On iteration of bijective functions with discontinuities

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During the ISFE54 Zygfryd Kominek raised discussion about the behavior of the iterates of real functions with discontinuities. **“Is it possible that the k-th iterate of such a function is continuous?”** During the problems and remarks sessions there were some remarks concerning this topic by Roman Ger, Peter Stadler and myself. Finally I was told that only **surjective** functions are interesting.

Therefore we discuss different types of bijective functions defined on a compact interval with finitely many removable and/or jump discontinuities.

Removable discontinuities

Functions of type I:
$I = [a,b]$ be a closed real interval, $a < b$,
$f: I \to I$ bijective with finitely many removable discontinuities
$\exists n \geq 2$ and $a \leq x_1 < \ldots < x_n \leq b$, so that $f(x) = x$ for
$x \in I \setminus \{x_i \mid 1 \leq i \leq n\}$ and f is not continuous in x_i, $1 \leq i \leq n$.

f bijective $\Rightarrow \forall j \; \exists! i \neq j$ so that $f(x_i) = x_j$.
Thus f defines a permutation $\pi \in S_n$ by

$$\pi(i) = j \iff f(x_i) = x_j.$$

Then π is free of fixed points, thus π is an derangement.

f^k is continuous, iff $f^k = \text{id}$.

$f^k(x_i) = x_{\pi^k(i)}$, $1 \leq i \leq n$, $k \in \mathbb{N}$.

f^k is continuous, iff $\pi^k = \text{id}$, iff $\operatorname{ord}(\pi) \mid k$.

Enumeration of derangements

Let d_n be the number of derangements in S_n:
recursive formulae:

$$d_0 = 1,\ d_1 = 0,\ d_n = (n-1)(d_{n-1} + d_{n-2}), \qquad n \geq 2.$$

$$d_0 = 1,\ d_n = nd_{n-1} + (-1)^n, \qquad n \geq 1.$$

Sieve formula:

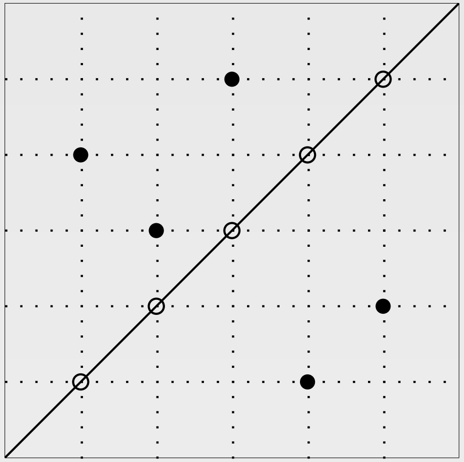
$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}, \qquad n \geq 0.$$

These numbers can be found as A000166 in the On-Line Encyclopedia of Integer Sequences.

Some numerical values:

n	d_n	$d_n/n!$
0	1	1
1	0	0
2	1	0.5
3	2	0.333333
4	9	0.375
5	44	0.366666
6	265	0.368055
7	1 854	0.367857
8	14 833	0.367881
9	133 496	0.367879
10	1 334 961	0.367879
11	14 684 570	0.367879
12	12 176 214 841	0.367879

Example

$f =$ 

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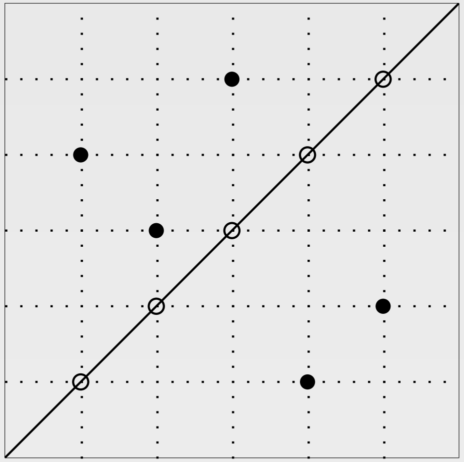
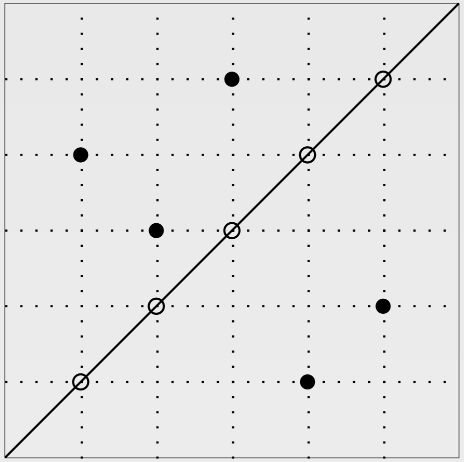
Example

$f =$ 

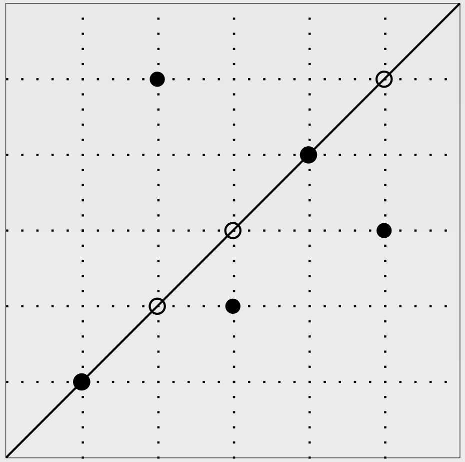
$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} = (1,4)(2,3,5),$$

Example

$f =$

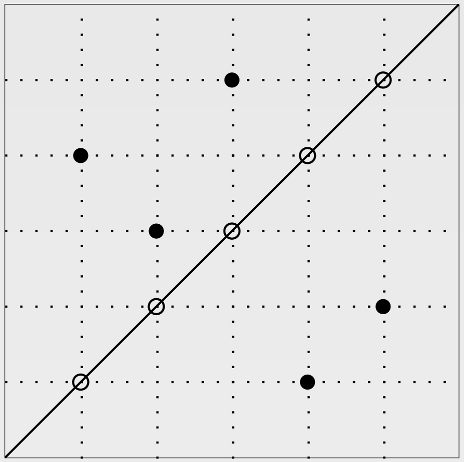
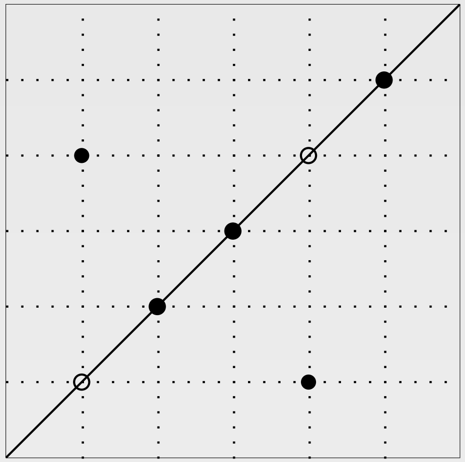


$f^2 =$



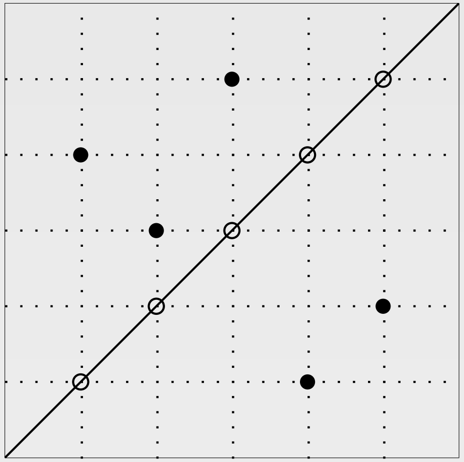
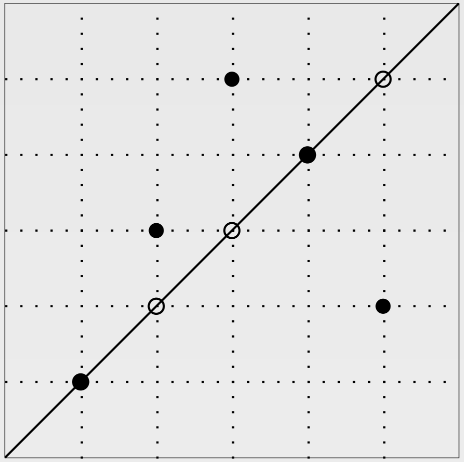
$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} = (1,4)(2,3,5), \quad \pi^2 = (2,5,3).$$

