On the formal second cocycle equation for iteration groups of type II

Harald Fripertinger
Karl-Franzens-Universität Graz
joint work with Ludwig Reich

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Bibliography

The first and second cocycle equation appeared while studying covariant embeddings of a linear functional equation with respect to an analytic iteration group. Cf.

[1] H.F. and L. Reich: *On covariant embeddings of a linear functional equation with respect to an analytic iteration group* International Journal of Bifurcation and Chaos, 13 No. 7: 1853–1875, 2003.

[2] H.F. and L. Reich: *On covariant embeddings of a linear functional equation with respect to an analytic iteration group in some non-generic cases*, Aequationes Math., 68, 60–87, 2004.

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The regularity conditions were omitted in

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Formal functional equations for iteration groups of type I. Cf.

[4] H.F. and L. Reich: *The formal translation equation and formal cocycle equations for iteration groups of type I*, Aequationes Math., 76: 54–91, 2008.

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The formal translation equation for iteration groups of type II. Cf.

[5] H.F. and L. Reich: *The formal translation equation for iteration groups of type II*, Aequationes Math., 79: 111–156, 2010.

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[4] H.F. and L. Reich: *The formal translation equation and formal cocycle equations for iteration groups of type I*, Aequationes Math., 76: 54–91, 2008.

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[5] H.F. and L. Reich: *The formal translation equation for iteration groups of type II*, Aequationes Math., 79: 111–156, 2010.

The formal first cocycle equation for iteration groups of type II. Cf.

[6] H.F. and L. Reich: *On the formal first cocycle equation for iteration groups of type II*, to appear in the Proceedings of ECIT 2010.

The translation equation

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We study solutions

$$F(s,x) = \sum_{n\geq 1} c_n(s)x^n \in \mathbb{C}[[x]]$$

of (T) in the ring of formal power series over $\mathbb{C}$ where $c_n\colon \mathbb{C} \to \mathbb{C}$, $n \geq 1$, $c_1(s) \neq 0$, $s \in \mathbb{C}$.

Solutions of (T) are called ***iteration groups***.

(T) implies $c_1(s+t) = c_1(s)c_1(t)$, $s,t \in \mathbb{C}$, whence c_1 is an exponential function.

Iteration groups of type I and II

Type I

If $c_1 \neq 1$, then $F(s,x)$ is of type I.
Then $c_n(s) = P_n(c_1(s))$, $s \in \mathbb{C}$, $P_n(y) \in \mathbb{C}[y]$, $n \geq 1$.

Type II

If $c_1 = 1$ and if $F(s,x) \neq x$, then $F(s,x)$ is of type II.
There exists an integer $k \geq 2$ so that

$$F(s,x) = x + c_k(s)x^k + \sum_{n>k} c_n(s)x^n,$$

where $c_k \neq 0$ is additive,
and $c_n(s) = P_n(c_k(s))$, $s \in \mathbb{C}$, $P_n(y) \in \mathbb{C}[y]$, $n \geq k$.

The cocycle equations

In connection with the problem of a covariant embedding of the linear functional equation $\varphi(p(x)) = a(x)\varphi(x) + b(x)$ with respect to an iteration group $(F(s,x))_{s\in\mathbb{C}}$ we have to solve the two cocycle equations

$$\alpha(s+t,x) = \alpha(s,x)\alpha\big(t,F(s,x)\big), \qquad s,t\in\mathbb{C}, \tag{Co1}$$

$$\beta(s+t,x) = \beta(s,x)\alpha\big(t,F(s,x)\big) + \beta\big(t,F(s,x)\big), \qquad s,t\in\mathbb{C}, \tag{Co2}$$

under the boundary conditions

$$\alpha(0,x) = 1, \qquad \beta(0,x) = 0, \tag{B1}$$

for

$$\alpha(s,x) = \sum_{n\geq 0}\alpha_n(s)x^n, \qquad \beta(s,x) = \sum_{n\geq 0}\beta_n(s)x^n.$$

Formal equations

If a is a nontrivial additive function, (or if e is a nontrivial exponential function) and if a polynomial relation $P\big(a(s), a(t)\big) = 0$ (or $P\big(e(s), e(t)\big) = 0$) holds true for all $s, t \in \mathbb{C}$, then $P = 0$.

