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Generalizations of the functional equation of the mean sun

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Abstract

Two generalizations $U(\lambda+\mu)y(\nu) = U(\lambda)y(\mu+\nu)$ $(\forall\lambda,\mu,\nu \in A)$ of the functional equation of the mean sun are studied, where $(A,+)$ is an Abelian group, K is a field, n is a positive integer, and both $y\colon A \to K^n$ and $U\colon A \to \mathrm{GL}(n,k)$ (or $U\colon A \to M_n(k)$ in the second case) are unknown functions, which will be determined by the equation.

1 Introduction

Local solar time is measured by a sundial. When the center of the sun is on an observer's meridian, the observer's local solar time is zero hours (noon). Because the earth moves with varying speed in its orbit at different times of the year and because the plane of the earth's equator is inclined to its orbital plane, the length of the solar day is different depending on the time of year. It is more convenient to define time in terms of the average of local solar time. Such time, called mean solar time, may be thought of as being measured relative to an imaginary sun (the mean sun) that lies in the earth's equatorial plane and about which the earth orbits with constant speed. Every mean solar day is of the same length.[1]

In [1, 4] it is shown that the mean sun satisfies the functional equation

$$M(\lambda+t,\phi)^T y(s) = M(\lambda,\phi)^T y(s+t), \qquad \forall s,t,\lambda \in \mathbb{R} \ \ -\pi/2 < \phi < \pi/2, \tag{1}$$

where $y(s)$ is a vector of length 1, which is the direction from the center of the earth to the sun at the time s (one day corresponds to 2π) expressed in a geocentric coordinate system. As a basis of this system we can choose two orthogonal vectors in the equatorial plane and one vector along the axis of the earth. $M(\lambda,\phi)$ is the matrix

$$M(\lambda,\phi) = \begin{pmatrix} -\sin\lambda & -\sin\phi\cos\lambda & \cos\phi\cos\lambda \\ \cos\lambda & -\sin\phi\sin\lambda & \cos\phi\sin\lambda \\ 0 & \cos\phi & \sin\phi \end{pmatrix}.$$

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[1] http://www.infoplease.com/ce6/society/A0845838.html

Then $M(\lambda,\phi)y(s)$ is the direction from the earth to the sun expressed in a local coordinate system on the surface of the earth in the point of longitude λ and latitude ϕ.

In the present paper we investigate generalizations of equation (1) for fixed ϕ. To be more precise, first we will solve the following functional equation

$$U(\lambda+\mu)y(\nu)=U(\lambda)y(\mu+\nu), \qquad \forall\lambda,\mu,\nu\in A, \tag{2}$$

where $(A,+)$ is an Abelian group, K is a field, n is a positive integer, and both $y\colon A\to K^n$ and $U\colon A\to \mathrm{GL}(n,K)$ are unknown functions, which will be determined by (2). In some situations we will additionally have to assume that $A=K$. Later on we will study the more general situation when we replace $\mathrm{GL}(n,k)$ by $M_n(K)$, the set of all $n\times n$ matrices over K.

The following types of questions can be asked in connection with (2):

1. Determine all solutions (U,y) of (2).
2. For given U determine all y, such that (U,y) is a solution of (2).
3. For given y determine all U, such that (U,y) is a solution of (2).
4. Find relations between U and y for a solution (U,y) of (2).

We will mainly deal with problems of the second and third kind.

In Theorem 6 we describe in an appropriate system of coordinates the structure of the space S_U of all solutions of (2) for a given $U\colon A\to \mathrm{GL}(n,K)$. We also state in this theorem how such a mapping U necessarily looks if a nontrivial solution y (i.e. $y\neq 0$) exists. A similar description of U-invariant subspaces of S_U is given in Theorem 8. We emphasize that by our result (and similarly by the following theorems) the problem of solving (2) can be reduced, at least to some extent, to the problem of finding all exponential functions $U_{11}\colon A\to \mathrm{GL}(k,K)$ (cf. the representation of U in Theorem 6), i.e. non singular matrices $U_{11}(\lambda)$ satisfying the equation

$$U_{11}(\lambda+\mu)=U_{11}(\lambda)U_{11}(\mu), \qquad \forall\lambda,\mu\in A.$$

Here we assume that these functions are known and we refer the reader to [3].

In Theorem 9 we construct to a given subspace S^0 of K^n the set of all mappings U and correspondingly the space S of all functions y, such that (U,y) satisfies (2) and S^0 is exactly the set of all initial values $y(0)$ for $y\in S$. It is clear that this yields together with Theorem 6 an implicit description of the set of all solutions (U,y) of (2) by varying the subspace S^0 of K^n. However, the space S and the mapping U obtained in this way from S^0 may have the property that S is a proper subset of S_U. Therefore we also deal with the problem to characterize the situation when $S=S_U$.

From a mathematical point of view it seems also interesting to study the functional equation (2) for mappings $U\colon A\to M_n(K)$. This situation is more complicated both with respect to the technical details and the construction (description) of the solutions U or y or (U,y). In Theorem 20 we start from a given mapping $U\colon A\to M_n(K)$ and describe completely in appropriate coordinates the set of all functions $y\colon A\to K^n$, such that (U,y) is a solution of (2). Again this theorem provides necessary conditions on U for the existence of nontrivial solutions y of (2). We also show in Theorem 21 how to construct all mappings $U\colon A\to M_n(K)$ and corresponding spaces S of functions $y\colon A\to K^n$, such that $\{y(0)\mid y\in S\}$ is a given subspace S^0 of K^n and (U,y) is a solution of (2), hence giving an implicit description of the general solution of (2) by varying S^0. However, we were not able to contribute to the problem when $S=S_U$.

The main difficulties in this last part seem to arise from the fact that there can occur solutions y (to a given U) with $y(0)=0$ but $y\neq 0$ (cf. Lemma 18).

2 Regular matrices $U(\lambda)$

Here in this part we always assume that U is a mapping from the abelian group A to $\mathrm{GL}(n,K)$.

Lemma 1. *Let B,C be matrices in $\mathrm{GL}(n,k)$. Then (U,y) is a solution of (2) if and only if (V,By) is a solution of (2), where $V(\lambda)=CU(\lambda)B^{-1}$.*

Proof. The pair (U,y) is a solution of (2) if $U(\lambda+\mu)y(\nu)=U(\lambda)y(\mu+\nu)$ for all $\lambda,\mu,\nu\in A$. Since B and C are regular matrices, this is equivalent to $CU(\lambda+\mu)B^{-1}By(\nu)=CU(\lambda)B^{-1}By(\mu+\nu)$ for all $\lambda,\mu,\nu\in A$. □

For $C=U(0)^{-1}$ we get $CU(0)=I_n$, the identity matrix. Hence, without loss of generality we will always assume that $U(0)=I_n$.

Lemma 2. *If (U,y) is a solution of (2), then*

$$y(\mu)=U(\mu)y(0),\qquad \forall\mu\in A, \tag{3}$$

$$[U(\lambda+\mu)-U(\lambda)U(\mu)]\,y(0)=0,\qquad \forall\lambda,\mu\in A. \tag{4}$$

Proof. Since $U(0)=I_n$, we get (3) from (2) for $\lambda=\nu=0$. And we get (4) from (2) and (3) for $\nu=0$. □

It is also possible to reverse the statement of Lemma 2.

