Non-trivial additive functions between vector spaces over not necessarily equal fields

Jointly written with Jens Schwaiger

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59-th ISFE, June 18–25, 2023, Hajdúszoboszló, Hungary

The general linear equation

$$f(ax+by+c) = Af(x)+Bf(y)+C$$

has been considered in section 2.2.6 of János Aczél's *Lectures on functional equations and their applications*, Academic Press, 1966, for functions from $\mathbb{R}$ to $\mathbb{R}$ and,

in more detail, for functions from $\mathbb{R}^N$ to $\mathbb{R}$ in section 13.10 of Marek Kuczma's book *An introduction to the theory of functional equations and inequalities*, Birkhäuser, 2009 (2nd ed.).

A generalization

Here we consider a generalization, i.e., the equation

$$f\Big(\sum_{i=1}^{n} a_i x_i + a_0\Big) = \sum_{i=1}^{n} A_i f(x_i) + A_0$$

with $f : V \to W$ and vector spaces V, W over not necessarily identical fields K and L.

The equation with $A_0, a_0 = 0$ was considered by Paolo Leonetti and Jens Schwaiger in *The general linear equation on open connected sets*, in *Acta Math. Hung.* vol. 161, number 1, pp. 201–211, (2020).

The last paper was motivated by D. Głazowska et al., *Commutativity of integral quasiarithmetic means on measure spaces*, in *Acta Math. Hung.*, vol. 153, number 2, pp. 350–355, (2017), where all continuous solutions of the equation $f(ax+by) = af(x)+bf(y)$ when $x, y > 0$ were found.

Additive functions between vector spaces

1 Theorem

Let V be a vector space over a field K and W a vector space over a field L where $V,W \neq \{0\}$. The following assertions are equivalent:

1. $\operatorname{char} K = \operatorname{char} L$.
2. There exists an additive function $f\colon V \to W$ such that $f \neq 0$.

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Assume 2. holds true: $f: V \to W$ non-zero and additive, $v_0 \in V$ such that $w_0 = f(v_0) \neq 0$. If $\operatorname{char} K = p > 0$, then $0 = f(0) = f((p \cdot 1_K)v_0) = (p \cdot 1_L)f(v_0) = (p \cdot 1_L)w_0$.

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Lemma 2
Consider two vector spaces V and W over K and L, respectively, and a subset $\emptyset \neq A \subseteq K$ together with a mapping $\varphi : A \to L$. If a non-trivial additive function $f : V \to W$ satisfies $f(ax) = \varphi(a) f(x)$ for all $a \in A$ and $x \in X$, then φ is injective.

3 Theorem
Consider two fields K and L over the same prime field P, a non-empty subset A of K, and an injective mapping $\varphi : A \to L$. Then the following assertions are equivalent:

1. There exists a field-isomorphism $\Phi : P(A) \to P(\varphi(A))$ such that $\Phi(a) = \varphi(a)$ for all $a \in A$.
2. For any vector space $V \neq \{0\}$ over K and any vector space $W \neq \{0\}$ over L there exists an additive function $f : V \to W$ such that $f \neq 0$ and $f(ax) = \varphi(a) f(x)$ for all $a \in A$ and $x \in V$.
3. For some vector space $V \neq \{0\}$ over K and some vector space $W \neq \{0\}$ over L there exists an additive function $f : V \to W$ such that $f \neq 0$ and $f(ax) = \varphi(a) f(x)$ for all $a \in A$ and $x \in V$.

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3. For some vector space $V \neq \{0\}$ over K and some vector space $W \neq \{0\}$ over L there exists an additive function $f\colon V \to W$ such that $f \neq 0$ and $f(ax) = \varphi(a)f(x)$ for all $a \in A$ and $x \in V$.

Proof. Assume 1. Let B be a basis of V over $P(A)$, $V \ni x = \sum_{b\in B} \lambda_b(x)b$, $w_0 \in W \setminus \{0\}$, then $f\colon V \to W$ defined by $f(x) = \Phi\Big(\sum_{b\in B} \lambda_b(x)\Big)w_0$ is non-zero, additive, and $f(ax) = \varphi(a)f(x)$, $a \in A$, $x \in V$.

Assume 3. $P(A)$ is the set of all rational functions

$$R(a_1,\ldots,a_n) = \frac{r(a_1,\ldots,a_n)}{s(a_1,\ldots,a_n)}$$

with polynomials $r(X_1,\ldots,X_n), s(X_1,\ldots,X_n) \in P[X_1,\ldots,X_n]$ such that $s(a_1,\ldots,a_n) \neq 0$, for $n \in \mathbb{N}$. Standard arguments prove that $f(R(a_1,\ldots,a_n)x) = R(\varphi(a_1),\ldots,\varphi(a_n))f(x)$ for all rational expressions $R(a_1,\ldots,a_n) \in P(A)$.

