Formal translation equation and formal cocycle equations for iteration groups of type I

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- Why **formal equations** in $\mathbb{C}[[x]]$?
- The formal translation equation.
- The formal cocycle equation.
- Formal autonomous, or partial differential equations, or Aczél–Jabotinsky type equations.

The translation equation

We study the translation equation

$$F(s+t,x) = F\big(s, F(t,x)\big), \qquad s,t \in \mathbb{C}, \tag{T}$$

where

$$F(s,x) = \sum_{n\geq 1} c_n(s)x^n \in \mathbb{C}[[x]], \qquad s \in \mathbb{C},$$

with $\mathrm{ord}(F(s,x)) = 1$ for all $s \in \mathbb{C}$, whence $c_1(s) \neq 0$ for all $s \in \mathbb{C}$.

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with $\mathrm{ord}(F(s,x)) = 1$ for all $s \in \mathbb{C}$, whence $c_1(s) \neq 0$ for all $s \in \mathbb{C}$.

$$\begin{aligned}
c_1(s+t) &= c_1(s)c_1(t) \\
c_2(s+t) &= c_1(s)c_2(t) + c_2(s)c_1(t)^2 \\
c_n(s+t) &= c_1(s)c_n(t) + c_n(s)c_1(t)^n \\
&\qquad + \tilde{P}_n\bigl(c_2(s), \ldots, c_{n-1}(s), c_1(t), \ldots, c_{n-1}(t)\bigr)
\end{aligned}$$

for all $s,t \in \mathbb{C}$.

Observations:

- $c_1(s+t) = c_1(s)c_1(t)$, $s,t \in \mathbb{C}$, thus c_1 is a generalized exponential function. We restrict ourselves to $c_1 \neq 1$.

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- $c_n(s) = P_n\big(c_1(s)\big)$, $s \in \mathbb{C}$, $n \geq 1$.
- $c_n(s+t) = P_n(c_1(s) \cdot c_1(t)) = c_1(s)P_n(c_1(t)) + P_n(c_1(s))c_1(t)^n + \tilde{P}_n\big(P_2(c_1(s)),\ldots,P_{n-1}(c_1(s)),c_1(t),\ldots,P_{n-1}(c_1(t))\big)$ for all $s,t \in \mathbb{C}$.

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- **Lemma.** Assume that $e \neq 1$ is a generalized exponential function and let $P(y,z) \in \mathbb{C}[y,z]$. If $P(e(s), e(t)) = 0$ for all $s,t \in \mathbb{C}$, then $P = 0$.

The formal translation equation

(T) is equivalent to the formal translation equation

$$G(y \cdot z, x) = G\big(y, G(z, x)\big) \qquad (\mathrm{T}_{\text{formal}})$$

in $(\mathbb{C}[y,z])[[x]]$ for

$$G(y,x) = yx + \sum_{n\geq 2} P_n(y)x^n,$$

$P_n(y) \in \mathbb{C}[y]$, $n \geq 2$, and the boundary condition

$$G(1,x) = x. \qquad (\mathrm{B})$$

Formal differential equations for (T)

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Differentiation of ($\mathrm{T_{formal}}$) with respect to y yields

$$z\frac{\partial}{\partial t}G(t,x)|_{t=yz}=\frac{\partial}{\partial y}G\big(y,G(z,x)\big).$$

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For $y = 1$ we get

$$z\frac{\partial}{\partial z}G(z,x) = H\big(G(z,x)\big), \qquad (\mathrm{D}_{\text{formal}})$$

where

$$H(x) = x + \sum_{n \geq 2} h_n x^n = \frac{\partial}{\partial y}G(y,x)|_{y=1}$$

is the infinitesimal generator of G.

Formal differential equations for (T) cont.