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In $(\mathbb{C}[y])[[x]]$ we have the formal derivation with respect to x.

Moreover the mixed chain rule is valid for formal derivations.

Differentiation is now a purely algebraic process!

The formal translation equation

Formal translation equation in $(\mathbb{C}[y,z])[[x]]$ for iteration groups of type II

$$G(y+z,x) = G\big(y, G(z,x)\big) \qquad (\mathrm{T_{formal}})$$

$$G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n, \quad P_n(y) \in \mathbb{C}[y], \quad n > k,$$

$$G(0,x) = x. \qquad (\mathrm{B})$$

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Theorem. Let $c_k \neq 0$ be an additive function. Then
$F(s,x) = x + c_k(s)x^k + \sum_{n>k} P_n(c_k(s))x^n$ is a solution of (T) if and only if
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Theorem. Let $c_k \neq 0$ be an additive function. Then $F(s,x) = x + c_k(s)x^k + \sum_{n>k} P_n(c_k(s))x^n$ is a solution of (T) if and only if $G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n$ is a solution of ($\mathrm{T}_{\text{formal}}$) and (B).

Theorem. For any formal generator $H(x) = x^k + \sum_{n>k} h_n x^n$ there exists exactly one solution $G(y,x)$ of ($\mathrm{T}_{\text{formal}}$) and (B) so that $\frac{\partial}{\partial y} G(y,x)|_{y=0} = H(x)$.

The first cocycle equation

For each generator $K(x) = \sum_{n\geq 1} \kappa_n x^n$ and each generalized exponential function α_0 there exists exactly one solution α of (Co1) which satisfies $\alpha(0,x) = 1$. It is given by

$$\alpha(s,x) = \alpha_0(s)\frac{E(G(c_k(s),x))}{E(x)} \underbrace{\prod_{j=1}^{k-1} \exp\left(\kappa_j \int [G(\sigma,x)]^j \, d\sigma|_{\sigma=c_k(s)}\right)}_{=:P(s,x)},$$

where $E(x) = \exp(\tilde{E}(x))$ and $\tilde{E}(x) = \frac{\sum_{n\geq k} \kappa_n x^n}{H(x)}$, and $G(U,x)$ is a solution of (T_{formal}) with generator $H(x)$ and $c_k \neq 0$ is additive. Conversely, each solution of (Co1) can be obtained in this form. $P(s,x)$ satisfies (Co1) and $P(0,x) = 1$.

The second cocycle equation

This representation of α motivates to study the series

$$\Delta(s,x) := \frac{\beta(s,x)}{E(x)\alpha(s,x)} = \frac{\beta(s,x)}{\alpha_0(s)E(G(c_k(s,x)))P(s,x)}$$

instead of β.

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Then β satisfies (Co2) and (B1) if and only if $\Delta(s,x) = \sum_{n\geq 0}\Delta_n(s)x^n$ satisfies

$$\Delta(s+t,x) = \Delta(s,x) + \alpha_0(s)^{-1}P(s,x)^{-1}\Delta(t,G(c_k(s),x)) \qquad \text{(Co2')}$$

and $\Delta(0,x) = 0$.

The inverse $P(s,x)^{-1}$ is a polynomial in $c_k(s)$ and we indicate it as $\tilde{P}(c_k(s),x)$.

