A remark on n-associative power series

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$$F(x_1,x_2,\ldots,x_n) \in \mathbb{C}[[x_1,x_2,\ldots,x_n]],\ n \geq 3,$$

$F(0,0,\ldots,0) = 0$, so F can be substituted into any formal power series

$$F(x_1,x_2\ldots,x_n) = \sum_{\substack{(i_1,\ldots,i_n)\in\mathbb{N}_0^n \\ i_1+\ldots+i_n\geq 1}} a_{i_1,\ldots,i_n} x_1^{i_1}\cdots x_n^{i_n}$$

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$$F(F(x_1, x_2, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1}) \qquad (1)$$

$$F(x_1, F(x_2, \ldots, x_{n+1}), x_{n+2}, \ldots, x_{2n-1}) \qquad (2)$$

$$\ldots$$

$$F(x_1, \ldots, x_{n-1}, F(x_n, x_{n+1}, \ldots, x_{2n-1})) \qquad (n)$$

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F is n-associative if $(1)=(2)=\ldots=(n)$. This is a system of $\binom{n}{2}$ equations in $\mathbb{C}[[x_1,x_2,\ldots,x_{2n-1}]]$.

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3. Solutions of the form $F(x_1,\ldots,x_n) = x_1 + x_n + \ldots$:

$$F(x_1,\ldots,x_n) = f^{-1}(f(x_1) + \rho f(x_2) + \rho^2 f(x_3) + \ldots + f(x_n))$$

$f(x) = x + \sum_{k\geq 2} f_k x^k \in \Gamma_1$.

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For $F(x_1,\dots,x_n) \in \mathbb{C}[[x_1,\dots,x_n]]$ let

$$(D_jF)(x_1,\dots,x_n) = \frac{\partial}{\partial x_j} F(x_1,\dots,x_n), \qquad 1 \le j \le n.$$

Similarly as in the last part of [2], it is possible to determine the power series f by differentiation. This will be shown next:

For $F(x_1,\ldots,x_n) \in \mathbb{C}[[x_1,\ldots,x_n]]$ let

$$(D_jF)(x_1,\ldots,x_n) = \frac{\partial}{\partial x_j} F(x_1,\ldots,x_n), \qquad 1 \le j \le n.$$

We will prove the following fact. If

$$F(x_1,\ldots,x_n) = x_1 + x_n + \sum_{\substack{(j_1,\ldots,j_n)\in\mathbb{N}_0^n \\ j_1+\ldots+j_n\ge 1}} a_{j_1,\ldots,j_n} x_1^{j_1}\cdots x_n^{j_n}$$

is an n-associative formal power series, then there exists a uniquely determined series $f(x) = x + \ldots \in \Gamma_1$, so that $F(x_1,\ldots,x_n) = f^{-1}\big(f(x_1) + \rho f(x_2) + \ldots + \rho^{n-2} f(x_{n-1}) + f(x_n)\big)$, where $\rho = a_{0,1,0,\ldots,0}$ which satisfies $\rho^{n-1} = 1$.

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Lemma

If F is n-associative and of the form $f^{-1}\big(f(x_1)+\rho f(x_2)+\ldots+\rho^{n-2}f(x_{n-1})+f(x_n)\big)$, where ρ is a root of unity of order $n-1$, and $f(x)=x+\ldots\in\Gamma_1$, then

$$f'(x)=\frac{1}{D_1F(0,\ldots,0,x)}.$$

Lemma

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$$f'(x)=\frac{1}{D_1F(0,\ldots,0,x)}.$$

Proof.

$$f\big(F(x_1,\ldots,x_n)\big)=f(x_1)+\rho f(x_2)+\ldots+\rho^{n-2}f(x_{n-1})+f(x_n)\Longrightarrow$$

$$f'\big(F(x_1,\ldots,x_n)\big)D_1F(x_1,\ldots,x_n)=f'(x_1)\Longrightarrow$$

$$f'\big(F(0,\ldots,0,x_n)\big)D_1F(0,\ldots,0,x_n)=f'(0)\Longrightarrow$$

$$f'(x_n)D_1F(0,\ldots,0,x_n)=1$$

Both $f'(x_n)=1+\ldots$ and $D_1F(0,\ldots,0,x_n)=1+\ldots$ are units in $\mathbb{C}[[x_1,\ldots,x_n]]$ and the assertion follows.