Example

$f =$  $f^3 =$ 

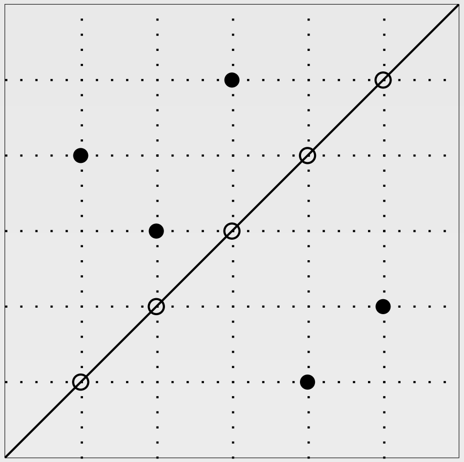
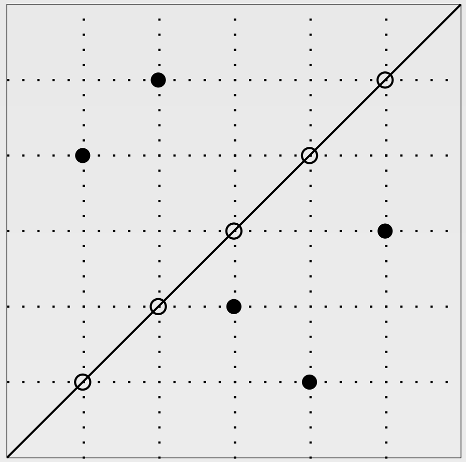
$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} = (1,4)(2,3,5), \;\; \pi^3 = (1,4).$$

Example

$f =$  $f^4 =$ 

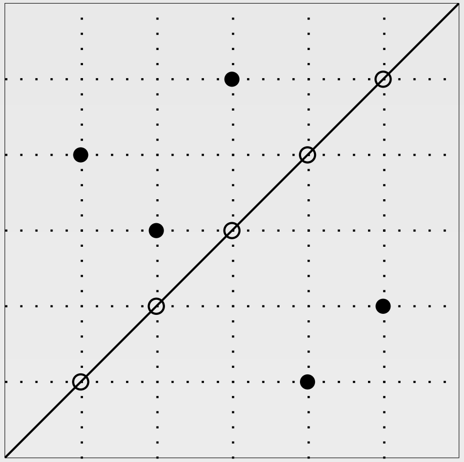
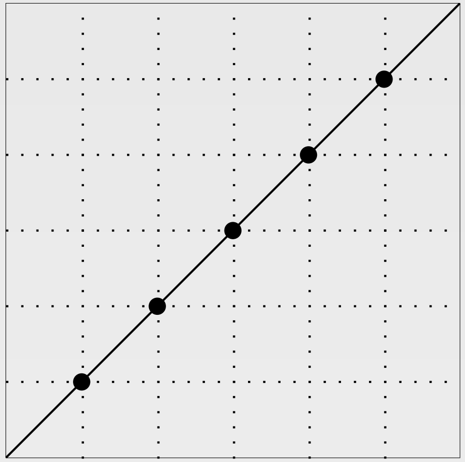
$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} = (1,4)(2,3,5), \quad \pi^4 = (2,3,5).$$

Example

$f =$  $f^5 =$ 

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} = (1,4)(2,3,5), \quad \pi^5 = (1,4)(2,5,3).$$

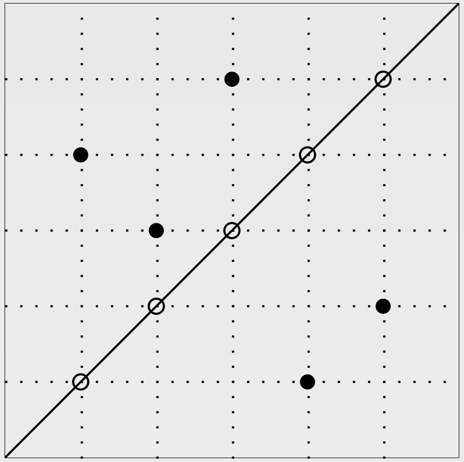
Example

$f =$  $f^6 =$ 

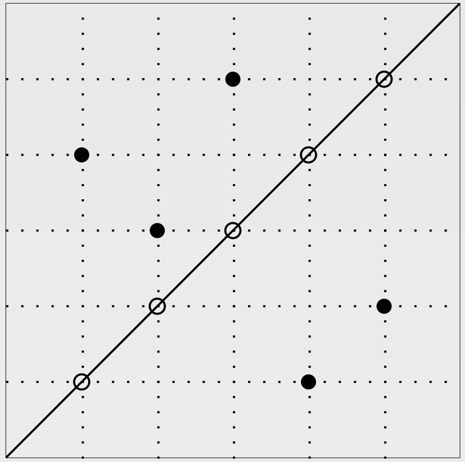
$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} = (1,4)(2,3,5), \quad \pi^6 = \text{id}.$$

Example

$f =$



$f^7 =$



$= f$

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} = (1,4)(2,3,5), \;\; \pi^7 = \pi.$$

Discontinuities in a cycle of length i disappear in f^k iff $i \mid k$.

Number of discontinuities of f^k, order of f

If π decomposes into a_i cycles of length i, then $a = (a_1, \ldots, a_n)$ is the cycle type of π. It satisfies

$$\sum_i i a_i = n.$$

The number of discontinuities of f^k is

$$n - \sum_{i|k} i a_i = \sum_{i \nmid k} i a_i.$$

The order of π is the least common multiple $\operatorname{ord}(\pi) = \operatorname{lcm}\{i \mid a_i \neq 0\}$. The maximum possible order of permutations in S_n is given by the Landau function

$$g(n) := \max\{\operatorname{ord}(\pi) \mid \pi \in S_n\}.$$

$$g(n) \leq g(n+1)$$

$$\tilde{g}(n) = \max\{\operatorname{ord}(\pi) \mid \pi \in S_n, \text{ a derangement }\}$$

$$\tilde{g}(n) \leq g(n), \qquad g(n) < g(n+1) \Rightarrow \tilde{g}(n+1) = g(n+1)$$

n	$g(n)$	$\tilde{g}(n)$
2	2	2
3	3	3
4	4	4
5	6	6
6	6	6
7	12	12
8	15	15
9	20	20
10	30	30
11	30	30
12	60	60
13	60	42
102	446185740	446185740
103	446185740	314954640
104	446185740	446185740

For $g(n)$ see A000793, for $\tilde{g}(n)$ see A123131 in the OEIS.

Conjugacy classes in S_n

Permutations which are conjugate in S_n lead to similar behavior.
Conjugacy classes in $S_n \leftrightarrow$ cycle types of n.
Cycle types of derangements in $S_n \leftrightarrow$ partitions of n having no parts of size 1.
A partition of n is a sequence $\alpha = (\alpha_1, \ldots, \alpha_h)$ with $\alpha_1 \geq \ldots \geq \alpha_h$ and $\alpha_1 + \ldots + \alpha_h = n$.