Lemma 3. *Assume $U(0)=I_n$ and let y be given by (3). If (U,y) satisfies (4), then (U,y) is a solution of (2).*

For any mapping $U\colon A\to\mathrm{GL}(n,k)$ let

$$S_U:=\{y\mid (U,y)\text{ is a solution of (2)}\}\quad\text{and}\quad S_U^0:=\{y(0)\mid y\in S_U\}\,.$$

Some basic properties of these two sets are collected in the following

Lemma 4. *Both S_U and S_U^0 are K-linear spaces and $\theta\colon S_U^0 \to S_U$, given by $\theta(y^0) := U(\cdot)y^0$, is a vector space isomorphism.*

Proof. It is clear that S_U and S_U^0 are linear spaces. Assume that $y^0 \in S_U^0$, then there is some $y \in S_U$, such that $y(0) = y^0$. Since (U, y) satisfies (3), the function θ is well defined. It is surjective, since for any $y \in S_U$ we have $\theta(y(0)) = U(\cdot)y(0) = y(\cdot)$ according to (3). The mapping θ is also injective, since from $\theta(y_1^0) = \theta(y_2^0)$ we derive $U(\lambda)y_1^0 = U(\lambda)y_2^0$, which implies for $\lambda = 0$ (and $U(0) = I_n$) that $y_1^0 = y_2^0$. Finally we have to prove that θ is a linear mapping. Let $y_1^0, y_2^0 \in S_U^0$ and let $\alpha_1, \alpha_2 \in K$, then $\alpha_1 y_1^0 + \alpha_2 y_2^0 \in S_U^0$ and $\theta(\alpha_1 y_1^0 + \alpha_2 y_2^0) = U(\cdot)(\alpha_1 y_1^0 + \alpha_2 y_2^0) = \alpha_1 U(\cdot)y_1^0 + \alpha_2 U(\cdot)y_2^0 = \alpha_1\theta(y_1^0) + \alpha_2\theta(y_2^0)$. □

In conclusion, both S_U and S_U^0 are m-dimensional linear spaces for some $0 \le m \le n$.

There are some more interesting properties of S_U and S_U^0.

Lemma 5. *Let $U\colon A \to \mathrm{GL}(n, k)$. Then:*

1. *S_U is $U(\lambda_0)$-invariant for all $\lambda_0 \in A$ (i.e. if $y \in S_U$, then also $U(\lambda_0)y \in S_U$).*
2. *S_U is invariant under translations (i.e. if $y \in S_U$, then also $y(\cdot + \lambda_0) \in S_U$ for all $\lambda_0 \in A$).*
3. *S_U^0 is $U(\lambda_0)$-invariant for all $\lambda_0 \in A$.*
4. *$S_U^0 = \{y(\lambda) \mid \lambda \in A,\ y \in S_U\}$*

Proof.

1. Assume that $z(\lambda) := U(\lambda_0)y(\lambda)$. Then (U, z) satisfies (2), since $U(\lambda+\mu)z(\nu) = U(\lambda+\mu)U(\lambda_0)y(\nu) = U(\lambda+\mu)U(\lambda_0)U(\nu)y(0) = U(\lambda+\mu)U(\lambda_0+\nu)y(0) = U(\lambda+\mu)y(\lambda_0+\nu) = U(\lambda)y(\mu+\lambda_0+\nu) = U(\lambda)U(0)y(\lambda_0+\mu+\nu) = U(\lambda)U(\lambda_0)y(\mu+\nu) = U(\lambda)z(\mu+\nu)$ by (3), (4), (3), (2), special form of $U(0)$ and (2).
2. Let $z(\lambda) := y(\lambda+\lambda_0)$, then (U, z) satisfies (2), since $U(\lambda+\mu)z(\nu) = U(\lambda+\mu)y(\nu+\lambda_0) = U(\lambda)y(\mu+\nu+\lambda_0) = U(\lambda)z(\mu+\nu)$ by (2).
3. If $y^0 \in S_U^0$, there exists some $y \in S_U$, such that $y^0 = y(0)$. From the first item of this lemma we know that $U(\lambda_0)y \in S_U$, hence $U(\lambda_0)y(0) = U(\lambda_0)y^0 \in S_U^0$.
4. According to the definition of S_U^0 we know that $S_U^0 \subseteq \{y(\lambda) \mid \lambda \in A,\ y \in S_U\}$. Let $y \in S_U$ and assume that $\lambda_0 \in A \setminus \{0\}$, then it follows from the second item of this lemma that $z(\cdot) := y(\cdot + \lambda_0) \in S_U$ and $y(\lambda_0) = z(0) \in S_U^0$. □

If $\{b_1, \ldots, b_m\}$ denotes a basis of S_U^0, then there exists a matrix $B \in \mathrm{GL}(n, k)$, such that $Bb_i = e_i$, the i-th unit vector in K^n. Applying this matrix B as a coordinate transformation on K^n as in Lemma 1 we get that $S_{UB^{-1}}^0 = \langle e_1, \ldots, e_m\rangle$, the m-dimensional linear space generated by the first m unit vectors in K^n. Thus without loss of generality we may assume that $S_U = \langle e_1, \ldots, e_m\rangle$.

Theorem 6. *Let $U\colon A \to \mathrm{GL}(n,K)$ be a mapping, such that $S_U^0 = \langle e_1, \ldots, e_m\rangle$ and $U(0) = I_n$. Then $U(\lambda)$ can be partitioned as a block matrix of the form*

$$U(\lambda) = \begin{pmatrix} U_{11}(\lambda) & U_{12}(\lambda) \\ (0)_{n-m,m} & U_{22}(\lambda) \end{pmatrix},$$

where $U_{11}(\lambda) \in \mathrm{GL}(m,K)$, $U_{22}(\lambda) \in \mathrm{GL}(n-m,K)$ and $U_{12}(\lambda) \in M_{m,n-m}(K)$. These matrices satisfy the boundary conditions $U_{11}(0) = I_m$, $U_{22}(0) = I_{n-m}$ and $U_{12}(0) = (0)_{m,n-m}$, the zero matrix. Moreover, U_{11} is an exponential function, i.e. $U_{11}(\lambda+\mu) = U_{11}(\lambda)U_{11}(\mu)$ for all $\lambda, \mu \in A$.

Each $y \in S_U$ can be expressed as

$$y(\lambda) = \begin{pmatrix} U_{11}(\lambda)\bar{y}(0) \\ 0 \end{pmatrix}, \qquad \forall \lambda \in A,$$

and $\bar{y}(0) \in K^m$.

Proof. Let $y \in S_U$, then $y(\lambda) \in S_U^0 = \langle e_1, \ldots, e_m\rangle$, so $y(\lambda) = \begin{pmatrix} \bar{y}(\lambda) \\ 0 \end{pmatrix}$ and $\bar{y}(\lambda) \in K^m$. We partition $U(\lambda)$ as a block matrix

$$U(\lambda) = \begin{pmatrix} U_{11}(\lambda) & U_{12}(\lambda) \\ U_{21}(\lambda) & U_{22}(\lambda) \end{pmatrix}, \tag{5}$$

such that $U_{11}(\lambda)$ is an $m \times m$-matrix. From (3) we deduce that

$$\begin{aligned} \bar{y}(\lambda) &= U_{11}(\lambda)\bar{y}(0) + U_{12}(\lambda)0 = U_{11}(\lambda)\bar{y}(0) \\ 0 &= U_{21}(\lambda)\bar{y}(0) + U_{22}(\lambda)0 = U_{21}(\lambda)\bar{y}(0). \end{aligned}$$

Since $\bar{y}(0)$ is an arbitrary element of K^m, it is clear that $U_{21}(\lambda) = (0)_{n-m,m}$ for all $\lambda \in A$. Because of the fact that $U(\lambda)$ is regular, both U_{11} and U_{22} are regular matrices as well. From $U(0) = I_n$ the boundary conditions follow. Inserting into (4) the form of U and y just described we get

$$\begin{pmatrix} U_{11}(\lambda+\mu) & U_{12}(\lambda+\mu) \\ (0)_{n-m,m} & U_{22}(\lambda+\mu) \end{pmatrix} \begin{pmatrix} \bar{y}(0) \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} U_{11}(\lambda) & U_{12}(\lambda) \\ (0)_{n-m,m} & U_{22}(\lambda) \end{pmatrix} \begin{pmatrix} U_{11}(\mu) & U_{12}(\mu) \\ (0)_{n-m,m} & U_{22}(\mu) \end{pmatrix} \begin{pmatrix} \bar{y}(0) \\ 0 \end{pmatrix},$$

which means that $U_{11}(\lambda+\mu)\bar{y}(0) = U_{11}(\lambda)U_{11}(\mu)\bar{y}(0)$ for all $\bar{y}(0) \in K^m$ and $\lambda, \mu \in A$, so that U_{11} is an exponential function. □

We are also interested in subspaces of S_U. First we present a generalization of Lemma 5.

