Formal Functional Equations and Lie–Gröbner Series

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Lie–Gröbner series

Let $F(x), H(x) \in \mathbb{C}[[x]]$ be formal series. Consider the differential operator

$$D: \mathbb{C}[[x]] \to \mathbb{C}[[x]], \qquad D(F) := F'(x)H(x).$$

Iterative powers of D are defined as

$$D^n(F) = \begin{cases} F & \text{if } n = 0 \\ D(D^{n-1}(F)) & \text{if } n > 0. \end{cases}$$

The Lie–Gröbner series of F is the series

$$LG(F) := \sum_{n\geq 0} \frac{y^n}{n!} D^n(F) \in \mathbb{C}[[x,y]].$$

Observations (1)

For $F_i \in \mathbb{C}[[x]]$, $c_i \in \mathbb{C}$, $1 \le i \le m$, $n, m \in \mathbb{N}$ we have

$$D^n\left(\sum_{i=1}^{m} c_i F_i\right) = \sum_{i=1}^{m} c_i D^n(F_i), \quad \text{(Linearity)}$$

$$D(F_1 \cdot F_2) = D(F_1)F_2 + F_1 D(F_2), \quad \text{(product rule)}$$

$$D^n(F_1 \cdot F_2) = \sum_{j=0}^{n} \binom{n}{j} D^j(F_1) D^{n-j}(F_2),$$

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$$LG\left(\sum_{i=1}^{m} c_i F_i\right) = \sum_{i=1}^{m} c_i LG(F_i), \quad \text{(Linearity)}$$

$$LG\left(\prod_{i=1}^{m} F_i\right) = \prod_{i=1}^{m} LG(F_i). \quad \text{(Multiplicativity)}$$

Commutation Theorem

For $F(x) = \sum_{n \geq 0} f_n x^n \in \mathbb{C}[[x]]$

let $F_k(x) = \sum_{n=0}^{k} f_n x^n \in \mathbb{C}[x]$. Then

$$LG(F_k(x)) = LG\left(\sum_{n=0}^{k} f_n x^n\right) = \sum_{n=0}^{k} f_n LG(x)^n = F_k(LG(x)).$$

Since F is the limit of F_k we obtain

Theorem 1. For any power series $F(x) \in \mathbb{C}[[x]]$ we have

$$LG(F(x)) = F(LG(x)). \tag{LG}$$

The translation equation

Translation equation

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We study solutions

$$F(s,x) = x + \sum_{n \geq k} c_n(s)x^n \in \mathbb{C}[[x]]$$

of (T) in the ring of formal power series over $\mathbb{C}$ where $c_n : \mathbb{C} \to \mathbb{C}$, $k \geq 2$, $c_k \neq 0$. The c_n are polynomials $P_n(c_k)$ in c_k.

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(T) implies that c_k is an additive function.

Formal iteration groups of type II

If c_k is a nontrivial additive function, then c_k takes infinitely many values. Each occurrence of $c_k(s)$ and $c_k(t)$ for indipendent values $s,t \in \mathbb{C}$ in (T) can be replaced by y and z, two independent variables. We obtain the formal translation equation in $(\mathbb{C}[y,z])[[x]]$:

$$G(y+z,x) = G\big(y, G(z,x)\big) \qquad (\mathrm{T_{formal}})$$

$$G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n, \quad P_n(y) \in \mathbb{C}[y], \quad n > k \geq 2,$$

$$G(0,x) = x. \qquad (\mathrm{B})$$

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$$G(0,x) = x. \qquad (\mathrm{B})$$

Theorem 2. $F(s,x) = x + c_k(s)x^k + \sum_{n>k} P_n(c_k(s))x^n$ is a solution of (T) if and only if $G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n$ is a solution of ($\mathrm{T_{formal}}$) and (B).

A differential equation for the translation equation

We are looking for relations between the solutions $G(y,x)$ of (T$_{\text{formal}}$) and the formal generator $H(x)$ of G

$$\frac{\partial}{\partial y} G(y,x)|_{y=0} = x^k + \sum_{n>k} h_n x^n = H(x).$$

Here $h_k := 1$. Notice that in the situation of an analytic iteration group the coefficient of x^k in $H(x)$ may be different from 1.

