



Formal Functional Equations and Lie–Gröbner Series

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Lie–Gröbner series

Let $F(x), H(x) \in \mathbb{C}[[x]]$ be formal series. Consider the differential operator

$$D: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]], \quad D(F) := F'(x)H(x).$$

Iterative powers of D are defined as

$$D^n(F) = \begin{cases} F & \text{if } n = 0 \\ D(D^{n-1}(F)) & \text{if } n > 0. \end{cases}$$

The Lie–Gröbner series of F is the series

$$LG(F) := \sum_{n \geq 0} \frac{y^n}{n!} D^n(F) \in \mathbb{C}[[x, y]].$$

Observations (1)

For $F_i \in \mathbb{C}[[x]]$, $c_i \in \mathbb{C}$, $1 \leq i \leq m$, $n, m \in \mathbb{N}$ we have

$$D^n \left(\sum_{i=1}^m c_i F_i \right) = \sum_{i=1}^m c_i D^n(F_i), \quad (\text{Linearity})$$

$$D(F_1 \cdot F_2) = D(F_1)F_2 + F_1 D(F_2), \quad (\text{product rule})$$

$$D^n(F_1 \cdot F_2) = \sum_{j=0}^n \binom{n}{j} D^j(F_1) D^{n-j}(F_2),$$

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$$D^n(F_1 \cdot F_2) = \sum_{j=0}^n \binom{n}{j} D^j(F_1) D^{n-j}(F_2),$$

$$LG \left(\sum_{i=1}^m c_i F_i \right) = \sum_{i=1}^m c_i LG(F_i), \quad (\text{Linearity})$$

$$LG \left(\prod_{i=1}^m F_i \right) = \prod_{i=1}^m LG(F_i). \quad (\text{Multiplicativity})$$

Commutation Theorem

For $F(x) = \sum_{n \geq 0} f_n x^n \in \mathbb{C}[[x]]$

let $F_k(x) = \sum_{n=0}^k f_n x^n \in \mathbb{C}[x]$. Then

$$LG(F_k(x)) = LG\left(\sum_{n=0}^k f_n x^n\right) = \sum_{n=0}^k f_n LG(x)^n = F_k(LG(x)).$$

Since F is the limit of F_k we obtain

Theorem 1. For any power series $F(x) \in \mathbb{C}[[x]]$ we have

$$LG(F(x)) = F(LG(x)). \quad (LG)$$



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$$F(s + t, x) = F(s, F(t, x)), \quad s, t \in \mathbb{C}. \quad (\text{T})$$

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$$F(s+t, x) = F(s, F(t, x)), \quad s, t \in \mathbb{C}. \quad (\text{T})$$

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We study solutions



$$F(s, x) = x + \sum_{n \geq k} c_n(s) x^n \in \mathbb{C}[[x]]$$

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of (T) in the ring of formal power series over \mathbb{C} where $c_n: \mathbb{C} \rightarrow \mathbb{C}$, $k \geq 2$, $c_k \neq 0$. The c_n are polynomials $P_n(c_k)$ in c_k .

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We study solutions

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Solutions of (T) are called ***iteration groups of type II***.

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Solutions of (T) are called ***iteration groups of type II***.

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(T) implies that c_k is an additive function.

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Formal iteration groups of type II

If c_k is a nontrivial additive function, then c_k takes infinitely many values. Each occurrence of $c_k(s)$ and $c_k(t)$ for independent values $s, t \in \mathbb{C}$ in (T) can be replaced by y and z , two independent variables. We obtain the formal translation equation in $(\mathbb{C}[y, z])[[x]]$:

$$G(y + z, x) = G(y, G(z, x)) \quad (\mathbb{T}_{\text{formal}})$$

$$G(y, x) = x + yx^k + \sum_{n>k} P_n(y)x^n, \quad P_n(y) \in \mathbb{C}[y], \quad n > k \geq 2,$$

$$G(0, x) = x. \quad (\mathbb{B})$$

Formal iteration groups of type II

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$$G(0, x) = x. \quad (\text{B})$$

Theorem 2. $F(s, x) = x + c_k(s)x^k + \sum_{n>k} P_n(c_k(s))x^n$ is a solution of (T) if and only if $G(y, x) = x + yx^k + \sum_{n>k} P_n(y)x^n$ is a solution of $(\mathbb{T}_{\text{formal}})$ and (B).

A differential equation for the translation equation

We are looking for relations between the solutions $G(y, x)$ of (T_{formal}) and the formal generator $H(x)$ of G

$$\frac{\partial}{\partial y} G(y, x)|_{y=0} = x^k + \sum_{n>k} h_n x^n = H(x).$$

Here $h_k := 1$. Notice that in the situation of an analytic iteration group the coefficient of x^k in $H(x)$ may be different from 1.

A differential equation for the translation equation

We are looking for relations between the solutions $G(y, x)$ of (T_{formal}) and the formal generator $H(x)$ of G

$$\frac{\partial}{\partial y} G(y, x)|_{y=0} = x^k + \sum_{n>k} h_n x^n = H(x).$$

Here $h_k := 1$. Notice that in the situation of an analytic iteration group the coefficient of x^k in $H(x)$ may be different from 1.

Differentiation of (T_{formal}) with respect to z together with the mixed chain rule and putting $z = 0$ yields

$$\frac{\partial}{\partial y} G(y, x) = H(x) \frac{\partial}{\partial x} G(y, x). \quad (\text{PD}_{\text{formal}})$$

In other words $\frac{\partial}{\partial y} G(y, x) = D(G(y, x))$, where $D(F(x)) := H(x) \frac{\partial}{\partial x} F(x)$.



Reordering the summands of G

Since the solutions of (T_{formal}) are elements of $(\mathbb{C}[y])[[x]]$ it is possible to write them in the form

$$G(y, x) = \sum_{n \geq 0} \phi_n(x) y^n \in (\mathbb{C}[[x]])[[y]].$$

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Reordering the summands of G

Since the solutions of (T_{formal}) are elements of $(\mathbb{C}[y])[[x]]$ it is possible to write them in the form

$$G(y, x) = \sum_{n \geq 0} \phi_n(x) y^n \in (\mathbb{C}[[x]])[[y]].$$

This allows us to rewrite (PD_{formal}) and (B) as

$$\sum_{n \geq 1} n \phi_n(x) y^{n-1} = \sum_{n \geq 0} D(\phi_n(x)) y^n \quad (1)$$

$$\phi_0(x) = x. \quad (2)$$

Reordering the summands of G

Since the solutions of (T_{formal}) are elements of $(\mathbb{C}[y])[[x]]$ it is possible to write them in the form

$$G(y, x) = \sum_{n \geq 0} \phi_n(x) y^n \in (\mathbb{C}[[x]])[[y]].$$

This allows us to rewrite (PD_{formal}) and (B) as

$$\sum_{n \geq 1} n \phi_n(x) y^{n-1} = \sum_{n \geq 0} D(\phi_n(x)) y^n \quad (1)$$

$$\phi_0(x) = x. \quad (2)$$

(1) is satisfied if and only if

$$\phi_{n+1}(x) = \frac{1}{n+1} D(\phi_n(x)) \quad (1_n)$$

holds true for all $n \geq 0$.

