On iterative roots of the identity and the groups $S_{n+1} \times (\{\pm 1\} \wr S_n)$

Harald Fripertinger
Karl-Franzens-Universität Graz
57-th ISFE, June 2–9, 2019, Jastarnia, Poland

During the last ISFE in Graz I was presenting a talk on iteration of bijective functions with discontinuities, which disappeared after some iterations. We were studying three types of functions defined on a compact interval $I = [0, n]$, $n \in \mathbb{N}$, $f\colon I \to I$ bijective with finitely many discontinuities.

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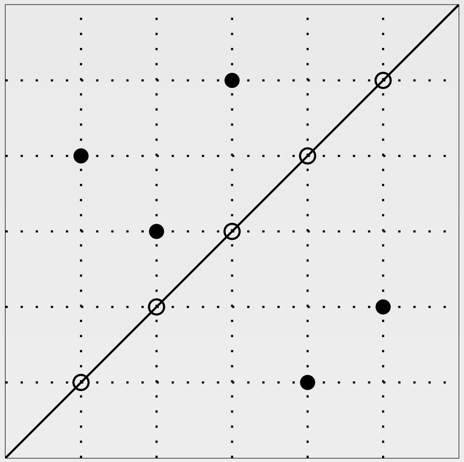
Consider an iterative root F of the identity on a compact real interval J. We will prove: If the union of the orbits of the discontinuities of F is finite, then there exists some $n \in \mathbb{N}$ and a continuous, bijective, and increasing function $\varphi\colon J \to [0,n]$, so that $\varphi \circ F \circ \varphi^{-1}$ corresponds to some $(\pi,(\varepsilon,\lambda)) \in S_{n+1} \times (\{\pm 1\} \wr S_n)$, S_n the symmetric group, i.e. $\varphi \circ f \circ \varphi^{-1}$ is a function of type III.

Functions of type I

Functions of type I have only removable discontinuities in $i \in \{1, \ldots, n-1\}$, e.g.

$f =$ 

They are totally described by the permutation $\pi \in S_{n-1}$, $\pi(i) = f(i)$.

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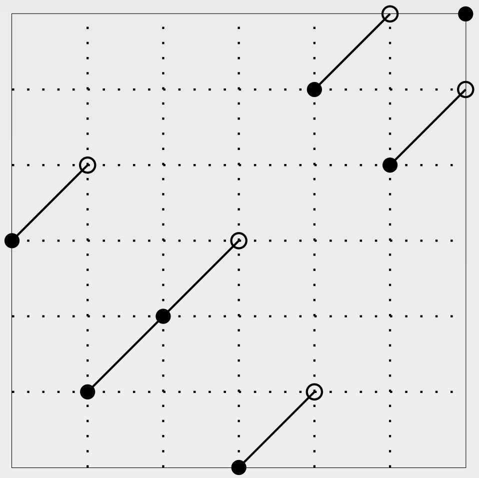
Functions of type II

Functions of type II have jump discontinuities in $i \in \{1, \dots, n-1\}$, maybe one removable discontinuity in n. On each interval $I_i = [i-1, i)$ they are strictly increasing and affine.

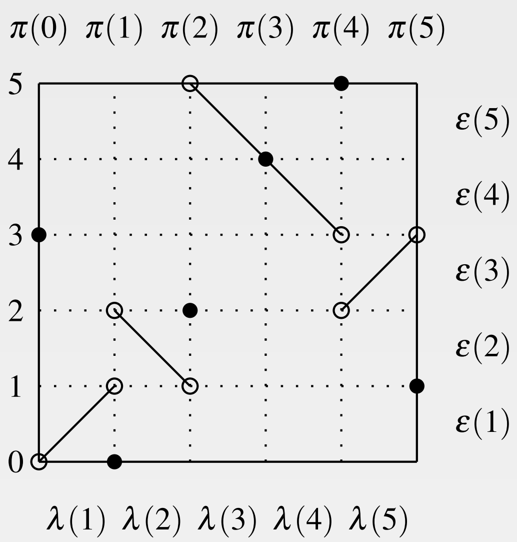
$f =$ 

They are totally described by the permutation $\pi \in S_n$, $\pi(i) = j$ if and only if $f(I_i) = I_j$.

Functions of type III

Functions of type III permute the integers $\{0,1,\dots,n\}$, permute the open intervals $I_i=(i-1,i)$, $i\in\{1,\dots,n\}$, on each interval they are affine and either strictly increasing or strictly decreasing.

E.g.,



$$\pi(0)=3$$
$$\pi(1)=0 \quad \lambda(1)=1 \quad \varepsilon(1)=1$$
$$\pi(2)=2 \quad \lambda(2)=2 \quad \varepsilon(2)=-1$$
$$\pi(3)=4 \quad \lambda(3)=5 \quad \varepsilon(3)=1$$
$$\pi(4)=5 \quad \lambda(4)=4 \quad \varepsilon(4)=-1$$
$$\pi(5)=1 \quad \lambda(5)=3 \quad \varepsilon(5)=-1$$

$\varepsilon(i)=1$ iff the values of I_i (in the range) appear in an increasing way, iff f is increasing on $I_{\lambda^{-1}(i)}$.

We identify f with $(\pi,(\varepsilon,\lambda))$, $\pi\in S_{n+1}$, $\varepsilon\in\{\pm 1\}^n$, $\lambda\in S_n$.

$\varepsilon(\lambda(i))$ is the direction of f on the interval I_i in the domain.

f is continuous in $i\in\{1,\ldots,n-1\}$, iff
either $\varepsilon(\lambda(i))=\varepsilon(\lambda(i+1))=1$, $\lambda(i+1)=\lambda(i)+1$, and $\pi(i)=\lambda(i)$,
or $\varepsilon(\lambda(i))=\varepsilon(\lambda(i+1))=-1$, $\lambda(i+1)=\lambda(i)-1$, and $\pi(i)=\lambda(i+1)$.

f is continuous in 0, iff
either $\varepsilon(\lambda(1))=1$ and $\pi(0)=\lambda(1)-1$
or $\varepsilon(\lambda(1))=-1$ and $\pi(0)=\lambda(1)$.

f is continuous in n must be studied accordingly.

f^k is continuous if either $f^k=\mathrm{id}$ or $f^k=n-\mathrm{id}$.

Structure theorem

Composition of $f \leftrightarrow (\pi, (\varepsilon, \lambda))$ and $f' \leftrightarrow (\pi', (\varepsilon', \lambda'))$ yields

$$f \circ f' \leftrightarrow (\pi \circ \pi', (\varepsilon \varepsilon'_\lambda, \lambda \circ \lambda'))$$

where

$$\varepsilon \varepsilon'_\lambda(i) = \varepsilon(i) \varepsilon'(\lambda^{-1}(i)), \qquad i \in \{1, \ldots, n\}.$$

The set of all functions of type III is the direct product

$$S_{n+1} \times (\{\pm 1\} \wr S_n)$$

where the factor on the right side is a wreath product

$$\{\pm 1\} \wr S_n = \{(\varepsilon, \lambda) \mid \varepsilon \in \{\pm 1\}^n,\ \lambda \in S_n\}$$

of order $n! \cdot 2^n$ with $(\varepsilon, \lambda)(\varepsilon', \lambda') = (\varepsilon \varepsilon'_\lambda, \lambda \circ \lambda')$.

The number of functions of type III on $[0,n]$ is

$$n!(n+1)!2^n$$

n	$n!(n+1)!2^n$
0	1
1	4
2	48
3	1152
4	46080
5	2764800
6	232243200
7	26011238400
8	3745618329600
9	674211299328000
10	148326485852160000

The order of f

$$\operatorname{ord}(f) = \operatorname{lcm}(\operatorname{ord}(\pi), \operatorname{ord}(\varepsilon, \lambda)).$$

From $f^{\operatorname{ord}(f)} = \operatorname{id}$ it follows that f is an iterative root of the identity.

General remarks

Theorem 1
Let J be a compact interval,
$\varphi : J \to [0,n]$ be continuous, bijective, and increasing,
and $f : [0,n] \to [0,n]$ be of type III with r discontinuities and $\operatorname{ord}(f) = k$,
then

$$F := \varphi^{-1} \circ f \circ \varphi : J \to J$$

is bijective, has r discontinuities, $F^k = \mathrm{id}_J$,
thus F is an iterative root of the identity of order k.