Lemma 7. *Let S be a subspace of S_U. Then the following statements are equivalent:*

1. *S is a $U(\lambda_0)$-invariant space for all $\lambda_0 \in A$.*
2. *S is invariant under translations.*
3. *S^0 is $U(\lambda_0)$-invariant for all $\lambda_0 \in A$, where $S^0 := \{y(0) \mid y \in S\}$.*

Proof. In order to prove that 1 implies 2, we set $z(\lambda) := y(\lambda+\lambda_0)$ for arbitrary $\lambda_0 \in A$. Since S is $U(\lambda_0)$-invariant, $U(\lambda_0)y \in S$. It is enough to prove that $z = U(\lambda_0)y$, since then $z \in S$. For $\lambda \in A$ we get $z(\lambda) = y(\lambda_0 + \lambda) = U(\lambda_0 + \lambda)y(0) = U(\lambda_0)U(\lambda)y(0) = U(\lambda_0)y(\lambda)$, so $z = U(\lambda_0)y$ by (3), (4) and (3).

To each $y^0 \in S^0$ there exists $y \in S$, such that $y^0 = y(0)$. Under the assumption 2, the function $z(\lambda) := y(\lambda + \lambda_0)$ belongs to S for any $\lambda_0 \in A$. So $z(0) \in S^0$ and $z(0) = y(\lambda_0) = U(\lambda_0)y(0) = U(\lambda_0)y^0$ by (3). Thus we proved that 2 implies 3.

In order to close the cycle of implications take $y \in S$. Then $y(0) \in S^0$. For arbitrary $\lambda_0 \in A$ also $U(\lambda_0)y(0)$ belongs to S^0. Hence, there exists $z \in S$ such that $z(0) = U(\lambda_0)y(0)$. Taking into account that S is a subspace of S_U we can write z as $z(\lambda) = U(\lambda)z(0) = U(\lambda)U(\lambda_0)y(0) = U(\lambda + \lambda_0)y(0) = U(\lambda_0)U(\lambda)y(0) = U(\lambda_0)y(\lambda)$ by (3), (4), (4) and (3). Thus $U(\lambda_0)y \in S$. □

A generalization of Theorem 6 is

Theorem 8. *Let S be a k-dimensional U-invariant subspace of S_U. Then there exist coordinates in K^n, such that $S^0 = \langle e_1, \ldots, e_k\rangle$, and $U(\lambda)$ is a block matrix of the form*

$$U(\lambda) = \begin{pmatrix} U_{11}(\lambda) & U_{12}(\lambda) \\ (0)_{n-k,k} & U_{22}(\lambda) \end{pmatrix}, \tag{6}$$

where $U_{11}(\lambda) \in \mathrm{GL}(k, K)$, $U_{22}(\lambda) \in \mathrm{GL}(n-k, K)$ and $U_{12}(\lambda) \in M_{k,n-k}(K)$, such that $U_{11}(0) = I_k$, $U_{22}(0) = I_{n-k}$, $U_{12}(0) = (0)_{k,n-k}$. Moreover, U_{11} is an exponential function, and $y \in S$ if and only if

$$y(\lambda) = U(\lambda)y(0) = \begin{pmatrix} U_{11}(\lambda)\xi \\ 0 \end{pmatrix}$$

for $\xi \in K^k$.

So far we described solutions (U, y) of (2) when the mapping U was given. Now we will assume that a linear subspace S^0 of K^n is given and we describe all solutions (U, y) of (2), such that $S^0_U = S^0$. Let S^0 be a k-dimensional U-invariant subspace of K^n, then without loss of generality $S^0 = \langle e_1, \ldots, e_k\rangle$.

Theorem 9. *Let $S^0 = \langle e_1, \ldots, e_k\rangle$ be a subspace of K^n, and let $U_{11}(\lambda) \in \mathrm{GL}(k, K)$, $U_{22}(\lambda) \in \mathrm{GL}(n-k, K)$ and $U_{12}(\lambda) \in M_{k,n-k}(K)$, such that $U_{11}(0) = I_k$, $U_{22}(0) = I_{n-k}$, $U_{12}(0) = (0)_{k,n-k}$. Moreover U_{11} is assumed to be an exponential function. Then*

$$S := \left\{ y \mid y(\lambda) = \begin{pmatrix} U_{11}(\lambda)\xi \\ 0 \end{pmatrix},\ \xi \in K^k \right\}$$

is a U-invariant subspace of S_U, where U is given by (6).

Proof.

$$U(\lambda+\mu)y(\nu) = \begin{pmatrix} U_{11}(\lambda+\mu) & U_{12}(\lambda+\mu) \\ (0)_{n-k,k} & U_{22}(\lambda+\mu) \end{pmatrix} \begin{pmatrix} U_{11}(\nu)\xi \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11}(\lambda+\mu)U_{11}(\nu)\xi \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} U_{11}(\lambda+\mu+\nu)\xi \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11}(\lambda)U_{11}(\mu+\nu)\xi \\ 0 \end{pmatrix} = U(\lambda)y(\mu+\nu)$$

□

When does $S = S_U$ hold?

Lemma 10. *The two spaces S and S_U coincide if and only if for all $\eta \in K^{n-k} \setminus \{0\}$ there exists $(\lambda_0, \mu_0) \in A^2$, such that*

$$\begin{pmatrix} [U_{11}(\lambda_0)U_{12}(\mu_0) + U_{12}(\lambda_0)U_{22}(\mu_0) - U_{12}(\lambda_0+\mu_0)]\,\eta \\ [U_{22}(\lambda_0)U_{22}(\mu_0) - U_{22}(\lambda_0+\mu_0)]\,\eta \end{pmatrix} \neq 0. \tag{7}$$

Proof. From Lemma 2 and Lemma 3 we know that S is a subspace of S_U different from S_U if and only if there exists $y^0 \notin S^0$, such that $[U(\lambda+\mu) - U(\lambda)U(\mu)]y^0 = 0$ for all $\lambda, \mu \in A$. In other words, $S = S_U$ if and only if for each $y^0 \notin S^0$ there exists $(\lambda_0, \mu_0) \in A^2$, such that $[U(\lambda_0+\mu_0) - U(\lambda_0)U(\mu_0)]y^0 \neq 0$. When writing y^0 in the form $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ for $\xi \in K^k$ and $\eta \in K^{n-k}$, then $y^0 \in K^n \setminus S^0$ if and only if $\eta \neq 0$. Together with (6) we get

$$[U(\lambda+\mu) - U(\lambda)U(\mu)]y^0 = \begin{pmatrix} [U_{11}(\lambda)U_{12}(\mu) + U_{12}(\lambda)U_{22}(\mu) - U_{12}(\lambda+\mu)]\,\eta \\ [U_{22}(\lambda)U_{22}(\mu) - U_{22}(\lambda+\mu)]\,\eta \end{pmatrix},$$

which finishes the proof. □

Now we are going to present several examples for the situation $S = S_U$, i.e. by Lemma 10 examples, where condition (7) is satisfied. Here we always assume that $A = K$. First we will deal with the second line of condition (7). Secondly, if this condition is not satisfied by all η, then let V denote the set

$$\left\{\eta \in K^{n-k} \mid [U_{22}(\lambda)U_{22}(\mu) - U_{22}(\lambda+\mu)]\,\eta = 0,\ \forall \lambda, \mu \in A\right\}.$$

Thus V is an r-dimensional subspace of K^{n-k} for $0 \le r \le n-k$. In order to satisfy the requirements of Lemma 10 in this situation as well, the first line in (7) must be satisfied for $\eta \in V$.

Now we describe some examples how to construct $U_{22}\colon K \to \mathrm{GL}(s, K)$ for $s \le n-k$, such that

$$\forall \eta \in K^s \setminus \{0\}\ \exists \lambda_0, \mu_0 \in K: \ [U_{22}(\lambda_0)U_{22}(\mu_0) - U_{22}(\lambda_0+\mu_0)]\,\eta \neq 0. \tag{8}$$

Case $\mathit{char} K \neq 2$**:** Set $U_{22}(\lambda) = cI_s$ for all $\lambda \in K \setminus \{0\}$ with $c \in K \setminus \{0,1\}$. Then $c^2 \neq c$. For $\lambda_0 = \mu_0 = 1$ we get $[U_{22}(1)U_{22}(1) - U_{22}(1+1)]\eta = [c^2 I_s - cI_s]\eta = (c^2 - c)I_s\eta \neq 0$ for all $\eta \neq 0$.

