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On barycentrically associative symmetric formal power series

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Barycentrically associative formal power series

For $n \in \mathbb{N} = \{1, 2, \ldots\}$ let $F_n = F_n(x_1, \ldots, x_n) \in \mathbb{C}[[x_1, \ldots, x_n]]$ of order greater or equal to 1. The family $(F_n)_{n \in \mathbb{N}}$ is called *barycentrically associative* if for all $n \in \mathbb{N}$ and for all $\ell \in \{1, \ldots, n\}$ and all nonnegative integers r so that $r + \ell \leq n$

$$F_n(x_1, \ldots, x_n) =$$
$$F_n(x_1, \ldots, x_r, \underbrace{F_\ell(x_{r+1}, \ldots, x_{r+\ell}), \ldots, F_\ell(x_{r+1}, \ldots, x_{r+\ell})}_{\ell-\text{times}}, x_{r+\ell+1}, \ldots x_n)$$

holds true.

Associativity of means

Decomposability

Some remarks

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4. Assume that $(F_n)_{n\in\mathbb{N}}$ is barycentrically associative.
   For $n \in \mathbb{N}$ let M_n be the homogeneous part of degree 1 of F_n (if $F_n = 0$ then $M_n = 0$), then the family $(M_n)_{n\in\mathbb{N}}$ is a barycentrically associative family of polynomials.

5. Classification of barycentrically associative polynomials [1]:
For $z \in \mathbb{C}$ and $n \in \mathbb{N}$ let

$$\Delta_n^z = \sum_{i=1}^{n} z^{n-i}(1-z)^{i-1}.$$

If $\Delta_n^z \neq 0$, then

$$M_n^z(x_1,\ldots,x_n) = \sum_{i=1}^{n} z^{n-i}(1-z)^{i-1}x_i/\Delta_n^z.$$

Let $n(z) = \inf\{n \in \mathbb{N} \mid \Delta_n^z = 0\}$ $(n(z) = \infty$ if $\Delta_n^z \neq 0$ for all $n)$.

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A family $(M_n)_{n\in\mathbb{N}}$ of polynomials is barycentrically associative if and only if:

Type I: there exists some $z \in \mathbb{C}$ and some $k \in \mathbb{N} \cup \{\infty\}$ with $k \leq n(z)$ so that $M_n = M_n^z$ for all $n < k$ and M_n constant for $n \geq k$;

Type II: $F_1(x) = x$, $F_2(x,y) \in \mathbb{C}[x,y]$ of degree at least 2, so that $F_2(x,x) = x$, and F_n constant for all $n \geq 3$.

6. Assume that $(F_n)_{n\in\mathbb{N}}$ is barycentrically associative.
If $f(x) = x + \ldots \in \mathbb{C}[[x]]$, then $(G_n)_{n\in\mathbb{N}}$ is barycentrically associative, where

$$G_n(x_1,\ldots,x_n) = (f^{-1}\circ F_n \circ f)(x_1,\ldots,x_n) = f^{-1}(F_n(f(x_1),\ldots,f(x_n)))$$

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7. Assume that $(M_n)_{n\in\mathbb{N}}$ is a family of barycentrically associative polynomials of type I, and $f(x) = x + \ldots \in \mathbb{C}[[x]]$, then $(f^{-1}\circ M_n \circ f)_{n\in\mathbb{N}}$ is a nontrivial family of barycentrically associative formal power series.

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6. Assume that $(F_n)_{n\in\mathbb{N}}$ is barycentrically associative. If $f(x) = x + \ldots \in \mathbb{C}[[x]]$, then $(G_n)_{n\in\mathbb{N}}$ is barycentrically associative, where

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8. Assume that $(F_n)_{n\in\mathbb{N}}$ is a family of barycentrically associative formal power series. Let $n_0 \geq 2$. If $F_{n_0} = M^z_{n_0}$, a polynomial, then F_{n_0+1} is a polynomial.

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9. Assume that $(F_n)_{n\in\mathbb{N}}$ is barycentrically associative. Let $n_0 \geq 2$. If $F_{n_0} = f^{-1}\circ M^z_{n_0}\circ f$ for some $f(x) = x + \ldots \in \mathbb{C}[[x]]$, $z \in \mathbb{C}$ where $n(z) > n_0 + 1$, and if $F_{n_0+1} \neq 0$, then $F_{n_0+1} = f^{-1}\circ M^z_{n_0+1}\circ f$.