General remarks

Theorem 1
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thus F is an iterative root of the identity of order k.

Problem.
Consider an iterative root $F\colon J \to J$ of the identity of order k on a compact interval J with finitely many discontinuities. Is it possible to find some $n \in \mathbb{N}$, a continuous, bijective, and increasing function $\varphi\colon J \to [0,n]$ and a function $f\colon [0,n] \to [0,n]$ of type III so that $F = \varphi^{-1} \circ f \circ \varphi$?

We will prove that the answer is

YES!

How to find n?

In general n is not uniquely determined, so we are looking for the smallest n.

Assume that $F^k = \mathrm{id}$ and F has r discontinuities $\xi_1, \ldots, \xi_r \in J = [a, b]$. Consider the union of orbits

$$U = \{a, b\} \cup \bigcup_{j=1}^{r} \{F^i(\xi_j) \mid 1 \leq i \leq k\},$$

then U is finite

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$$U = \{a,b\} \cup \bigcup_{j=1}^{r} \{F^i(\xi_j) \mid 1 \leq i \leq k\},$$

then U is finite and we determine n by

$$n = |U| - 1.$$

We label the $n+1$ elements of U by $a = x_0 < \ldots < x_n = b$. Since $F(U) = U$ we have $F(x_i) \in U$ for all $0 \leq i \leq n$, thus F is a permutation of U.

Let J_i be the open interval (x_{i-1}, x_i), $1 \leq i \leq n$, then

$$[a,b] = U \cup J_1 \cup \cdots \cup J_n.$$

For all $i \in \{1, \ldots, n\}$ it is obvious that:

- F is continuous on J_i,
- there exists some $j \in \{1, \ldots, n\}$ so that $F(J_i) = J_j$, thus F permutes the intervals J_i.

This particular n will be called $n(F)$.

How to find φ?

We will construct φ in several steps:

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We will construct φ in several steps:

1. We determine some $\varphi : J \to [0, n]$ so that $\varphi(J_i) = (i-1, i)$ for $1 \le i \le n$.

   Let $\varphi(x_i) = i$, $0 \le i \le n$.

   For $x \in J_i = (x_{i-1}, x_i)$ let

   $$\varphi(x) = i - 1 + \frac{x - x_{i-1}}{x_i - x_{i-1}},$$

   then φ is continuous in J_i, and $\lim_{x \to x_{i-1}^+} \varphi(x) = i - 1 = \varphi(x_{i-1})$ and $\lim_{x \to x_i^-} \varphi(x) = i = \varphi(x_i)$. Therefore φ is continuous on J.

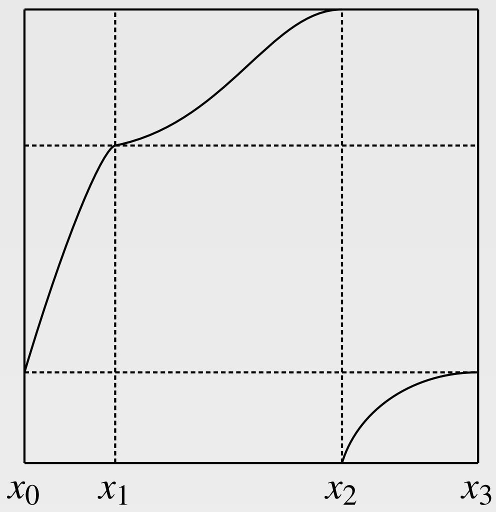
   Moreover φ is strictly increasing and bijective.

Let $\tilde{F} = \varphi \circ F \circ \varphi^{-1} \colon [0,n] \to [0,n]$, then

- $\tilde{F}$ is bijective,
- $\tilde{F}^j = \mathrm{id}_{[0,n]}$, iff $F^j = \mathrm{id}_J$,
- $\tilde{F}$ is an iterative root of the identity of order k,
- $\tilde{F}$ has discontinuities in $\varphi(\xi_i)$, $1 \leq i \leq r$,
- $\tilde{F}(i) \in \{0,\ldots,n\}$, $i \in \{0,\ldots,n\}$, $\tilde{F}$ permutes these elements,
- $\tilde{F}$ is continuous on $I_i = (i-1,i)$, $1 \leq i \leq r$,
- $\tilde{F}$ is a permutation of the intervals I_i, $1 \leq i \leq r$,
- $\tilde{F}$ is increasing on I_i, iff F is increasing on J_i, $1 \leq i \leq r$.

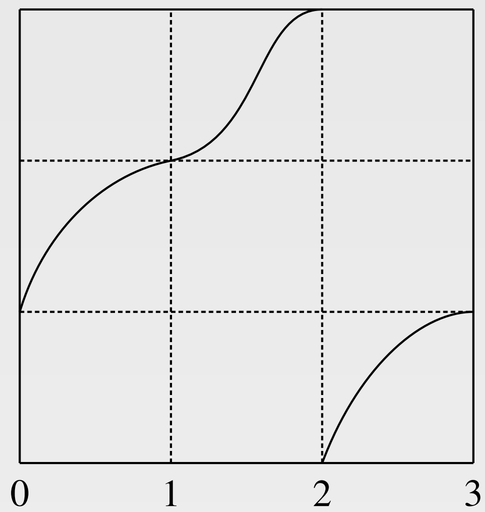
From

$$F$$



we get

$$\tilde{F} = \varphi \circ F \circ \varphi^{-1}$$



2. Now we must find some $\psi\colon[0,n]\to[0,n]$ so that $\psi\circ\tilde{F}\circ\psi^{-1}$ is affine on each interval $I_i=(i-1,i)$.

Lemma 2

Assume that $f:=\tilde{F}|_{I_i}$ is a mapping $I_i\to I_j$ for $i\neq j$.

If f is strictly increasing, then there exists some $\psi_j\colon I_j\to I_j$ bijective and increasing, so that $\psi_j(f(x))=j+x-i$, $x\in I_i$.

If f is strictly decreasing, then there exists some $\psi_j\colon I_j\to I_j$ bijective and increasing, so that $\psi_j(f(x))=j-x+i-1$, $x\in I_i$.

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2. Now we must find some $\psi\colon[0,n]\to[0,n]$ so that $\psi\circ\tilde F\circ\psi^{-1}$ is affine on each interval $I_i=(i-1,i)$.

Lemma 2
Assume that $f:=\tilde F|_{I_i}$ is a mapping $I_i\to I_j$ for $i\neq j$.
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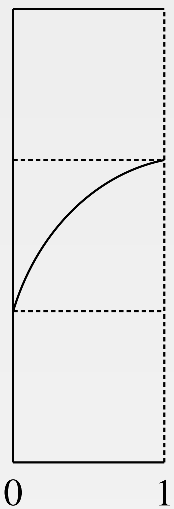
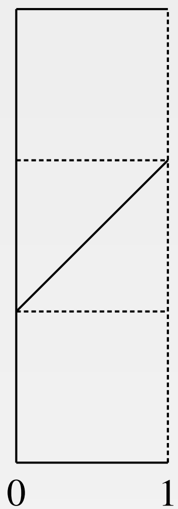
If f is strictly decreasing, then there exists some $\psi_j\colon I_j\to I_j$ bijective and increasing, so that $\psi_j(f(x))=j-x+i-1$, $x\in I_i$.

$\tilde F|_{I_1}$ $\qquad\qquad$ $\psi_2\circ\tilde F|_{I_1}$

From



we get



Proof.

1. Let $\psi_j(x) = j + f^{-1}(x) - i$, for $x \in I_j$, then ψ_j is a bijective and increasing mapping $I_j \to I_j$, and
$\psi_j(f(x)) = j + f^{-1}(f(x)) - i = j + x - i$, $x \in I_i$.

2. Let $\psi_j(x) = j - f^{-1}(x) + i - 1$, for $x \in I_j$, then ψ_j is a bijective and increasing mapping $I_j \to I_j$, and $\psi_j(f(x)) = j - f^{-1}(f(x)) + i - 1$, $x \in I_i$.