Case $\mathit{char} K = 2$ **and** $|K| > 2$**:** There exists $c \in K \setminus \{0\}$, such that $c^2 \neq 1$. Let $U_{22}(\lambda) = cI_s$ for all $\lambda \in K \setminus \{0\}$, then for $\lambda_0 = \mu_0 = 1$ we get $[U_{22}(1)U_{22}(1) - U_{22}(1+1)]\eta = [U_{22}(1)^2 - U_{22}(0)]\eta = [c^2 I_s - I_s]\eta = (c^2 - 1)I_s\eta \neq 0$ for all $\eta \neq 0$.

Case $|K| = 2$**:** If $s = 1$ each mapping $U_{22}\colon K \to \mathrm{GL}(1, K) = \{1\}$ is a homomorphism, so (8) cannot be satisfied. If $s > 1$ there exist matrices $M \in \mathrm{GL}(s, K)$ of order $2^s - 1$. As a permutation of the vectors in K^s the cycle decomposition of M consists of one fixed point, the 0-vector, and a cycle of length $2^s - 1$. (Actually, cf. [2] 3.5 Theorem, there are $\phi(2^s - 1)/s$ irreducible polynomials of degree s over $K = GF(2)$, such that the companion matrix of these polynomials is of order $2^s - 1$.) If $U_{22}(1) = M$, then for $\lambda_0 = \mu_0 = 1$ we get $[U_{22}(1)U_{22}(1) - U_{22}(1+1)]\eta = [M^2 - I_s]\eta \neq 0$ for all $\eta \neq 0$, since $2 < 2^s - 1$.

Now we describe examples how to construct $U_{12}\colon K \to M_{k,n-k}(K)$, such that

$$\forall \eta \in K^{n-k} \setminus \{0\}\ \exists \lambda_0, \mu_0 \in K: \ [U_{11}(\lambda_0)U_{12}(\mu_0) + U_{12}(\lambda_0)U_{22}(\mu_0) - U_{12}(\lambda_0 + \mu_0)]\,\eta \neq 0. \tag{9}$$

Again we assume that $A = K$. Furthermore we assume that both U_{11} and U_{22} are exponential functions. Hence $r = n - k$. From the preceding considerations we already know that $U_{11}(\lambda) \in \mathrm{GL}(k, K)$, $U_{22}(\lambda) \in \mathrm{GL}(r, K)$, $U_{11}(0) = I_k$, $U_{22}(0) = I_r$ and $U_{12}(0) = (0)_{k,r}$. Again we describe several different cases:

Case $\mathit{char} K \neq 2$**:** If $k \geq r$, then assume that $U_{12}(1) = (0)_{k,r}$ and

$$U_{12}(2) = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & -1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & -1 \\ 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} = \begin{pmatrix} -I_r \\ (0)_{k-r,r} \end{pmatrix},$$

where the upper part is $-I_r$ and the lower part is a 0-matrix of the dimension $(k - r) \times r$. Then for $\lambda_0 = \mu_0 = 1$ we get that $U_{11}(1)U_{12}(1) + U_{12}(1)U_{22}(1) - U_{12}(1 + 1) = -U_{12}(2)$ and it is obvious that $-U_{12}(2)\eta \neq 0$ for all $\eta \in K^r \setminus \{0\}$.

For $k < r$ one possible way to proceed is indicated in

Lemma 11. *If there are enough elements in K, to be more precise, if $|K| \geq 2\left\lceil \frac{r}{k} \right\rceil + 1$, then it is always possible to find λ_0 and μ_0 satisfying (9).*

Proof. There exist uniquely determined integers q, s, such that $r = kq + s$ and $0 \leq s < k$. If $q > 0$ assume that $\lambda_1 \in K \setminus \{0\}$, then $-\lambda_1 \in K \setminus \{0, \lambda_1\}$. Let $U_{12}(\pm\lambda_1)$ be given by

$$U_{12}(\lambda_1) = (\, I_k \quad (0)_{k,r-k}\,) \cdot U_{22}(\lambda_1) \text{ and } U_{12}(-\lambda_1) = U_{11}(-\lambda_1) \cdot (\, I_k \quad (0)_{k,r-k}\,)\,.$$

If $q > 1$ and $|K|$ is big enough, then there exists $\lambda_2 \in K \setminus \{0, \pm\lambda_1\}$ and we assume that

$$\begin{aligned} U_{12}(\lambda_2) &= (\,(0)_{k,k} \quad I_k \quad (0)_{k,r-2k}\,) \cdot U_{22}(\lambda_2) \text{ and} \\ U_{12}(-\lambda_2) &= U_{11}(-\lambda_2) \cdot (\,(0)_{k,k} \quad I_k \quad (0)_{k,r-2k}\,)\,. \end{aligned}$$

Going on like this we can find elements $\lambda_1, \ldots, \lambda_q \in K$ and matrices $U_{12}(\pm\lambda_i)$. If $s > 0$ and $|K|$ is big enough, then there exists $\lambda_{q+1} \in K \setminus \{0, \pm\lambda_1, \ldots, \pm\lambda_q\}$ and we assume that

$$U_{12}(\lambda_{q+1}) = \begin{pmatrix} (0)_{s,qk} & I_s \\ (0)_{k-s,qk} & (0)_{k-s,s} \end{pmatrix} \cdot U_{22}(\lambda_{q+1}) \text{ and}$$

$$U_{12}(-\lambda_{q+1}) = U_{11}(-\lambda_{q+1}) \cdot \begin{pmatrix} (0)_{s,qk} & I_s \\ (0)_{k-s,qk} & (0)_{k-s,s} \end{pmatrix}.$$

When $\eta \in K^r \setminus \{0\}$, then there exists $1 \leq i \leq r$, such that $\eta_i \neq 0$. Hence, there exists $j \in \{1, 2, \ldots, q+1\}$, such that $(j-1)k \leq i < jk$. For $\lambda_0 = \lambda_j$ and $\mu_0 = -\lambda_j$ we have $U_{11}(\lambda_j)U_{12}(-\lambda_j) + U_{12}(\lambda_j)U_{22}(-\lambda_j) - U_{12}(0) = 2U_{12}(\lambda_j)U_{22}(-\lambda_j)$. According to the choice of i and j it is clear that (9) is satisfied. $\square$

This is a very general result, but it is not the best result which is possible.

Example 12. Let K be the prime field of characteristic 3, $k = 1$ and $r = 2$. In this case $|K| < 2\left\lceil \frac{r}{k} \right\rceil + 1$, but it is also possible to find U_{12}, such that (9) is satisfied. For instance U given by

$$U(1) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad U(2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

satisfies (9).

Case $char K = 2$ **and** $|K| > 2$**:** Assume first $k \geq r$. There exists $\lambda \in K \setminus \{0, 1\}$ and then $\lambda + 1 \notin \{0, 1, \lambda\}$. Let furthermore $U_{12}(1) = U_{12}(\lambda) = (0)_{k,r}$ and

$$U_{12}(\lambda + 1) = \begin{pmatrix} I_r \\ (0)_{k-r,r} \end{pmatrix},$$

then for $\lambda_0 = 1$ and $\mu_0 = \lambda$ we get $U_{11}(1)U_{12}(\lambda) + U_{12}(1)U_{22}(\lambda) - U_{12}(\lambda + 1) = U_{12}(\lambda + 1)$ and (9) is satisfied. For $k < r$ the following lemma holds:

Lemma 13. *If there are enough elements in K, to be more precise, if $|K| \geq 2\left\lceil \frac{r}{k} \right\rceil + 2$, then it is always possible to find λ_0 and μ_0 satisfying (9).*

Proof. There exist uniquely determined integers q, s, such that $r = kq + s$ and $0 \leq s < k$. Assume $U_{12}(1) = (0)_{k,r}$. For $1 \leq i \leq q$ there exists $\lambda_i \in K \setminus \{0, 1, \lambda_j, \lambda_j + 1 \mid j < i\}$, such that $\lambda_i + 1 \notin \{0, 1, \lambda_j, \lambda_j + 1, \lambda_i \mid j < i\}$. Let $U_{12}(\lambda_i)$ and $U_{12}(\lambda_i + 1)$ be given by

$$\begin{aligned} &U_{12}(\lambda_i) = (0)_{k,r} \text{ and} \\ &U_{12}(\lambda_i + 1) = \begin{pmatrix} (0)_{k,(i-1)k} & I_k & (0)_{k,r-ik} \end{pmatrix}. \end{aligned}$$