Symmetric barycentrically associative formal series

$F_n(x_1,\ldots,x_n)$ is symmetric if $F_n(x_1,\ldots,x_n) = F_n(x_{\pi(1)},\ldots,x_{\pi(n)})$ for all $\pi \in S_n$.

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Assume that $(F_n)_{n\in\mathbb{N}}$ is a barycentrically associative family of symmetric formal series. Then:

$$F_1(x) = x$$

$F_2(x,y) = \frac{1}{2}(x+y)+\ldots = \varphi(x)+\varphi(y)+xyh(x,y)$, where $\varphi(x) = \frac{1}{2}x+\ldots$ and $h(x,y) = h(y,x)$.

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There exists some $f(x) = x+\ldots \in \mathbb{C}[[x]]$ so that $(f^{-1}\circ\varphi\circ f)(x) = \frac{1}{2}x$.

Let $\tilde{F}_n = f^{-1}\circ F_n \circ f$, then $(\tilde{F}_n)_{n\in\mathbb{N}}$ is a barycentrically associative family of symmetric formal series, and $\tilde{F}_2(x,y) = f^{-1}(F(f(x),f(y))) = \frac{1}{2}(x+y)+xy\tilde{h}(x,y)$. We want to prove that if $\tilde{F}_3 \neq 0$, then $\tilde{h}(x,y) = 0$, thus $\tilde{F}_2(x,y) = \frac{1}{2}(x+y) = M_2^{1/2}(x,y)$.

Let $P(x,y) = \tilde{F}_3(x,x,y)$ and $Q(x,y) = \tilde{F}_3(x,y,y)$, then $x = \tilde{F}_3(x,x,x) = P(x,x) = Q(x,x)$ and

$$\begin{aligned}\tilde{F}_3(x,y,z) &= \tilde{F}_3(\tilde{F}_2(x,y), \tilde{F}_2(x,y), z) = P(\tilde{F}_2(x,y), z) \\ &= \ldots = Q(x, \tilde{F}_2(y,z)) = P(\tilde{F}_2(y,z), x).\end{aligned}$$

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Let $P(x,y)=\tilde{F}_3(x,x,y)$ and $Q(x,y)=\tilde{F}_3(x,y,y)$, then $x=\tilde{F}_3(x,x,x)=P(x,x)=Q(x,x)$ and

$$\begin{aligned}\tilde{F}_3(x,y,z) &= \tilde{F}_3(\tilde{F}_2(x,y),\tilde{F}_2(x,y),z) = P(\tilde{F}_2(x,y),z)\\ &= \ldots = Q(x,\tilde{F}_2(y,z)) = P(\tilde{F}_2(y,z),x).\end{aligned}$$

Ansatz: $P(x,y)=\sum_{n\geq 1}\sum_{i=0}^{n} a_{i,n-i}x^i y^{n-i}$ and $xy\tilde{h}(x,y)=\sum_{n\geq 2}\sum_{i=1}^{n-1} h_{i,n-i}x^i y^{n-i}$.

Comparing coefficients in $P(\tilde{F}_2(x,y),z)=P(\tilde{F}_2(y,z),x)$ we obtain that $a_{1,0}=2/3$, $a_{0,1}=1/3$, and $a_{i,j}=0$ for all $i,j\geq 0$, $i+j\geq 2$, and $h_{i,j}=0$ for all $i,j\geq 1$, $i+j\geq 2$.

Consequently $\tilde{h}(x,y)=0$.

Theorem. Let $(F_n)_{n\in\mathbb{N}}$ be a family of symmetric formal power series of order at least 1. This family is barycentrically associative if and only if

$F_1(x) = x$, $F_2(x,y) \in \mathbb{C}[[x,y]]$ symmetric, so that $F_2(x,x) = x$, and $F_n = 0$ for all $n \geq 3$;

or there exists some $f(x) = x + \ldots \in \mathbb{C}[[x]]$ and some $k \in \mathbb{N} \cup \{\infty\}$ so that $F_n = f^{-1} \circ M_n^{1/2} \circ f$ for all $n < k$ and $F_n = 0$ for $n \geq k$.

Bibliography

[1] Jean-Luc Marichal, Pierre Mathonet and Jörg Tomaschek. A classification of barycentrically associative polynomial functions *Aequationes mathematicae*, 2015.

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