Let $\psi_j(x) = x$ for $x \notin I_j$, then ψ_j is bijective and increasing on $[0, n]$.

$\tilde{F}$ is a permutation of the intervals I_i. Consider a cycle $I_{i_1} \to I_{i_2} \to \ldots \to I_{i_\ell} \to I_{i_1}$ of length $\ell \geq 1$. Then $F^\ell(I_{i_j}) = I_{i_j}$, $1 \leq j \leq \ell$.

Composition of two increasing or two decreasing functions yields an increasing function, composition of one increasing and one decreasing function yields a decreasing function.

Therefore, if $\tilde{F}$ is decreasing on an even number of intervals in this cycle, then $\tilde{F}^\ell$ is increasing on all I_{i_j}, otherwise $\tilde{F}^\ell$ is decreasing on all I_{i_j}.

Since ψ_j restricted to I_i is a bijective mapping $I_i \to I_i$, $1 \leq i \leq n$, the restriction $\psi_j \circ \tilde{F} \circ \psi_j^{-1}$ to I_i involves only $\tilde{F}|_{I_i}$.

First case

$\tilde{F}$ contains a cycle of intervals of length ℓ with an even number of decreasing functions in this cycle, then $\tilde{F}^{\ell} = \mathrm{id}$ on this cycle.

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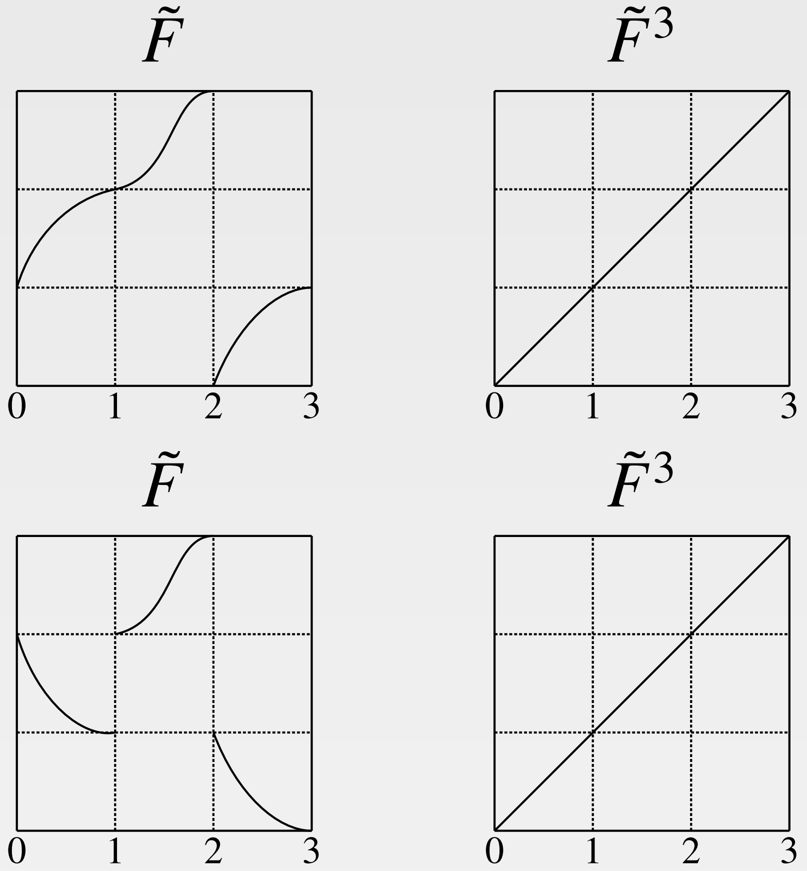


In the **first case** $\tilde{F}^{\ell}|_{I_{i_j}} = \mathrm{id}$ for all $1 \le j \le \ell$.

$\tilde{F}|_{I_{i_1}}: I_{i_1} \to I_{i_2}$
According to Lemma 2 there exists a bijective and increasing mapping ψ_{i_2} on $[0,n]$ so that $\psi_{i_2} \circ \tilde{F}|_{I_{i_1}}$ is affine, i.e. it is either $x \mapsto i_2 + x - i_1$ or $x \mapsto i_2 - 1 + i_1 - x$. Then also $\psi_{i_2} \circ \tilde{F} \circ \psi_{i_2}^{-1}$ is affine on I_{i_1}.

$\psi_{i_2} \circ \tilde{F} \circ \psi_{i_2}^{-1}|_{I_{i_2}}: I_{i_2} \to I_{i_3}$
According to Lemma 2 there exists a bijective and increasing mapping ψ_{i_3} on $[0,n]$ so that $\psi_{i_3} \circ \psi_{i_2} \circ \tilde{F} \circ \psi_{i_2}^{-1}$ is affine on I_{i_2}. Then also $\psi_{i_3} \circ \psi_{i_2} \circ \tilde{F} \circ \psi_{i_2}^{-1} \circ \psi_{i_3}^{-1}$, $j = 1, 2$, is affine on I_{i_j}.

Continuing in the same way:
$\psi_{i_{\ell-1}} \circ \cdots \circ \psi_{i_2} \circ \tilde{F} \circ \psi_{i_2}^{-1} \circ \cdots \circ \psi_{i_{\ell-1}}^{-1}|_{I_{i_{\ell-1}}}: I_{i_{\ell-1}} \to I_{i_\ell}$
There exists a bijective and increasing mapping ψ_{i_ℓ} on $[0,n]$ so that $\psi_{i_\ell} \circ \psi_{i_{\ell-1}} \circ \cdots \circ \psi_{i_2} \circ \tilde{F} \circ \psi_{i_2}^{-1} \circ \cdots \circ \psi_{i_{\ell-1}}^{-1}$ is affine on $I_{i_{\ell-1}}$. Then also $\psi_{i_\ell} \circ \cdots \circ \psi_{i_2} \circ \tilde{F} \circ \psi_{i_2}^{-1} \circ \cdots \circ \psi_{i_\ell}^{-1}$, $1 \le j \le \ell - 1$, is affine on I_{i_j}.

The mapping $\psi = \psi_{i_\ell} \circ \cdots \circ \psi_{i_2}$ is bijective and increasing on $[0,n]$, $\psi \circ \tilde{F} \circ \psi^{-1}|_{I_{i_j}}$ is affine, $1 \le j \le \ell - 1$, and $\psi(x) = x$ for $x \in I_{i_1}$.

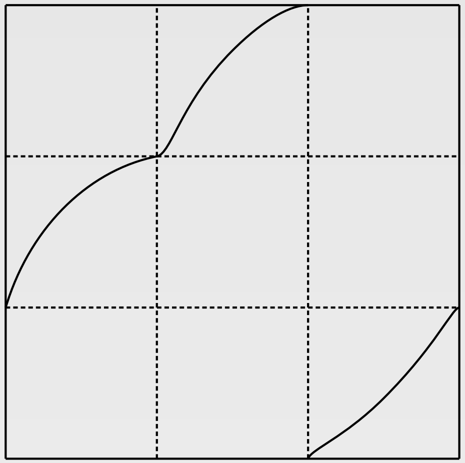
We have $\mathrm{id}\,|_{I_{i_1}} = \tilde{F}^\ell|_{I_{i_1}} = \tilde{F}|_{I_{i_\ell}} \circ \cdots \circ \tilde{F}|_{I_{i_1}}$. Therefore

$$\mathrm{id}\,|_{I_{i_1}} = \psi \circ \mathrm{id} \circ \psi^{-1}|_{I_{i_1}} = \psi \circ \tilde{F}^\ell \circ \psi^{-1}|_{I_{i_1}} = (\psi \circ \tilde{F} \circ \psi^{-1})^\ell|_{I_{i_1}} = (\psi \circ \tilde{F} \circ \psi^{-1}) \circ \Big[(\psi \circ \tilde{F} \circ \psi^{-1}) \circ \cdots \circ (\psi \circ \tilde{F} \circ \psi^{-1}) \Big] \Big|_{I_{i_1}} = (\psi \circ \tilde{F} \circ \psi^{-1})|_{I_{i_\ell}} \circ \Big[(\psi \circ \tilde{F} \circ \psi^{-1})|_{I_{i_{\ell-1}}} \circ \cdots \circ (\psi \circ \tilde{F} \circ \psi^{-1})|_{I_{i_1}} \Big].$$

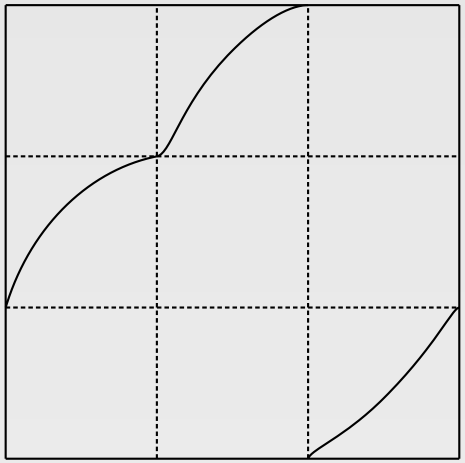
The term between $[$ and $]$ is a composition of affine function, thus it is affine, whence also $\psi \circ \tilde{F} \circ \psi^{-1}|_{I_{i_\ell}}$ is affine.