If $s > 0$ and $|K|$ is big enough, then there exists $\lambda_{q+1} \in K$, such that $\lambda_{q+1}, \lambda_{q+1}+1 \in K \setminus \{0, 1, \lambda_j, \lambda_j + 1 \mid j \leq q\}$ and we assume that

$$U_{12}(\lambda_{q+1}) = (0)_{k,r} \text{ and}$$

$$U_{12}(\lambda_{q+1} + 1) = \begin{pmatrix} (0)_{s,qk} & I_s \\ (0)_{k-s,qk} & (0)_{k-s,s} \end{pmatrix}.$$

Given $\eta \in K^r \setminus \{0\}$ there exists $1 \leq i \leq r$, such that $\eta_i \neq 0$. Hence, there exists $j \in \{1, 2, \ldots, q + 1\}$, such that $(j - 1)k \leq i < jk$. For $\lambda_0 = \lambda_j$ and $\mu_0 = 1$ we have $U_{11}(\lambda_j)U_{12}(1) + U_{12}(\lambda_j)U_{22}(1) - U_{12}(\lambda_j + 1) = -U_{12}(\lambda_j + 1)$. According to the choice of i and j it is clear that (9) is satisfied. □

Case $|K| = 2$: In the situation $r \geq k$ we can only give partial results. If $\lambda_0 = 0$ or $\mu_0 = 0$, then it is impossible to satisfy (9). Since U_{11} and U_{22} are exponential functions, the orders of $U_{11}(1)$ and $U_{22}(1)$ are divisors of 2. If both $U_{11}(1) = I_k$ and $U_{22}(1) = I_r$, then it is also impossible to satisfy (9). If $r = 1$, then $U_{22}(1)$ is the identity matrix I_1. From the previous statements it is clear that necessarily $k > 1$ and $U_{11}(1)$ must be a matrix of order 2. If U is defined by

$$U_{11}(1) = \begin{pmatrix} 1 & 0 & \ldots & 0 & 1 \\ 0 & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad U_{12}(1) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

then (9) is satisfied. For $r = 2$ assume that $U_{11}(1)$ is given as above and

$$U_{22}(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad U_{12}(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

then again (9) is satisfied. For $k = r = 3$ and for any choice of $U_{11}(1), U_{22}(1) \in \mathrm{GL}(3, K)$ of order dividing 2 the computer did not find a matrix $U_{12}(1)$ in $M_3(K)$ such that (9) is satisfied. Other cases were not studied so far.

If $k > r$ it is not possible to satisfy (9), since there is only one possible choice $\lambda_0 = \mu_0 = 1$, which determines exactly one matrix $U_{11}(1)U_{12}(1) + U_{12}(1)U_{22}(1)$. This matrix describes a homomorphism from K^k to K^r, which has a kernel of dimension $\geq k - r > 0$.

3 The general situation

In this part we generalize the functional equation (2) by assuming that $U(\lambda)$ is not necessarily a regular matrix, i.e. $U\colon A \to M_n(K)$. Also in this situation Lemma 1 holds. When we define S_U and S_U^0 as it was done earlier, then S_U and S_U^0 are K-linear spaces (cf. Lemma 4). Again S_U^0 is an m-dimensional subspace of K^n for $0 \leq m \leq n$, and S_U is invariant under translations, and $S_U^0 = \{y(\lambda) \mid \lambda \in A,\ y \in S_U\}$ (cf. Lemma 5). Without loss of generality we can assume (as in the earlier case) that there exists a basis of K^n, such that $S_U^0 = \langle e_1, \ldots, e_m \rangle$.

Since $U(0)$ need not be a regular matrix, we do not get the results of Lemma 2, and in general there is no isomorphism between S_U and S_U^0.

For $\lambda = \nu = 0$ or $\nu = 0$ we derive from (2)

Lemma 14. *Let (U, y) be a solution of (2), then*

$$U(0)y(\mu) = U(\mu)y(0), \qquad \forall \mu \in A, \tag{10}$$

$$U(\lambda)y(\mu) = U(0)y(\lambda + \mu), \qquad \forall \lambda, \mu \in A. \tag{11}$$

If $U(\lambda)$ is partitioned as in (5) and $y(\lambda)$ is written as $\begin{pmatrix} \bar{y}(\lambda) \\ 0 \end{pmatrix}$ for $\bar{y}(\lambda) \in K^m$, then from (10) we get

$$\begin{pmatrix} U_{11}(0) & U_{12}(0) \\ U_{21}(0) & U_{22}(0) \end{pmatrix} \begin{pmatrix} \bar{y}(\mu) \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11}(\mu) & U_{12}(\mu) \\ U_{21}(\mu) & U_{22}(\mu) \end{pmatrix} \begin{pmatrix} \bar{y}(0) \\ 0 \end{pmatrix},$$

which leads to the system of equations

$$\begin{aligned} U_{11}(0)\bar{y}(\mu) &= U_{11}(\mu)\bar{y}(0) \qquad \forall \mu \in A \\ U_{21}(0)\bar{y}(\mu) &= U_{21}(\mu)\bar{y}(0) \qquad \forall \mu \in A. \end{aligned} \tag{12}$$

Lemma 15. *Let (U, y) be a solution of (2). Then there exists a system of coordinates of K^n, such that*

$$U(\lambda) = \begin{pmatrix} U_{11}(\lambda) & U_{12}(\lambda) \\ (0)_{n-m,m} & U_{22}(\lambda) \end{pmatrix}, \qquad \forall \lambda \in A, \tag{13}$$

where the $m \times m$*-matrix* $U_{11}(0)$ *is the block matrix of the form*

$$U_{11}(0) = \begin{pmatrix} I_k & (0)_{k,m-k} \\ (0)_{m-k,k} & (0)_{m-k,m-k} \end{pmatrix} \tag{14}$$

for some $k \leq m$.

Proof. According to Lemma 1 choose matrices $C \in \mathrm{GL}(n,K)$ and $B' \in \mathrm{GL}(m,K)$, such that

$$CU(0) \begin{pmatrix} B' & (0)_{m,n-m} \\ (0)_{n-m,m} & I_{n-m} \end{pmatrix} = \begin{pmatrix} V_{11}(0) & V_{12}(0) \\ (0)_{n-m,m} & V_{22}(0) \end{pmatrix}$$

and

$$V_{11}(0) = \begin{pmatrix} I_k & (0)_{k,m-k} \\ (0)_{m-k,k} & (0)_{m-k,m-k} \end{pmatrix}.$$

Without loss of generality assume that $U = V$. From the second line of (12) we deduce that $0 = 0\bar{y}(\mu) = U_{21}(\mu)\bar{y}(0)$ for all $\mu \in A$. Since $\bar{y}(0)$ can arbitrarily be chosen in K^m, it is clear that $U_{21}(\mu) = (0)_{n-m,m}$ for all $\mu \in A$. □

Since $S_U^0 = \langle e_1, \ldots, e_m \rangle$, there exist $y_1, \ldots, y_m \in S_U$, such that $y_j(0) = e_j$, the j-th unit vector in K^n, for $1 \leq j \leq m$. Let $S_U' := \langle y_1, \ldots, y_m \rangle$, then S_U' is an m-dimensional subspace of S_U. In order to prove this, it is only necessary to show that $y_1, \ldots, y_m$ are linearly independent. Let $\alpha_1, \ldots, \alpha_m \in K$, such that $\sum_{i=1}^m \alpha_i y_i = 0$, then also $\sum_{i=1}^m \alpha_i y_i(0) = 0$, which implies $\sum_{i=1}^m \alpha_i e_i = 0$, so that $\alpha_1 = \ldots = \alpha_m = 0$.

For $y \in S_U'$ there exist uniquely defined $\alpha_1, \ldots, \alpha_m \in K$ such that $y = \sum_{i=1}^m \alpha_i y_i$. These α_i can be read from $y(0)$, since $y(0) = \sum_{i=1}^m \alpha_i e_i$.