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Similarly, differentiation of ($\mathrm{T}_{\text{formal}}$) with respect to z, application of the mixed chain rule and setting $z=1$ yields

$$y\frac{\partial}{\partial y}G(y,x)=H(x)\frac{\partial}{\partial x}G(y,x). \qquad (\mathrm{PD}_{\text{formal}})$$

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Combining ($\mathrm{D}_{\mathrm{formal}}$) and ($\mathrm{PD}_{\mathrm{formal}}$) yields an Aczél–Jabotinsky differential equation of the form

$$H(x)\frac{\partial}{\partial x}G(y,x)=H\big(G(y,x)\big). \qquad (\mathrm{AJ}_{\mathrm{formal}})$$

Solutions of (PD$_{\text{formal}}$)

Theorem.

1. For any generator $H(x) = x + h_2x^2 + \ldots$ the differential equation (PD$_{\text{formal}}$) together with (B) has exactly one solution. It is given by

$$G(y,x) = yx + \sum_{n\geq 2} P_n(y)x^n \in (\mathbb{C}[y])[[x]].$$

Solutions of (PD_{formal})

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$$G(y,x) = yx + \sum_{n\geq 2} P_n(y)x^n \in (\mathbb{C}[y])[[x]].$$

2. The polynomials P_n, $n \geq 2$, are of formal degree n and they are of the form

$$P_n(y) = \frac{h_n}{n-1}(y^n - y) + \sum_{j=1}^{n-1} \frac{\Phi_j^{(n)}(h_2,\dots,h_{n-1})}{n-j}(y^n - y^j),$$

where the polynomials $\Phi_j^{(n)}$, $1 \leq j \leq n-1$, are (recursively) determined

by

$$\sum_{r=2}^{n-1} h_r(n-r+1)P_{n-r+1}(y) = \sum_{j=1}^{n-1} \Phi_j^{(n)}(h_2,\dots,h_{n-1})y^j.$$

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3. Each solution $G(y,x)$ of (PD$_{\text{formal}}$) and (B) is a solution of (T$_{\text{formal}}$).

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4. Rearranging the solution of (T$_{\text{formal}}$) as

$$G(y,x) = \sum_{n\geq 1} \phi_n(x)y^n,$$

we obtain: $\phi_1 \in \Gamma_1$,

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$$G(y,x) = \sum_{n\geq 1} \phi_n(x)y^n,$$

we obtain: $\phi_1 \in \Gamma_1$, and $(\phi_n(x)y^n)_{n\geq 1}$ is a summable family in $(\mathbb{C}[y])[[x]]$. This allows us to rewrite (PD$_{\text{formal}}$) and (B) as

$$\sum_{n\geq 1} n\phi_n(x)y^n = H(x)\sum_{n\geq 1} \phi_n'(x)y^n, \tag{1}$$

$$\sum_{n\geq 1} \phi_n(x) = x, \tag{2}$$

where $(\phi_n'(x))_{n\geq 1}$ is also a summable family. We note that (1) is satisfied if and only if

$$n\phi_n(x) = H(x)\phi_n'(x) \qquad (1_n)$$

holds true for all $n \geq 1$. This is a Briot–Bouquet differential equation.

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5. Let $n \geq 1$. For any $\varphi_n^{(n)} \in \mathbb{C}$, there is exactly one solution ϕ_n of (1_n), so that

$$\phi_n(x) \equiv \varphi_n^{(n)} x^n \bmod x^{n+1}.$$

Let $\phi_{n,0}$ be the solution corresponding to $\varphi_n^{(n)} = 1$.

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$$\phi_n(x) = \varphi_n^{(n)} \phi_{n,0}(x).$$

7. $\phi_{n,0}(x) = [\phi_{1,0}(x)]^n$ for $n \geq 1$.

Normal forms of iteration groups

Theorem.

1. If $G(y,x) = \sum_{n\geq 1} \phi_n(x) y^n$ is a solution of (T$_{\text{formal}}$) and (B), then

$$G(y,x) = S^{-1}\big(yS(x)\big)$$

for some $S \in \Gamma_1$.

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of ($\mathrm{T}_{\text{formal}}$) and (B).