E.g., partitions of $n = 8$ with no parts of size 1:
$8 = 6+2 = 5+3 = 4+4 = 4+2+2 = 3+3+2 = 2+2+2+2$.
These are 7 different types.

For given n the set of $\{k \in \mathbb{N} \mid f$ is of type I and has n discontinuities, $f^k = \text{id}$, $f^j \neq \text{id}$, $1 \leq j < k\}$ is finite. It is a subset of $\{2, \ldots, \tilde{g}(n)\}$.

E.g., for $n = 8$ it is $\{8, 6, 15, 4, 4, 6, 2\}$.

There is a well known formula for the number of permutations in the conjugacy class of cycle type $(a_1, \ldots, a_n)$.

n	d_n	$\tilde{p}_n$	p_n
0	1	1	1
1	0	0	1
2	1	1	2
3	2	1	3
4	9	2	5
5	44	2	7
6	265	4	11
7	1 854	4	15
8	14 833	7	22
9	133 496	8	30
10	1 334 961	12	42
11	14 684 570	14	56
12	12 176 214 841	21	77

For $\tilde{p}_n$, the partition numbers without 1, see A002865, for p_n, the partition numbers, see A000041 in the OEIS.

Summary for type I

The behavior of a function f of type I with n discontinuities is totally described by the permutation $\pi \in S_n$ which is a derangement.

f^k is continuous, iff k is a multiple of $\mathrm{ord}(\pi)$.

The number of discontinuities of f^k can be described in terms of the cycle type of π. Thus it depends only on the conjugacy class of π.

There are no functions of type I with n removable discontinuities so that the minimum $k > 0$ with f^k is continuous is greater than $\tilde{g}(n)$. E.g., there are no functions with 2 removable discontinuities so that f^3 is continuous.

There are no functions of type I with exactly one removable discontinuity.

The iterates f^k have at most as many discontinuities as f.

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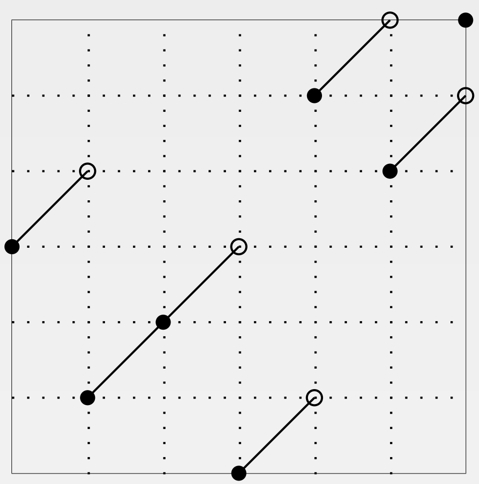
Jump discontinuities

Functions of type II:
Now we consider bijective functions $f\colon [0,n] \to [0,n]$, $n \geq 2$, so that for each $i \in \{1,\dots,n\}$ there exists one $j \in \{1,\dots,n\}$ so that

$$f(t) = t - (i-1) + (j-1) = t - i + j, \qquad t \in [i-1, i),$$

and $f(n) = n$. Therefore f is continuous in each interval $I_i := [i-1, i)$ (in $i-1$ continuous from the right).

$f =$ 

Since f is bijective, it defines a permutation $\pi \in S_n$ by

$$\pi(i) = j \Longleftrightarrow f(I_i) = I_j.$$

$$\pi = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 4\,2\,3\,1\,6\,5 \end{pmatrix}, \ \pi(2+1) = \pi(2) + 1.$$

Successions of a permutation

Then

$$f(t) = \pi(i) + t - i, \qquad t \in I_i, \quad i \in \{1, \ldots, n\}.$$

f is continuous in i, iff $\pi(i+1) = \pi(i) + 1$, $1 \leq i < n$.

f is continuous in n, iff $\pi(n) = n$.

f^k is continuous, iff $f^k = \mathrm{id}$.

$$f^k(t) = \pi^k(i) + t - i, \qquad t \in I_i, \quad i \in \{1, \ldots, n\}.$$

f^k is continuous, iff $\pi^k = \mathrm{id}$.

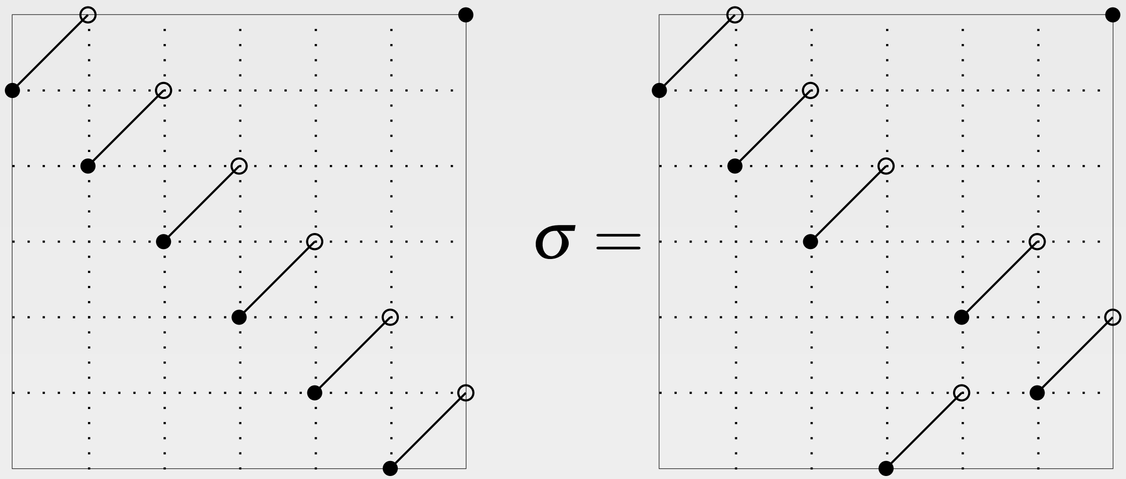
$i \in \{1, \ldots, n-1\}$ is called a succession (or a small ascent) of π, iff $\pi(i+1) = \pi(i) + 1$.

The number of discontinuities of f among $\{1, \ldots, n-1\}$ is the number of i-s which are no successions of π.

A permutation π without successions satisfying $\pi(n) < n$ defines a function with n discontinuities.

E.g., $\pi = (1,n)(2,n-1)\ldots$ or $\sigma = (1,n,2,n-1,\ldots)$ lead to n discontinuities of f.

$\pi =$ $\qquad$ $\sigma =$



Discontinuities can appear only in the positions $1,\ldots,n$. These functions have the maximum number of discontinuities.

Permutations without successions

Let a_n be the number of permutations in S_n having no successions and b_n the number of permutations in S_n having exactly one succession, then

$$a_1 = 1, \quad a_2 = 1, \quad b_1 = 0, \quad b_2 = 1,$$

and

$$a_n = (n-1)a_{n-1} + b_{n-1}, \qquad n \geq 2,$$
$$b_n = (n-1)a_{n-1}, \qquad n \geq 2,$$

thus

$$a_n = (n-1)a_{n-1} + (n-2)a_{n-2} = b_n + b_{n-1}, \qquad n \geq 3.$$
$$b_n = (n-1)(b_{n-1} + b_{n-2}), \qquad n \geq 3.$$

Thus $b_n = d_n$, $n \geq 1$.

For a_n see A000255 in the OEIS.

