On formal cocycle equations

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joint work with Ludwig Reich

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The translation equation

Translation equation

\[ F(s + t, x) = F(s, F(t, x)), \quad s, t \in \mathbb{C}. \]  \tag{T}
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(T)

We study solutions

\[ F(s, x) = \sum_{n \geq 1} c_n(s)x^n \in \mathbb{C}[x] \]

of (T) in the ring of formal power series over \( \mathbb{C} \) where \( c_n : \mathbb{C} \to \mathbb{C}, n \geq 1, \)
\( c_1(s) \neq 0, s \in \mathbb{C}. \)
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of \((T)\) in the ring of formal power series over \(\mathbb{C}\) where \(c_n: \mathbb{C} \to \mathbb{C}, n \geq 1, c_1(s) \neq 0, s \in \mathbb{C}\).

Solutions of \((T)\) are called **iteration groups**.
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of (T) in the ring of formal power series over \( \mathbb{C} \) where \( c_n: \mathbb{C} \to \mathbb{C}, n \geq 1, \) \( c_1(s) \neq 0, s \in \mathbb{C}. \)

Solutions of (T) are called **iteration groups**.

(T) implies \( c_1(s + t) = c_1(s)c_1(t), s, t \in \mathbb{C}, \) whence \( c_1 \) is an exponential function.
Iteration groups of type I and II

Type I

If $c_1 \neq 1$, then $F(s, x)$ is of type I. Then $c_n(s) = P_n(c_1(s))$, $s \in \mathbb{C}$, $P_n(y) \in \mathbb{C}[y]$, $n \geq 1$. 
Iteration groups of type I and II

**Type I**

If $c_1 \neq 1$, then $F(s, x)$ is of type I.

Then $c_n(s) = P_n(c_1(s))$, $s \in \mathbb{C}$, $P_n(y) \in \mathbb{C}[y]$, $n \geq 1$.

**Type II**

If $F(s, x) \neq x$ and if $c_1 = 1$, then $F(s, x)$ is of type II.

There exists an integer $k \geq 2$ so that $F(s, x) = x + c_k(s)x^k + \sum_{n>k} c_n(s)x^n$,

where $c_k(s + t) = c_k(s) + c_k(t)$, $s, t \in \mathbb{C}$, whence $c_k$ is additive,

and $c_n(s) = P_n(c_k(s))$, $s \in \mathbb{C}$, $P_n(y) \in \mathbb{C}[y]$, $n \geq k$. 
The cocycle equations

In connection with the problem of a covariant embedding of the linear functional equation $\varphi(p(x)) = a(x)\varphi(x) + b(x)$ with respect to an iteration group $(F(s,x))_{s \in \mathbb{C}}$ we have to solve the two cocycle equations

$$\alpha(s+t,x) = \alpha(s,x)\alpha(t,F(s,x)), \quad s, t \in \mathbb{C}, \quad (Co1)$$

$$\beta(s+t,x) = \beta(s,x)\alpha(t,F(s,x)) + \beta(t,F(s,x)), \quad s, t \in \mathbb{C}, \quad (Co2)$$

under the boundary conditions

$$\alpha(0,x) = 1, \quad \beta(0,x) = 0, \quad (B1)$$

for

$$\alpha(s,x) = \sum_{n \geq 0} \alpha_n(s)x^n, \quad \beta(s,x) = \sum_{n \geq 0} \beta_n(s)x^n.$$
Formal equations

If $a$ is a nontrivial additive function, (or if $e$ is a nontrivial exponential function) and if a polynomial relation $P(a(s), a(t)) = 0$ (or $P(e(s), e(t)) = 0$) holds true for all $s, t \in \mathbb{C}$, then $P = 0$. 
Formal equations

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This observations allows to study formal equations by replacing $a(s)$ and $a(t)$ (or $e(s)$ and $e(t)$) by indeterminates $y$ and $z$. 
Formal equations

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In \(\mathbb{C}[y]\) we have the formal derivation with respect to \(y\).
Formal equations

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In $\mathbb{C}[y]$ we have the formal derivation with respect to $y$.