Proof. Let $S(x) := \phi_{1,0}(x)$, $\Sigma(x) := \sum_{n\geq 1} \varphi_n^{(n)} x^n$, then $x = G(1,x) = \sum_{n\geq 1} \varphi_n^{(n)} [S(x)]^n = \Sigma(S(x))$, thus $\sigma = S^{-1}$, and $G(y,x) = \sum_{n\geq 1} \varphi_n^{(n)} [y\phi_{1,0}(x)]^n = \Sigma(yS(x))$.

Series as coefficients

Now we replace y by $1+u$ and z by $1+v$ in (T$_{\text{formal}}$) and obtain

$$G(1+u+v+uv,x) = G\big(1+u, G(1+v,x)\big).$$

Denoting $G(y,x)$ which is $G(1+u,x)$ by $K(u,x)$ yields

$$K(u+v+uv,x) = K\big(u, K(v,x)\big) \tag{3}$$

in $(\mathbb{C}[u,v])[[x]]$ for

$$K(u,x) = (1+u)x + \sum_{n\geq 2} Q_n(u)x^n,$$

$Q_n(u) \in \mathbb{C}[u]$, $n \geq 2$. The boundary condition (B) is replaced by

$$K(0,x) = x. \tag{B'}$$

Since substitution of 0 into a formal power series is possible, we are able to determine the solutions even in the ring $(\mathbb{C}[[u,v]])[[x]]$. Here we consider

$$K(u,x) = \sum_{n\geq 0} Q_n(u)x^n,$$

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Theorem.

1. For any generator $H(x) = x + h_2x^2 + \ldots$ the differential equation (3) together with (B') has exactly one solution

$$K(u,x) = (1+u)x + \sum_{n\geq 2} Q_n(u)x^n \in (\mathbb{C}[[u]])[[x]].$$

2. The formal series Q_n, $n \geq 2$, are polynomials of formal degree n and they are of the form

$$Q_n(u) = \frac{h_n}{n-1}\left(\left(1+u\right)^n - \left(1+u\right)\right) + \sum_{j=1}^{n-1} \frac{\Phi_j^{(n)}(h_2,\dots,h_{n-1})}{n-j}\left(\left(1+u\right)^n - \left(1+u\right)^j\right)$$

where the polynomials $\Phi_j^{(n)}$, $1 \leq j \leq n-1$, are (recursively) determined by

$$\sum_{r=2}^{n-1} h_r(n-r+1)Q_{n-r+1}(u) = \sum_{j=1}^{n-1} \Phi_j^{(n)}(h_2,\dots,h_{n-1})(1+u)^j.$$

The other differential equations

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Similar results can be obtained for the system (AJ_{formal}) together with $P_1(y) = y$.

The cocycle equations

In connection with the problem of a covariant embedding of the linear functional equation $\varphi(p(x)) = a(x)\varphi(x) + b(x)$ with respect to an iteration group $(F(s,x))_{s\in\mathbb{C}}$ we have to solve the two cocycle equations

$$\alpha(s+t,x) = \alpha(s,x)\alpha\big(t,F(s,x)\big), \qquad s,t\in\mathbb{C}, \tag{Co1}$$

$$\beta(s+t,x) = \beta(s,x)\alpha\big(t,F(s,x)\big) + \beta\big(t,F(s,x)\big), \qquad s,t\in\mathbb{C}, \tag{Co2}$$

under the boundary conditions

$$\alpha(0,x) = 1, \qquad \beta(0,x) = 0, \tag{B1}$$

for

$$\alpha(s,x) = \sum_{n\geq 0} \alpha_n(s)x^n, \qquad \beta(s,x) = \sum_{n\geq 0} \beta_n(s)x^n.$$

Again we restrict ourselves to iteration groups F of type I. Moreover, it is possible to consider just the normal forms $F(s,x) = c_1(s)x$.

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The first cocycle equation

I don't present any details. It is straight forward to see (in the usual way or using formal equations) that

$$\alpha(s,x) = \alpha_0(s)\frac{E(c_1(s)x)}{E(x)}$$

for a generalized exponential function $\alpha_0(s)$ and a formal series $E(x)$ with $E(x) \equiv 1 \bmod x$.