Functions with maximum number of discontinuities

Let c_n be the number of permutations π in S_n having no successions satisfying $\pi(n) = n$. Then

$$c_n = a_{n-1} - c_{n-1}, \qquad n \geq 2.$$

Therefore $a_{n-1} = c_n + c_{n-1}$ and since $c_2 = b_1$ and $c_3 = b_2$ we deduce $c_n = b_{n-1}$, $n \geq 2$.

The number of permutations π in S_n having no successions and satisfying $\pi(n) < n$ is therefore

$$a_n - c_n = a_n - b_{n-1} = b_n = (n-1)a_{n-1}, \qquad n \geq 2.$$

This is the number of functions $f\colon [0,n] \to [0,n]$ of type II having n discontinuities (in the points $1, \dots, n$).

Permutations with prescribed number of successions

Let $a_{n,k}$ be the number of permutations $\pi \in S_n$ having exactly k successions, $0 \le k < n$, then $a_{n,0} = a_n$ and $a_{n,1} = b_n$.

$$a_{n,k} = \frac{(n-1)!}{k!} \sum_{j=0}^{n-k-1} (-1)^j \frac{n-k-j}{j!} = \binom{n-1}{k} a_{n-k}$$

Therefore

$$n! = \sum_{k=0}^{n-1} a_{n,k} = \sum_{k=0}^{n-1} \binom{n-1}{k} a_{n-k}.$$

By binomial inversion we obtain

$$a_n = \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} (k+1)!$$

n	$a_{n,0}$	$a_{n,1}$	$a_{n,2}$	$a_{n,3}$	$a_{n,4}$	$a_{n,5}$	$a_{n,6}$	$a_{n,7}$	$a_{n,8}$
3	3	2	1						
4	11	9	3	1					
5	53	44	18	4	1				
6	309	265	110	30	5	1			
7	2119	1854	795	220	45	6	1		
8	16687	14833	6489	1855	385	63	7	1	
9	148329	133496	59332	17304	3710	616	84	8	1
10	1468457	1334961	600732	177996	38934	6678	924	108	9

See A123513 in the OEIS.

$a_{n,n-1} = 1$, $\pi = \text{id}$,

$a_{n,n-2} = n-1$, $\pi = (1,\dots,n)^j$, $j = 1,\dots,n-1$,

$$a_{n,n-3} = 3\sum_{j=3}^{n}(j-2).$$ A045943

$a_{n,n-4}$. A111080

Cycles with many successions

We consider a cycle of length $k \geq 2$ with $k-2$ successions,

$$\pi = (1,2,\ldots,k) = \begin{pmatrix} 1 & 2 & \ldots & k-1 & k \\ 2 & 3 & \ldots & k & 1 \end{pmatrix}.$$

Then for $1 \leq j < k$

$$\pi^j = \begin{pmatrix} 1 & 2 & \ldots & k-j & k-j+1 & \ldots & k \\ j+1 & j+2 & \ldots & k & 1 & \ldots & j \end{pmatrix}$$

has

$$\begin{cases} k-2 \text{ successions} & \text{if } k \nmid j \\ k-1 \text{ successions} & \text{if } k \mid j. \end{cases}$$

Let $f_{1,k}\colon [0,k] \to [0,k]$ be the function of type II determined by π, then the iterates $f_{1,k}^j$ have

$$\begin{cases} 2 \text{ discontinuities} & \text{if } k \nmid j \\ 0 \text{ discontinuities} & \text{if } k \mid j. \end{cases}$$

Discontinuities in $k-(j \bmod k)$ and k.

The iterates $f_{s,k}^{j}$ of the functions $f_{s,k}\colon [0,sk] \to [0,sk]$ corresponding to the product of s cycles of length k

$$(1,2,\ldots,k)(k+1,k+2,\ldots,2k)\cdots((s-1)k+1,\ldots,sk)$$

have
$$\begin{cases} 2s \text{ discontinuities} & \text{if } k \nmid j \\ 0 \text{ discontinuities} & \text{if } k \mid j. \end{cases}$$

Discontinuities in $rk-(j \bmod k)$ and rk for $1 \le r \le s$.

The iterates $f_{s,k}^{j}$ of the functions $f_{s,k}\colon [0,sk] \to [0,sk]$ corresponding to the product of s cycles of length k

$$(1,2,\ldots,k)(k+1,k+2,\ldots,2k)\cdots((s-1)k+1,\ldots,sk)$$

have
$$\begin{cases} 2s \text{ discontinuities} & \text{if } k \nmid j \\ 0 \text{ discontinuities} & \text{if } k \mid j. \end{cases}$$
Discontinuities in $rk-(j \bmod k)$ and rk for $1 \le r \le s$.

Similarly we consider the iterates $g_{s,k}^{j}$ of the functions $g_{s,k}\colon [0,sk+1] \to [0,sk+1]$ corresponding to the product of s cycles and one fixed point

$$(1)(2,3,\ldots,k+1)(k+2,k+3,\ldots,2k+1)\cdots((s-1)k+2,\ldots,sk+1).$$

They have
$$\begin{cases} 2s+1 \text{ discontinuities} & \text{if } k \nmid j \\ 0 \text{ discontinuities} & \text{if } k \mid j. \end{cases}$$
Discontinuities in 1 and $rk+1-(j \bmod k)$ and $rk+1$ for $1 \le r \le s$.

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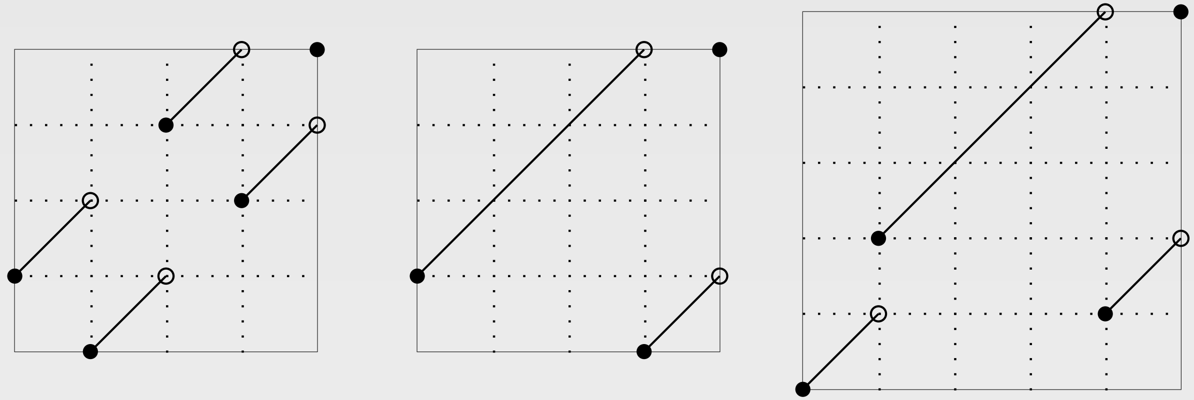
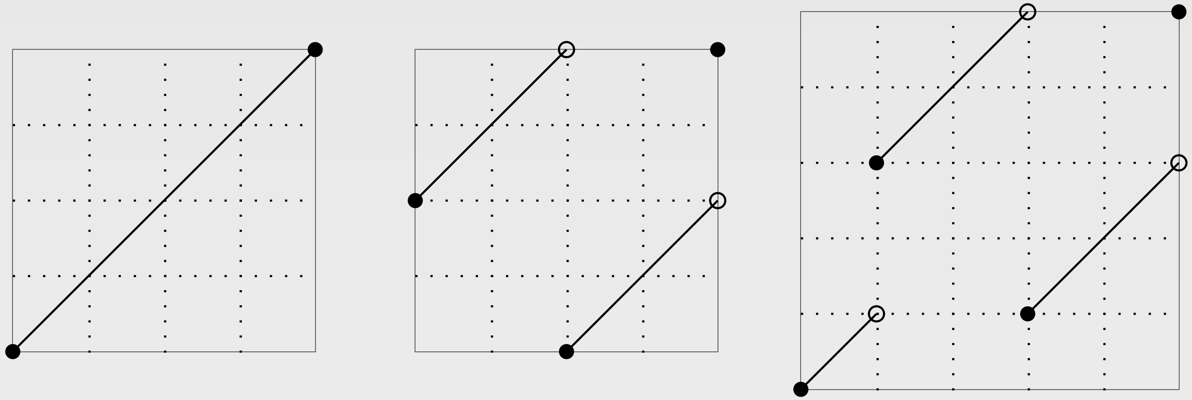
E.g., the iterates