In $(\mathbb{C}[y])[x]$ we have the formal derivation with respect to $x$. 
If $a$ is a nontrivial additive function, (or if $e$ is a nontrivial exponential function) and if a polynomial relation $P(a(s), a(t)) = 0$ (or $P(e(s), e(t)) = 0$) holds true for all $s, t \in \mathbb{C}$, then $P = 0$.

This observations allows to study formal equations by replacing $a(s)$ and $a(t)$ (or $e(s)$ and $e(t)$) by indeterminates $y$ and $z$.

In $\mathbb{C}[y]$ we have the formal derivation with respect to $y$.

In $(\mathbb{C}[y])[x]$ we have the formal derivation with respect to $x$.

Moreover the mixed chain rule is valid for formal derivations.
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This observations allows to study formal equations by replacing $a(s)$ and $a(t)$ (or $e(s)$ and $e(t)$) by indeterminates $y$ and $z$.

In $\mathbb{C}[y]$ we have the formal derivation with respect to $y$.

In $(\mathbb{C}[y])[x]$ we have the formal derivation with respect to $x$.

Moreover the mixed chain rule is valid for formal derivations.

Differentiation is now a purely algebraic process!
The formal translation equation

Formal translation equation in \((\mathbb{C}[y,z])[x]\) for iteration groups of type II

\[
G(y + z, x) = G(y, G(z, x)) \quad (T_{\text{formal}})
\]

\[
G(y, x) = x + yx^k + \sum_{n>k} P_n(y)x^n, \quad P_n(y) \in \mathbb{C}[y], \quad n > k,
\]

\[
G(0, x) = x. \quad (B)
\]
The formal translation equation

Formal translation equation in \((\mathbb{C}[y,z])[[x]]\) for iteration groups of type II

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\]

\[
G(0, x) = x. \quad \text{(B)}
\]

**Theorem.** Let \(c_k \neq 0\) be an additive function. Then

\[
F(s, x) = x + c_k(s)x^k + \sum_{n > k} P_n(c_k(s))x^n
\]

is a solution of \((T)\) if and only if

\[
G(y, x) = x + yx^k + \sum_{n > k} P_n(y)x^n
\]

is a solution of \((T)_\text{formal}\) and \((B)\).
The first cocycle equation

Let $F(s, x)$ be an iteration group. If $\alpha(s, x) = \sum_{n \geq 0} \alpha_n(s)x^n$ is a solution of $(\text{Co1})$, then $\alpha_0$ is an exponential function and

$$\hat{\alpha}(s, x) := \frac{\alpha(s, x)}{\alpha_0(s)} = 1 + \frac{\alpha_1(s)}{\alpha_0(s)}x + \cdots$$

is also a solution of $(\text{Co1})$ and $(\text{B1})$. 
The first cocycle equation

Let \( F(s, x) \) be an iteration group. If
\[
\alpha(s, x) = \sum_{n \geq 0} \alpha_n(s)x^n
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is a solution of (\text{Co1}), then \( \alpha_0 \) is an exponential function and
\[
\hat{\alpha}(s, x) := \frac{\alpha(s, x)}{\alpha_0(s)} = 1 + \frac{\alpha_1(s)}{\alpha_0(s)}x + \cdots
\]
is also a solution of (\text{Co1}) and (\text{B1}).

Substitution into the logarithmic series yields
\[
\gamma(s, x) := \log(\hat{\alpha}(s, x)) = \sum_{n \geq 1} \gamma_n(s)x^n
\]
satisfying
\[
\gamma(s + t, x) = \gamma(s, x) + \gamma(t, F(s, x)) \tag{\text{Co1}_{\log}}
\]
together with \( \gamma(0, x) = 0 \).
If \( F(s, x) = x + \sum_{n \geq k} P_n(c_k(s))x^n \) is an iteration group of type II, then each coefficient function \( \gamma_n(s) \) is a polynomial \( \tilde{P}_n(c_k(s)) \) and for all \( s, t \in \mathbb{C} \)

\[
\sum_{n \geq 1} \tilde{P}_n(c_k(s) + c_k(t))x^n = \\
\sum_{n \geq 1} \tilde{P}_n(c_k(s))x^n + \sum_{n \geq 1} \tilde{P}_n(c_k(t)) \left[ x + \sum_{r \geq k} P_r(c_k(s))x^r \right]_n.
\]
If $F(s, x) = x + \sum_{n \geq k} P_n(c_k(s))x^n$ is an iteration group of type II, then each coefficient function $\gamma_n(s)$ is a polynomial $\tilde{P}_n(c_k(s))$ and for all $s, t \in \mathbb{C}$

