} \sum_{g : gx = x} 1 = \sum_{x \in X} |G_x| =$$

$$= |G| \sum_{x \in X} \frac{1}{|G(x)|} = |G| \sum_{\omega \in G \backslash X} \sum_{x \in \omega} \frac{1}{|\omega|} = |G||G \backslash X|.$$
Symmetry types of mappings

The most important applications of classification under group actions can be described as operations on the set of mappings between two sets. Group actions on the domain $X$ or range $Y$ induce group actions on $Y^X$. 
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Then $G$ acts on $Y^X$ by

$$G \times Y^X \rightarrow Y^X, \quad (g, f) \mapsto f \circ \bar{g}^{-1}.$$
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— Then $G$ acts on $Y^X$ by

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$$G \times Y^X \to Y^X, \quad (g, f) \mapsto f \circ \bar{g}^{-1}.$$ 

— Then $H$ acts on $Y^X$ by

$$H \times Y^X \to Y^X, \quad (h, f) \mapsto \bar{h} \circ f.$$ 

— Then the direct product $H \times G$ acts on $Y^X$ by

$$(H \times G) \times Y^X \to Y^X, \quad ((h, g), f) \mapsto \bar{h} \circ f \circ \bar{g}^{-1}.$$
Bibliography on combinatorial methods using group actions


Classification of tone rows

Equivalent tone rows are collected in one orbit of tone rows under the group action of \((\ast)\). If we know all the orbits then we know all non-equivalent tone rows.

How many orbits are there?
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How many orbits are there?

<table>
<thead>
<tr>
<th>acting group</th>
<th># of orbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle T \rangle \times \langle R \rangle$</td>
<td>19,960,320</td>
</tr>
<tr>
<td>$\langle T, I \rangle \times \langle R \rangle$</td>
<td>9,985,920</td>
</tr>
<tr>
<td>$\langle T \rangle \times \langle S \rangle$</td>
<td>3,326,788</td>
</tr>
<tr>
<td>$\langle T, I \rangle \times \langle S \rangle$</td>
<td>1,664,354</td>
</tr>
<tr>
<td>$\langle T, I \rangle \times \langle S, R \rangle$</td>
<td>836,017</td>
</tr>
<tr>
<td>$\langle T, I, Q \rangle \times \langle S, R \rangle$</td>
<td>419,413</td>
</tr>
<tr>
<td>$\langle T, I \rangle \times \langle S, R, F \rangle$</td>
<td>419,413</td>
</tr>
<tr>
<td>$\langle T, I, Q \rangle \times \langle S, R, F \rangle$</td>
<td>211,012</td>
</tr>
</tbody>
</table>
Arnold Schöenberg considered tone rows as equivalent if they can be determined by transposing, inversion and/or retrograde from a single tone row. Thus, his model corresponds to the settings in 2.
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According to Theorem the dihedral group is the biggest group which preserves the neighbor relations in $\mathbb{Z}_{12}$. Therefore, we consider the settings of 5. as the standard settings for our classification. There is also big evidence that Josef Matthias Hauer was considering this equivalence relation. Also Ron C. Read considers in [7, page 546] this notion as the natural equivalence relation on the set of all tone rows. He also determines 836 017 as the number of pairwise non-equivalent tone-rows. Previously, this number was already determined in a geometric problem by Solomon W. Golomb and Lloyd R. Welch in [4]. In their manuscript [5] David J. Hunter and Paul T. von Hippel also consider the cyclic shift as a symmetry operation for tone rows.
The orbit of a tone row

In connection with the orbit of $f$ we solve the following problems:
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- Determine the \textit{set of all elements} of the orbit $G(f)$. 
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- Given two tone rows $f_1$ and $f_2$ belonging to the same orbit, determine an element $g \in G$ so that $f_2 = gf_1$. 
The normal form