Theorem

Let $F(x_1,\dots,x_n)$ be n-associative, $\rho = a_{0,1,0,\dots,0}$, and let $f(x)$ be the unique primitive series of $1/D_1F(0,\dots,0,x)$ satisfying $f(0)=0$, then $f(x) = x+\dots \in \Gamma_1$, and
$F(x_1,\dots,x_n) = f^{-1}\big(f(x_1)+\rho f(x_2)+\dots+\rho^{n-2}f(x_{n-1})+f(x_n)\big)$.

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Proof.

1. If we write $F(x_1,\dots,x_n) = \sum_{j=1}^n \varphi_j(x_j) + \hat{F}(x_1,\dots,x_n)$, then we can show that there exists a root ρ of unity of order $n-1$ and a series $\psi(x)\in\Gamma_1$ so that $\varphi_2(x) = \psi(\rho\psi^{-1}(x))$ and $\varphi_j(x) = \varphi_2^{j-1}(x)$, $1\le j\le n$.

$$D_jF(0,\dots,0) = \frac{\partial}{\partial x_j}\left(\sum_{i=1}^n \varphi_i(x_i) + \hat{F}(x_1,\dots,x_n)\right)\Bigg|_{x_1=\ldots=x_n=0}$$

$$= \varphi_j'(0) + 0 = \rho^{j-1}, \qquad 1\le j\le n.$$

2. Now we prove that

$$D_jF(0,\dots,0,x_n) = \frac{\rho^{j-1}}{f'(x_n)}, \qquad 1 \le j \le n-1.$$

For $j = 1$ the assertion is clear due to the construction of f. Consider some $2 \le j \le n-1$. Differentiation of (1) = (j) with respect to x_j we have

$$D_1F\big(F(x_1,\dots,x_n),x_{n+1},\dots,x_{2n-1}\big)D_jF(x_1,\dots,x_n) =$$

$$D_jF\big(x_1,\dots,x_{j-1},F(x_j,\dots,x_{j+n-1}),x_{j+n},\dots,x_{2n-1}\big)D_1F(x_j,\dots,x_{j+n-1}).$$

Setting $x_1 = \ldots = x_{2n-2} = 0$ we obtain

$$D_1F(0,\dots,0,x_{2n-1})D_jF(0,\dots,0) = D_jF(0,\dots,0,x_{2n-1})D_1F(0,\dots,0)$$

which means that

$$D_jF(0,\dots,0,x_{2n-1}) = \frac{\rho^{j-1}}{f'(x_{2n-1})}.$$

3. For $1 \le j < n$ we have

$$D_jF(\underbrace{0,\ldots,0}_{j-1\text{-times}},x_j,\ldots,x_n) = \frac{\rho^{j-1}f'(x_j)}{f'\big(F(0,\ldots,0,x_j,\ldots,x_n)\big)}. \qquad (*)$$

We differentiate (j) = (n) with respect to x_j, and set $x_1 = \ldots = x_{j+n-2} = 0$ to get

$$D_jF\big(0,\ldots,0,F(0,\ldots,0,x_{j+n-1}),x_{j+n},\ldots,x_{2n-1}\big)D_1F(0,\ldots,0,x_{j+n-1}) =$$
$$D_jF\big(0,\ldots,0,F(0,\ldots,0,x_{j+n-1},\ldots,x_{2n-1})\big),$$

which means by 2. that

$$D_jF(0,\ldots,0,x_{j+n-1},\ldots,x_{2n-1}) = \frac{\rho^{j-1}f'(x_{j+n-1})}{f'\big(F(0,\ldots,0,x_{j+n-1},\ldots,x_{2n-1})\big)}.$$

4. Let $G(x_1,\dots,x_n) = f\big(F(x_1,\dots,x_n)\big) - f(x_1) - \rho f(x_2) - \dots - f(x_n)$. We prove that $G = 0$. Differentiating G with respect to x_1 we obtain

$$D_1G(x_1,\dots,x_n) = f'\big(F(x_1,\dots,x_n)\big)D_1F(x_1,\dots,x_n) - f'(x_1).$$

From $(*)$ for $j = 1$ we get $D_1G(x_1,\dots,x_n) = 0$, thus $G(x_1,\dots,x_n)$ does not depend on x_1, whence $G(x_1,\dots,x_n) = G(0,x_2,\dots,x_n)$.

Using the same arguments we show for $2 \le j < n$ that $G(0,\dots,0,x_j,\dots,x_n) = G(0,\dots,0,x_{j+1},\dots,x_n)$.

Finally we have $G(x_1,\dots,x_n) = G(0,\dots,0,x_n) = f\big(F(0,\dots,0,x_n)\big) - f(x_n) = f(x_n) - f(x_n) = 0$. Thus $G = 0$ and F has the given representation.

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