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Here $h_k := 1$. Notice that in the situation of an analytic iteration group the coefficient of x^k in $H(x)$ may be different from 1.

Differentiation of (T$_{\text{formal}}$) with respect to z together with the mixed chain rule and putting $z = 0$ yields

$$\frac{\partial}{\partial y}G(y,x) = H(x)\frac{\partial}{\partial x}G(y,x). \qquad (\text{PD}_{\text{formal}})$$

In other words $\frac{\partial}{\partial y}G(y,x) = D(G(y,x))$, where $D(F(x)) := H(x)\frac{\partial}{\partial x}F(x)$.

Reordering the summands of G

Since the solutions of (T_{formal}) are elements of $(\mathbb{C}[y])[[x]]$ it is possible to write them in the form

$$G(y,x) = \sum_{n\geq 0} \phi_n(x) y^n \in (\mathbb{C}[[x]])[[y]].$$

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This allows us to rewrite (PD$_{\text{formal}}$) and (B) as

$$\sum_{n\geq 1} n\phi_n(x)y^{n-1} = \sum_{n\geq 0} D(\phi_n(x))y^n \tag{1}$$

$$\phi_0(x) = x. \tag{2}$$

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(1) is satisfied if and only if

$$\phi_{n+1}(x) = \frac{1}{n+1} D(\phi_n(x)) \tag{1_n}$$

holds true for all $n \geq 0$.

Solution as a Lie–Gröbner series

By induction we derive from (1_n) that

$$\phi_0(x) = \frac{D^0(x)}{0!} \quad \text{and} \quad \phi_n(x) = \frac{D^n(x)}{n!}, \qquad n \geq 1.$$

Thus

$$G(y,x) = \sum_{n\geq 0} \phi_n(x) y^n = \sum_{n\geq 0} \frac{1}{n!} D^n(x) y^n = LG(x),$$

where

$$D\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]], \qquad D(F(x)) := H(x)F'(x).$$

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Theorem 3. 1. If G is a solution of ((T_{formal}),(B)), then it is a solution of (PD_{formal}), whence it is the Lie-Gröbner series $LG(x)$ where H is the formal generator of G.

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Theorem 3. 1. If G is a solution of ((T_{formal}),(B)), then it is a solution of (PD_{formal}), whence it is the Lie-Gröbner series $LG(x)$ where H is the formal generator of G.

2. For any generator $H(x) = x^k + \sum_{n>k} h_n x^n$, $k \geq 2$, the unique solution $G(y,x) = LG(x)$ of the system ((1),(2)) is a solution of (T_{formal}).

Proof

$$
\begin{aligned}
G(y,G(z,x)) &= \sum_{n\geq 0}\frac{1}{n!}y^nD^n\Big(\sum_{\nu\geq 0}\frac{1}{\nu!}z^\nu D^\nu(x)\Big)\\
&= \sum_{n\geq 0}\sum_{\nu\geq 0}\frac{1}{n!}\frac{1}{\nu!}y^nz^\nu D^{n+\nu}(x)\\
&= \sum_{N\geq 0}\sum_{n=0}^{N}\frac{1}{N!}\frac{N!}{n!(N-n)!}y^nz^{N-n}D^N(x)\\
&= \sum_{N\geq 0}\frac{1}{N!}\Big(\sum_{n=0}^{N}\binom{N}{n}y^nz^{N-n}\Big)D^N(x)\\
&= \sum_{N\geq 0}\frac{1}{N!}(y+z)^ND^N(x)\\
&= G(y+z,x).
\end{aligned}
$$