We define a **total order** on $\mathbb{Z}_{12}$ by assuming that $0 < 1 < \ldots < 11$. We represent the tone row $f$ as a vector of length 12 of the form $(f(1), \ldots, f(12))$. Using this total order, the set of tone rows written as vectors is totally ordered by the **lexicographical order**. We say the tone row $f_1$ is smaller than the tone row $f_2$, and we write $f_1 < f_2$, if there exists an integer $i \in \{1, \ldots, 12\}$ so that $f_1(i) < f_2(i)$ and $f_1(j) = f_2(j)$ for all $1 \leq j < i$. 
The normal form

We define a **total order** on $\mathbb{Z}_{12}$ by assuming that $\overline{0} < \overline{1} < \ldots < \overline{11}$. We represent the tone row $f$ as a vector of length 12 of the form $(f(1), \ldots, f(12))$. Using this total order, the set of tone rows written as vectors is totally ordered by the **lexicographical order**. We say the tone row $f_1$ is smaller than the tone row $f_2$, and we write $f_1 < f_2$, if there exists an integer $i \in \{1, \ldots, 12\}$ so that $f_1(i) < f_2(i)$ and $f_1(j) = f_2(j)$ for all $1 \leq j < i$.

Given a group $G$ which describes the equivalence of tone rows and a tone row $f$, we compute the orbit $G(f)$ of $f$ by applying all elements of $G$ to $f$. By doing this we obtain the set $\{gf \mid g \in G\}$ which contains at most $|G|$ tone rows. As the **standard representative** of this orbit, or as the **normal form** of $f$, we choose the smallest element in $G(f)$ with respect to the lexicographical order.

Database: The normal form $(9,2,8,3,10,6,4,0,1,11,5,7)$
The stabilizer type of a tone row

Let $f$ be a tone row and let $G$ be a group describing the equivalence classes of tone rows. From Theorem we already know that the size of the orbit of $f$ depends on the size of its stabilizer $G_f$. The stabilizer $G_f$ is a subgroup of $G$. 
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The **stabilizer type** of the orbit $G(f)$ is the conjugacy class $\tilde{G}_f = \{gGfg^{-1} \mid g \in G\}$, since the stabilizer of $gf$ is $gGfg^{-1}$, $g \in G$. 

The stabilizer type of a tone row

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The **stabilizer type** of the orbit \( G(f) \) is the conjugacy class
\[
\tilde{G}_f = \{ gG_fg^{-1} \mid g \in G \},
\]
since the stabilizer of \( gf \) is \( gGfg^{-1} \), \( g \in G \).

In our standard situation we have 17 different stabilizer types. We list all these stabilizer types \( \tilde{U}_i \) and the number of \( D_{12} \times D_{12} \)-orbits of tone rows which have stabilizer type \( \tilde{U}_i \). These numbers were computed by applying Burnside’s Lemma. For doing this, we used the computer algebra system GAP.
Examples:

- Consider \( f = (0, 1, 10, 8, 9, 11, 5, 3, 2, 4, 7, 6) \), then \( T^6 f = (6, 7, 4, 2, 3, 5, 11, 9, 8, 10, 1, 0) \) and \( f R = (6, 7, 4, 2, 3, 5, 11, 9, 8, 10, 1, 0) \), thus \( (T^6, R) \ast f = f \).
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- Consider \( f = (5, 4, 0, 7, 3, 2, 8, 9, 1, 6, 10, 11) \), then \( TI f = (8, 9, 1, 6, 10, 11, 5, 4, 0, 7, 3, 2) = fS^6 \) and \( T^6 f = (11, 10, 6, 1, 9, 8, 2, 3, 7, 0, 4, 5) = fR \). The stabilizer of \( f \) is \( \{ \text{id}, (TI, S^6), (T^6, R), (T^7 I, S^6 R) \} \).
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- The chromatic scale has a stabilizer of order 24.

Database: The stabilizer of the chromatic scale
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• The chromatic scale has a stabilizer of order 24.