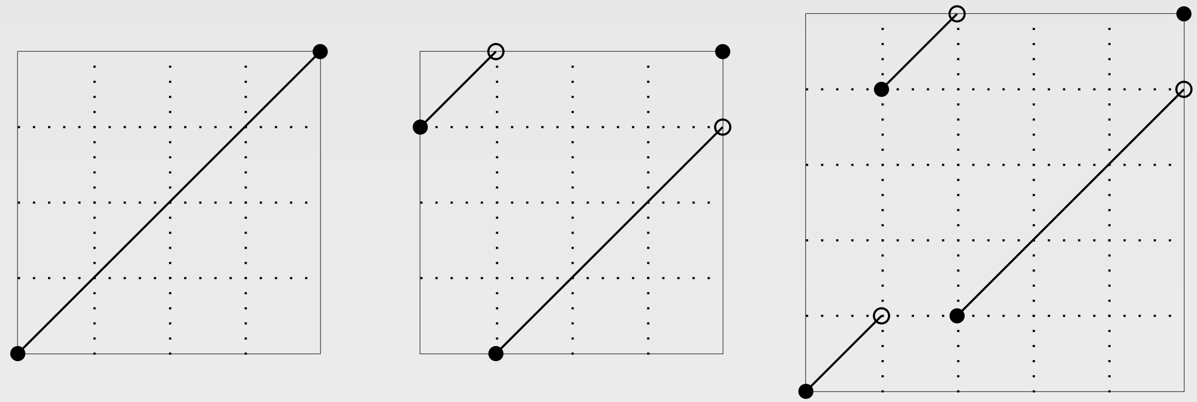
E.g., the iterates $f_{2,2}^{1}$, $f_{1,4}^{1}$ and $g_{1,4}^{1}$ of the functions $f_{2,2}$, $f_{1,4}$ and $g_{1,4}$ are:



E.g., the iterates $f_{2,2}^2$, $f_{1,4}^2$ and $g_{1,4}^2$ of the functions $f_{2,2}$, $f_{1,4}$ and $g_{1,4}$ are:



E.g., the iterates $f_{2,2}^2$, $f_{1,4}^3$ and $g_{1,4}^3$ of the functions $f_{2,2}$, $f_{1,4}$ and $g_{1,4}$ are:



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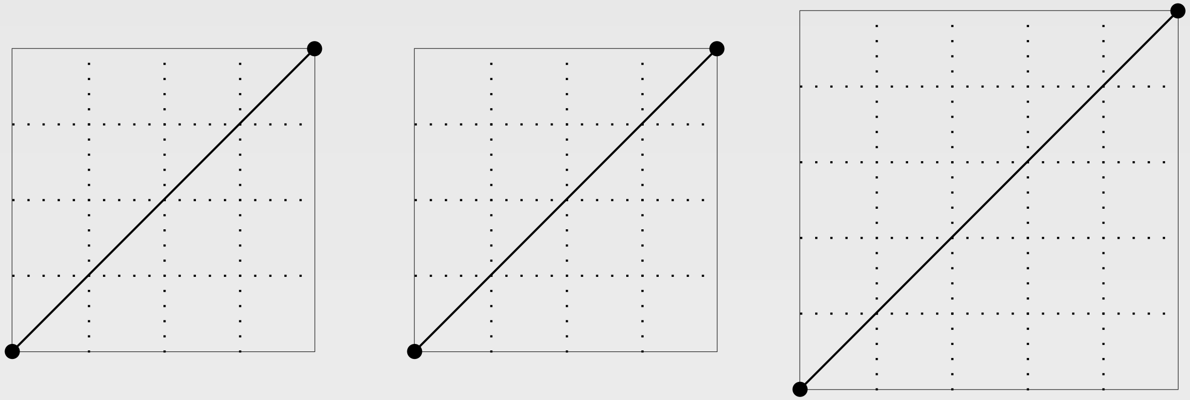
E.g., the iterates $f_{2,2}^2$, $f_{1,4}^4$ and $g_{1,4}^4$ of the functions $f_{2,2}$, $f_{1,4}$ and $g_{1,4}$ are:



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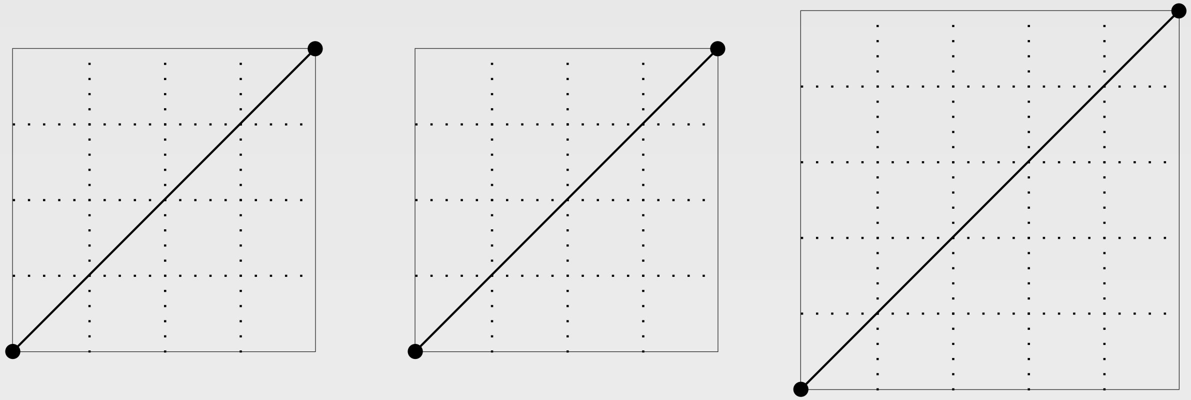
E.g., the iterates $f^2_{2,2}$, $f^4_{1,4}$ and $g^4_{1,4}$ of the functions $f_{2,2}$, $f_{1,4}$ and $g_{1,4}$ are:



Theorem

For any $n \geq 2$ and $k \geq 2$ the iterates $f^j_{n/2,k}$ (for even n) or $g^j_{(n-1)/2,k}$ (for odd n) of the functions $f_{n/2,k}$, or $g_{(n-1)/2,k}$ have

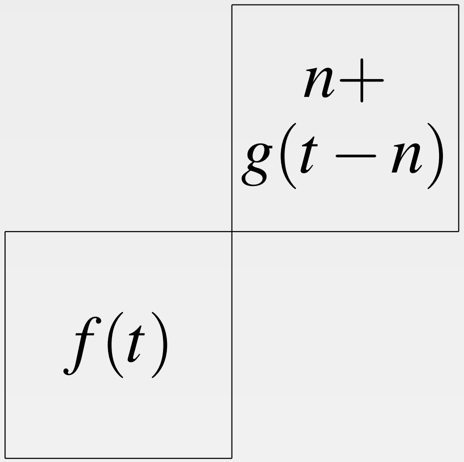
$$\begin{cases} n \text{ discontinuities} & \text{if } k \not| \, j \\ 0 \text{ discontinuities} & \text{if } k \mid j. \end{cases}$$

Concatenation of functions

Given two functions $f\colon [0,n] \to [0,n]$ and $g\colon [0,m] \to [0,m]$ of type II, then $f \bullet g\colon [0,n+m] \to [0,n+m]$

$$(f \bullet g)(t) = \begin{cases} f(t) & \text{if } t \in [0,n) \\ n+g(t-n) & \text{if } t \in [n,n+m) \\ n+m & \text{if } t = n+m. \end{cases}$$

is of type II.