Consequently $\psi \circ \tilde{F} \circ \psi^{-1}$ is affine on I_{i_j} for each $1 \le j \le \ell$.

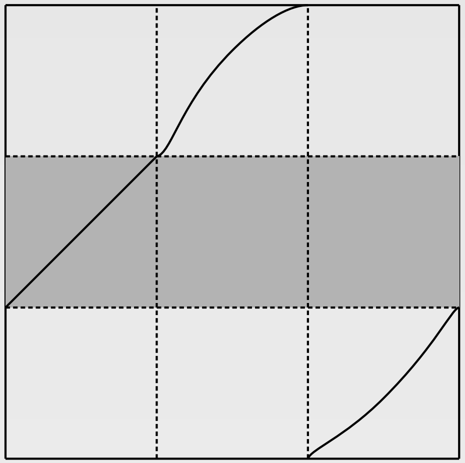
$\tilde{F}$



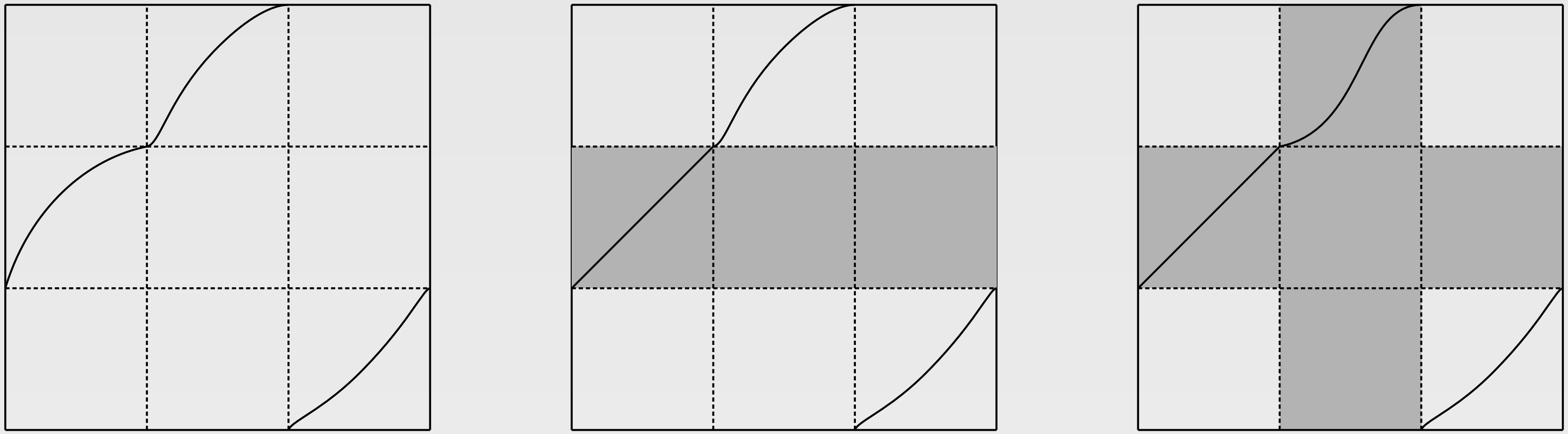
$\tilde{F}$



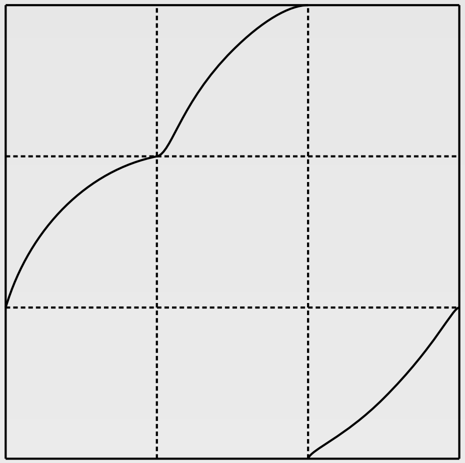
$\psi_{I_2} \circ \tilde{F}$



$\tilde{F}$ $\quad$ $\psi_{I_2} \circ \tilde{F}$ $\quad$ $\psi_{I_2} \circ \tilde{F} \circ \psi_{I_2}^{-1}$



$$\tilde{F}$$



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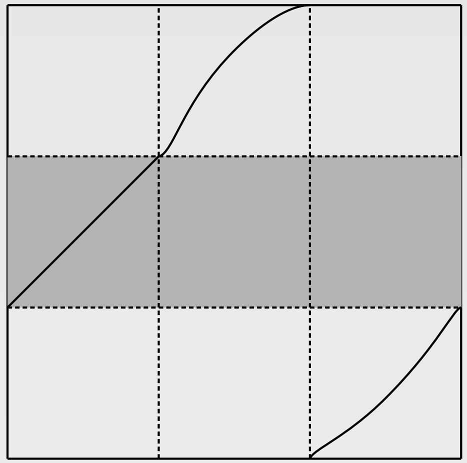
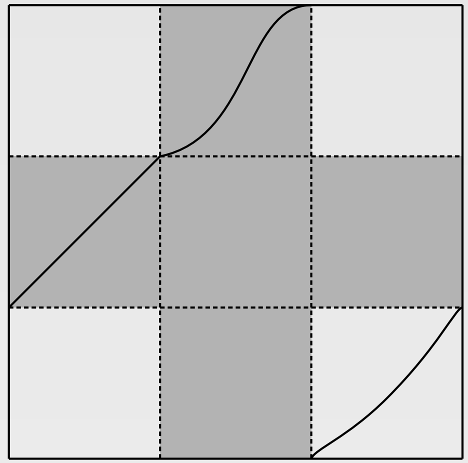
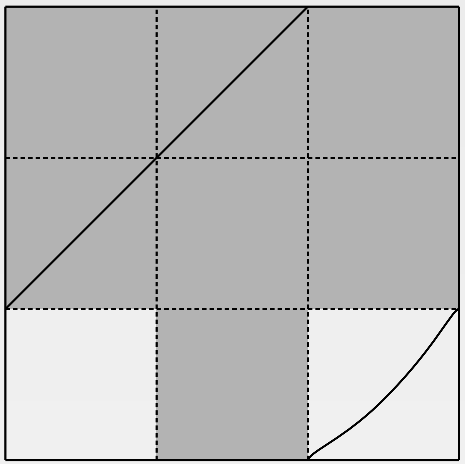


$$\psi_{I_2} \circ \tilde{F} \circ \psi_{I_2}^{-1}$$



$$\psi_{I_3} \circ \psi_{I_2} \circ \tilde{F} \circ \psi_{I_2}^{-1}$$



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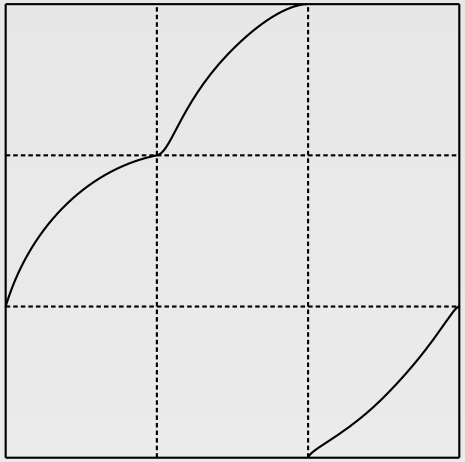