Solution as a Lie–Gröbner series

By induction we derive from (1_n) that

$$\phi_0(x) = \frac{D^0(x)}{0!} \quad \text{and} \quad \phi_n(x) = \frac{D^n(x)}{n!}, \quad n \geq 1.$$

Thus

$$G(y, x) = \sum_{n \geq 0} \phi_n(x) y^n = \sum_{n \geq 0} \frac{1}{n!} D^n(x) y^n = LG(x),$$

where

$$D: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]], \quad D(F(x)) := H(x)F'(x).$$

Solution as a Lie–Gröbner series

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where

$$D: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]], \quad D(F(x)) := H(x)F'(x).$$

Theorem 3. 1. If G is a solution of $((T_{\text{formal}}, (B)))$, then it is a solution of (PD_{formal}) , whence it is the Lie-Gröbner series $LG(x)$ where H is the formal generator of G .

Solution as a Lie–Gröbner series

By induction we derive from (1_n) that

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Thus

$$G(y, x) = \sum_{n \geq 0} \phi_n(x) y^n = \sum_{n \geq 0} \frac{1}{n!} D^n(x) y^n = LG(x),$$

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$$D: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]], \quad D(F(x)) := H(x)F'(x).$$

Theorem 3. 1. If G is a solution of $((T_{\text{formal}}, (B)))$, then it is a solution of (PD_{formal}) , whence it is the Lie-Gröbner series $LG(x)$ where H is the formal generator of G .

2. For any generator $H(x) = x^k + \sum_{n > k} h_n x^n$, $k \geq 2$, the unique solution $G(y, x) = LG(x)$ of the system $((1), (2))$ is a solution of (T_{formal}) .

Proof

$$\begin{aligned}
 G(y, G(z, x)) &= \sum_{n \geq 0} \frac{1}{n!} y^n D^n \left(\sum_{v \geq 0} \frac{1}{v!} z^v D^v(x) \right) \\
 &= \sum_{n \geq 0} \sum_{v \geq 0} \frac{1}{n!} \frac{1}{v!} y^n z^v D^{n+v}(x) \\
 &= \sum_{N \geq 0} \sum_{n=0}^N \frac{1}{N!} \frac{N!}{n!(N-n)!} y^n z^{N-n} D^N(x) \\
 &= \sum_{N \geq 0} \frac{1}{N!} \left(\sum_{n=0}^N \binom{N}{n} y^n z^{N-n} \right) D^N(x) \\
 &= \sum_{N \geq 0} \frac{1}{N!} (y+z)^N D^N(x) \\
 &= G(y+z, x).
 \end{aligned}$$

Commutation Theorem for iteration groups of type II

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As an immediate consequence of the commutation theorem (LG) we get

Theorem 4. Let

$$G(y, x) = \sum_{n \geq 0} \frac{1}{n!} D^n(x) y^n, \quad D(F(x)) = F'(x)H(x),$$

be a formal iteration group of type II with formal generator $H(x)$. Then for any power series $K(x)$ of order at least 1 we have

$$G(y, K(x)) = K(G(y, x)).$$



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functional equation

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formal equation

1 replacing independent values by independent variables



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formal equation



differential equation



differentiation, mixed chain rule, purely algebraic differentiation



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reordering of the series

3 reordering the summands

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reordering of the series



solution as a (generalized) Lie–Gröbner series

4 solving by comparing coefficients

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formal equation



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reordering of the series



solution as a (generalized) Lie–Gröbner series



this (generalized) Lie–Gröbner series is a solution of the formal equation



relative simple proof

The first cocycle equation



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The first cocycle equation

$$\alpha(s+t, x) = \alpha(s, x)\alpha(t, F(s, x)), \quad s, t \in \mathbb{C}, \quad (\text{Co1})$$

for

$$\alpha(s, x) = \sum_{n \geq 0} \alpha_n(s)x^n, \quad \text{with } \alpha(0, x) = 1.$$

F is an iteration group of type II. α_0 is an exponential function.

Substitution into the logarithmic series yields

$$\gamma(s, x) := \log \left(\frac{\alpha(s, x)}{\alpha_0(s)} \right) = \sum_{n \geq 1} \gamma_n(s)x^n,$$

$$\gamma(s+t, x) = \gamma(s, x) + \gamma(t, F(s, x)), \quad \gamma(0, x) = 0. \quad (\text{Co1}_{\log})$$

We obtain the formal first cocycle equation in $(\mathbb{C}[y, z])[[x]]$

$$\Gamma(y+z, x) = \Gamma(y, x) + \Gamma(z, G(y, x)), \quad \Gamma(0, x) = 0. \quad (\text{Co1}_{\text{formal}})$$

The first cocycle equation

$$\alpha(s+t, x) = \alpha(s, x)\alpha(t, F(s, x)), \quad s, t \in \mathbb{C}, \quad (\text{Co1})$$

for

$$\alpha(s, x) = \sum_{n \geq 0} \alpha_n(s)x^n, \quad \alpha(0, x) = 1.$$

F is an iteration group of type (λ, μ) and α is an exponential function.