Since f and g are bijective and $f(n) = n$, the concatenation $f \bullet g$ is bijective, thus $f \bullet g$ is of type II.

If furthermore f is continuous in n and $g(0) = 0$, then $f \bullet g$ is continuous in n since g is continuous from the right side in 0.

$f \bullet g$ is not continuous in n, iff f is not continuous in n or $g(0) \neq 0$.

Assume that f and g of type II have r respectively s discontinuities. Then the number of discontinuities of $f \bullet g$ is

$$\begin{cases} r+s+1 & \text{if } f \text{ is continuous in } n \text{ and } g(0) \neq 0, \\ r+s & \text{else.} \end{cases}$$

Actually $f_{s,k} = f_{s-1,k} \bullet f_{1,k}$ and $g_{s,k} = g_{s-1,k} \bullet f_{1,k}$ for $s > 1$.

Even though $f_{1,k}(0) \neq 0$ the function f_{sk} has $2s$ (and g_{sk} has $2s+1$) discontinuities since $f_{s-1,k}$ and $g_{s-1,k}$ are not continuous at the end of their domains.

Combining cycles of different length the discontinuities at positions between two cycles must be studied separately.

The functions $g_{s,k}$ satisfy $g_{s,k}(0) = 0$, thus the j-th iterate of the concatenation of $g_{s_1,k_1} \bullet \ldots \bullet g_{s_r,k_r}$ has

$$\sum_{\substack{i=1 \\ k_i \nmid j}}^{r} (2s_i + 1)$$

discontinuities. Concatenation of $g_{s,k}$ does not introduce new discontinuities.

Concatenation of the functions $f_{s,k}$ is more complicated, since $f_{s,k}(0) = 2 \neq 0$, and $f_{s,k}^{j}(0) = 0$ whenever j is a multiple of k.

E.g., let $h = f_{1,2} \bullet f_{1,3}$ and $h' = f_{1,3} \bullet f_{1,2}$, then the number of discontinuities of h^j and h'^j is

j	1	2	3	4	5	6
discontinuities of h^j	4	3	2	3	4	0
discontinuities of h'^j	4	2	3	2	4	0

h: discontinuities in 1 and 2 (from $f_{1,2}$) and in 4 and 5 (from $f_{1,3}$).
h^2: discontinuities in 2 ($f_{1,3}^2(0) \neq 0$) and in 3 and 5 (from $f_{1,3}^2$).
h^3: discontinuities in 1 and 2 (from $f_{1,2}^3$).
h^4: discontinuities in 2 ($f_{1,3}^4(0) \neq 0$) and in 4 and 5 (from $f_{1,3}^4$).
h^5: discontinuities in 1 and 2 (from $f_{1,2}^5$) and in 3 and 5 (from $f_{1,3}^5$).

We study permutations starting and ending with a fixed point, thus functions which have their discontinuities in the interior of the domain.

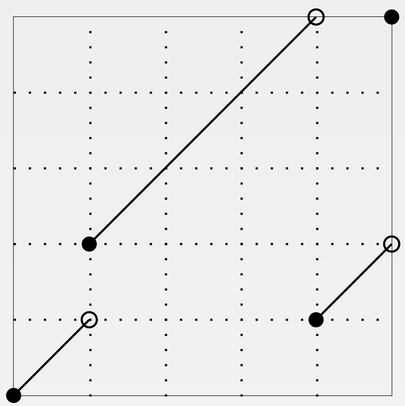
We study functions with an even number ℓ of discontinuities.

If ℓ is even, $\ell \geq 6$, then $\ell = (\ell - 3) + 3$, and the iterates f^j of the function $f = g_{(\ell-4)/2,k} \bullet g_{1,k}$ have

$$\begin{cases} \ell \text{ discontinuities} & \text{if } k \not| \, j \\ 0 \text{ discontinuities} & \text{if } k \mid j. \end{cases}$$

It can be used instead of $f_{\ell/2,k}$.

2 discontinuities: There is no function $f\colon [0,n] \to [0,n]$ so that $f(0) = 0$ and $f(n) = n$ which has exactly two discontinuities.



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4 discontinuities: The permutation $\pi = (1)(2,4)(3)(5)$ has order 2 and yields 4 discontinuities.

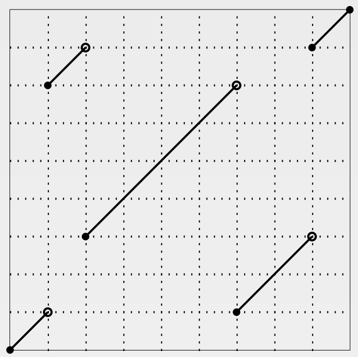
There is no permutation of order 3 which yields 4 discontinuities.

A family of permutations of order $2k+1$, $k \geq 2$, which yields a function f having 4 discontinuities.

$\pi = (1)(2,6,3,4,5)(7)$,

$\pi = (1)(2,8,3,4,5,6,7)(9)$,

$\pi = (1)(2,2k+2,3,4,\dots,2k+1)(2k+3)$



The number of discontinuities of the iterates f^j:

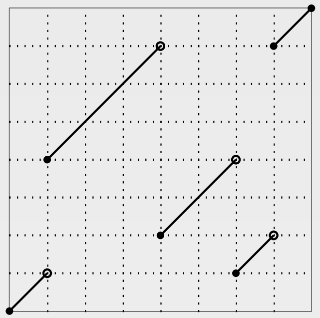
j	1	2	$3,\dots,2k-2$	$2k-1$	$2k$	$2k+1$
discontinuities of f^j	4	6	7	6	4	0

A family of permutations of order $2k$, $k \geq 2$, which yields a function f having 4 discontinuities.

$\pi = (1)(2,4,3,5)(6)$,

$\pi = (1)(2,5,3,6,4,7)(8)$,

$\pi = (1)(2,k+2,3,k+3,\ldots,k+1,2k+1)(2k+2)$.



The number of discontinuities of the iterates f^j:

j	1	$2,\ldots,2k-2$	$2k-1$	$2k$
discontinuities of f^j	4	5	4	0

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Theorem For $\ell \in \{3,5,6,7,\dots\}$ and $k \geq 2$ we have found functions $h_{\ell,k}\colon [0,n] \to [0,n]$ of type II, $h_{\ell,k}(0) = 0$, $h_{\ell,k}(n) = n$, continuous in 0 and n so that their iterates $h^j_{\ell,k}$ have

$$\begin{cases} \ell \text{ discontinuities} & \text{if } k \not| \, j \\ 0 \text{ discontinuities} & \text{if } k \mid j. \end{cases}$$

Then the j-th iterate of the concatenation

$$h_{\ell_1,k_1} \bullet \dots \bullet h_{\ell_r,k_r}$$

$\ell_i \in \{3,5,6,7,\dots\}$, $k_i \geq 2$, $1 \leq i \leq r$, has exactly

$$\sum_{\substack{i=1 \\ k_i \not| \, j}}^{r} \ell_i$$

discontinuities. $h_{\ell,k}$ corresponds to $g_{(\ell-1)/2,k}$ if $\ell \equiv 1 \bmod 2$, and to $g_{(\ell-4)/2,k} \bullet g_{1,k}$ if $\ell \equiv 0 \bmod 2$.