Define the $m \times m$-matrix $Y(\lambda)$ corresponding to the chosen $y_1, \ldots, y_m$ by

$$Y(\lambda) = (\bar{y}_1(\lambda), \ldots, \bar{y}_m(\lambda)), \tag{15}$$

i.e. the j-th column of $Y(\lambda)$ is the vector $\bar{y}_j(\lambda) \in K^m$. Then for $y \in S_U'$ we have

$$\bar{y}(\lambda) = \sum_{i=1}^m \alpha_i \bar{y}_i(\lambda) = Y(\lambda) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} = Y(\lambda)\bar{y}(0). \tag{16}$$

Replacing y by y_j in the first line of (12) we get for all $\mu \in A$

$$U_{11}(0)\bar{y}_j(\mu) = U_{11}(\mu)\bar{y}_j(0) = U_{11}(\mu)\bar{e}_j, \qquad j = 1, \ldots, m.$$

These equations are collected to the matrix equation

$$U_{11}(0)Y(\mu) = U_{11}(\mu), \qquad \forall \mu \in A. \tag{17}$$

The special form of U from Lemma 15 inserted into (11) yields for $y = y_j$ the equation

$$U_{11}(\lambda)\bar{y}_j(\mu) = U_{11}(0)\bar{y}_j(\lambda + \mu), \qquad \forall \lambda, \mu \in A.$$

Again these equations can be collected for $j = 1, \dots, m$ and we derive

$$U_{11}(\lambda)Y(\mu) = U_{11}(0)Y(\lambda+\mu), \qquad \forall \lambda, \mu \in A. \tag{18}$$

Equations (17) and (18) together yield

$$U_{11}(0)\left[Y(\lambda+\mu) - Y(\lambda)Y(\mu)\right] = (0)_{m,m}, \qquad \forall \lambda, \mu \in A. \tag{19}$$

According to the special form of $U_{11}(0)$ described in Lemma 15 we partition $Y(\lambda)$ as a block matrix

$$Y(\lambda) = \begin{pmatrix} Y_{11}(\lambda) & Y_{12}(\lambda) \\ Y_{21}(\lambda) & Y_{22}(\lambda) \end{pmatrix},$$

such that $Y_{11}(\lambda)$ is a $k\times k$-matrix. We note that the "auxiliary" matrix function $Y\colon A \to M_m(K)$, which will help us to describe the space S_U of solutions y (for given U), is in general not uniquely determined. However, from (17), from the decomposition of $U_{11}(0)$ in Lemma 15 and the corresponding decomposition of $Y(\lambda)$ we see that $Y_{11}(\lambda)$ and $Y_{12}(\lambda)$ are uniquely determined by $U_{11}(\lambda)$, namely

$$U_{11}(\mu) = U_{11}(0)Y(\mu) = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Y_{11}(\mu) & Y_{12}(\mu) \\ Y_{21}(\mu) & Y_{22}(\mu) \end{pmatrix} = \begin{pmatrix} Y_{11}(\mu) & Y_{12}(\mu) \\ (0)_{m-k,k} & (0)_{m-k,m-k} \end{pmatrix}. \tag{20}$$

Then (19) can be rewritten as

$$\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}\left[\begin{pmatrix} Y_{11}(\lambda+\mu) & Y_{12}(\lambda+\mu) \\ Y_{21}(\lambda+\mu) & Y_{22}(\lambda+\mu) \end{pmatrix} - \begin{pmatrix} Y_{11}(\lambda) & Y_{12}(\lambda) \\ Y_{21}(\lambda) & Y_{22}(\lambda) \end{pmatrix}\begin{pmatrix} Y_{11}(\mu) & Y_{12}(\mu) \\ Y_{21}(\mu) & Y_{22}(\mu) \end{pmatrix}\right] = (0)_{m,m}$$

and we end up with the system of equations

$$\begin{array}{rcl} Y_{11}(\lambda+\mu) & = & Y_{11}(\lambda)Y_{11}(\mu) + Y_{12}(\lambda)Y_{21}(\mu) \\ Y_{12}(\lambda+\mu) & = & Y_{11}(\lambda)Y_{12}(\mu) + Y_{12}(\lambda)Y_{22}(\mu) \end{array} \qquad \forall \lambda, \mu \in A. \tag{21}$$

From $Y(0) = I_m$ we deduce that $Y_{11}(0) = I_k$, $Y_{21}(0) = (0)_{m-k,k}$, $Y_{12}(0) = (0)_{k,m-k}$ and $Y_{22}(0) = I_{m-k}$. If μ is replaced by $\mu_1+\mu_2$ and taking into account that $+$ is an associative composition we get from the first line of (21) that $Y_{11}(\lambda+(\mu_1+\mu_2)) = Y_{11}(\lambda)Y_{11}(\mu_1+\mu_2)+Y_{12}(\lambda)Y_{21}(\mu_1+\mu_2) = Y_{11}(\lambda)[Y_{11}(\mu_1)Y_{11}(\mu_2)+Y_{12}(\mu_1)Y_{21}(\mu_2)]+Y_{12}(\lambda)Y_{21}(\mu_1+\mu_2)$ is equal to $Y_{11}((\lambda+\mu_1)+\mu_2) = Y_{11}(\lambda+\mu_1)Y_{11}(\mu_2)+Y_{12}(\lambda+\mu_1)Y_{21}(\mu_2) = [Y_{11}(\lambda)Y_{11}(\mu_1)+Y_{12}(\lambda)Y_{21}(\mu_1)]Y_{11}(\mu_2) + [Y_{11}(\lambda)Y_{12}(\mu_1) + Y_{12}(\lambda)Y_{22}(\mu_1)]Y_{21}(\mu_2)$, which yields

$$Y_{12}(\lambda)[Y_{21}(\mu_1+\mu_2) - Y_{21}(\mu_1)Y_{11}(\mu_2) - Y_{22}(\mu_1)Y_{21}(\mu_2)] = 0, \quad \forall \lambda, \mu_1, \mu_2 \in A. \tag{22}$$

In the same way we can derive from the second line of (21) that

$$Y_{12}(\lambda)[Y_{22}(\mu_1+\mu_2) - Y_{21}(\mu_1)Y_{12}(\mu_2) - Y_{22}(\mu_1)Y_{22}(\mu_2)] = 0, \quad \forall \lambda, \mu_1, \mu_2 \in A. \tag{23}$$

Each $Y_{12}(\lambda)$ determines a homomorphism from K^{m-k} to K^k. Let $W := \bigcap_{\lambda\in A} \ker Y_{12}(\lambda)$. Then W is an r-dimensional subspace of K^{m-k} for $0 \le r \le m-k$ with basis $\left\{\hat{d}_1, \dots, \hat{d}_r\right\}$.

Moreover, there exists an $m-k-r$-dimensional subspace V of K^{m-k}, such that $K^{m-k} = V \oplus W$. Let $\{\hat{c}_1, \dots, \hat{c}_{m-k-r}\}$ be a basis of V. We embed K^{m-k} in a natural way into K^m by placing k zeros in front of each vector, i.e.

$$\bar{c}_i := \begin{pmatrix} 0 \\ \hat{c}_i \end{pmatrix} \in K^m \quad \text{and} \quad \bar{d}_i := \begin{pmatrix} 0 \\ \hat{d}_i \end{pmatrix} \in K^m.$$

Then it is possible to find a matrix $B'' \in \mathrm{GL}(m-k, K)$, such that the coordinate transformation on K^m induced by

$$B' := \begin{pmatrix} I_k & (0)_{k,m-k} \\ (0)_{m-k,k} & B'' \end{pmatrix} \in \mathrm{GL}(m, K)$$

satisfies

$$\begin{array}{ll} B'\bar{e}_i = \bar{e}_i & 1 \le i \le k \\ B'\bar{c}_i = \bar{e}_{k+i} & 1 \le i \le m-k-r \\ B'\bar{d}_i = \bar{e}_{m-r+i} & 1 \le i \le r. \end{array}$$

Let B be the corresponding coordinate transformation on K^n

$$B := \begin{pmatrix} B' & (0)_{m,n-m} \\ (0)_{n-m,m} & I_{n-m} \end{pmatrix} \in \mathrm{GL}(n, K).$$

If U is decomposed as in (13) and (14), then also UB has this property.