Database: The stabilizer of the chromatic scale

• 99.93\% of all orbits of tone rows have the trivial stabilizer type.
<table>
<thead>
<tr>
<th>name</th>
<th>generators</th>
<th>$U$</th>
<th>$\tilde{U}$</th>
<th>$\tilde{U}_i \parallel R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{U}_1$</td>
<td>identity</td>
<td>1</td>
<td>1</td>
<td>827 282</td>
</tr>
<tr>
<td>$\tilde{U}_2$</td>
<td>$(TI, S^6)$</td>
<td>2</td>
<td>6</td>
<td>912</td>
</tr>
<tr>
<td>$\tilde{U}_3$</td>
<td>$(T^6, R)$</td>
<td>2</td>
<td>3</td>
<td>912</td>
</tr>
<tr>
<td>$\tilde{U}_4$</td>
<td>$(T^6, S^6)$</td>
<td>2</td>
<td>1</td>
<td>130</td>
</tr>
<tr>
<td>$\tilde{U}_5$</td>
<td>$(I, SR)$</td>
<td>2</td>
<td>36</td>
<td>942</td>
</tr>
<tr>
<td>$\tilde{U}_6$</td>
<td>$(TI, R)$</td>
<td>2</td>
<td>36</td>
<td>5649</td>
</tr>
<tr>
<td>$\tilde{U}_7$</td>
<td>$(T^4, S^4)$</td>
<td>3</td>
<td>2</td>
<td>11</td>
</tr>
<tr>
<td>$\tilde{U}_8$</td>
<td>$(T^3, S^3)$</td>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\tilde{U}_9$</td>
<td>$(TI, S^6), (T^6, R)$</td>
<td>4</td>
<td>36</td>
<td>96</td>
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<tr>
<td>$\tilde{U}_{10}$</td>
<td>$(I, SR), (T^6, S^6)$</td>
<td>4</td>
<td>18</td>
<td>12</td>
</tr>
<tr>
<td>$\tilde{U}_{11}$</td>
<td>$(TI, R), (T^6, S^6)$</td>
<td>4</td>
<td>18</td>
<td>42</td>
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<tr>
<td>$\tilde{U}_{12}$</td>
<td>$(I, SR), (T^4, S^4)$</td>
<td>6</td>
<td>24</td>
<td>2</td>
</tr>
<tr>
<td>$\tilde{U}_{13}$</td>
<td>$(TI, R), (T^4, S^4)$</td>
<td>6</td>
<td>24</td>
<td>15</td>
</tr>
<tr>
<td>$\tilde{U}_{14}$</td>
<td>$(I, SR), (T^3, S^3)$</td>
<td>8</td>
<td>36</td>
<td>6</td>
</tr>
<tr>
<td>$\tilde{U}_{15}$</td>
<td>$(TI, R), (T^2, S^2)$</td>
<td>12</td>
<td>12</td>
<td>2</td>
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<tr>
<td>$\tilde{U}_{16}$</td>
<td>$(I, SR), (T, S)$</td>
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<td>12</td>
<td>1</td>
</tr>
<tr>
<td>$\tilde{U}_{17}$</td>
<td>$(I, SR), (T, S^5)$</td>
<td>24</td>
<td>12</td>
<td>1</td>
</tr>
</tbody>
</table>

Database: Search for tone rows of given stabilizer type
The interval structure

The **interval** from $a$ to $b$ in $\mathbb{Z}_{12}$ is the difference $b - a \in \mathbb{Z}_{12}$. This is the minimum number of steps in clockwise direction from $a$ to $b$ in the regular 12-gon. The tone row $f$ determines the following sequence of eleven intervals

$$(f(2) - f(1), f(3) - f(2), \ldots, f(12) - f(11)). \tag{**}$$

Since we consider a tone-row as a closed polygon we also have to add the closing interval $f(1) - f(12)$. Consequently, the **interval structure** of the tone-row $f: \{1, \ldots, 12\} \to \mathbb{Z}_{12}$ is the function $g: \{1, \ldots, 12\} \to \mathbb{Z}_{12} \setminus \{0\}$, defined by

$$g(i) := \begin{cases} 
    f(i + 1) - f(i) & \text{for } 1 \leq i \leq 11 \\
    f(1) - f(12) & \text{for } i = 12.
\end{cases}$$
The interval structure of \( f = (9, 2, 8, 3, 10, 6, 4, 0, 1, 11, 5, 7) \)

is \( g = (5, 6, 7, 7, 8, 10, 8, 1, 10, 6, 2, 2) \).