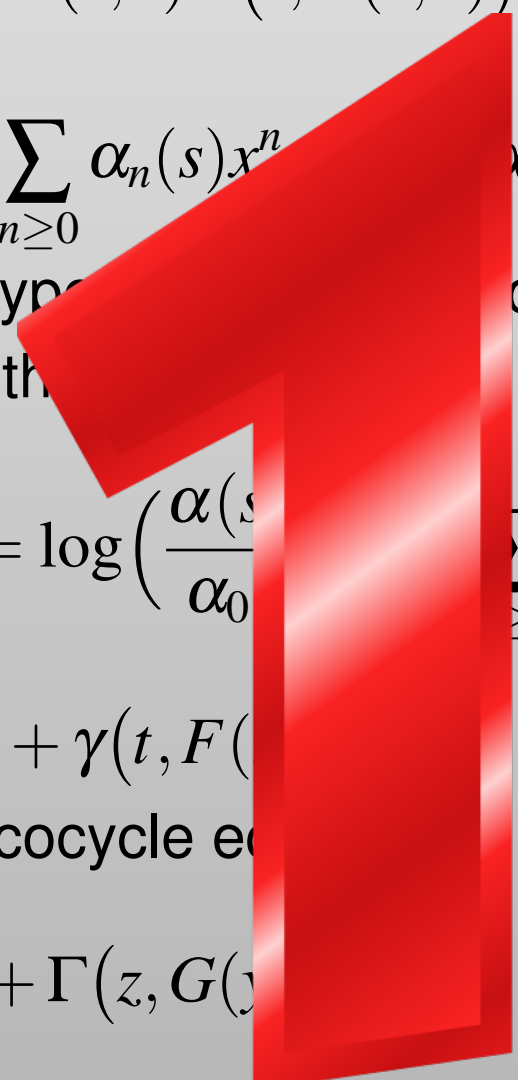
Substitution into the logarithm

$$\gamma(s, x) := \log \left(\frac{\alpha(s, x)}{\alpha_0(s)} \right) = \sum_{n \geq 1} \gamma_n(s)x^n,$$

$$\gamma(s+t, x) = \gamma(s, x) + \gamma(t, F(s, x)), \quad \gamma(0, x) = 0. \quad (\text{Co1}_{\log})$$

We obtain the formal first cocycle equation in $(\mathbb{C}[y, z])[[x]]$

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F is an iteration group of type II. α_0 is an exponential function.

Substitution into the logarithmic series yields

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We obtain the formal first cocycle equation in $(\mathbb{C}[y, z])[[x]]$

$$\Gamma(y+z, x) = \Gamma(y, x) + \Gamma(z, G(y, x)), \quad \Gamma(0, x) = 0. \quad (\text{Co1}_{\text{formal}})$$

A differential equation for the first cocycle equation

Differentiation of $(\text{Co1}_{\text{formal}})$ with respect to z together with the mixed chain rule and putting $z = 0$ yields

$$\frac{\partial}{\partial y} \Gamma(y, x) = K(x) + H(x) \frac{\partial}{\partial x} \Gamma(y, x), \quad (\text{Co1PD}_{\text{formal}})$$

where $K(x) := \frac{\partial}{\partial y} \Gamma(y, x)|_{y=0}$ and $H(x)$ is the formal generator of the formal iteration group $G(y, x)$.

Thus $\frac{\partial}{\partial y} \Gamma(y, x) = K(x) + D(\Gamma(y, x))$, where $D(F(x)) := H(x) \frac{\partial}{\partial x} F(x)$.

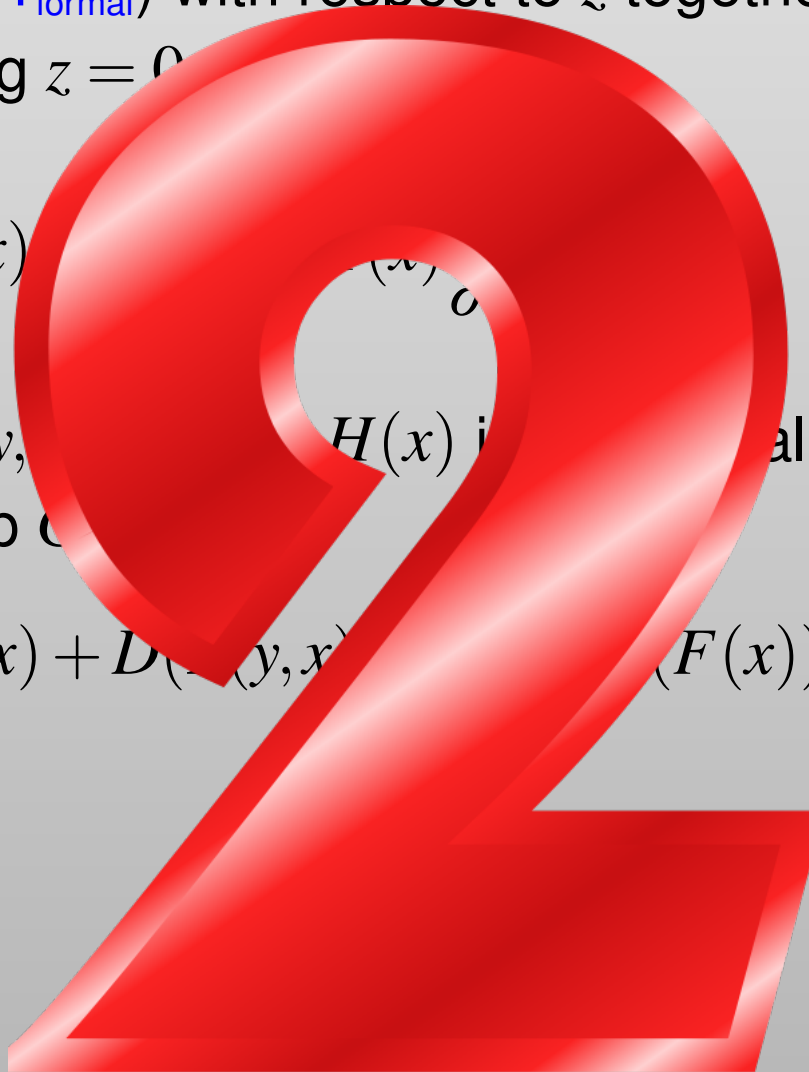
A differential equation for the first cocycle equation

Differentiation of (Co1_{formal}) with respect to z together with the mixed chain rule and putting $z = 0$

$$\frac{\partial}{\partial y} \Gamma(y, x) = K(x) + D_x \Gamma(y, x) \cdot H(x) \quad (\text{Co1PD}_{\text{formal}})$$

where $K(x) := \frac{\partial}{\partial y} \Gamma(y, x)|_{y=0}$ and $H(x)$ is the infinitesimal generator of the formal iteration group $\Gamma(y, x)$.

Thus $\frac{\partial}{\partial y} \Gamma(y, x) = K(x) + D_x \Gamma(y, x) \cdot H(x)$ and $(F(x)) := H(x) \frac{\partial}{\partial x} F(x)$.



A differential equation for the first cocycle equation

Differentiation of $(\text{Co1}_{\text{formal}})$ with respect to z together with the mixed chain rule and putting $z = 0$ yields

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where $K(x) := \frac{\partial}{\partial y} \Gamma(y, x)|_{y=0}$ and $H(x)$ is the formal generator of the formal iteration group $G(y, x)$.