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\]

This yields the formal first cocycle equation in $(\mathbb{C}[y, z])[x]$ for iteration groups of type II

\[
\Gamma(y + z, x) = \Gamma(y, x) + \Gamma(z, G(y, x)) \quad (\text{Co1}_{\text{formal}})
\]

together with $\Gamma(0, x) = 0$ for $\Gamma(y, x) = \sum_{n \geq 1} \tilde{P}_n(y)x^n$, where

$G(y, x) = x + yx^k + \sum_{n > k} P_n(y)x^n$ is a formal iteration group of type II.
If \( F(s, x) = x + \sum_{n \geq k} P_n(c_k(s))x^n \) is an iteration group of type II, then each coefficient function \( \gamma_n(s) \) is a polynomial \( \tilde{P}_n(c_k(s)) \) and for all \( s, t \in \mathbb{C} \)

\[
\sum_{n \geq 1} \tilde{P}_n(c_k(s) + c_k(t))x^n = \sum_{n \geq 1} \tilde{P}_n(c_k(s))x^n + \sum_{n \geq 1} \tilde{P}_n(c_k(t)) \left[ x + \sum_{r \geq k} P_r(c_k(s))x^r \right]^n.
\]

This yields the formal first cocycle equation in \((\mathbb{C} [y, z])[[x]]\) for iteration groups of type II

\[
\Gamma(y + z, x) = \Gamma(y, x) + \Gamma(z, G(y, x)) \quad \text{(Co1 formal)}
\]

together with \( \Gamma(0, x) = 0 \) for \( \Gamma(y, x) = \sum_{n \geq 1} \tilde{P}_n(y)x^n \), where

\[
G(y, x) = x + yx^k + \sum_{n > k} P_n(y)x^n
\]
is a formal iteration group of type II.

**Theorem.** Let \( c_k \neq 0 \) be an additive function. Then \( \gamma(s, x) = \sum_{n \geq 1} \tilde{P}_n(c_k(s))x^n \) is a solution of \((\text{Co1}_\log)\) satisfying \( \gamma(0, x) = 0 \) if and only if \( \Gamma(y, x) = \sum_{n \geq 1} \tilde{P}_n(y)x^n \) is a solution of \((\text{Co1}_\text{formal})\) satisfying \( \Gamma(0, x) = 0 \).
Three equations derived from $(\text{Co1}_{\text{formal}})$

Differentiation of $(\text{Co1}_{\text{formal}})$ with respect to $y$ and setting $y = 0$ yields

$$\frac{\partial}{\partial z} \Gamma(z, x) = K(\Gamma(z, x)),$$  \hspace{1cm} (Co1D$_{\text{formal}}$)

where $K(x) := \frac{\partial}{\partial y} \Gamma(y, x)|_{y=0}$. 
Three equations derived from \((\text{Co1}_{\text{formal}})\)

Differentiation of \((\text{Co1}_{\text{formal}})\) with respect to \(y\) and setting \(y = 0\) yields

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where \(K(x) := \frac{\partial}{\partial y} \Gamma(y, x)|_{y=0}\).