Commutation Theorem for iteration groups of type II

As an immediate conseqeunce of the commutation theorem (LG) we get

Theorem 4. Let

$$G(y,x)=\sum_{n\geq 0}\frac{1}{n!}D^n(x)y^n, \qquad D(F(x))=F'(x)H(x),$$

be a formal iteration group of type II with formal generator $H(x)$. Then for any power series $K(x)$ of order at least 1 we have

$$G(y,K(x))=K(G(y,x)).$$

The general idea

functional equation

The general idea

functional equation

$\Updownarrow$

formal equation

---

1 replacing indepenedent values by independent variables

The general idea

functional equation

$\Updownarrow$

formal equation

$\Downarrow$

differential equation

2 differentiation, mixed chain rule, purely algebraic differentiation

The general idea

functional equation

$\Updownarrow$

formal equation

$\Downarrow$

differential equation

$\Downarrow$

reordering of the series

---

3 reordering the summands

The general idea

functional equation

$\Updownarrow$

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$\Downarrow$

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$\Downarrow$

reordering of the series

$\Downarrow$

solution as a (generalized) Lie–Gröbner series

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4 solving by comparing coefficients

The general idea

functional equation

$\Updownarrow$

formal equation $\Longleftarrow$

$\Downarrow$

differential equation

$\Downarrow$

reordering of the series

$\Downarrow$

solution as a (generalized) Lie–Gröbner series

$\Downarrow$

this (generalized) Lie–Gröbner series is a solution of the formal equation

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5 relative simpel proof

The first cocycle equation

The first cocycle equation

$$\alpha(s+t,x) = \alpha(s,x)\alpha\big(t,F(s,x)\big), \qquad s,t \in \mathbb{C}, \tag{Co1}$$

for

$$\alpha(s,x) = \sum_{n\geq 0} \alpha_n(s)x^n, \ \text{ with } \ \alpha(0,x) = 1.$$

F is an iteration group of type II. α_0 is an exponential function. Substitution into the logarithmic series yields

$$\gamma(s,x) := \log\left(\frac{\alpha(s,x)}{\alpha_0(s)}\right) = \sum_{n\geq 1} \gamma_n(s)x^n,$$

$$\gamma(s+t,x) = \gamma(s,x) + \gamma\big(t,F(s,x)\big), \qquad \gamma(0,x) = 0. \tag{Co1$_{\text{log}}$}$$

We obtain the formal first cocycle equation in $(\mathbb{C}[y,z])[[x]]$

$$\Gamma(y+z,x) = \Gamma(y,x) + \Gamma\big(z,G(y,x)\big), \qquad \Gamma(0,x) = 0. \tag{Co1$_{\text{formal}}$}$$

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A differential equation for the first cocycle equation

Differentiation of ($\text{Co1}_{\text{formal}}$) with respect to z together with the mixed chain rule and putting $z = 0$ yields

$$\frac{\partial}{\partial y}\Gamma(y,x) = K(x) + H(x)\frac{\partial}{\partial x}\Gamma(y,x), \qquad (\text{Co1PD}_{\text{formal}})$$

where $K(x) := \frac{\partial}{\partial y}\Gamma(y,x)|_{y=0}$ and $H(x)$ is the formal generator of the formal iteration group $G(y,x)$.

Thus $\frac{\partial}{\partial y}\Gamma(y,x) = K(x) + D(\Gamma(y,x))$, where $D(F(x)) := H(x)\frac{\partial}{\partial x}F(x)$.

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Reordering the summands of Γ

Since the solution of (Co1$_{\text{formal}}$) are elements of $(\mathbb{C}[y])[[x]]$ it is possible to write them in the form

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(3) is satisfied if and only if

$$\psi_1(x) = K(x)$$

$$\psi_{n+1}(x) = \frac{1}{n+1} D(\psi_n(x)), \quad n \geq 1,$$

holds true.

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Solution as a generalized Lie–Gröbner series

The unique solution of (3) with $\Gamma(0,x) = 0$ is

$$\Gamma(y,x) = \sum_{n\geq 1} \psi_n(x) y^n = \sum_{n\geq 1} \frac{1}{n!} D^{n-1}(K(x)) y^n,$$

where

$$D\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]], \qquad D(F(x)) := H(x)F'(x).$$

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Theorem 5. 1. If Γ is a solution of ($\text{Co1}_{\text{formal}}$), then it is a solution of ($\text{Co1PD}_{\text{formal}}$), it has a representation as a generalized Lie–Gröbner series where H is the formal generator of the iteration group G.