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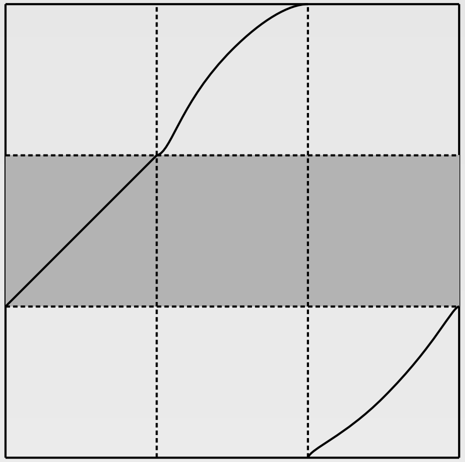

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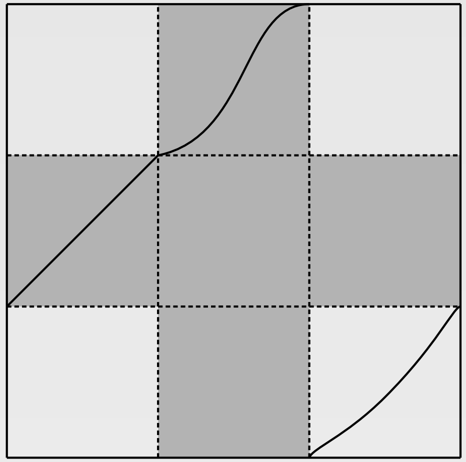

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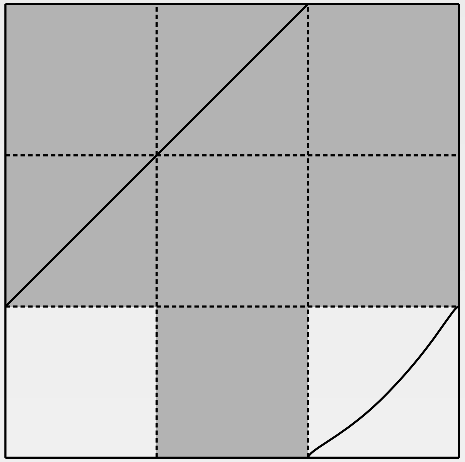

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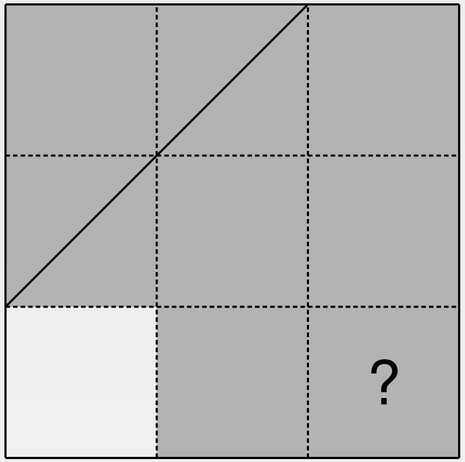
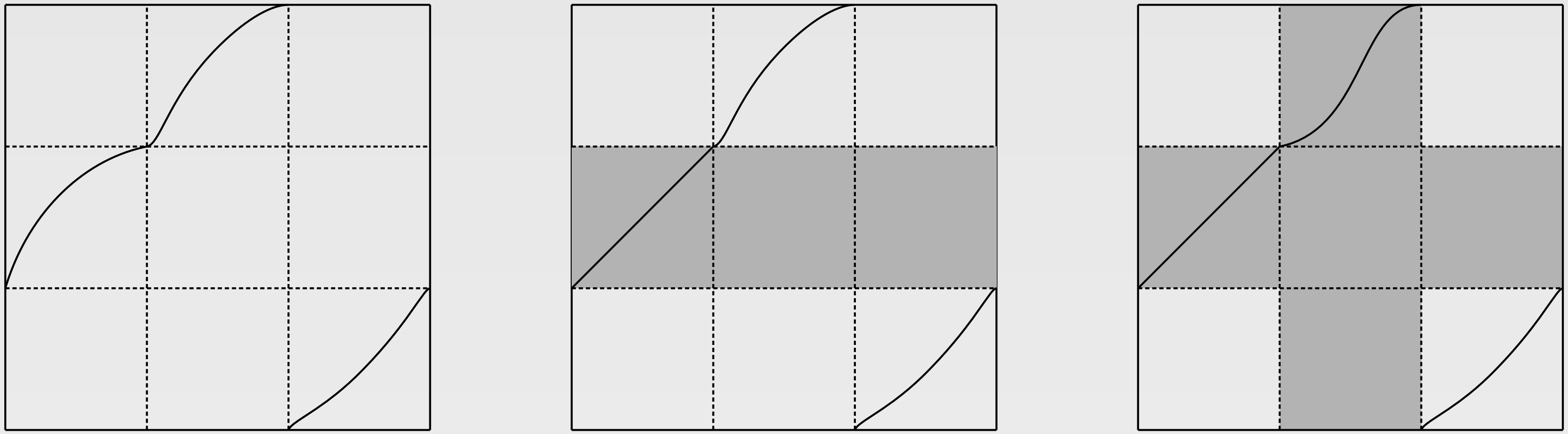
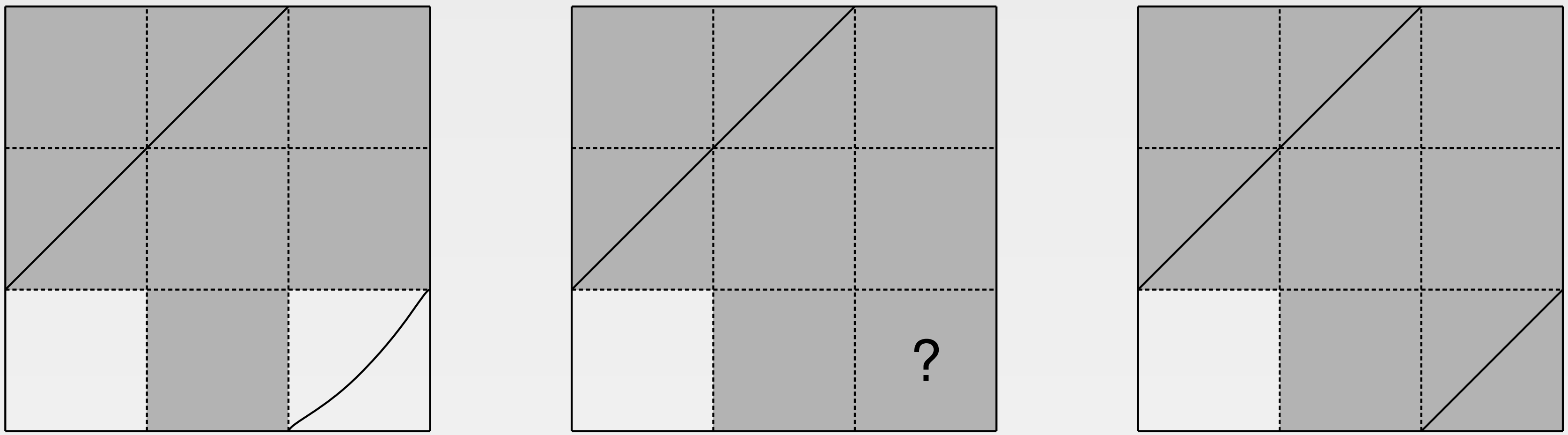


$\tilde{F}$ $\qquad$ $\psi_{I_2} \circ \tilde{F}$ $\qquad$ $\psi_{I_2} \circ \tilde{F} \circ \psi_{I_2}^{-1}$