Differentiation with respect to \(z\) together with the mixed chain rule yields

\[
\frac{\partial}{\partial y} \Gamma(y, x) = K(x) + H(x) \frac{\partial}{\partial x} \Gamma(y, x), \quad (\text{Co1PD}_{\text{formal}})
\]

where \(H(x) = \frac{\partial}{\partial y} G(y, x)|_{y=0}\) is the formal generator of the formal iteration group \(G(y, x)\).
Three equations derived from \((\text{Co1}_{\text{formal}})\)

Differentiation of \((\text{Co1}_{\text{formal}})\) with respect to \(y\) and setting \(y = 0\) yields

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where \(H(x) = \frac{\partial}{\partial y} G(y, x)|_{y=0}\) is the formal generator of the formal iteration group \(G(y, x)\). There is also an Aczél–Jabotinsky differential equation

\[
K(x) + H(x) \frac{\partial}{\partial x} \Gamma(y, x) = K(\Gamma(y, x)). \tag{\text{Co1AJ}_{\text{formal}}}\]
Theorem. Each solution of $(\text{Co1PD}_{\text{formal}})$ or $(\text{Co1D}_{\text{formal}})$ together with $\Gamma(0, x) = 0$ is a solution of $(\text{Co1})_{\text{formal}}$. 
**Theorem.** Each solution of \((\text{Co1PD}_{\text{formal}})\) or \((\text{Co1D}_{\text{formal}})\) together with \(\Gamma(0, x) = 0\) is a solution of \((\text{Co1}_{\text{formal}})\).

**Theorem.** Let \(K(x) = \sum_{n \geq 1} \kappa_n x^n\) be a formal series of order at least 1. Then the solution of \((\text{Co1D}_{\text{formal}})\) with \(\Gamma(0, x) = 0\) is

\[
\Gamma(y, x) = \sum_{j=1}^{k-1} \int \kappa_j [G(\sigma, x)]^j d\sigma + E(G(y, x)) - E(x),
\]

where \(E(x)\) is given by

\[
\frac{\partial}{\partial x} E(x) = \frac{\sum_{n \geq k} \kappa_n x^n}{H(x)}.
\]
Theorem. Each solution of \((\text{Co1PD}_{\text{formal}})\) or \((\text{Co1D}_{\text{formal}})\) together with \(\Gamma(0,x) = 0\) is a solution of \((\text{Co1}_{\text{formal}})\).

Theorem. Let \(K(x) = \sum_{n \geq 1} \kappa_n x^n\) be a formal series of order at least 1. Then the solution of \((\text{Co1D}_{\text{formal}})\) with \(\Gamma(0,x) = 0\) is

\[
\Gamma(y,x) = \sum_{j=1}^{k-1} \int \kappa_j [G(\sigma,x)]^j d\sigma + E(G(y,x)) - E(x),
\]

where \(E(x)\) is given by \(\frac{\partial}{\partial x} E(x) = \frac{\sum_{n \geq k} \kappa_n x^n}{H(x)}\).

By applying the exponential series we obtain the solutions of \((\text{Co1})\) as

\[
\alpha_1(s) \frac{\tilde{E}(G(c_k(s),x))}{\tilde{E}(x)} \prod_{j=1}^{k-1} \exp \left( \int \kappa_j [G(\sigma,x)]^j d\sigma \bigg|_{\sigma = c_k(s)} \right),
\]

where \(\tilde{E}(x) = \exp(E(x)) = 1 + \ldots\).
Theorem. Let $K(x) = \sum_{n \geq 1} \kappa_n x^n$ be a formal series of order at least 1. The polynomials $\tilde{P}_n(y)$ describing the coefficients of the solution of $(\text{Co1D}_\text{formal})$ and $\Gamma(0, x) = 0$ are universal polynomials of the form

$$\tilde{P}_n(y) = \begin{cases} 
\kappa_n y & n < k \\
\kappa_k y + \frac{\kappa_1}{2} y^2 & n = k \\
\kappa_n y + \frac{(n-k+1)\kappa_{n-k+1}}{2} y^2 + \tilde{Q}_n(y, \kappa_1, \ldots, \kappa_{n-k}) & n > k,
\end{cases}$$

and of a formal degree $1 + \left\lfloor \frac{n-1}{k-1} \right\rfloor$. 
Theorem. Let $K(x) = \sum_{n \geq 1} \kappa_n x^n$ be a formal series of order at least 1. The polynomials $\tilde{P}_n(y)$ describing the coefficients of the solution of $(\text{Co1D}_{\text{formal}})$ and $\Gamma(0, x) = 0$ are universal polynomials of the form

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\end{cases}$$

and of a formal degree $1 + \left\lfloor \frac{n-1}{k-1} \right\rfloor$.