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2. Let G be a formal iteration group of type II with formal generator H. For any series $K(x)$ of order at least 1 the unique solution $\Gamma(y,x)$ of (3) with $\Gamma(0,x)=0$ is a solution of ($\text{Co1}_{\text{formal}}$). (The proof is based on the Commutation Theorem for iteration groups of type II.)

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$$D\colon \mathbb{C}[\![x]\!]\to\mathbb{C}[\![x]\!],\qquad D(F(x)):=H(x)F'(x).$$

Theorem 5. 1. If Γ is a solution of ($\text{Co1}_{\text{formal}}$), then it is a solution of ($\text{Co1PD}_{\text{formal}}$), it has a representation as a generalized Lie–Gröbner series where H is the formal generator of the iteration group G.
2. Let G be a formal iteration group of type II with formal generator H. For any series $K(x)$ of order at least 1 the unique solution $\Gamma(y,x)$ of (3) with $\Gamma(0,x)=0$ is a solution of ($\text{Co1}_{\text{formal}}$). (The proof is based on the Commutation Theorem for iteration groups of type II.)

The second cocycle equation

The second cocycle equation

$$\beta(s+t,x)=\beta(s,x)\alpha\big(t,F(s,x)\big)+\beta\big(t,F(s,x)\big),\qquad s,t\in\mathbb{C},\qquad \text{(Co2)}$$

for

$$\beta(s,x)=\sum_{n\geq 0}\beta_n(s)x^n,\ \text{ with }\ \beta(0,x)=0.$$

α is a solution of (Co1), F is an iteration group of type II.

$$R(ST,U+V,\sigma+\tau,x)=R(S,U,\sigma,x)+S^{\lambda}\tilde{P}(U,x)R(T,V,\tau,G(U,x)) \qquad (\text{Co2}_{\text{formal}})$$

$$R(1,0,0,x)=0. \qquad \text{(B2)}$$

where $R(S,U,\sigma,x)\in(\mathbb{C}[S,U,\sigma])[\![x]\!]$ and $\lambda\in\{0,1\}$.

$\tilde{P}(U,x)=\sum_{j=1}^{k-1}(-\kappa_j)[G(U,x)]^j$ is a solution of ($\text{Co1}_{\text{formal}}$), and $G(U,x)$ is a formal iteration group of type II with formal generator H.

The second cocycle equation

$$\beta(s+t,x) = \beta(s,x)\alpha\big(t,F(s,x)\big) + \beta\big(t, [illegible] s,x)\big), \qquad s,t \in \mathbb{C}, \qquad \text{(Co2)}$$

for

$$\beta(s,x) = \sum_{n \geq [illegible]} [illegible] \beta(0,x) = 0.$$

α is a solution of (Co1), F is a [illegible] itera[illegible] [illegible]p of type II.

$$R(ST,U+V,\sigma+\tau,x) = R(S,U,\sigma [illegible] \tilde{P}(U,x)R(T,V,\tau,G(U,x)) \qquad \text{(Co2}_{\text{formal}}\text{)}$$

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where $R(S,U,\sigma,x) \in (\mathbb{C}[S,U,\sigma])[[x$ [illegible] $\in \{0,1\}$.

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Differential equations for the second cocycle equation

Consider the generalized operator

$$\mathscr{D}\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]], \qquad \mathscr{D}(F(x)) = \Big(\sum_{j=1}^{k-1} -\kappa_j x^j\Big) F(x) + F'(x)H(x).$$

If $\kappa_1 = \ldots = \kappa_{k-1} = 0$, then $\mathscr{D} = D$.

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By differentiating (Co2$_{\text{formal}}$) with respect to S (U and σ) and setting $S = 1$, $U = 0$, and $\sigma = 0$ we get

$$T\frac{\partial}{\partial T}R(T,V,\tau,x) = N_1(x) + \lambda R(T,V,\tau,x), \qquad \text{(Co2PD1)}$$

$$\frac{\partial}{\partial V}R(T,V,\tau,x) = N_2(x) + \mathscr{D}(R(T,V,\tau,x)), \qquad \text{(Co2PD2)}$$

$$\frac{\partial}{\partial \tau}R(T,V,\tau,x) = N_3(x), \qquad \text{(Co2PD3)}$$

where N_1, N_2 and N_3 are generators.