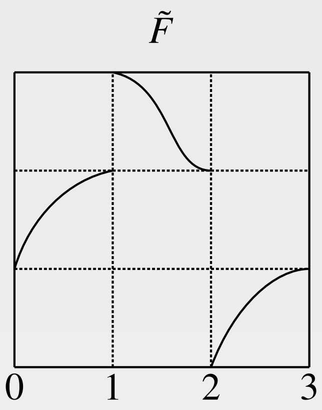
$\psi_{I_3} \circ \psi_{I_2} \circ \tilde{F} \circ \psi_{I_2}^{-1}$ $\qquad$ $\psi_{I_3} \circ \psi_{I_2} \circ \tilde{F} \circ \psi_{I_2}^{-1} \circ \psi_{I_3}^{-1}$

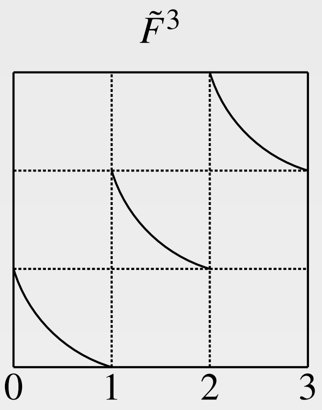


since $\mathrm{id} = \tilde{F}^3|_{I_1} = \psi \circ \tilde{F} \circ \psi^{-1}|_{I_3} \circ (\psi \circ \tilde{F} \circ \psi^{-1}|_{I_2} \circ \psi \circ \tilde{F} \circ \psi^{-1}|_{I_1})$.

Second case

$\tilde{F}$ contains a cycle of intervals of length ℓ with an odd number of decreasing functions in this cycle, then $\tilde{F}^\ell$ need not be affine on these intervals, but there exists some bijective increasing function $\tilde{\psi}$ so that $\tilde{\psi} \circ \tilde{F}^\ell \circ \tilde{\psi}^{-1}$ is affine on these intervals.





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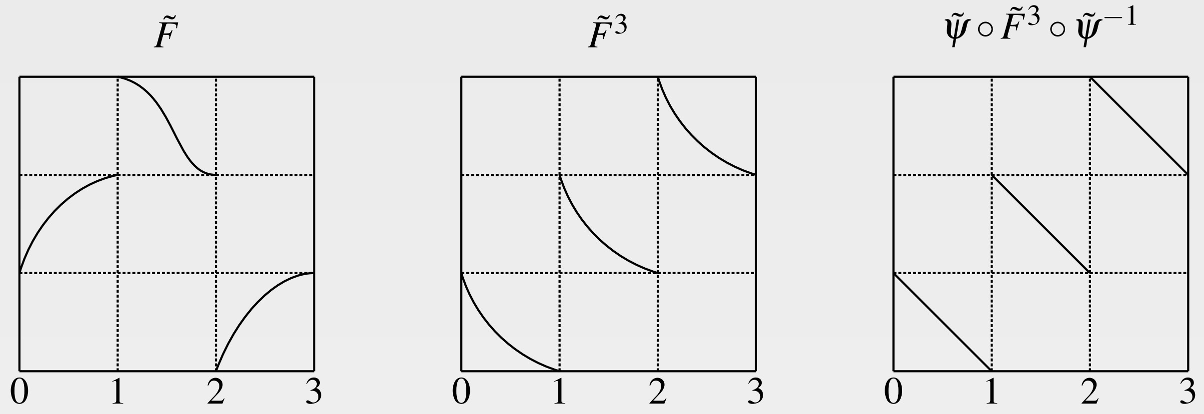

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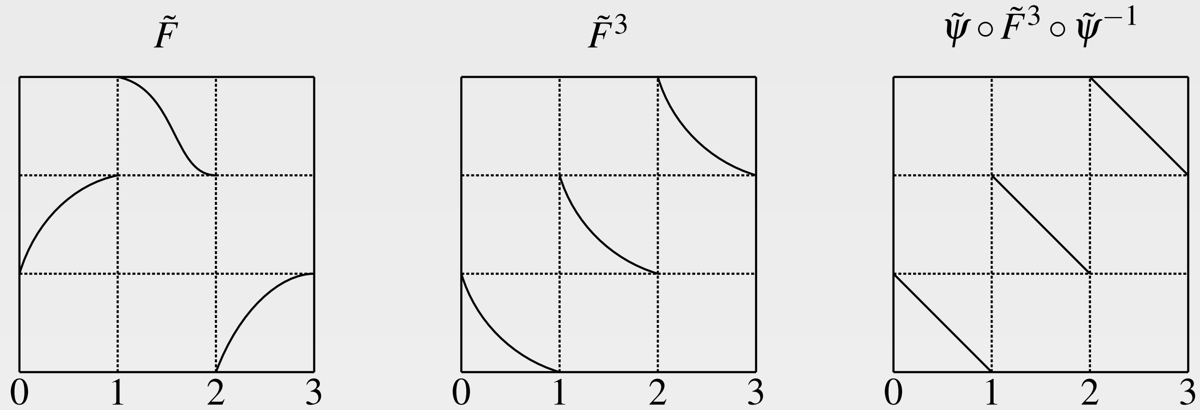

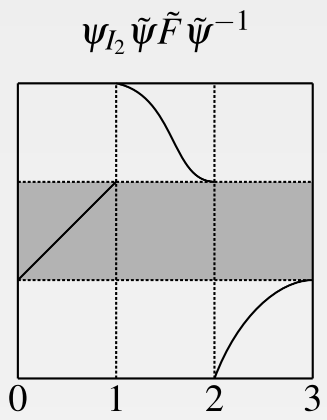

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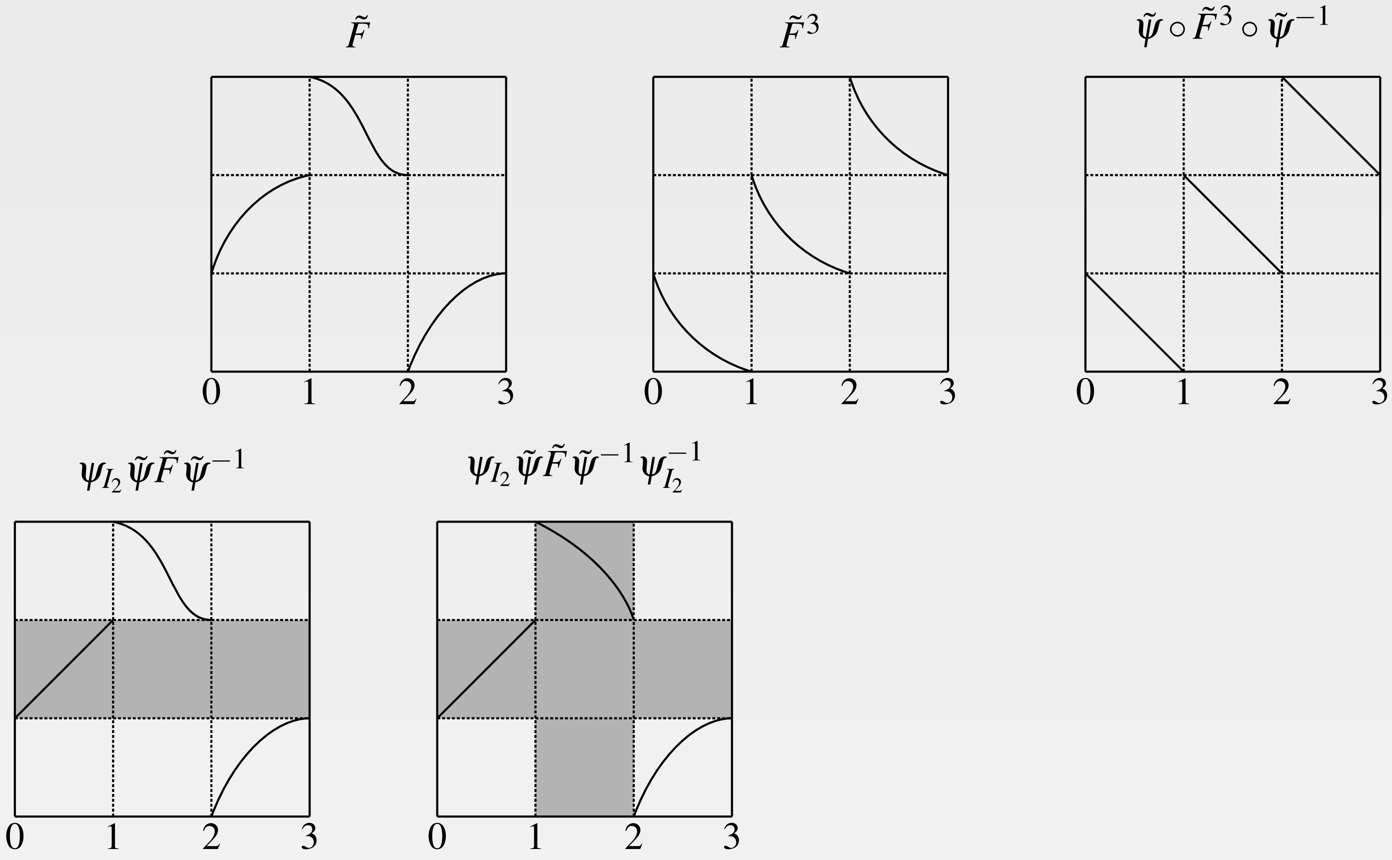