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The second cocycle equation

Simple substitutions show that instead of (Co2) it is enough to solve

$$\Delta(s+t,x) = \Delta(s,x) + \frac{1}{\alpha_0(s)}\Delta(t,c_1(s)x) \qquad \text{(Co2')}$$

and

$$\Delta(0,x) = 0 \qquad \text{(B1')}$$

for

$$\Delta(s,x) = \frac{\beta(s,x)}{\alpha_0(s)E(c_1(s)x)}.$$

If we write $\Delta(s,x)$ as $\sum_{n\geq 0}\Delta_n x^n$, then $\Delta(s,x)$ satisfies (Co2') if and only if

$$\Delta_n(s+t) = \Delta_n(s) + \alpha_0(s)^{-1}\Delta_n(t)c_1(s)^n, \qquad n \geq 0. \qquad (4)$$

Different situations

- If α_0 and c_1 are algebraically independent, then the coefficients $\Delta_n(s)$ are polynomials $D_n(c_1(s), \alpha_0(s)^{-1})$ and we obtain

$$D(ST, \sigma\tau, x) = D(S, \sigma, x) + \sigma D(T, \tau, Sx) \qquad (\mathrm{Co2}_{\mathrm{formal}_1})$$

in $(\mathbb{C}[S, T, \sigma, \tau])[[x]]$ for

$$D(S, \sigma, x) = \sum_{n \geq 0} D_n(S, \sigma) x^n,$$

$D_n(S, \sigma) \in \mathbb{C}[S, \sigma]$, $n \geq 0$, and the boundary condition

$$D(1, 1, x) = 0. \qquad (\mathrm{B2}_1)$$

If α_0^{-1} and c_1 are algebraically dependent, then they are multiplicatively dependent and there exist a generalized exponential function $e \neq 1$ and integers $r_0, r_1 \in \mathbb{Z}$, such that $\alpha_0^{-1} = e^{r_0}$ and $c_1 = e^{r_1}$. Without loss of generality we may assume that $r_1 > 0$.

- If $\alpha_0^{-1} c_1^n = e^{r_0 + n r_1} \neq 1$ for all $n \geq 0$, then the coefficients $\Delta_n(s)$ are rational functions $\tilde{D}_n(e(s))$ and we obtain

$$D(ST, x) = D(S, x) + S^{r_0} D(T, S^{r_1} x) \qquad (\text{Co2}_{\text{formal}_2})$$

in $(\mathbb{C}(S,T))[[x]]$ for

$$D(S,x) = \sum_{n \geq 0} \tilde{D}_n(S) x^n,$$

$\tilde{D}_n(S) \in \mathbb{C}(S)$, $n \geq 0$, and the boundary condition

$$D(1,x) = 0. \qquad (\text{B2}_2)$$

- If $\alpha_0 = c_1^{n_0}$, $n_0 \geq 0$, then $\Delta_n(s) = D_n(e(s), A(s))$ is a rational function in $e(s)$ and a polynomial in $A(s)$. If $A \neq 0$, then we obtain

$$D(ST, U+V, x) = D(S, U, x) + S^{r_0} D(T, V, S^{r_1} x) \qquad (\text{Co2}_{\text{formal}_3})$$

in $(\mathbb{C}(S,T)[U,V])[[x]]$ for

$$D(S,U,x) = \sum_{n \geq 0} D_n(S,U) x^n,$$

$D_n(S,U) \in \mathbb{C}(S)[U]$, $n \geq 0$, and the boundary condition

$$D(1,0,x) = 0. \qquad (\text{B2}_3)$$

The formal version of the second cocycle equation

$$D(ST,\sigma\tau,U+V,x) = D(S,\sigma,U,x) + \sigma^{\lambda} S^{\mu} D(T,\tau,V,S^{\nu}x) \qquad (\text{Co2}_{\text{formal}})$$

for $\lambda,\mu,\nu \in \mathbb{Z}$, $\nu > 0$, where

$$D(S,\sigma,U,x) = \sum_{n\geq 0} D_n(S,\sigma,U)x^n \in (\mathbb{C}(S)[\sigma,U])[[x]],$$