Thus $\frac{\partial}{\partial y} \Gamma(y, x) = K(x) + D(\Gamma(y, x))$, where $D(F(x)) := H(x) \frac{\partial}{\partial x} F(x)$.



Reordering the summands of Γ

Since the solution of (Co1_{formal}) are elements of $(\mathbb{C}[y])[[x]]$ it is possible to write them in the form

$$\Gamma(y, x) = \sum_{n \geq 1} \psi_n(x) y^n \in (\mathbb{C}[[x]])[[y]].$$

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Reordering the summands of Γ

Since the solution of (Co1_{formal}) are elements of $(\mathbb{C}[y])[[x]]$ it is possible to write them in the form

$$\Gamma(y, x) = \sum_{n \geq 1} \psi_n(x) y^n \in (\mathbb{C}[[x]])[[y]].$$

This allows us to rewrite (Co1PD_{formal}) as

$$\sum_{n \geq 1} n \psi_n(x) y^{n-1} = K(x) + \sum_{n \geq 1} D(\psi_n(x)) y^n. \quad (3)$$

Reordering the summands of Γ

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This allows us to rewrite (Co1PD_{formal}) as

$$\sum_{n \geq 1} n \psi_n(x) y^{n-1} = K(x) + \sum_{n \geq 1} D(\psi_n(x)) y^n. \quad (3)$$

(3) is satisfied if and only if

$$\begin{aligned} \psi_1(x) &= K(x) \\ \psi_{n+1}(x) &= \frac{1}{n+1} D(\psi_n(x)), \quad n \geq 1, \end{aligned}$$

holds true.

Reordering the summands of Γ

Since the solution of (Co1_{formal}) are elements of $(\mathbb{C}[y])[[x]]$ it is possible to write them in the form

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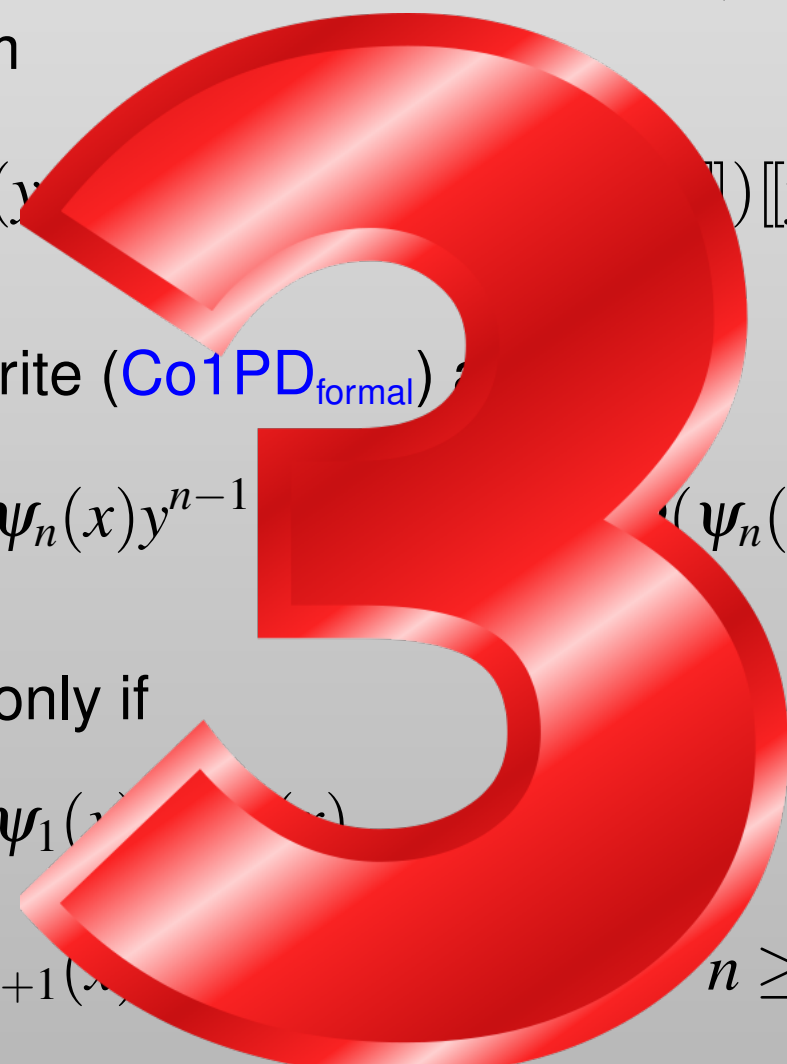
This allows us to rewrite (Co1PD_{formal}) as

$$\sum_{n \geq 1} n \psi_n(x) y^{n-1} = \sum_{n \geq 1} (\psi_n(x)) y^n. \quad (3)$$

(3) is satisfied if and only if

$$\psi_{n+1}(x) = \psi_n(x) \quad n \geq 1,$$

holds true.



Reordering the summands of Γ

Since the solution of (Co1_{formal}) are elements of $(\mathbb{C}[y])[[x]]$ it is possible to write them in the form

$$\Gamma(y, x) = \sum_{n \geq 1} \psi_n(x) y^n \in (\mathbb{C}[[x]])[[y]].$$

This allows us to rewrite (Co1PD_{formal}) as

$$\sum_{n \geq 1} n \psi_n(x) y^{n-1} = K(x) + \sum_{n \geq 1} D(\psi_n(x)) y^n. \quad (3)$$

(3) is satisfied if and only if

$$\begin{aligned} \psi_1(x) &= K(x) \\ \psi_{n+1}(x) &= \frac{1}{n+1} D(\psi_n(x)), \quad n \geq 1, \end{aligned}$$

holds true.

Solution as a generalized Lie–Gröbner series



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where

$$D: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]], \quad D(F(x)) := H(x)F'(x).$$

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Solution as a general Puiseux–Gröbner series



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The unique solution of (3) with $\Gamma(y, x) = \sum_{n \geq 1} \gamma_n(x) y^n$ is

$$\Gamma(y, x) = \sum_{n \geq 1} \gamma_n(x) y^n, \quad \gamma_n(x) = D^{n-1}(K(x))y^n,$$

where

$$D: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]], \quad D(F(x)) := H(x)F'(x).$$

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Solution as a generalized Lie–Gröbner series



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The unique solution of (3) with $\Gamma(0, x) = 0$ is

$$\Gamma(y, x) = \sum_{n \geq 1} \psi_n(x) y^n = \sum_{n \geq 1} \frac{1}{n!} D^{n-1}(K(x)) y^n,$$

where

$$D: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]], \quad D(F(x)) := H(x)F'(x).$$

Theorem 5. 1. If Γ is a solution of (Co1_{formal}), then it is a solution of (Co1PD_{formal}), it has a representation as a generalized Lie–Gröbner series where H is the formal generator of the iteration group G .