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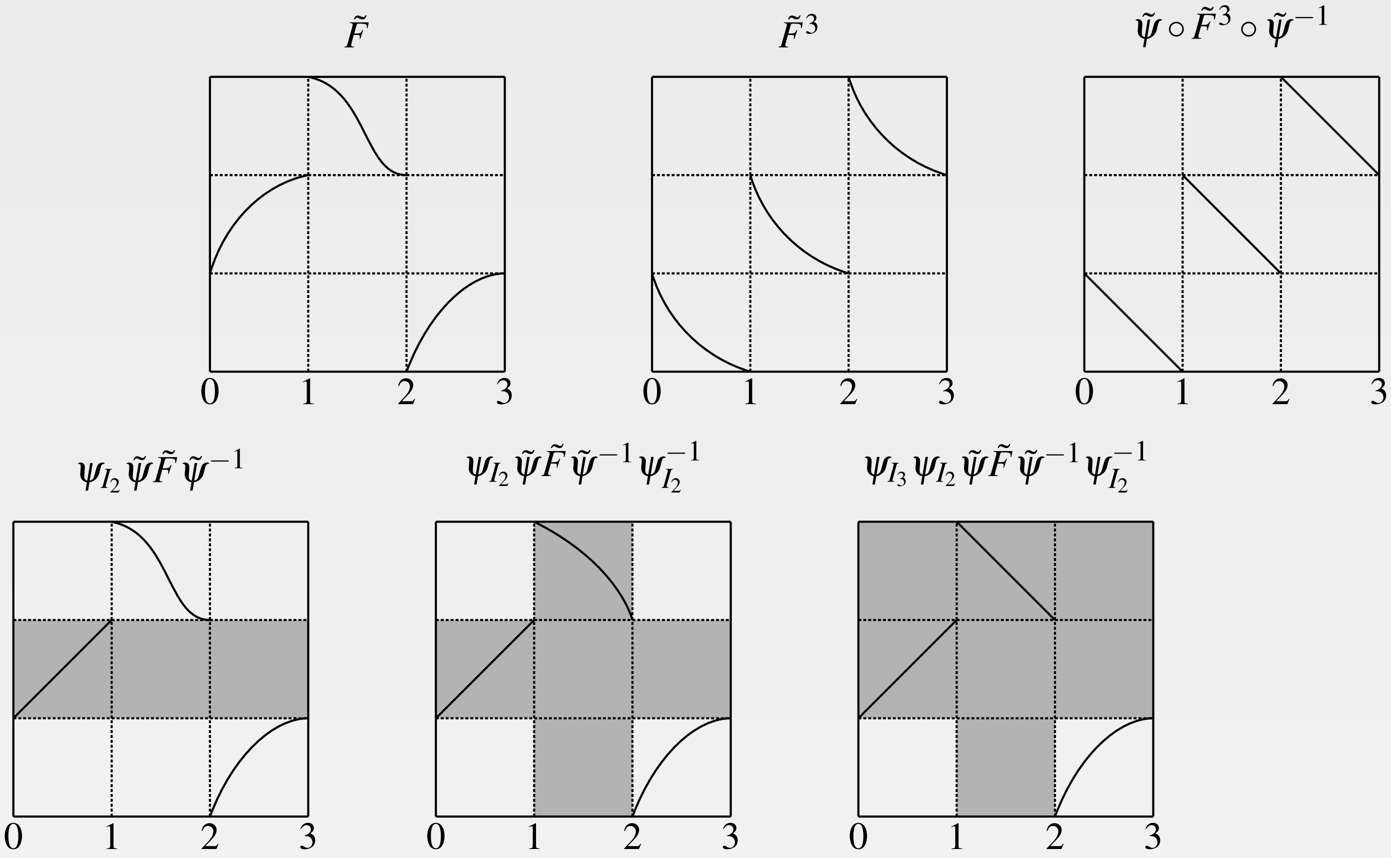

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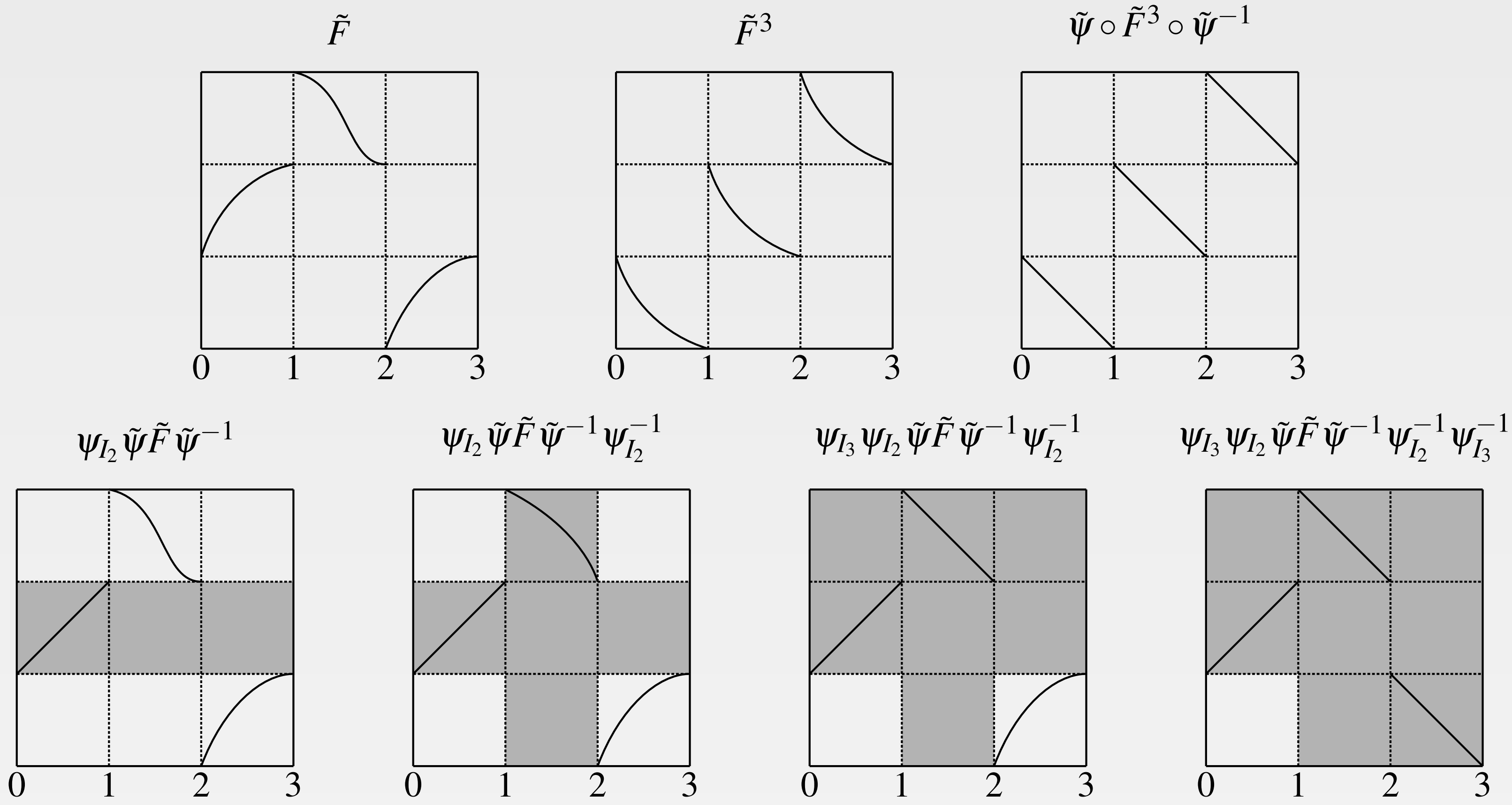

In the **second case** $\tilde{F}^{\ell}|_{I_{i_j}}$ is a bijective and decreasing mapping on I_{I_j}, thus a decreasing involution, for each $1 \le j \le \ell$. There exists a bijective and increasing function $\tilde{\psi}$ so that $(\tilde{\psi} \circ \tilde{F}^{\ell} \circ \tilde{\psi}^{-1})|_{I_{i_j}}$ is also affine, i.e. $x \mapsto 2i_j - 1 - x$, $x \in I_{i_j}$ for each $1 \le j \le \ell$.