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In all situations $\Delta_n(s)$ is a polynomial in $c_k(s)$.In some cases also a polynomial of degree 1 in $\alpha_0^{-1}(s)$or a polynomial of degree 1 in an arbitrary additive function $A(s)$.Since (Co2') holds for all $s, t \in \mathbb{C}$ we can replace the values $c_k(s), c_k(t)$ by indeterminates U, V, $\alpha_0^{-1}(s), \alpha_0^{-1}(t)$ by S, T and $A(s), A(t)$ by σ, τ. This yields the formal equation

$$R(ST, U+V, \sigma+\tau, x) = R(S, U, \sigma, x) + S^{\lambda}\tilde{P}(U,x)R(T, V, \tau, G(U,x)) \qquad (\mathrm{Co2}_{\mathrm{formal}})$$

where $\lambda \in \{0,1\}$. The series $\tilde{P}(U,x)$ and $G(U,x)$ are solutions of the formal version of (Co1) respectively of ($\mathrm{T}_{\mathrm{formal}}$).

We study solutions of ($\mathrm{Co2}_{\mathrm{formal}}$) satisfying the boundary condition

$$R(1,0,0,x) = 0. \qquad (\mathrm{B})$$

Three differential equations derived from ($Co2_{formal}$)

Differentiation of ($Co2_{formal}$) with respect to T and setting $T = 1$, $V = 0$, and $\tau = 0$ yields

$$S\frac{\partial}{\partial S}R(S,U,\sigma,x) = S^{\lambda}\tilde{P}(U,x)N_S(G(U,x)), \tag{D1}$$

where $N_S(x) := \frac{\partial}{\partial S}R(S,0,0,x)|_{S=1}$.

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Similarly by differentiating with respect to V or τ we obtain

$$\frac{\partial}{\partial U}R(S,U,\sigma,x) = S^{\lambda}\tilde{P}(U,x)N_U(G(U,x)), \tag{D2}$$

and

$$\frac{\partial}{\partial \sigma}R(S,U,\sigma,x) = S^{\lambda}\tilde{P}(U,x)N_\sigma(G(U,x)), \tag{D3}$$

where $N_U(x) := \frac{\partial}{\partial U}R(1,U,0,x)|_{U=0}$, and $N_\sigma(x) := \frac{\partial}{\partial \sigma}R(1,0,\sigma,x)|_{\sigma=0}$.

Solutions of (D1), (D2), (D3) and (B) for $\lambda = 1$

$$R(S,U,\sigma,x) = S\tilde{P}(U,x)N_S(G(U,x)) - N_S(x)$$

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Relations for the generators N_S, N_U, and N_σ.

$$N_\sigma = 0$$

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Each solution of of (D1), (D2), (D3), and (B) for $\lambda = 1$ is a solution of ($Co2_{formal}$) for $\lambda = 1$.

This representation of the solutions coincides with the representation found in our previous papers.

Solutions of (D1), (D2), (D3) and (B) for $\lambda = 0$

$$R(S,U,\sigma,x) = \sigma N_\sigma(x) + \int \tilde{P}(U,x) N_U(G(U,x))\,dU - c,$$

where

$$c = \int \tilde{P}(U,x) N_U(G(U,x))\,dU \Big|_{U=0}.$$

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If $\tilde{P}(U,x) = 1$, then $N_\sigma(x) \in \mathbb{C}$ (constant).

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If $\tilde{P}(U,x) = 1 - \kappa_{k-1} x^{k-1} + \ldots$ and $\kappa_{k-1} = n_1 \in \mathbb{Z}_{>0}$, then

$$N_\sigma(x) = \sum_{j \geq n_1} N_{\sigma,j} x^j$$

where N_{σ,n_1} is not determined, $N_{\sigma,j}$, $j > n_1$, uniquely determined by (∗) and N_{σ,n_1}.

Comparing the solutions for $\lambda = 0$ with the solutions for $\lambda = 1$ we analyze when we can write

$$\int \tilde{P}(U,x)N_U(G(U,x))dU = F(G(U,x)) \qquad (\circ)$$

for some $F(x) = \sum_{n\geq 0} f_n x^n \in \mathbb{C}[[x]]$.