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The function $\Phi\colon P(A) \to P(\varphi(A))$defined by

$$\Phi(R(a_1,\dots,a_n)) = R(\varphi(a_1),\dots,\varphi(a_n))$$

is well defined, bijective, and a homomorphism. $\square$

4 Remark
Consider two fields K and L over the same prime field P, a non-empty subset A of K, and an injective mapping $\varphi\colon A \to L$. Moreover, there exists a field isomorphism $\Phi\colon P(A) \to P(\varphi(A))$ such that $\Phi(a) = \varphi(a)$ for all $a \in A$.

Then for any vector space V over K, any vector space W over L, and any basis B of V over $P(A)$, as well as any mapping $\alpha'\colon B \to W$ there is exactly one additive function $\alpha\colon V \to W$ with $\alpha|_B = \alpha'$ and $\alpha(ax) = \Phi(a)\alpha(x)$ for all $x \in V$ and all $a \in P(A)$.

The proof is similar to the first part of the previous proof.

Solving the general linear functional equation

$$f\Big(\sum_{i=1}^{n} a_i x_i + a_0\Big) = \sum_{i=1}^{n} A_i f(x_i) + A_0, \qquad x_i \in V,\ 1 \le i \le n, \tag{1}$$

for $f\colon V \to W$, where V is a vector space over K and W is a vector space over L, with given scalars $a_i \in K$ and $A_i \in L$, $1 \le i \le n$, $n \ge 2$, and given vectors $a_0 \in V$ and $A_0 \in W$.

Assume that $a_i \ne 0$ for $1 \le i \le n$. We define mappings $f_i\colon V \to W$, $0 \le i \le n$, by

$$f_0(x) = f(x + a_0) - A_0, \qquad f_i(x) = A_i f\Big(\frac{x}{a_i}\Big),\ 1 \le i \le n,$$

then (1) is equivalent to

$$f_0\Big(\sum_{i=1}^{n} x_i\Big) = \sum_{i=1}^{n} f_i(x_i). \tag{2}$$

Let $g_i(x) = f_i(x) - f_i(0)$, $0 \leq i \leq n$, then $g_i(0) = 0$, $0 \leq i \leq n$, and

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$$g_0\Big(\sum_{i=1}^{n} x_i\Big) = \sum_{i=1}^{n} g_i(x_i) \tag{3}$$

since $f_0(0) = \sum_{i=1}^{n} f_i(0)$.

From $g_i(0) = 0$ for all i we obtain by (3) that $g_j(x) = g_0(x)$, $x \in V$, $1 \leq j \leq n$. Therefore, the function $\alpha := g_0$ is additive.

Let $g_i(x) = f_i(x) - f_i(0)$, $0 \le i \le n$, then $g_i(0) = 0$, $0 \le i \le n$, and

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From $g_i(0) = 0$ for all i we obtain by (3) that $g_j(x) = g_0(x)$, $x \in V$, $1 \le j \le n$. Therefore, the function $\alpha := g_0$ is additive. Defining $\alpha_i = f_i(0)$, $0 \le i \le n$, we have proven the following consequence of Theorem 3.

5 Corollary
If $f\colon V \to W$ satisfies (1) with $a_i \neq 0$ for $1 \le i \le n$, then there exist constant vectors $\alpha_i \in W$, $0 \le i \le n$, and an additive mapping $\alpha\colon V \to W$, such that

$$f(x + a_0) - A_0 = \alpha(x) + \alpha_0, \qquad A_i f\Big(\frac{x}{a_i}\Big) = \alpha(x) + \alpha_i,\ 1 \le i \le n, \quad x \in V. \tag{4}$$

We immediately get from (4)

$$f(x) = \alpha(x) + \alpha_0', \qquad \alpha_0' = \alpha_0 + A_0 - \alpha(a_0), \tag{5}$$