Similarities to the solutions $G(y, x) = x + yx^k + \sum_{n > k} P_n(y)x^n$ of $(\text{T}_{\text{formal}})$ of type II depending on the formal generator $H(x) = \sum_{n \geq k} h_n x^n$:

The polynomials $P_n(y)$ are universal polynomials of the form

$$P_n(y) = \begin{cases} 
h_n y & \text{if } k \leq n < 2k - 1 \\
h_{2k-1} y + \frac{k}{2} y^2 & \text{if } n = 2k - 1 \\
h_n y + \frac{n+1}{2} h_{n-k+1} y^2 + \Phi_n(y, h_{k+1}, \ldots, h_{n-k}) & \text{if } n \geq 2k,
\end{cases}$$

and of a formal degree $\left\lfloor \frac{n-1}{k-1} \right\rfloor$. 
Reordering the summands

Solution of \((\text{Co}1_{\text{formal}})\): \(\Gamma(y, x) = \sum_{n \geq 1} \tilde{P}_n(y)x^n \in (\mathbb{C}[y])[[x]]\)

\[
\tilde{P}_n(y) = \sum_{j=1}^{d_n} \tilde{P}_{n,j}y^j \in \mathbb{C}[y], \quad d_n = 1 + \left\lfloor \frac{n - 1}{k - 1} \right\rfloor, \quad n \geq 1,
\]
Reordering the summands

Solution of $\text{(Co1}_{\text{formal}})$: \[\Gamma(y, x) = \sum_{n \geq 1} \tilde{P}_n(y)x^n \in (\mathbb{C}[y])[x]\]

\[\tilde{P}_n(y) = \sum_{j=1}^{d_n} \tilde{P}_{n,j}y^j \in \mathbb{C}[y], \quad d_n = 1 + \left\lfloor \frac{n - 1}{k - 1} \right\rfloor, \quad n \geq 1,\]

\[\Gamma(y, x) = \sum_{n \geq 1} \psi_n(x)y^n \in (\mathbb{C}[x])[y]\]

\[\psi_n(x) = \sum_{r \geq 1} \tilde{P}_{r,n}x^r, \quad n \geq 1.\]

$\left(\psi_n(x)\right)_{n \geq 1}$ and $\left(\psi_n(x)y^n\right)_{n \geq 1}$ are summable families.
Reordering the summands

Solution of \((\text{Co1}_{\text{formal}})\): \(\Gamma(y, x) = \sum_{n \geq 1} \tilde{P}_n(y)x^n \in (\mathbb{C}[y])[x]\)

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\tilde{P}_n(y) = \sum_{j=1}^{d_n} \tilde{P}_{n,j}y^j \in \mathbb{C}[y], \quad d_n = 1 + \left\lfloor \frac{n - 1}{k - 1} \right\rfloor, \quad n \geq 1,
\]

\[
\Gamma(y, x) = \sum_{n \geq 1} \psi_n(x)y^n \in (\mathbb{C}[x])[y]
\]

\[
\psi_n(x) = \sum_{r \geq 1} \tilde{P}_{r,n}x^r, \quad n \geq 1.
\]

\((\psi_n(x))_{n \geq 1}\) and \((\psi_n(x)y^n)_{n \geq 1}\) are summable families.

This allows us to rewrite \((\text{Co1PD}_{\text{formal}})\) as

\[
\sum_{n \geq 1} n\psi_n(x)y^{n-1} = K(x) + H(x) \sum_{n \geq 1} \psi'_n(x)y^n.
\]
(1) is satisfied if and only if

\[ \psi_1(x) = K(x) \]

\[ \psi_{n+1}(x) = \frac{1}{n+1} H(x) \psi'_n(x), \quad n \geq 1, \]

holds true.