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The unique solution of (3) with $\Gamma(0, x) = 0$ is

$$\Gamma(y, x) = \sum_{n \geq 1} \psi_n(x) y^n = \sum_{n \geq 1} \frac{1}{n!} D^{n-1}(K(x)) y^n,$$

where

$$D: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]], \quad D(F(x)) := H(x)F'(x).$$

Theorem 5. 1. If Γ is a solution of ([Co1_{formal}](#)), then it is a solution of ([Co1PD_{formal}](#)), it has a representation as a generalized Lie–Gröbner series where H is the formal generator of the iteration group G .

2. Let G be a formal iteration group of type II with formal generator H . For any series $K(x)$ of order at least 1 the unique solution $\Gamma(y, x)$ of (3) with $\Gamma(0, x) = 0$ is a solution of ([Co1_{formal}](#)). (The proof is based on the [Commutation Theorem for iteration groups of type II.](#))

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The unique solution of (3) with $\Gamma(0, x) = 0$ is

$$\Gamma(y, x) = \sum_{n \geq 1} \psi_n(x) y^n = \sum_{n \geq 1} \frac{1}{n!} D^{n-1}(K(x)) y^n,$$

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$$D: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]], \quad D(F(x)) := H(x)F'(x).$$

Theorem 5. 1. If Γ is a solution of (3), then it is a solution of (Co1PD_{formal}), it has a representation as a generalized Lie–Gröbner series where H is the formal generator of the iteration group G .

2. Let G be a formal iteration group of type II with formal generator H . For any series $K(x)$ of order at least 1, the unique solution $\Gamma(y, x)$ of (3) with $\Gamma(0, x) = 0$ is a solution of (3). (The proof is based on the Commutation Theorem for iteration groups of type II.)

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The unique solution of (3) with $\Gamma(0, x) = 0$ is

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The second cocycle equation



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The second cocycle equation

$$\beta(s+t, x) = \beta(s, x)\alpha(t, F(s, x)) + \beta(t, F(s, x)), \quad s, t \in \mathbb{C}, \quad (\text{Co2})$$

for

$$\beta(s, x) = \sum_{n \geq 0} \beta_n(s)x^n, \quad \text{with } \beta(0, x) = 0.$$

α is a solution of (Co1), F is an iteration group of type II.

$$R(ST, U + V, \sigma + \tau, x) = R(S, U, \sigma, x) + S^\lambda \tilde{P}(U, x)R(T, V, \tau, G(U, x)) \quad (\text{Co2}_{\text{formal}})$$

$$R(1, 0, 0, x) = 0. \quad (\text{B2})$$

where $R(S, U, \sigma, x) \in (\mathbb{C}[S, U, \sigma])[[x]]$ and $\lambda \in \{0, 1\}$.

$\tilde{P}(U, x) = \sum_{j=1}^{k-1} (-\kappa_j)[G(U, x)]^j$ is a solution of (Co1_{formal}), and $G(U, x)$ is a formal iteration group of type II with formal generator H .

The second cocycle equation

$$\beta(s+t, x) = \beta(s, x)\alpha(t, F(s, x)) + \beta(t, F(s, x)), \quad s, t \in \mathbb{C}, \quad (\text{Co2})$$

for

$$\beta(s, x) = \sum_{n \geq 0} \beta_n(s, x) x^n, \quad \beta(0, x) = 0.$$

α is a solution of (Co1), F is a formal iteration group of type II.

$$R(ST, U + V, \sigma + \tau, x) = R(S, U, \sigma, x) \tilde{P}(U, x) R(T, V, \tau, G(U, x)) \quad (\text{Co2}_{\text{formal}})$$

$$R(1, 0, 0, x) = 1 \quad (\text{B2})$$

where $R(S, U, \sigma, x) \in (\mathbb{C}[S, U, \sigma])[[x]]$ and $\sigma \in \{0, 1\}$.

$\tilde{P}(U, x) = \sum_{j=1}^{k-1} (-\kappa_j) [G(U, x)]^j$ is a formal power series of (Co1_{formal}), and $G(U, x)$ is a formal iteration group of type II with formal generator H .

The second cocycle equation

$$\beta(s+t, x) = \beta(s, x)\alpha(t, F(s, x)) + \beta(t, F(s, x)), \quad s, t \in \mathbb{C}, \quad (\text{Co2})$$

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Consider the generalized operator

$$\mathcal{D}: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]], \quad \mathcal{D}(F(x)) = \left(\sum_{j=1}^{k-1} -\kappa_j x^j \right) F(x) + F'(x)H(x).$$

If $\kappa_1 = \dots = \kappa_{k-1} = 0$, then $\mathcal{D} = D$.



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Consider the generalized operator

$$\mathcal{D}: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]], \quad \mathcal{D}(F(x)) = \left(\sum_{j=1}^{k-1} -\kappa_j x^j \right) F(x) + F'(x)H(x).$$

If $\kappa_1 = \dots = \kappa_{k-1} = 0$, then $\mathcal{D} = D$.

By differentiating (Co2_{formal}) with respect to S (U and σ) and setting $S = 1$, $U = 0$, and $\sigma = 0$ we get

$$T \frac{\partial}{\partial T} R(T, V, \tau, x) = N_1(x) + \lambda R(T, V, \tau, x), \quad (\text{Co2PD1})$$

$$\frac{\partial}{\partial V} R(T, V, \tau, x) = N_2(x) + \mathcal{D}(R(T, V, \tau, x)), \quad (\text{Co2PD2})$$

$$\frac{\partial}{\partial \tau} R(T, V, \tau, x) = N_3(x), \quad (\text{Co2PD3})$$

where N_1 , N_2 and N_3 are generators.

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Consider the generalized operator

$$\mathcal{D}: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]], \quad \mathcal{D} = \sum_{i=0}^{\infty} \kappa_i D^i + F(x) + F'(x)H(x).$$

If $\kappa_1 = \dots = \kappa_{k-1} = 0$, then $\mathcal{D} = D$.

By differentiating (Co2) with respect to T and σ and setting

$S = 1, U = 0$, and $\sigma =$

$$T \frac{\partial}{\partial T} R(T, V, \tau, x) = N_1(T, V, \tau, x), \quad (\text{Co2PD1})$$

$$\frac{\partial}{\partial V} R(T, V, \tau, x) = N_2(T, V, \tau, x), \quad (\text{Co2PD2})$$

$$\frac{\partial}{\partial \tau} R(T, V, \tau, x) = N_3(T, V, \tau, x), \quad (\text{Co2PD3})$$

where N_1, N_2 and N_3 are generators.