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Differential equations for the second cocycle equation

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Differential equations for the second cocycle equation

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where N_1, N_2 and N_3 are generators.

Reordering the summands of R

We consider the solutions $R(S,U\sigma,x)$ of (Co2$_{\text{formal}}$) as elements of $(\mathbb{C}[S,\sigma])[[U,x]]$, and write them in the form

$$R(S,U\sigma,x) = \sum_{n\geq 0} R_n(S,\sigma,x)U^n$$

with $R_n(S,\sigma,x) \in (\mathbb{C}[S,\sigma])[[x]]$.

(Co2PD2) is rewritten as

$$\sum_{n\geq 1} nR_n(S,\sigma,x)U^{n-1} = N_2(x) + \sum_{n\geq 0} \mathscr{D}(R_n(S,\sigma,x))U^n.$$

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Solutions of the second cocycle equation for $\lambda = 0$

The solutions of ((Co2PD1),(Co2PD2),(Co2PD3)) are

$$R(S,U,\sigma,x) = \sigma N_3(x) + \sum_{n\geq 1} \frac{1}{n!} \mathscr{D}^{n-1}(N_2(x))U^n,$$

where $N_2(x)$ is arbitrary, $N_1(x) = 0$, and $\mathscr{D}(N_3(x)) = 0$.

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Theorem 6. 1. If R is a solution of ($\text{Co2}_{\text{formal}}$) and (B2), then R satisfies ((Co2PD1),(Co2PD2),(Co2PD3)), whence is has a representation as a generalized Lie–Gröbner series, where $N_1(x) = \frac{\partial}{\partial S} R(S,0,0,x)|_{S=1} = 0$, $N_2(x) = \frac{\partial}{\partial U} R(1,U,0,x)|_{U=0}$, $N_3(x) = \frac{\partial}{\partial \sigma} R(1,0,\sigma,x)|_{\sigma=0}$, $\mathscr{D}(N_3(x)) = 0$.

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2. If $N_1(x) = 0$, $\mathscr{D}(N_3(x)) = 0$, then the unique solution of ((Co2PD1),(Co2PD2),(Co2PD3)) as a generalized Lie–Gröbner series is a solution of ($\text{Co2}_{\text{formal}}$) and (B2).

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Solutions of the second cocycle equation for $\lambda = 0$

The solutions of ((Co2PD1),(Co2PD2),(Co2PD3)) are

$$R(S,U,\sigma,x) = \sigma N_3(x) + \sum_{n \geq 1} \frac{1}{n!} \mathscr{D}^{n-1}(N_2(x))U^n,$$

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Solutions of the second cocycle equation for $\lambda = 1$

The solutions of ((Co2PD1),(Co2PD2),(Co2PD3)) are

$$R(S,U,\sigma,x) = S\sum_{n\geq 0}\frac{1}{n!}\mathscr{D}^n(N_1(x))U^n - N_1(x),$$

where $N_1(x)$ is arbitrary, $N_2(x) = \mathscr{D}(N_1(x))$, and $N_3(x) = 0$.

Solutions of the second cocycle equation for $\lambda = 1$

The solutions of ((Co2PD1),(C[illegible]2[illegible]2PD3)) are

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where $N_1(x)$ is arbitrary, $N_2(x) = \mathscr{D}(N_1(x))$, and $N_3(x) = 0$.

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where $N_1(x)$ is arbitrary, $N_2(x) = \mathscr{D}(N_1(x))$, and $N_3(x) = 0$.

Theorem 7. 1. If R is a solution of (Co2$_{\text{formal}}$) and (B2), then R satisfies ((Co2PD1),(Co2PD2),(Co2PD3)), whence is has a representation as a generalized Lie–Gröbner series, where $N_1(x) = \frac{\partial}{\partial S}R(S,0,0,x)|_{S=1}$, $N_2(x) = \frac{\partial}{\partial U}R(1,U,0,x)|_{U=0}$, $\mathscr{D}(N_1(x)) = N_2(x)$, $N_3(x) = \frac{\partial}{\partial\sigma}R(1,0,\sigma,x)|_{\sigma=0} = 0$.