and the boundary condition

$$D(1,1,0,x) = 0. \qquad (\text{B2})$$

Three systems of formal differential equations

1. Differentiation of (Co2$_{\text{formal}}$) with respect to T, τ, or V respectively, and substituting $T=1$, $\tau=1$, $V=0$ yields

$$S\frac{\partial}{\partial S}D(S,\sigma,U,x)=\sigma^{\lambda}S^{\mu}K(S^{\nu}x), \tag{Co2D1$_{\text{formal}}$}$$

$$\sigma\frac{\partial}{\partial \sigma}D(S,\sigma,U,x)=\sigma^{\lambda}S^{\mu}L(S^{\nu}x), \tag{Co2D2$_{\text{formal}}$}$$

and

$$\frac{\partial}{\partial U}D(S,\sigma,U,x)=\sigma^{\lambda}S^{\mu}M(S^{\nu}x), \tag{Co2D3$_{\text{formal}}$}$$

where $K(x)=\frac{\partial}{\partial T}D(T,1,0,x)|_{T=1}$, $L(x)=\frac{\partial}{\partial \tau}D(1,\tau,0,x)|_{\tau=1}$ and $M(x)=\frac{\partial}{\partial V}D(1,1,V,x)|_{V=0}$.

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$$\sigma\frac{\partial}{\partial \sigma}D(S,\sigma,U,x)=\sigma^{\lambda}S^{\mu}L(S^{\nu}x), \qquad \text{(Co2D2}_{\text{formal}}\text{)}$$

and

$$\frac{\partial}{\partial U}D(S,\sigma,U,x)=\sigma^{\lambda}S^{\mu}M(S^{\nu}x), \qquad \text{(Co2D3}_{\text{formal}}\text{)}$$

where $K(x)=\frac{\partial}{\partial T}D(T,1,0,x)|_{T=1}$, $L(x)=\frac{\partial}{\partial \tau}D(1,\tau,0,x)|_{\tau=1}$ and $M(x)=\frac{\partial}{\partial V}D(1,1,V,x)|_{V=0}$.

It is possible to solve these equations. There exists a unique solution which is determined by $K(x)$, $L(x)$ and $M(x)$. The coefficient functions are explicitly described.

2. Differentiation of (Co2$_{\text{formal}}$) with respect to S, σ or U, respectively, and substituting $S = 1$, $\sigma = 1$, $U = 0$ yields the equations

$$T\frac{\partial}{\partial T}D(T,\tau,V,x) = K(x) + \mu D(T,\tau,V,x) + \nu x\frac{\partial}{\partial x}D(T,\tau,V,x), \tag{Co2PD1$_{\text{formal}}$}$$

$$\tau\frac{\partial}{\partial \tau}D(T,\tau,V,x) = L(x) + \lambda D(T,\tau,V,x), \tag{Co2PD2$_{\text{formal}}$}$$

$$\frac{\partial}{\partial V}D(T,\tau,V,x) = M(x). \tag{Co2PD3$_{\text{formal}}$}$$

3. Combining the last six differential equations we obtain

$$\sigma^{\lambda} S^{\mu} K(S^{\nu} x) = K(x) + \mu D(S, \sigma, U, x) + \nu x \frac{\partial}{\partial x} D(S, \sigma, U, x), \tag{Co2AJ1$_{\text{formal}}$}$$

$$\sigma^{\lambda} S^{\mu} L(S^{\nu} x) = L(x) + \lambda D(S, \sigma, U, x), \tag{Co2AJ2$_{\text{formal}}$}$$

$$\sigma^{\lambda} S^{\mu} M(S^{\nu} x) = M(x). \tag{Co2AJ3$_{\text{formal}}$}$$

If $\lambda \neq 0$ or $\mu + n\nu \neq 0$ for all $n \geq 0$, then this system has a unique solution.

If $\lambda = 0$ and $\mu + n_0 \nu = 0$ for some $n_0 \geq 0$, then this system together with

$$D_{n_0}(ST, \sigma\tau, U + V) = D_{n_0}(S, \sigma, U) + D_{n_0}(T, \tau, V)$$

has a unique solution.

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