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Consider the generalized operator

$$\mathcal{D}: \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]], \quad \mathcal{D}(F(x)) = \left(\sum_{j=1}^{k-1} -\kappa_j x^j \right) F(x) + F'(x)H(x).$$

If $\kappa_1 = \dots = \kappa_{k-1} = 0$, then $\mathcal{D} = D$.

By differentiating (Co2_{formal}) with respect to S (U and σ) and setting $S = 1$, $U = 0$, and $\sigma = 0$ we get

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$$\frac{\partial}{\partial \tau} R(T, V, \tau, x) = N_3(x), \quad (\text{Co2PD3})$$

where N_1 , N_2 and N_3 are generators.

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Reordering the summands of R

We consider the solutions $R(S, U\sigma, x)$ of (Co2_{formal}) as elements of $(\mathbb{C}[S, \sigma])[[U, x]]$, and write them in the form

$$R(S, U\sigma, x) = \sum_{n \geq 0} R_n(S, \sigma, x) U^n$$

with $R_n(S, \sigma, x) \in (\mathbb{C}[S, \sigma])[[x]]$.

(Co2PD2) is rewritten as

$$\sum_{n \geq 1} n R_n(S, \sigma, x) U^{n-1} = N_2(x) + \sum_{n \geq 0} \mathcal{D}(R_n(S, \sigma, x)) U^n.$$

Reordering the summands of R



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We consider the solutions $R(S, U, \sigma, x)$ of (Co2_{formal}) as elements of $(\mathbb{C}[S, \sigma])[[U, x]]$, and write them as

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(Co2PD2) is rewritten as

$$\sum_{n \geq 1} n R_n(S, \sigma, x) U^{n-1} = N_2(x) + \dots (S, \sigma, x) U^n.$$

Reordering the summands of R

We consider the solutions $R(S, U\sigma, x)$ of (Co2_{formal}) as elements of $(\mathbb{C}[S, \sigma])[[U, x]]$, and write them in the form

$$R(S, U\sigma, x) = \sum_{n \geq 0} R_n(S, \sigma, x) U^n$$

with $R_n(S, \sigma, x) \in (\mathbb{C}[S, \sigma])[[x]]$.

(Co2PD2) is rewritten as

$$\sum_{n \geq 1} n R_n(S, \sigma, x) U^{n-1} = N_2(x) + \sum_{n \geq 0} \mathcal{D}(R_n(S, \sigma, x)) U^n.$$

Solutions of the second cocycle equation for $\lambda = 0$

The solutions of ((Co2PD1),(Co2PD2),(Co2PD3)) are

$$R(S, U, \sigma, x) = \sigma N_3(x) + \sum_{n \geq 1} \frac{1}{n!} \mathcal{D}^{n-1}(N_2(x)) U^n,$$

where $N_2(x)$ is arbitrary, $N_1(x) = 0$, and $\mathcal{D}(N_3(x)) = 0$.

Solutions of the second cycle equation for $\lambda = 0$

The solutions of ((Co2PD1),(Co2PD2),(Co2PD3)) are

$$R(S, U, \sigma, x) = \mathcal{D}^{n-1}(N_2(x))U^n,$$

where $N_2(x)$ is arbitrary, $N_1(x) = 0$, and $\mathcal{D}(N_3(x)) = 0$.

Solutions of the second cocycle equation for $\lambda = 0$

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where $N_2(x)$ is arbitrary, $N_1(x) = 0$, and $\mathcal{D}(N_3(x)) = 0$.

Theorem 6. 1. If R is a solution of (Co2_{formal}) and (B2), then R satisfies ((Co2PD1),(Co2PD2),(Co2PD3)), whence it has a representation as a generalized Lie–Gröbner series, where $N_1(x) = \frac{\partial}{\partial S} R(S, 0, 0, x)|_{S=1} = 0$, $N_2(x) = \frac{\partial}{\partial U} R(1, U, 0, x)|_{U=0}$, $N_3(x) = \frac{\partial}{\partial \sigma} R(1, 0, \sigma, x)|_{\sigma=0}$, $\mathcal{D}(N_3(x)) = 0$.

Solutions of the second cocycle equation for $\lambda = 0$

The solutions of ((Co2PD1),(Co2PD2),(Co2PD3)) are

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2. If $N_1(x) = 0$, $\mathcal{D}(N_3(x)) = 0$, then the unique solution of ((Co2PD1),(Co2PD2),(Co2PD3)) as a generalized Lie–Gröbner series is a solution of (Co2_{formal}) and (B2).

Solutions of the second cocycle equation for $\lambda = 0$

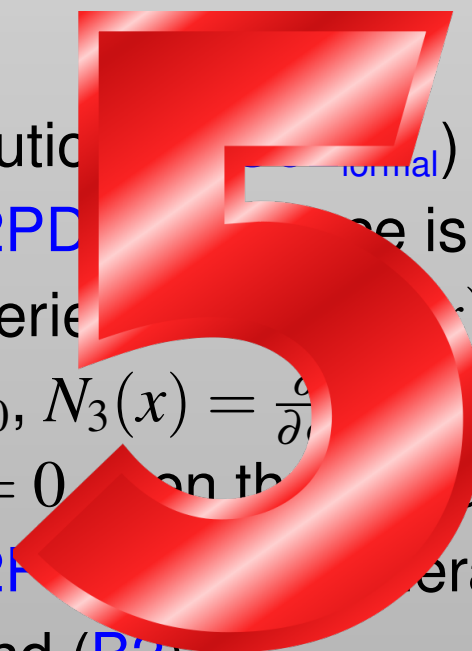
The solutions of ((Co2PD1),(Co2PD2),(Co2PD3)) are

$$R(S, U, \sigma, x) = \sigma N_3(x) + \sum_{n \geq 1} \frac{1}{n!} \mathcal{D}^{n-1}(N_2(x)) U^n,$$

where $N_2(x)$ is arbitrary, $N_1(x) = 0$, and $\mathcal{D}(N_3(x)) = 0$.

Theorem 6. 1. If R is a solution of ((Co2PD1),(Co2PD2),(Co2PD3)) and (B2), then R satisfies ((Co2PD1),(Co2PD2),(Co2PD3)) and has a representation as a generalized Lie–Gröbner series $R(S, U, \sigma, x) = \frac{\partial}{\partial S} R(S, 0, 0, x)|_{S=1} = 0$, $N_2(x) = \frac{\partial}{\partial U} R(1, U, 0, x)|_{U=0}$, $N_3(x) = \frac{\partial}{\partial \sigma} R(1, 0, \sigma, x)|_{\sigma=0}$, $\mathcal{D}(N_3(x)) = 0$.