$A_i\alpha_0' = \alpha_i$, $1 \leq i \leq n$, and

$$\alpha(a_i x) = A_i\alpha(x), \quad x \in V,\ 1 \leq i \leq n. \tag{6}$$

Then (1) reads as

$$\sum_{i=1}^{n} A_i\alpha(x_i) + \alpha_0' + \alpha(a_0) = \sum_{i=1}^{n} A_i\alpha(x_i) + A\alpha_0' + A_0,$$

where $A := \sum_{i=1}^{n} A_i$. Consequently

$$(1-A)\alpha_0' = A_0 - \alpha(a_0). \tag{7}$$

This way we have proven one implication in

6 Theorem
Let V and W be vector spaces over K and L, respectively, $a_i \in K \setminus \{0\}$ and $A_i \in L$, $1 \le i \le n$, $n \in \mathbb{N}, n \ge 2$, and $f: V \to W$. Moreover let $a_0 \in V$ and $A_0 \in W$. Then the following assertions are equivalent:

1. The function $f: V \to W$ is a solution of (1).
2. There exists an additive function $\alpha: V \to W$ and a constant $\alpha_0' \in W$, such that f is of the form (5), and α, α_0' satisfy (6) and (7).

Constant solutions are of the form (5) with $\alpha = 0$. Thus (1) has constant solutions iff $(1-A)\alpha_0' = A_0$. More exactly in the case $A \neq 1$ this constant is unique and given by $\alpha_0' = \frac{1}{1-A} A_0$. If $A = 1$ and $A_0 = 0$ the constant α_0' is arbitrary. If $A = 1$ and $A_0 \neq 0$ there are no constant solutions.

Non-constant solutions

This means that (6) has non-zero additive solutions.

Lemma 7
Let V and W be vector spaces over K and L, respectively, and assume that $\alpha: V \to W$ is additive, different from 0, and satisfies (6) with $a_i \neq 0$ for all $1 \leq i \leq n$. Then for all i, j we have $a_i = a_j$ if, and only if $A_i = A_j$.

Thus with $S_a := \{a_i \mid 1 \leq i \leq n\}$ and $S_A := \{A_i \mid 1 \leq i \leq n\}$ the mapping $\varphi: S_a \to S_A$, $\varphi(a_i) := A_i$, is well-defined and bijective.

Proof. Assume $a_i = a_j$. Then $A_i\alpha(x) = \alpha(a_i x) = \alpha(a_j x) = A_j\alpha(x)$ for all x. Thus $(A_i - A_j)\alpha = 0$ implying $A_i - A_j = 0$ as $\alpha \neq 0$.

Now assume $A_i = A_j$. Then $0 = A_i\alpha(x) - A_j\alpha(x) = \alpha(a_i x) - \alpha(a_j x) = \alpha((a_i - a_j)x)$ for all x. If $a_i \neq a_j$ this would imply $\alpha = 0$, a contradiction. □

Now we describe all situations guaranteeing the existence of non-constant solutions of (1). Since a solution f of (1) is non-constant iff the corresponding additive function α is non-constant, we may assume that Lemma 7 holds true which ensures that we may apply Remark 4.

8 Theorem
Let $V \neq \{0\}$ and $W \neq \{0\}$ be vector spaces over K and L, respectively. Assume that K and L have the same prime field P. Assume moreover (the necessary condition for the existence of a non-trivial additive α), that there is a field isomorphism $\Phi: P(a_1, a_2, \dots, a_n) \to P(A_1, A_2, \dots, A_n)$ such that $\Phi(a_i) = A_i$ for all $1 \leq i \leq n$, $n \geq 2$. Then the following holds true.

1. If $A := \sum_{i=1}^{n} A_i \neq 1$, then there are non-constant (and also constant) solutions of (1). If we fix a basis B of V over $P(a_1, a_2, \dots, a_n)$ and if we choose any mapping $\alpha': B \to W$ in order to define α as in Remark 4, then the function $f = \alpha + \frac{1}{1-A}(A_0 - \alpha(a_0))$ is a solution of (1) which is not constant iff $\alpha' \neq 0$.

2. If $A = 1$ and $a_0 \neq 0$ we may fix a basis B of V over $P(a_1, a_2, \ldots, a_n)$ with $a_0 \in B$ and define α with any $\alpha' : B \to W$ such that $\alpha'(a_0) = A_0$ and arbitrarily on $B \setminus \{a_0\}$. Then for each of these α the sum $\alpha + \alpha'_0$ with arbitrary $\alpha'_0 \in W$ satisfies (1). If $A_0 \neq 0$, this α is different from 0 and thus f non-constant. If $A_0 = 0$ and $B \setminus \{a_0\} \neq \emptyset$ we get non-constant solutions by choosing some $b \in B \setminus \{a_0\}$ and choosing $\alpha'(b) \neq 0$.