Without loss of generality we assume that the basis of K^n was chosen in such a way that Lemma 15 is satisfied and that $\{e_{k+1}, \dots, e_{m-r}\}$ and $\{e_{m-r+1}, \dots, e_m\}$ are a basis of V or W respectively. Then it is useful and important to partition $Y(\lambda)$ further as a 3×3 block matrix of the form

$$Y(\lambda) = \begin{pmatrix} Z_{11}(\lambda) & Z_{12}(\lambda) & Z_{13}(\lambda) \\ Z_{21}(\lambda) & Z_{22}(\lambda) & Z_{23}(\lambda) \\ Z_{31}(\lambda) & Z_{32}(\lambda) & Z_{33}(\lambda) \end{pmatrix},$$

such that $Z_{11}(\lambda) = Y_{11}(\lambda) \in M_k(K)$, $Z_{22}(\lambda) \in M_{m-k-r}(K)$ and $Z_{33}(\lambda) \in M_r(K)$. Hence

$$Y_{12}(\lambda) = \begin{pmatrix} Z_{12}(\lambda) & Z_{13}(\lambda) \end{pmatrix}, \; Y_{21}(\lambda) = \begin{pmatrix} Z_{21}(\lambda) \\ Z_{31}(\lambda) \end{pmatrix}, \; Y_{22}(\lambda) = \begin{pmatrix} Z_{22}(\lambda) & Z_{23}(\lambda) \\ Z_{32}(\lambda) & Z_{33}(\lambda) \end{pmatrix}.$$

Let $x = \begin{pmatrix} \tilde{x} \\ \hat{x} \end{pmatrix}$ denote a vector in K^{m-k}, where $\tilde{x} \in K^{m-k-r}$ and $\hat{x} \in K^r$. Then x belongs to W if and only if $\tilde{x} = 0$. Moreover $Y_{12}(\lambda)|_W = 0$ for all $\lambda \in A$, which means $Z_{13}(\lambda) = (0)_{k,r}$ for all $\lambda \in A$. From the definition of W it is clear that $Z_{12}(\lambda)\tilde{x} = 0$ for all $\lambda \in A$ is equivalent to $\tilde{x} = 0$.

The first line of (21) reads now as

$$Z_{11}(\lambda + \mu) = Z_{11}(\lambda)Z_{11}(\mu) + Z_{12}(\lambda)Z_{21}(\mu) \quad \forall \lambda, \mu \in A.$$

From the second line of (21) we derive

$$Z_{12}(\lambda+\mu) = Z_{11}(\lambda)Z_{12}(\mu) + Z_{12}(\lambda)Z_{22}(\mu), \quad \forall\lambda,\mu \in A$$

and

$$(0)_{k,r} = Z_{13}(\lambda+\mu) = Z_{11}(\lambda)(0)_{k,r} + Z_{12}(\lambda)Z_{23}(\mu), \quad \forall\lambda,\mu \in A.$$

Hence, each column of $Z_{23}(\mu)$ is $0 \in K^{m-k-r}$, so that $Z_{23}(\mu) = (0)_{m-k-r,m-k-r}$ for all $\mu \in A$.

From (22) we deduce

$$Z_{12}(\lambda)[Z_{21}(\mu_1+\mu_2) - Z_{21}(\mu_1)Z_{11}(\mu_2) - Z_{22}(\mu_1)Z_{21}(\mu_2)] = (0)_{k,k}, \quad \forall\lambda,\mu_1,\mu_2 \in A.$$

Let M denote the matrix between the two braces [and], then each column of M is $0 \in K^{m-k-r}$ and consequently $M = (0)_{m-k-r,k}$. Hence, we proved that

$$Z_{21}(\mu_1+\mu_2) - Z_{21}(\mu_1)Z_{11}(\mu_2) - Z_{22}(\mu_1)Z_{21}(\mu_2) = (0)_{m-k-r,k}, \quad \forall\mu_1,\mu_2 \in A.$$

The same way we deduce from (23) that

$$Z_{12}(\lambda)[Z_{22}(\mu_1+\mu_2) - Z_{21}(\mu_1)Z_{12}(\mu_2) - Z_{22}(\mu_1)Z_{22}(\mu_2)] = (0)_{k,m-k-r}, \quad \forall\lambda,\mu_1,\mu_2 \in A$$

and correspondingly

$$Z_{22}(\mu_1+\mu_2) - Z_{21}(\mu_1)Z_{12}(\mu_2) - Z_{22}(\mu_1)Z_{22}(\mu_2) = (0)_{m-k-r,m-k-r}, \quad \forall\mu_1,\mu_2 \in A.$$

This finishes the proof of

Theorem 16. *There exists a coordinate system of K^m, such that $Y(\lambda)$ is a solution of (19) if and only if $Y(\lambda)$ can be written as*

$$Y(\lambda) = \begin{pmatrix} Z_{11}(\lambda) & Z_{12}(\lambda) & (0)_{k,r} \\ Z_{21}(\lambda) & Z_{22}(\lambda) & (0)_{m-k-r,r} \\ Z_{31}(\lambda) & Z_{32}(\lambda) & Z_{33}(\lambda) \end{pmatrix},$$

where

$$\begin{pmatrix} Z_{11}(\lambda) & Z_{12}(\lambda) \\ Z_{21}(\lambda) & Z_{22}(\lambda) \end{pmatrix}$$

is an exponential function, $Z_{11}(\lambda) \in M_k(K)$, $Z_{22}(\lambda) \in M_{m-k-r}(K)$, $Z_{33}(\lambda) \in M_r(K)$, satisfying the conditions $Z_{11}(0) = I_k$, $Z_{22}(0) = I_{m-k-r}$, $Z_{33}(0) = I_r$, $Z_{12}(0) = (0)_{k,m-k-r}$, $Z_{21}(0) = (0)_{m-k-r,k}$, $Z_{31}(0) = (0)_{r,k}$ and $Z_{32}(0) = (0)_{r,m-k-r}$. For $\lambda \neq 0$ the matrices $Z_{31}(\lambda), Z_{32}(\lambda), Z_{33}(\lambda)$ can be arbitrarily chosen.

Next we describe the structure of S_U in more details.

Lemma 17. *For each $y \in S_U$ with $y(0) \neq 0$ there exists a subspace S'_U of S_U, such that $y \in S'_U$.*

Proof. Since $y(0) \neq 0$ also $\bar{y}(0) \neq 0$. Hence there exist $z_2^0, \ldots, z_m^0 \in K^n$, such that $\{y(0), z_2^0, \ldots, z_m^0\}$ is a basis of $S_U^0 = \langle e_1, \ldots, e_m \rangle$. Consequently, there exist $z_2, \ldots, z_m \in S_U$, such that $z_j(0) = z_j^0$ for $j = 2, \ldots, m$. Then $y, z_2, \ldots, z_m$ are linearly independent, which implies that $\langle y, z_2, \ldots, z_m \rangle$ is an m-dimensional subspace of S_U. Hence, there exist $y_1, \ldots, y_m \in \langle y, z_2, \ldots, z_m \rangle$, such that $y_j(0) = e_j$ for $j = 1, \ldots, m$ and $y \in \langle y, z_2, \ldots, z_m \rangle = \langle y_1, \ldots, y_m \rangle =: S'_U$. □

Let $N(S_U)$ denote the set $\{z \in S_U \mid z(0) = 0\}$. Then $N(S_U)$ is a subspace of S_U. The appearance of this subspace $N(S_U)$ of S_U, which is in general not $\{0\}$, is one of the main differences to the case of mappings $U: A \to \mathrm{GL}(n, K)$. We will see that $N(S_U)$ is closely related to the space W. This is described in

Lemma 18. *A function $z: A \to K^n$ belongs to $N(S_U)$ if and only if*

$$\bar{z}(0) = 0 \text{ and } \bar{z}(\lambda) = \begin{pmatrix} 0 \\ \hat{z}(\lambda) \end{pmatrix} \text{ for } \hat{z}(\lambda) \in W. \tag{24}$$