2. If $N_1(x) = 0$, $\mathcal{D}(N_3(x)) = 0$ then the generalized Lie–Gröbner series is a solution of (Co2_{formal}) and (B2).



Solutions of the second cocycle equation for $\lambda = 0$

The solutions of ((Co2PD1),(Co2PD2),(Co2PD3)) are

$$R(S, U, \sigma, x) = \sigma N_3(x) + \sum_{n \geq 1} \frac{1}{n!} \mathcal{D}^{n-1}(N_2(x)) U^n,$$

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Theorem 6. 1. If R is a solution of (Co2_{formal}) and (B2), then R satisfies ((Co2PD1),(Co2PD2),(Co2PD3)), whence it has a representation as a generalized Lie–Gröbner series, where $N_1(x) = \frac{\partial}{\partial S} R(S, 0, 0, x)|_{S=1} = 0$, $N_2(x) = \frac{\partial}{\partial U} R(1, U, 0, x)|_{U=0}$, $N_3(x) = \frac{\partial}{\partial \sigma} R(1, 0, \sigma, x)|_{\sigma=0}$, $\mathcal{D}(N_3(x)) = 0$.

2. If $N_1(x) = 0$, $\mathcal{D}(N_3(x)) = 0$, then the unique solution of ((Co2PD1),(Co2PD2),(Co2PD3)) as a generalized Lie–Gröbner series is a solution of (Co2_{formal}) and (B2).

Solutions of the second cocycle equation for $\lambda = 1$

The solutions of ((Co2PD1),(Co2PD2),(Co2PD3)) are

$$R(S, U, \sigma, x) = S \sum_{n \geq 0} \frac{1}{n!} \mathcal{D}^n(N_1(x)) U^n - N_1(x),$$

where $N_1(x)$ is arbitrary, $N_2(x) = \mathcal{D}(N_1(x))$, and $N_3(x) = 0$.

Solutions of the second cocycle equation for $\lambda = 1$

The solutions of $((Co2PD1), (Co2PD2), (Co2PD3))$ are

$$R(S, U, \sigma, x) = \sum_{n \geq 0} \dots U^n - N_1(x),$$

where $N_1(x)$ is arbitrary, $N_2(x) = \mathcal{D}(N_1(x))$, and $N_3(x) = 0$.

Solutions of the second cocycle equation for $\lambda = 1$

The solutions of ((Co2PD1),(Co2PD2),(Co2PD3)) are

$$R(S, U, \sigma, x) = S \sum_{n \geq 0} \frac{1}{n!} \mathcal{D}^n(N_1(x)) U^n - N_1(x),$$

where $N_1(x)$ is arbitrary, $N_2(x) = \mathcal{D}(N_1(x))$, and $N_3(x) = 0$.

Theorem 7. 1. If R is a solution of (Co2_{formal}) and (B2), then R satisfies ((Co2PD1),(Co2PD2),(Co2PD3)), whence it has a representation as a generalized Lie–Gröbner series, where $N_1(x) = \frac{\partial}{\partial S} R(S, 0, 0, x)|_{S=1}$, $N_2(x) = \frac{\partial}{\partial U} R(1, U, 0, x)|_{U=0}$, $\mathcal{D}(N_1(x)) = N_2(x)$, $N_3(x) = \frac{\partial}{\partial \sigma} R(1, 0, \sigma, x)|_{\sigma=0} = 0$.

Solutions of the second cocycle equation for $\lambda = 1$

The solutions of ((Co2PD1),(Co2PD2),(Co2PD3)) are

$$R(S, U, \sigma, x) = S \sum_{n \geq 0} \frac{1}{n!} \mathcal{D}^n(N_1(x)) U^n - N_1(x),$$

where $N_1(x)$ is arbitrary, $N_2(x) = \mathcal{D}(N_1(x))$, and $N_3(x) = 0$.

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generalized Lie–Gröbner series, where $N_1(x) = \frac{\partial}{\partial S} R(S, 0, 0, x)|_{S=1}$,

$N_2(x) = \frac{\partial}{\partial U} R(1, U, 0, x)|_{U=0}$, $\mathcal{D}(N_1(x)) = N_2(x)$,

$N_3(x) = \frac{\partial}{\partial \sigma} R(1, 0, \sigma, x)|_{\sigma=0} = 0$.

2. If $N_3(x) = 0$, $\mathcal{D}(N_1(x)) = N_2(x)$, then the unique solution of ((Co2PD1),(Co2PD2),(Co2PD3)) as a generalized Lie–Gröbner series is a solution of (Co2_{formal}) and (B2).

Solutions of the second cocycle equation for $\lambda = 1$

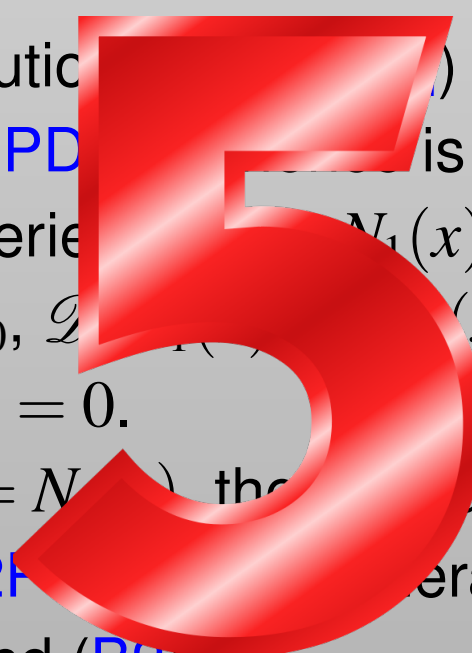
The solutions of ((Co2PD1),(Co2PD2),(Co2PD3)) are

$$R(S, U, \sigma, x) = S \sum_{n \geq 0} \frac{1}{n!} \mathcal{D}^n(N_1(x)) U^n - N_1(x),$$

where $N_1(x)$ is arbitrary, $N_2(x) = \mathcal{D}(N_1(x))$, and $N_3(x) = 0$.