\[ \psi_1(x) = K(x), \]

\[ \psi_2(x) = H(x)K'(x)/2, \]

\[ \psi_3(x) = \left( H(x)H'(x)K'(x) + H(x)^2K''(x) \right)/6. \]
(1) is satisfied if and only if
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\[ \psi_{n+1}(x) = \frac{1}{n+1} H(x)\psi'_n(x), \quad n \geq 1, \]
holds true. \[ \psi_1(x) = K(x), \]
\[ \psi_2(x) = H(x)K'(x)/2, \]
\[ \psi_3(x) = \left( H(x)H'(x)K'(x) + H(x)^2K''(x) \right)/6. \]

Similarities to formal iteration groups \( G(y, x) = \sum_{n \geq 0} \phi_n(x)y^n \) of type II:
\[ \phi_{n+1}(x) = \frac{1}{n+1} H(x)\phi'_n(x), \quad n \geq 0. \]
\[ \phi_1(x) = H(x), \]
\[ \phi_2(x) = H(x)H'(x)/2, \]
\[ \phi_3(x) = \left( H(x)H'(x)^2 + H(x)^2H''(x) \right)/6. \]
Some results

Generalizing the representations given on the previous slide:

\[ \psi_n(x) = \frac{1}{n!} \sum_{i \in J_n} L(i) K^{(i-1)}(x) \prod_{j=0}^{n-2} \left[ H^{(j)}(x) \right]^{i_j}, \quad n \geq 2, \]

\[ J_n = \left\{ (i_j)_{j \geq -1} \mid i_{-1} \geq 1, \ i_j \geq 0, \ \sum_{j \geq 0} i_j = n - 1, \ \sum_{j \geq 1} j i_j = n - 1 - i_{-1} \right\}, \]

where \( L: \bigcup_{n \geq 2} J_n \to \mathbb{N} \) is recursively determined.
Some results

Generalizing the representations given on the previous slide:

$$\psi_n(x) = \frac{1}{n!} \sum_{i \in J_n} L(i) K^{(i-1)}(x) \prod_{j=0}^{n-2} \left[ H^{(j)}(x) \right]^{i_j}, \quad n \geq 2,$$

$$J_n = \left\{ (i_j)_{j \geq -1} \mid i_{-1} \geq 1, i_j \geq 0, \sum_{j \geq 0} i_j = n-1, \sum_{j \geq 1} ji_j = n-1-i_{-1} \right\},$$

where $L: \bigcup_{n \geq 2} J_n \to \mathbb{N}$ is recursively determined.

Similarities to iteration groups of type II:

$$\phi_n(x) = \frac{1}{n!} \sum_{i \in I_n} L(i) \prod_{j=0}^{n-1} \left[ H^{(j)}(x) \right]^{i_j}, \quad n \geq 1,$$

$$I_n = \left\{ (i_j)_{j \geq 0} \mid i_j \geq 0, i_0 \geq 1, \sum_{j \geq 0} i_j = n, \sum_{j \geq 1} ji_j = n-1 \right\},$$

where $L: \bigcup_{n \geq 1} I_n \to \mathbb{N}$ is recursively determined.
Solution as a Lie–Gröbner-series

\[ \Gamma(y, x) := \sum_{n \geq 1} \frac{1}{n!} D^{n-1}(K(x)) y^n, \]

where

\[ D : \mathbb{C}[x] \to \mathbb{C}[x], \quad D(f(x)) := H(x) f'(x). \]
Solution as a Lie–Gröbner-series

\[ \Gamma(y,x) := \sum_{n \geq 1} \frac{1}{n!} D^{n-1}(K(x)) y^n, \]

where

\[ D: \mathbb{C}[[x]] \to \mathbb{C}[[x]], \quad D(f(x)) := H(x)f'(x). \]

Similarities to iteration groups of type II:

\[ G(y,x) := \sum_{n \geq 0} \frac{1}{n!} D^n(x) y^n, \]

where

\[ D: \mathbb{C}[[x]] \to \mathbb{C}[[x]], \quad D(f(x)) := H(x)f'(x). \]
On formal cocycle equations
The translation equation
Iteration groups of type I and II
The cocycle equations
Formal equations
The formal translation equation
The first cocycle equation
Three equations derived from \( (\text{Co}_1_{\text{formal}}) \)
Reordering the summands
Some results
Solution as a Lie–Gröbner-series