   If $B = \{a_0\}$ and $A_0 = 0$ there are only constant solutions.

3. If $A = 1$ and $a_0 = 0$ there is no solution at all for $A_0 \neq 0$. If $A_0 = 0$ we may choose α as in 1. and an arbitrary $\alpha'_0 \in W$ which results in a solution $f = \alpha + \alpha'_0$ of (1).

For $V = K$, $W = L$ this implies

9 Corollary
Consider two fields K and L over the same prime field P, $a_i \in K$ and $A_i \in L$, $0 \leq i \leq n$, $n \in \mathbb{N}$, $n \geq 2$, where $1 = \sum_{i=1}^{n} A_i$ and $a_i \neq 0$ for $1 \leq i \leq n$. The following two assertions are equivalent:

1. There exists a non constant solution $f\colon K \to L$ of (1).
2. There exists an additive mapping $\alpha\colon K \to L$ such that $\alpha(a_0) = A_0$ and the restriction $\Phi := \alpha|_{P(a_1,\ldots,a_n)}$ is a field-isomorphism $P(a_1,\ldots,a_n) \to P(A_1,\ldots,A_n)$ satisfying $\Phi(a_i) = A_i$, $1 \leq i \leq n$.

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The case $n = 1$

The assumption that all $a_i \neq 0$ in Theorem 6 is not very important. If, for example, $a_n = 0$, (1) with $x_1 = x_2 = \ldots = x_{n-1}$ shows that either f is constant or $A_n = 0$, which gives (1) with $n-1$ instead of n.

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The case $n = 1$ however is of different taste. In this case (1) is a functional equation in one variable of the form $f(\varphi(x)) = \psi(f(x))$ with affine functions φ and ψ. To investigate solvability and general solution in this case asks for the analysis of the fixed points of the iterates of that functions.

Here $\varphi(x)$ is of the form $ax + v$, for $a \in K^*$ and $x, v \in V$. We will indicate this φ as (a, v). The set $\{(a, v) \mid a \in K^*,\ v \in V\}$ of all these φ together with multiplication $(a, v)(a', v') = (aa', av' + v)$ can be considered as a subgroup of the group of all affine mappings on V. The mapping $((a, v), x) \mapsto ax + v$ defines an action of this group on V.

Now (1) for $n = 1$ reads as

$$f((a_1, a_0)x) = (A_1, A_0)f(x), \qquad x \in V. \tag{8}$$

If $a_1 = 1$, then we assume that $a_0 \neq 0$ since $(1, 0)$ is the identity on V. Let G and H be the cyclic groups generated by (a_1, a_0) and (A_1, A_0), respectively.

10 Theorem
Let V and W be vector spaces over K and L, respectively, $a_1 \in K^*$, $A_1 \in L^*$, $a_0 \in V$ and $A_0 \in W$.

1. If $f\colon V \to W$ satisfies (8), then for any $x \in V$ the function f maps the G-orbit $G(x)$ surjectively on the H-orbit $H(f(x))$. If, moreover, $(a_1, a_0)^n x = x$ for some $n \in \mathbb{Z}$, then $(A_1, A_0)^n f(x) = f(x)$.

2. In order to construct solutions $f\colon V \to W$ of (8), determine a system of representatives of the G-orbits on V. On each G-orbit we define f separately in the following way. If the G-orbit of a representative x is infinite, choose any $y \in W$ and define $f(x) := y$. If the size of the G-orbit of x is $n \in \mathbb{N}$, then choose some $y \in W$ such that the size of the H-orbit of y is a divisor of n, and define $f(x) := y$. On the remaining elements of $G(x)$ define $f(x') = (A_1, A_0)^n y$ if x' is of the form $(a_1, a_0)^n x$ for some $n \in \mathbb{Z}$. Then f satisfies (8).

This yields the following description of the situations when (8) has constant or non-constant solutions.

1. If $a_1 = 1$ and char$(K) = p$, then there exist solutions of (8) if and only if there exist H-orbits on W of size 1 or p. If there are no H-orbits of size p but one orbit of size 1, then we have only constant solutions of (8).

2. If $a_1 = 1$ and char$(K) = 0$, then there exist both constant and non-constant solutions of (8).

3. If a_1 is of infinite order, then there exist both constant and non-constant solutions of (8).

4. If $a_1 \neq 1$ is of finite order, then there exist solutions of (8) if and only if $A_1 \neq 1$. In this situation there exist non-constant solutions of (8) if and only if there exist H-orbits on W of size $d > 1$ such that d is a divisor of ord(a_1).

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