To the \( D_{12} \times D_{12} \)-orbit of the tone row \( f \) corresponds to the \( \langle I \rangle \times D_{12} \)-orbit of the interval structure \( g \). It is given by its standard representative \((1, 8, 10, 8, 7, 7, 6, 5, 2, 2, 6, 10)\).

Database: Search for tone rows of given interval structure
A tone row \( f \) is called an **all-interval row** if all elements of \( \mathbb{Z}_{12} \setminus \{0\} \) occur in the sequence (**`). Then each element of \( \mathbb{Z}_{12} \setminus \{0\} \) occurs exactly once in (**`). Hence, \( \{g(j) \mid 1 \leq j \leq 11\} = \{1, \ldots, 11\} \) and, therefore,

\[
g(12) = - \sum_{j=1}^{11} g(j) = - \sum_{i \in \mathbb{Z}_{12} \setminus \{0\}} i = 6.
\]

Generalizing this notion, the \( D_{12} \times D_{12} \)-orbit of \( f \) contains all-interval rows if and only if each possible value occurs in the interval structure of \( f \). In this situation the interval 6 occurs exactly twice and all other intervals exactly once in the interval structure of \( f \). Thus the interval type of \( f \) looks like \((1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1)\). In this situation we call the orbit \( D_{12} \times D_{12}(f) \) an all-interval orbit.
Example:

Consider the vector \((1, -2, 3, -4, 5, -6, 7, -8, 9, -10, 11, 6)\) in \(\mathbb{Z}_{12}\) which is the interval structure \((1, 10, 3, 8, 5, 6, 7, 4, 9, 2, 11, 6)\) of the tone row \((0, 1, 11, 2, 10, 3, 9, 4, 8, 5, 7, 6)\). This is an all interval-row.

On enumeration and listing of all-interval rows see [2, 6, 1, 3].

Database: Search for all-interval rows
Orbits of hexachords

A **hexachord** in the 12-scale \( \mathbb{Z}_{12} \) is a 6-subset of \( \mathbb{Z}_{12} \). There exist \( \binom{12}{6} = 924 \) different hexachords in the set \( \mathcal{H} = \{ A \subset \mathbb{Z}_{12} \mid |A| = 6 \} \).

If a group \( G \) acts on \( \mathbb{Z}_{12} \), then the induced action of \( g \in G \) on a \( k \)-subset \( A \) of \( \mathbb{Z}_{12} \), \( k \in \{1, \ldots, 12\} \), is given by \( g \ast A := \{ g \ast a \mid a \in A \} \).

The number of orbits of hexachords under the action of a group is easily computed by an application of Pólya’s Theorem. We get

| \( G \)      | \( |G \setminus \mathcal{H}| \) |
|-------------|-------------------------------|
| \( C_{12} \) | 80                            |
| \( D_{12} \) | 50                            |
| \( \text{Aff}_1(\mathbb{Z}_{12}) \) | 34                           |

Database: **List all \( k \)-chords**
Tropes

Let $A$ be a hexachord, then its complement $A' := \mathbb{Z}_{12} \setminus A$ is also a hexachord. Now we consider “pairs” $\{A, A'\}$ of hexachords which we call tropes. (We use quotation marks around the word pair, since $\{A, A'\}$ is actually not a pair, but a 2-set of hexachords!)