Summary for type II

The behavior of a function f of type II is totally described by the permutation $\pi \in S_n$.

f^k is continuous, iff k is a multiple of $\mathrm{ord}(\pi)$.

The number of discontinuities of f^k can be described in terms of successions of π^k and the value $\pi^k(n)$, but not in terms of the cycle type of π.

There is no functions of type II

- with exactly one discontinuity,
- with exactly two discontinuities in the interior of the domain,
- of order 3 with 4 discontinuities in the interior of the domain.

Iterates f^k can have more discontinuities than f.

A generalization

Functions of type III

A bijective function $f\colon [0,n] \to [0,n]$
f permutes the integers $\{0,1,\ldots,n\}$,
$\forall i \in \{1,\ldots,n\}\ \exists j \in \{1,\ldots,n\}$ so that either

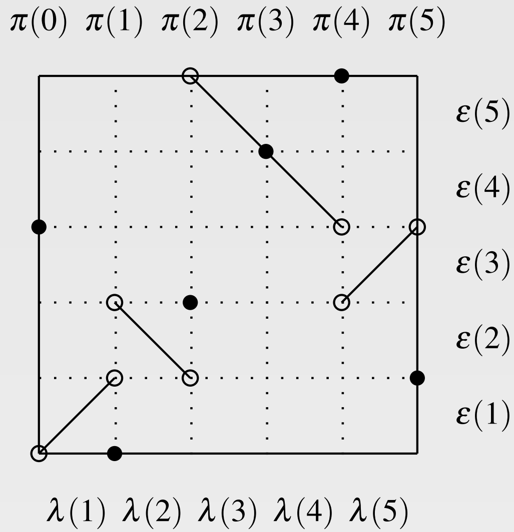
$$f(t) = t-(i-1)+(j-1) = t-i+j, \qquad t \in (i-1,i),$$

or

$$f(t) = j-(t-(i-1)) = j+i-1-t, \qquad t \in (i-1,i).$$

f permutes the open intervals $I_i = (i-1,i)$, $1 \le i \le n$.
First case f monotonically increasing, second case f decreasing on I_i.

E.g.,



$$\begin{array}{lll}
\pi(0)=3 & & \\
\pi(1)=0 & \lambda(1)=1 & \varepsilon(1)=1 \\
\pi(2)=2 & \lambda(2)=2 & \varepsilon(2)=-1 \\
\pi(3)=4 & \lambda(3)=5 & \varepsilon(3)=1 \\
\pi(4)=5 & \lambda(4)=4 & \varepsilon(4)=-1 \\
\pi(5)=1 & \lambda(5)=3 & \varepsilon(5)=-1
\end{array}$$

$\varepsilon(i)=1$ iff the values of I_i (in the range) appear in an increasing way, iff f is increasing on $I_{\lambda^{-1}(i)}$.

We identify f with $(\pi,(\varepsilon,\lambda))$, $\pi\in S_{n+1}$, $\varepsilon\in\{\pm1\}^n$, $\lambda\in S_n$.

$$f(t)=\lambda(i)-\frac{1}{2}+\varepsilon(\lambda(i))(t-i+\frac{1}{2}), \qquad t\in I_i,\ 1\le i\le n$$

$\varepsilon(\lambda(i))$ is the direction of f on the interval I_i in the domain.

f is continuous in $i \in \{1, \ldots, n-1\}$, iff
either $\varepsilon(\lambda(i)) = \varepsilon(\lambda(i+1)) = 1$, $\lambda(i+1) = \lambda(i)+1$, and $\pi(i) = \lambda(i)$,
or $\varepsilon(\lambda(i)) = \varepsilon(\lambda(i+1)) = -1$, $\lambda(i+1) = \lambda(i)-1$, and $\pi(i) = \lambda(i+1)$.

f is continuous in 0 iff
either $\varepsilon(\lambda(1)) = 1$ and $\pi(0) = \lambda(1)-1$
or $\varepsilon(\lambda(1)) = -1$ and $\pi(0) = \lambda(1)$.

f is continuous in n must be studied accordingly.

f^k is continuous if either $f^k = \mathrm{id}$ or $f^k = n - \mathrm{id}$.

Structure theorem

Composition of $f \leftrightarrow (\pi, (\varepsilon, \lambda))$ and $f' \leftrightarrow (\pi', (\varepsilon', \lambda'))$ yields

$$f \circ f' \leftrightarrow (\pi \circ \pi', (\varepsilon\varepsilon'_\lambda, \lambda \circ \lambda'))$$

where

$$\varepsilon\varepsilon'_\lambda(i) = \varepsilon(i)\varepsilon'(\lambda^{-1}(i)), \qquad i \in \{1, \ldots, n\}.$$

The set of all functions of type III is the direct product

$$S_{n+1} \times (\{\pm 1\} \wr S_n)$$

where the factor on the right side is a wreath product

$$\{\pm 1\} \wr S_n = \{(\varepsilon, \lambda) \mid \varepsilon \in \{\pm 1\}^n,\ \lambda \in S_n\}$$

of order $n! \cdot 2^n$ with $(\varepsilon, \lambda)(\varepsilon', \lambda') = (\varepsilon\varepsilon'_\lambda, \lambda \circ \lambda')$.

The number of functions of type III on $[0,n]$ is

$$n!(n+1)!2^n$$

n	$n!(n+1)!2^n$
0	1
1	4
2	48
3	1152
4	46080
5	2764800
6	232243200
7	26011238400
8	3745618329600
9	674211299328000
10	148326485852160000

Functions of type I or type II are particular cases of these functions.

The order of f

With each cycle of $\lambda = \prod_\nu (j_\nu, \lambda(j_\nu), \ldots, \lambda^{l_\nu - 1}(j_\nu))$ we associate the ν-th cycle product
$h_\nu(\varepsilon, \lambda) = \varepsilon(j_\nu)\varepsilon(\lambda^{-1}(j_\nu)) \cdots \varepsilon(\lambda^{-l_\nu+1}(j_\nu)) = \varepsilon \cdots \varepsilon_{\lambda^{l_\nu - 1}}(j_\nu)$.

It is the direction of f^{l_ν} on the intervals I_j for $j \in \{j_\nu, \lambda(j_\nu), \ldots, \lambda^{l_\nu - 1}(j_\nu)\}$.

$f^k = \text{id}$, iff $(\pi^k, (\varepsilon, \lambda)^k) = (\text{id}, (1, \text{id}))$, iff $\pi^k = \text{id}$, $\lambda^k = \text{id}$, (thus $l_\nu \mid k$ for all ν) and $h_\nu^{k/l_\nu}(\varepsilon, \lambda) = 1$ for all ν.

Thus k is a multiple of $\text{ord}(\pi)$ and $\text{ord}(\varepsilon, \lambda)$. The latter is either $\text{ord}(\lambda)$ or $2\,\text{ord}(\lambda)$.