Proof. The function z belongs to $N(S_U)$ if and only if $\bar{z}(0) = 0$ and $U_{11}(\lambda + \mu)\bar{z}(\nu) = U_{11}(\lambda)\bar{z}(\mu + \nu)$ for all $\lambda, \mu, \nu \in A$. Especially for $\lambda = \nu = 0$ and because of the particular form of U_{11} given in (20) we get

$$\begin{pmatrix} Y_{11}(\mu) & Y_{12}(\mu) \\ (0)_{m-k,k} & (0)_{m-k,m-k} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} Y_{11}(0) & Y_{12}(0) \\ (0)_{m-k,k} & (0)_{m-k,m-k} \end{pmatrix} \begin{pmatrix} \tilde{z}(\mu) \\ \hat{z}(\mu) \end{pmatrix} =$$

$$\begin{pmatrix} I_k & (0)_{k,m-k} \\ (0)_{m-k,k} & (0)_{m-k,m-k} \end{pmatrix} \begin{pmatrix} \tilde{z}(\mu) \\ \hat{z}(\mu) \end{pmatrix}, \quad \forall \mu \in A,$$

so that $\tilde{z}(\mu) = 0$ for all $\mu \in A$. For $\nu = 0$ we derive

$$\begin{pmatrix} Y_{11}(\lambda + \mu) & Y_{12}(\lambda + \mu) \\ (0)_{m-k,k} & (0)_{m-k,m-k} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} Y_{11}(\lambda) & Y_{12}(\lambda) \\ (0)_{m-k,k} & (0)_{m-k,m-k} \end{pmatrix} \begin{pmatrix} 0 \\ \hat{z}(\mu) \end{pmatrix},$$

so that $Y_{12}(\lambda)\hat{z}(\mu) = 0$ for all $\lambda, \mu \in A$. This however implies that $\hat{z}(\mu) \in W$ for all $\mu \in A$.

Assuming conversely that $\bar{z}(0) = 0$, $\tilde{z}(\lambda) = 0$ and $\hat{z}(\lambda) \in W$ for all $\lambda \in A$, then it is obvious that $z \in N(S_U)$. □

In conclusion we get the following result:

Lemma 19. *Let S'_U be an m-dimensional subspace of S_U (constructed as above), then $S_U = S'_U \oplus N(S_U)$.*

Proof. It is clear that $S'_U \cap N(S_U) = \{0\}$ and that $S'_U \oplus N(S_U)$ is a subspace of S_U. Assume that x is an element of S_U, then there exists $y \in S'_U$, such that $y(0) = x(0)$. Then $z(\lambda) := x(\lambda) - y(\lambda)$ belongs to $N(S_U)$. Hence $x(\lambda) = z(\lambda) + y(\lambda)$, which finishes the proof. □

We notice that in the decomposition $S_U = S'_U \oplus N(S_U)$ the space S'_U is in general not uniquely determined, whereas $N(S_U)$ is unique by definition. However, S'_U can be any m-dimensional subspace of S_U, such that the space of initial values $y(0)$ (for $y \in S'_U$) is already S^0_U.

As an immediate consequence we get

$$\dim S_U = \dim S'_U + \dim N(S_U) = m + \dim N(S_U).$$

The following Theorem 20 yields together with Theorem 16 the structure of the space S_U, of all solutions of (2), for a given function $U\colon A \to M_n(K)$. These theorems also contain necessary conditions on U in order to admit a nontrivial solution y.

Theorem 20. *Let $U\colon A \to M_n(K)$ be given and assume that $\dim S^0_U = m$. Then there exist coordinates in K^n and solutions (U, y_j) of (2) for $j = 1, \ldots, m$, such that $y_j(0) = e_j$. Moreover, $U(\lambda)$ can be written as in (13) and $U_{11}(\lambda)$ satisfies (20). $Y_{11}(\lambda)$ and $Y_{22}(\lambda)$ are the blocks in the first row of the matrix $Y(\lambda)$ given by (15). This matrix is also a solution of (19) and each element y of S_U can be expressed as $y(\lambda) = \begin{pmatrix} \bar{y}(\lambda) + \bar{z}(\lambda) \\ 0 \end{pmatrix}$ for $\bar{y}$ given by (16) with arbitrary $\bar{y}(0) \in K^m$ and $\bar{z}(\lambda)$ given by (24).*

We finish by Theorem 21, which provides a construction of all solutions (U, y) of (2) by starting from an arbitrary subspace of initial values $y(0)$ of K^n. This choice then leads via the block matrix $Y(\lambda)$ satisfying (19) to a matrix valued function U and a space S of solutions corresponding to U. In this general situation we do not discuss the problem when $S = S_U$.

Theorem 21. *If $Y(\lambda)$ satisfies (19), $U_{11}(\mu)$ is given by (20), $\bar{y}$ is given by (16) for arbitrary $\bar{y}(0) \in K^m$ and $\bar{z}(\lambda)$ given by (24), then (U, y) is a solution of (2) for*

$$y(\lambda) = \begin{pmatrix} \bar{y}(\lambda) + \bar{z}(\lambda) \\ 0 \end{pmatrix}.$$

Proof. From the special form of $U(\lambda)$ and $y(\lambda)$ it is clear that (2) is satisfied if and only if $U_{11}(\lambda + \mu)[\bar{y}(\nu) + \bar{z}(\nu)] = U_{11}(\lambda)[\bar{y}(\mu + \nu) + \bar{z}(\mu + \nu)]$ for all $\lambda, \mu, \nu \in A$. This is equivalent to $U_{11}(\lambda + \mu)[Y(\nu)\bar{y}(0) + \bar{z}(\nu)] = U_{11}(\lambda)[Y(\mu + \nu)\bar{y}(0) + \bar{z}(\mu + \nu)]$. Due to the definition of $\bar{z}(\lambda)$ this is equivalent to $U_{11}(0)[Y(\lambda + \mu)Y(\nu) - Y(\lambda)Y(\mu + \nu)]\bar{y}(0) = 0$ for all $\lambda, \mu, \nu \in A$. Since $\bar{y}(0)$ is an arbitrary element of K^m we derive

$$U_{11}(0)[Y(\lambda + \mu)Y(\nu) - Y(\lambda)Y(\mu + \nu)] = (0)_{m,m}, \qquad \forall \lambda, \mu, \nu \in A.$$

We can rewrite $U_{11}(0)[Y(\lambda + \mu)Y(\nu) - Y(\lambda)Y(\mu + \nu)]$ as $U_{11}(0)[Y(\lambda + \mu)Y(\nu) - Y(\lambda + \mu + \nu) + Y(\lambda + \mu + \nu) - Y(\lambda)Y(\mu + \nu)] = U_{11}(0)[Y(\lambda + \mu)Y(\nu) - Y(\lambda + \mu + \nu)] + U_{11}(0)[Y(\lambda + \mu + \nu) - Y(\lambda)Y(\mu + \nu)]$, which is equal to $(0)_{m,m}$ since (19) holds. □

In order to determine all solutions (U, y) of (2) we start with an arbitrary m-dimensional subspace S^0 of K^n for some $0 \leq m \leq n$. Let $\{b_1, \ldots, b_m\}$ be a basis of S^0, then there exists a matrix $B \in \mathrm{GL}(n, K)$, such that $Bb_i = e_i$ for $1 \leq i \leq m$. Hence $BS^0 = \langle e_1, \ldots, e_m \rangle$. For each solution $Y(\lambda)$ of (19) described in Theorem 16 let $U_{11}(\lambda)$ be given by (20) and $U(\lambda)$ be given by (13) with arbitrary matrices $U_{12}(\lambda)$ and $U_{22}(\lambda)$. Then each element y of

$$T := \left\{ y(\lambda) = \begin{pmatrix} \bar{y}(\lambda) + \bar{z}(\lambda) \\ 0 \end{pmatrix} \mid \bar{y}(0) \in K^m,\ \bar{y}(\lambda) = Y(\lambda)\bar{y}(0),\ \bar{z}(\lambda) \text{ given by (24)} \right\}$$

is together with U a solution of (2). Due to this construction T^0, the space of initial values $y(0)$ for $y \in T$, is equal to $\langle e_1, \ldots, e_m \rangle$. According to Lemma 1 each pair $(UB, B^{-1}y)$ for $y \in T$ is a solution of (2) and $\{B^{-1}y(0) \mid y \in T\} = S^0$. Hence, by varying S^0 over all subspaces of K^n we determine all solutions (U, y) of (2).

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