If $\tilde{P}(U,x) = 1$, then $(\circ)$ is true whenever $\operatorname{ord} N_U \geq k$. If so, then f_0 is arbitrary in $\mathbb{C}$.

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$$\int \tilde{P}(U,x) N_U(G(U,x))dU = F(G(U,x)) \qquad (\circ)$$

for some $F(x) = \sum_{n \geq 0} f_n x^n \in \mathbb{C}[[x]]$.

If $\tilde{P}(U,x) = 1$, then $(\circ)$ is true whenever $\operatorname{ord} N_U \geq k$. If so, then f_0 is arbitrary in $\mathbb{C}$.

If $\tilde{P}(U,x) = 1 - \kappa_r x^r + \ldots$, where $r < k-1$ and $\kappa_r \neq 0$, or $r = k-1$ and $\kappa_{k-1} \notin \mathbb{Z}_{\geq 0}$, then $(\circ)$ is true whenever $\operatorname{ord} N_U \geq r$. If so, then F is uniquely determined.

If $\tilde{P}(U,x) = 1 - \kappa_{k-1} x^{k-1} + \ldots$ and $\kappa_{k-1} = n_1 \in \mathbb{Z}_{>0}$, then $(\circ)$ is true whenever $\operatorname{ord} N_U \geq k-1$ and N_{U,n_1+k-1} satisfies a polynomial relation in $N_{U,j}$ for $k \leq j < n_1 + k - 1$, which is also a polynomial relation in f_j for $1 \leq j < n_1$. If so, then f_{n_1} is arbitrary, $f_0, \ldots, f_{n_1-1}$ are uniquely determined and f_j, $j > n_1$, are uniquely determined depending on f_{n_1}.

Reordering the summands

Now we consider the solutions $R(S,U\sigma,x)$ of (Co2$_{\text{formal}}$) as elements of $(\mathbb{C}[S,\sigma])[[U,x]]$, and rewrite them in the form

$$R(S,U\sigma,x) = \sum_{n\geq 0} Q_n(S,\sigma,x)U^n$$

with $Q_n(S,\sigma,x) \in (\mathbb{C}[S,\sigma])[[x]]$.

In this situation we study another system of differential equations:

Three (partial) differential equations from (Co2$_{\text{formal}}$)

We determine another system of differential equations by differentiating (Co2$_{\text{formal}}$) with respect to S (U and σ) and setting $S = 1$, $U = 0$, and $\sigma = 0$:

$$T\frac{\partial}{\partial T}R(T,V,\tau,x) = N_T(x) + \delta_{\lambda,1}R(T,V,\tau,x), \tag{PD1}$$

$$\frac{\partial}{\partial V}R(T,V,\tau,x) = N_V(x) + \Big(\sum_{j=r}^{k-1} -\kappa_j\Big)R(T,V,\tau,x) + \frac{\partial}{\partial x}R(T,V,\tau,x)H(x), \tag{PD2}$$

$$\frac{\partial}{\partial \tau}R(T,V,\tau,x) = N_\tau(x), \tag{PD3}$$

where N_T, N_V and N_τ are generators.

Integration by Differentiation

Using the reordered series $R(S,U\sigma,x) = \sum_{n\geq 0} Q_n(S,\sigma,x)U^n$, it is easy to solve (PD1), (PD2) (PD3), and (B):

E.g., the main computation is to solve equations of the form

$$n\hat{R}_n(x) = \sum_{j=r}^{k-1} (-\kappa_j)x^j \hat{R}_{n-1}(x) + \hat{R}'_{n-1}(x)H(x), \qquad n \geq 2,$$

where $\hat{R}_{n-1}(x)$ is already computed.

This method yields new representations of solutions of ($\text{Co2}_{\text{formal}}$).

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