The smallest positive k with these properties is the order of f

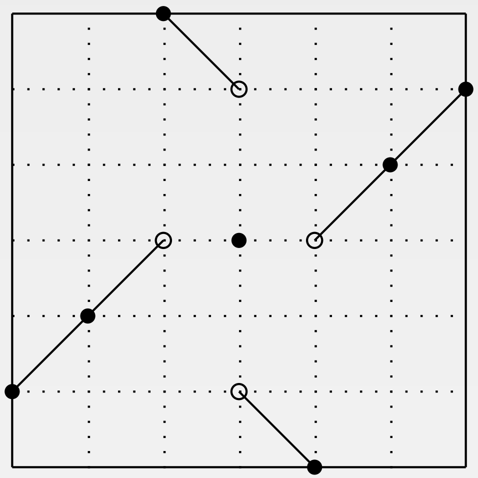
$$\text{ord}(f) = \text{lcm}(\text{ord}(\pi), \text{ord}(\varepsilon, \lambda)).$$

Decreasing continuous iterate

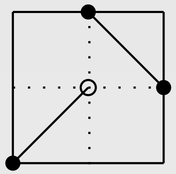
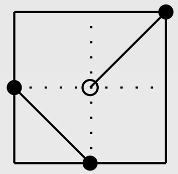
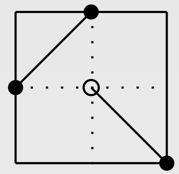
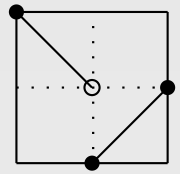
$f^k = (\pi^k, (\lambda^k, \tilde{\varepsilon})) = n -$ id, iff
$\pi^k = (0,n)(1,n-1)\ldots$, $\lambda^k = (1,n)(2,n-1)\ldots$, and $\tilde{\varepsilon} = -1$.
Thus π and λ are iterative roots of order k of permutations of cycle type

$$\begin{cases} (0, \frac{n+1}{2}) \text{ and } (1, \frac{n-1}{2}), & \text{if } n+1 \equiv 0 \bmod 2, \\ (1, \frac{n}{2}) \text{ and } (0, \frac{n}{2}), & \text{if } n+1 \equiv 1 \bmod 2. \end{cases}$$

They can be enumerated and also constructed.

E.g., $f =$  satisfies $f^3 = 6 -$ id.

Functions with exactly one discontinuity

Four forms:  or  or  or  where the interval $[0,n]$ can be partitioned into two intervals $[0,n_1]$ and $[n_1,n]$ so that f is strictly monotonic on both intervals. Let $n_2 = n - n_1$.

A function of the **first form** is the concatenation of id_{n_1} and $n_2 - \mathrm{id}_{n_2}$ both of which are continuous, and the discontinuity disappears with the second iteration.

For functions of the **second form** the discontinuity also disappears with the second iteration.

For functions of the **third form**:

If $n_1 = n_2 = 1$, then
$\pi = (0, 1, 2)$ a cycle of length 3,
$\lambda = (1, 2)$ a cycle of length 2 with cycle product -1,
$\text{ord}(\varepsilon, \lambda) = 4$,
$\text{ord}(f) = \text{lcm}(3, 4) = 12$.

If $n_1 = 1$, $n_2 > 1$, then
$\pi = (0, n-1, 1, n)(2, n-2)(3, n-3)\ldots$ a product of cycles of length 4, 2, (and 1)
$\lambda = (1, n)(2, n-1)\ldots$ a product of cycles of length 2 (and 1) where the first cycle product is -1 (and the last cycle product is -1 in case the last cycle has length 1),
$\text{ord}(\varepsilon, \lambda) = 4$,
$\text{ord}(f) = 4$.

If $n_1 = 2$ and $n_2 = 1$ or $n_2 = 2$, then $\mathrm{ord}(f) = 12$.

If $n_1 = 2$, $n_2 > 2$, then
$\pi = (0, n-2, 2, n)(1, n-1)(3, n-3)(4, n-4)\ldots$ a product of cycles of length 4, 2, (and 1)
$\lambda = (1, n-1, 2, n)(3, n-2)(4, n-3)\ldots$ a product of cycles of length 4, 2 (and 1) where only the last cycle product is -1 in case the last cycle has length 1,
$\mathrm{ord}(\varepsilon, \lambda) = 4$,
$\mathrm{ord}(f) = 4$.

For functions of the **forth form**:

If $n_1 = 1$, then
$\pi = (0, n, n-1, \ldots, 2, 1)$ a cycle of length $n+1$,
$\lambda = (1, n, n-1, \ldots, 3, 2)$ a cycle of length n with cycle product -1,
$\operatorname{ord}(\varepsilon, \lambda) = 2n$,
$\operatorname{ord}(f) = \operatorname{lcm}(n+1, 2n)$.

If $n_1 = 2$ and n is odd, then
$\pi = (0, n, n-2, \ldots, 3, 1, n-1, n-3, \ldots, 4, 2)$ a cycle of length $n+1$,
$\lambda = (1, n, n-2, \ldots, 5, 3)(2, n-1, n-3, \ldots, 6, 4)$ a product of two cycles of length $(n+1)/2$ and $(n-1)/2$, with both cycle products -1,
$\operatorname{ord}(\varepsilon, \lambda) = 2\operatorname{lcm}((n+1)/2, (n-1)/2) = (n^2-1)/2$,
$\operatorname{ord}(f) = \operatorname{lcm}(n+1, (n^2-1)/2) = (n^2-1)/2$.

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If $n_1 = 2$ and n is even, then

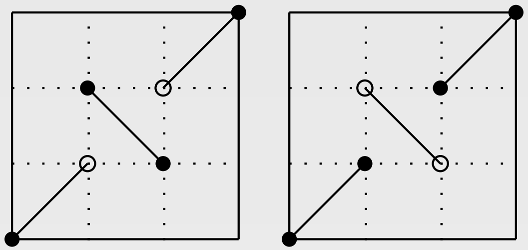
$\pi = (0, n, n-2, \ldots, 4, 2)(1, n-1, n-3, \ldots, 5, 3)$a product of two cycles of length $n/2+1$ and $n/2$,

$\lambda = (1, n, n-2, \ldots, 4, 2, n-1, n-3, \ldots, 5, 3)$ a cycle of length n with cycle product 1,

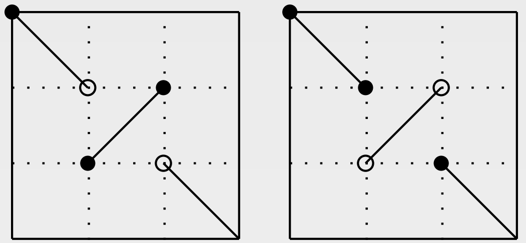
ord$(\varepsilon, \lambda) = n$,

ord$(f) =$ lcm(lcm$(n/2+1, n/2), n) =$ lcm$(n/2+1, n)$.

Functions with exactly two discontinuities in the interior of the interval



have exactly two discontinuities in the interior of the interval and are of order 2. They correspond to the first and second form.



have exactly two discontinuities in the interior of the interval and are of order 2. They correspond to the third and fourth form.

The interval can be partitioned into 3 parts $[0, n_1]$, $[n_1, n_2]$, $[n_2, n_3]$ with $n_1 < n_2 < n_3 \in \mathbb{N}$.

General remarks

Let J be a compact interval and $F\colon J \to J$ a bijective mapping with finitely many discontinuities, then they must be removeable or jump discontinuities.

General remarks

Let J be a compact interval and $F\colon J \to J$ a bijective mapping with finitely many discontinuities, then they must be removeable or jump discontinuities.

Let $\varphi\colon J \to [0,n]$ be continuous, bijective, and increasing, and $f\colon [0,n] \to [0,n]$ be of type III with r discontinuities and $\operatorname{ord}(f) = k$, then

$$F := \varphi^{-1} \circ f \circ \varphi\colon J \to J$$

is bijective, has r discontinuities, $F^k = \mathrm{id}_J$, and F is an iterative root of the identity of order k.

An iterative root of order 6 of the identity with 3 discontinuities and an iterative root of order 3 of $1 - \mathrm{id}$ constructed from the function with decreasing continuous iterate.

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