Without loss of generality $\tilde{F}^{\ell}|_{I_{i_j}}$ is affine for each $1 \le j \le \ell$.

Similar to the first case, there exists $\psi = \psi_{i_\ell} \circ \ldots \circ \psi_{i_2}$ so that $\psi \circ \tilde{F}|_{I_{i_j}} \circ \psi|_{I_{i_j}}$ is affine on I_{i_j} for $1 \le j \le \ell - 1$.
By construction $\psi(x) = x$ for $x \in I_{i_1}$.

Therefore we have
$\psi \circ \tilde{F}^{\ell}|_{I_{i_1}} \circ \psi^{-1}|_{I_{i_1}} = \psi(2i_1 - 1 - \psi^{-1}(x)) = 2i_1 - 1 - x =$
$\psi \circ \tilde{F}|_{I_{i_\ell}} \circ \psi^{-1}|_{I_{i_\ell}} \circ \Big(\psi \circ \tilde{F}|_{I_{i_{\ell-1}}} \circ \psi^{-1}|_{I_{i_{\ell-1}}} \circ \cdots \circ \psi \circ \tilde{F}|_{I_{i_1}} \circ \psi^{-1}|_{I_{i_1}}\Big)(x)$.
The term between $\big($ and $\big)$ is a composition of affine function, thus it is affine, whence also $\psi \circ \tilde{F}|_{I_{i_\ell}} \circ \psi^{-1}|_{I_{i_\ell}}$ is affine.

Consequently $\psi \circ \tilde{F}|_{I_{i_j}} \circ \psi^{-1}$ is affine on I_{i_j} for each $1 \le j \le \ell$.

Equivalence

Two bijective functions $F_1\colon J_1 \to J_1$ and $F_2\colon J_2 \to J_2$ defined on compact intervals J_1 and J_2 are considered in relation

$$F_1 \sim F_2$$

iff there exists a bijective increasing function $\varphi\colon J_1 \to J_2$ so that

$$F_2 = \varphi \circ F_1 \circ \varphi^{-1}.$$

We have:

$$F_1 \sim F_1$$

$$F_1 \sim F_2 \Leftrightarrow F_2 \sim F_1$$

$$F_1 \sim F_2 \text{ and } F_2 \sim F_3 \Rightarrow F_1 \sim F_3$$

Theorem 3

Consider $f_1, f_2 \in S_{n+1} \times (\{\pm 1\} \wr S_n)$ with $n = n(f_1) = n(f_2)$. Then

$$f_1 \sim f_2 \Leftrightarrow f_1 = f_2.$$

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Theorem 4

Consider an iterative root $F : J \to J$ of the identity of order k on a compact interval J with finitely many discontinuities. Let $n = n(F)$. Then there exists exactly one $f \in S_{n+1} \times (\{\pm 1\} \wr S_n)$ so that

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Remark.

Consider $f \in S_{n+1} \times (\{\pm 1\} \wr S_n)$ with $m = n(f) < n$. Then there exists $f' \in S_{m+1} \times (\{\pm 1\} \wr S_m)$ so that $f \sim f'$. It is possible that there exists $f'' \in S_{n+1} \times (\{\pm 1\} \wr S_n)$, $f'' \neq f$, so that $f \sim f''$.

Number of non-equivalent functions with $n(f) = n$

n	$n!(n+1)!2^n$	$n(f) < n$	$n(f) = n$
0	1		
1	4		
2	48	4	44
3	1152	40	1112
4	46080	892	45188
5	2764800	37708	2727092
6	232243200	2337808	229905392
7	26011238400	201311920	25809926480
8	3745618329600	22951808356	3722666521244
9	674211299328000		
10	148326485852160000		

How to enumerate them? How to construct them?

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Stetigkeiten in [1,3,6]
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A permutation λ for $n = 7$, where neither 1, 3 nor 6 occur in the orbit of a discontinuity of f if $f(1) = 3$, $f(3) = 6$ and $f(6) = 1$. Thus they could be omitted in order to get a function on $n = 4$.

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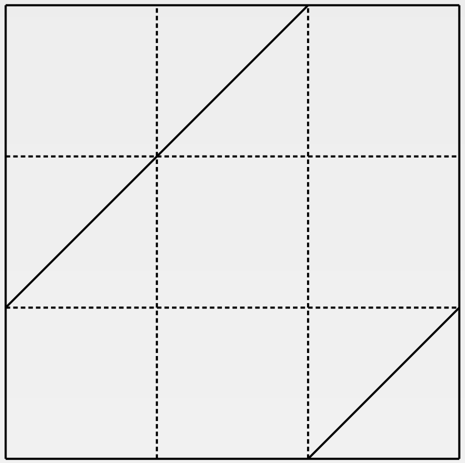
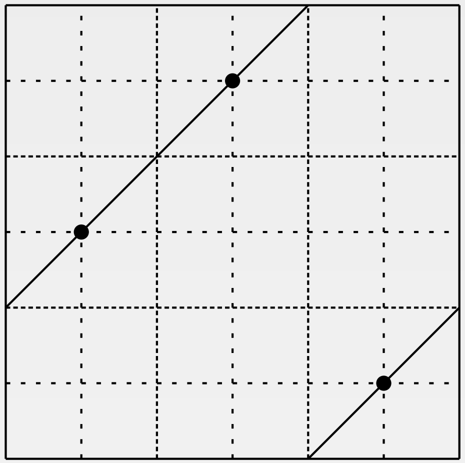
Depending on $f(i)$, $i \in \{0,2,4,5,7\}$ there are $(8-3)!$ functions of this particular form with $n(f) = 4$.

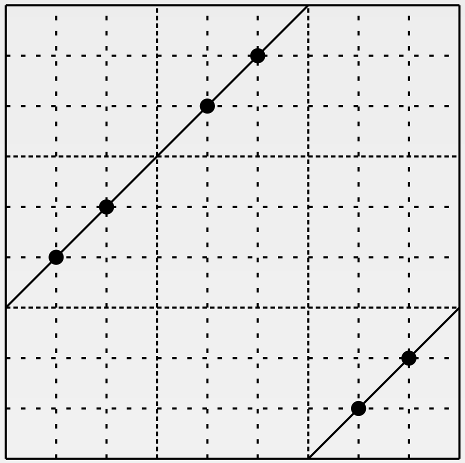
How to construct functions with $n(f) < n$?

Divide all intervals $(i-1,i)$ belonging to a cycle of λ into k intervals of length $1/k$, then form a cycle of length ℓ we obtain $k \cdot \ell$ intervals.

E.g. a function f with $n(f) = 3$, where $\lambda = (1,2,3)$, and $k = 1$, $k = 2$, $k = 3$, which yield $n = 3$, $n = 6$, $n = 9$.







Open questions

Do the functions f with $n(f) = n$ generate the set of all functions of type III on n?

If so, is it true that for each f with $n(f) < n$ there exists some $\tilde{f}$ with $n(\tilde{f}) = n$ and some j so that $f = \tilde{f}^j$?

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