Solutions of the second cocycle equation for $\lambda = 1$

The solutions of ((Co2PD1),(Co2PD2),(Co2PD3)) are

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2. If $N_3(x) = 0$, $\mathscr{D}(N_1(x)) = N_2(x)$, then the unique solution of ((Co2PD1),(Co2PD2),(Co2PD3)) as a generalized Lie–Gröbner series is a solution of (Co2$_{\text{formal}}$) and (B2).

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Solutions of the second cocycle equation for $\lambda = 1$

The solutions of ((Co2PD1),(Co2PD2),(Co2PD3)) are

$$R(S,U,\sigma,x) = S\sum_{n\geq 0}\frac{1}{n!}\mathcal{D}^n(N_1(x))U^n - N_1(x),$$

where $N_1(x)$ is arbitrary, $N_2(x) = \mathcal{D}(N_1(x))$, and $N_3(x) = 0$.

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$N_3(x) = \frac{\partial}{\partial \sigma}R(1,0,\sigma,x)|_{\sigma=0} = 0$.
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Solutions of the second cocycle equation for $\lambda = 1$

The solutions of ((Co2PD1),(Co2PD2),(Co2PD3)) are

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Observations (2)

$$\mathscr{D}(F_1 \cdot F_2) = \mathscr{D}(F_1) \cdot F_2 + F_1 \cdot H(x) \cdot \frac{\partial}{\partial x} F_2(x), \quad \text{(product rule)}$$

$$\mathscr{D}(F) = \frac{\partial}{\partial U} \left(\tilde{P}(U,x) F(G(U,x)) \right) \Big|_{U=0},$$

$$\sum_{n \geq 0} \frac{1}{n!} \mathscr{D}^n(F(x)) U^n = \tilde{P}(U,x) F(G(U,x)),$$

$$\frac{\partial}{\partial U} \tilde{P}(U,x) = \mathscr{D}(\tilde{P}(U,x)),$$

where $\tilde{P}(U,x) = \sum_{j=1}^{k-1} (-\kappa_j) [G(U,x)]^j$, $G(U,x)$ is a formal iteration group of type II with formal generator H.

Observations (3)

$$\tilde{P}(U,x)F(G(U,x)) = F(x) \Longleftrightarrow \frac{\partial}{\partial U}(\tilde{P}(U,x)F(G(U,x))) = 0 \Longleftrightarrow$$
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- if $\kappa_1 = \ldots = \kappa_{k-1} = 0$: $F(x) = F_0 \in \mathbb{C}$,
- if $\kappa_1 = \ldots = \kappa_{r-2} = 0$ and $\kappa_r \neq 0$ and $r < k-1$ or $r = k-1$ and $\kappa_{k-1} \notin \mathbb{N}$: $F(x) = 0$,

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- if $\kappa_1 = \ldots = \kappa_{k-2} = 0$ and $\kappa_{k-1} = n_0 \in \mathbb{N}$: $F(x) = F_{n_0}\left(x^{n_0} + \sum_{n>n_0} F_n x^n\right)$, where $F_{n_0} \in \mathbb{C}$.

Observations (3)

$\tilde{P}(U,x)F(G(U,x)) = F(x) \Longleftrightarrow \frac{\partial}{\partial U}(\tilde{P}(U,x)F(G(U,x))) = 0 \Longleftrightarrow$
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- if $\kappa_1 = \ldots = \kappa_{k-2} = 0$ and $\kappa_{k-1} = n_0 \in \mathbb{N}$: $F(x) = F_{n_0}\left(x^{n_0} + \sum_{n>n_0} F_n x^n\right)$, where $F_{n_0} \in \mathbb{C}$.

$\mathscr{D}(F) = R \Longleftrightarrow F(x) = \int \tilde{P}(U,x)R(G(U,x))\,dU|_{U=0}$ (primitive function)

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The first and second cocycle equation appeared while studying covariant embeddings of a linear functional equation with respect to an analytic iteration group. Cf.

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