Theorem 7. 1. If R is a solution of ((Co2PD1),(Co2PD2),(Co2PD3)) and (B2), then R satisfies ((Co2PD1),(Co2PD2),(Co2PD3)) and has a representation as a generalized Lie–Gröbner series $R(S, U, \sigma, x) = S N_1(x) + \mathcal{D}(N_1(x)) U + \frac{\mathcal{D}^2(N_1(x))}{2!} U^2 + \dots$ with $N_1(x) = \frac{\partial}{\partial S} R(S, 0, 0, x)|_{S=1}$, $N_2(x) = \frac{\partial}{\partial U} R(1, U, 0, x)|_{U=0} = \mathcal{D}(N_1(x))$, $N_3(x) = \frac{\partial}{\partial \sigma} R(1, 0, \sigma, x)|_{\sigma=0} = 0$.

2. If $N_3(x) = 0$, $\mathcal{D}(N_1(x)) = N_2(x)$, then the unique solution of ((Co2PD1),(Co2PD2),(Co2PD3)) is a generalized Lie–Gröbner series $R(S, U, \sigma, x) = S N_1(x) + \mathcal{D}(N_1(x)) U + \frac{\mathcal{D}^2(N_1(x))}{2!} U^2 + \dots$ is a solution of (Co2_{formal}) and (B2).



Solutions of the second cocycle equation for $\lambda = 1$

The solutions of ((Co2PD1),(Co2PD2),(Co2PD3)) are

$$R(S, U, \sigma, x) = S \sum_{n \geq 0} \frac{1}{n!} \mathcal{D}^n(N_1(x)) U^n - N_1(x),$$

where $N_1(x)$ is arbitrary, $N_2(x) = \mathcal{D}(N_1(x))$, and $N_3(x) = 0$.

Theorem 7. 1. If R is a solution of (Co2_{formal}) and (B2), then R satisfies ((Co2PD1),(Co2PD2),(Co2PD3)), whence it has a representation as a generalized Lie–Gröbner series, where $N_1(x) = \frac{\partial}{\partial S} R(S, 0, 0, x)|_{S=1}$, $N_2(x) = \frac{\partial}{\partial U} R(1, U, 0, x)|_{U=0}$, $\mathcal{D}(N_1(x)) = N_2(x)$, $N_3(x) = \frac{\partial}{\partial \sigma} R(1, 0, \sigma, x)|_{\sigma=0} = 0$.

2. If $N_3(x) = 0$, $\mathcal{D}(N_1(x)) = N_2(x)$, then the unique solution of ((Co2PD1),(Co2PD2),(Co2PD3)) as a generalized Lie–Gröbner series is a solution of (Co2_{formal}) and (B2).

Observations (2)

$$\mathcal{D}(F_1 \cdot F_2) = \mathcal{D}(F_1) \cdot F_2 + F_1 \cdot H(x) \cdot \frac{\partial}{\partial x} F_2(x), \quad (\text{product rule})$$

$$\mathcal{D}(F) = \frac{\partial}{\partial U} \left(\tilde{P}(U, x) F(G(U, x)) \right) \Big|_{U=0},$$

$$\sum_{n \geq 0} \frac{1}{n!} \mathcal{D}^n(F(x)) U^n = \tilde{P}(U, x) F(G(U, x)),$$

$$\frac{\partial}{\partial U} \tilde{P}(U, x) = \mathcal{D}(\tilde{P}(U, x)),$$

where $\tilde{P}(U, x) = \sum_{j=1}^{k-1} (-\kappa_j) [G(U, x)]^j$, $G(U, x)$ is a formal iteration group of type II with formal generator H .



Observations (3)

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$$\tilde{P}(U, x)F(G(U, x)) = F(x) \iff \frac{\partial}{\partial U} (\tilde{P}(U, x)F(G(U, x))) = 0 \iff \mathcal{D}(F) = 0 \iff$$

Observations (3)

$$\tilde{P}(U, x)F(G(U, x)) = F(x) \iff \frac{\partial}{\partial U} (\tilde{P}(U, x)F(G(U, x))) = 0 \iff \mathcal{D}(F) = 0 \iff$$

- if $\kappa_1 = \dots = \kappa_{k-1} = 0$: $F(x) = F_0 \in \mathbb{C}$,

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Observations (3)

$$\tilde{P}(U, x)F(G(U, x)) = F(x) \iff \frac{\partial}{\partial U} (\tilde{P}(U, x)F(G(U, x))) = 0 \iff \mathcal{D}(F) = 0 \iff$$

- if $\kappa_1 = \dots = \kappa_{k-1} = 0$: $F(x) = F_0 \in \mathbb{C}$,
- if $\kappa_1 = \dots = \kappa_{r-2} = 0$ and $\kappa_r \neq 0$ and $r < k - 1$ or $r = k - 1$ and $\kappa_{k-1} \notin \mathbb{N}$:
 $F(x) = 0$,

Observations (3)

$$\tilde{P}(U, x)F(G(U, x)) = F(x) \iff \frac{\partial}{\partial U} (\tilde{P}(U, x)F(G(U, x))) = 0 \iff \mathcal{D}(F) = 0 \iff$$

- if $\kappa_1 = \dots = \kappa_{k-1} = 0$: $F(x) = F_0 \in \mathbb{C}$,
- if $\kappa_1 = \dots = \kappa_{r-2} = 0$ and $\kappa_r \neq 0$ and $r < k - 1$ or $r = k - 1$ and $\kappa_{k-1} \notin \mathbb{N}$:
 $F(x) = 0$,
- if $\kappa_1 = \dots = \kappa_{k-2} = 0$ and $\kappa_{k-1} = n_0 \in \mathbb{N}$: $F(x) = F_{n_0} (x^{n_0} + \sum_{n > n_0} F_n x^n)$,
 where $F_{n_0} \in \mathbb{C}$.

Observations (3)

$$\tilde{P}(U, x)F(G(U, x)) = F(x) \iff \frac{\partial}{\partial U} (\tilde{P}(U, x)F(G(U, x))) = 0 \iff \mathcal{D}(F) = 0 \iff$$

- if $\kappa_1 = \dots = \kappa_{k-1} = 0$: $F(x) = F_0 \in \mathbb{C}$,
- if $\kappa_1 = \dots = \kappa_{r-2} = 0$ and $\kappa_r \neq 0$ and $r < k - 1$ or $r = k - 1$ and $\kappa_{k-1} \notin \mathbb{N}$:
 $F(x) = 0$,
- if $\kappa_1 = \dots = \kappa_{k-2} = 0$ and $\kappa_{k-1} = n_0 \in \mathbb{N}$: $F(x) = F_{n_0} (x^{n_0} + \sum_{n > n_0} F_n x^n)$,
where $F_{n_0} \in \mathbb{C}$.

$$\mathcal{D}(F) = R \iff F(x) = \int \tilde{P}(U, x)R(G(U, x)) dU|_{U=0} \quad (\text{primitive function})$$



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