The set of all tropes will be indicated by $\mathcal{T} := \{\{A, \mathbb{Z}_{12} \setminus A\} | A \in \mathcal{H}\}$. In total there exist $924/2 = 462$ tropes in the 12-scale. If a group $G$ acts on $\mathbb{Z}_{12}$, then the induced action of $g \in G$ on a trope $\{A, A'\}$ is given by $g \ast \{A, A'\} := \{g \ast A, g \ast A'\}$.

Number of orbits of tropes:

|     | $|G\backslash\mathcal{T}|$ |
|-----|-------------------|
| $C_{12}$ | 44                             |
| $D_{12}$ | 35                             |
| $\text{Aff}_1(\mathbb{Z}_{12})$ | 26                             |

Database: List all tropes
List of all 35 orbits of tropes

1 [000000111111]
2 [000001011111]
3 [000001101111]
4 [000001110111]
5 [000010101111]
6 [000010110111]
7 [000010111011]
8 [000010111101]
9 [000011001111]
10 [000011010111]
11 [000011011011]
12 [000011100111]
13 [000100110111]
14 [000100111011]
15 [000101010111]
16 [000101011011]
17 [000101011101]
18 [000101100111]
19 [000101101011]
20 [000101101101]
Consider a tone row $f$. We obtain in a natural way six “pairs” of hexachords defined by $f$:

\[ \tau_1 := \{ \{f(1), f(2), f(3), f(4), f(5), f(6)\}, \{f(7), f(8), f(9), f(10), f(11), f(12)\} \} \]

\[ \tau_2 := \{ \{f(2), f(3), f(4), f(5), f(6), f(7)\}, \{f(8), f(9), f(10), f(11), f(12), f(1)\} \} \]

\[ \tau_3 := \{ \{f(3), f(4), f(5), f(6), f(7), f(8)\}, \{f(9), f(10), f(11), f(12), f(1), f(2)\} \} \]

\[ \tau_4 := \{ \{f(4), f(5), f(6), f(7), f(8), f(9)\}, \{f(10), f(11), f(12), f(1), f(2), f(3)\} \} \]

\[ \tau_5 := \{ \{f(5), f(6), f(7), f(8), f(9), f(10)\}, \{f(11), f(12), f(1), f(2), f(3), f(4)\} \} \]

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Trope structure of a tone row

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\end{align*}
\]

From $\tau_i$ to $\tau_{i+1}$ exactly 2 pitch classes are changing their hexachord. They form the \textit{moving pair} $\{f(i), f(i+6)\}$. The two tropes $\tau_i$ and $\tau_{i+1}$ are called \textit{connectable}.
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\end{align*}
\]

From $\tau_i$ to $\tau_{i+1}$ exactly 2 pitch classes are changing their hexachord. They form the \textbf{moving pair} $\{f(i), f(i + 6)\}$. The two tropes $\tau_i$ and $\tau_{i+1}$ are called \textit{connectable}.
Theorem. There exists a tone row $f$ so that $\sigma: \{1, \ldots, 6\} \rightarrow \{1, \ldots, 35\}$ is the trope number sequence of $f$, if and only if for $1 \leq r \leq 6$ there exists a representative $\tau_r$ of the $\sigma(r)$-th $D_{12}$-orbit of tropes, so that

- $\tau_r$ and $\tau_{r+1}$ are connectable with the moving pair $\{i_r, j_r\}$, $1 \leq r \leq 5$, and
- $\tau_6$ and $\tau_1$ are connectable with the moving pair $\{i_6, j_6\}$, and
- each element of $\mathbb{Z}_{12}$ is moving exactly once, i.e.

$$\bigcup_{r=1}^{6} \{i_r, j_r\} = \mathbb{Z}_{12}.$$
Let $f$ be a tone row.

**Trope sequence:** $t_f: \{1, \ldots, 6\} \rightarrow \mathcal{T}$, $t_f(i) = \tau_i$, $1 \leq i \leq 6$. 
Let $f$ be a tone row.

**Trope sequence**: $t_f: \{1, \ldots, 6\} \to \mathcal{T}, t_f(i) = \tau_i, 1 \leq i \leq 6$.

**Trope number sequence**: Replace the tropes by the numbers of their $D_{12}$-orbits, $s_f: \{1, \ldots, 6\} \to \{1, \ldots, 35\}$, $s_f(i)$ is the number of the orbit $D_{12}(\tau_i), 1 \leq i \leq 6$. 
Let $f$ be a tone row.

**Trope sequence**: $t_f : \{1, \ldots, 6\} \rightarrow \mathcal{T}$, $t_f(i) = \tau_i$, $1 \leq i \leq 6$.

**Trope number sequence**: Replace the tropes by the numbers of their $D_{12}$-orbits, $s_f : \{1, \ldots, 6\} \rightarrow \{1, \ldots, 35\}$, $s_f(i)$ is the number of the orbit $D_{12}(\tau_i)$, $1 \leq i \leq 6$.

We associate the $D_{12} \times D_{12}$-orbit of the tone row $f$ with the $D_{12}$-orbit of $s_f$ where the dihedral group $D_{12}$ acts on the domain of $s_f$. We call this orbit of $s_f$ the **trope structure** of the orbit $(D_{12} \times D_{12})(f)$. 
Let $f$ be a tone row.

**Trope sequence:** $t_f: \{1, \ldots, 6\} \rightarrow \mathcal{T}, t_f(i) = \tau_i, \ 1 \leq i \leq 6$.

**Trope number sequence:** Replace the tropes by the numbers of their $D_{12}$-orbits, $s_f: \{1, \ldots, 6\} \rightarrow \{1, \ldots, 35\}$, $s_f(i)$ is the number of the orbit $D_{12}(\tau_i), \ 1 \leq i \leq 6$.

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From the database it is possible to deduce that there are 538,139 different trope structures. There are trope structures, e.g. $(1,1,1,1,1,1)$, which determine a unique $D_{12} \times D_{12}$-orbit of tone rows. But there exist also two trope structures namely $(10,18,22,14,22,18)$ and $(10,18,22,14,22,27)$ which belong to 48 $D_{12} \times D_{12}$-orbits of tone rows.

**Database:** Compute the trope structure or Search for the trope structure
Tropes and stabilizers of tone rows

There is a close connection between the stabilizer type of a tone row and its trope structure. E.g.

**Theorem.** Let $f$ be a tone row. The pair $(T^6, R)$ belongs to the stabilizer of $f$, if and only if the following assertions hold true.

- $f$ has exactly four different trope numbers.
- The trope number sequence of $f$ is of the form $(t_1, t_2, t_3, t_4, t_3, t_2)$, where $t_1$ belongs to $\{1, 8, 14, 31, 34\}$, which are the numbers of those tropes $\{A, A'\}$ so that $T^6(A) = A'$, and $t_4$ is an element of $\{25, 32, 35\}$, which are the numbers of those tropes so that $T^6(A) = A$.
- There exists a trope sequence $(\tau_1, \ldots, \tau_6)$ where $\tau_r$ belongs to the $t_r$-th $D_{12}$-orbit of tropes, $1 \leq r \leq 6$, which satisfies the properties of Theorem and $\tau_6 = T^6(\tau_2)$ and $\tau_5 = T^6(\tau_3)$. 
Data were computed with SYMMETRICA [9] and GAP [8].


Database: **Database on tone rows and tropes**

1. We will check for Le Merle Noir.

2. We determine all information on this tone row.

3. Search for all-interval rows and retrieve musical information on those used by Alban Berg. Who else used similar tone rows?
References


[9] SYMMETRICA. A program system devoted to representation theory, invariant theory and combinatorics of finite symmetric groups and related classes of groups. Copyright by “Lehrstuhl II für Mathematik, Universität Bayreuth, 95440 Bayreuth”.

http://www.algorithm.uni-bayreuth.de/en/research/SYMMETRICA/.
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