Enumeration and construction in music theory

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Abstract

In this paper we describe in a more or less complete way how to apply methods from algebraic combinatorics to the classification of different objects in music theory. Among these objects there are intervals, chords, tone-rows, motives, mosaics etc. The methods we are using can be described in a very general way so that they can be applied for the classification of objects in different sciences. For instance for the isomer enumeration in chemistry, for spin analysis in physics, for the classification of isometry classes of linear codes, in general for investigating isomorphism classes of objects (cf. [27, 28]). Here we present an application of these methods to music theory.

The main aim of this paper is to show how the number of essentially different (i. e. not similar) objects can be computed and how to construct a (complete) system of representatives of them. Similarity is defined by certain symmetry operations which are motivated by music theory.

In other words, we try to enumerate or construct a list of objects such that objects of this list are pairwise not similar and each possible object is similar to (exactly) one object of this list.

The objects we are interested in belong to (or are constructed in) the n-scale Z_n consisting of exactly n pitch-classes, which generalizes the concept of 12 tones in one octave to n tones in one octave. After describing the symmetry operations as permutations we can apply different methods from combinatorics under group action (e. g. Pólya's Theory of counting cf. [33, 32], Read's method of orderly generation cf. [2, 3] etc.) for the classification of these objects.

The reader of this article should have some knowledge about basic mathematical notions and definitions concerning sets, functions, permutations and elementary number theory. Furthermore about elementary group theory, number theory and basic knowledge about rings and fields. For everybody not familiar with these subjects it will be necessary to consult some other textbooks in order to understand all the details. As a standard reference for number theory I would suggest to read [31], as an introduction to algebra, groups, rings and fields look at [42] or [30]. Combinatorics under group actions is nicely

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described in [27, 28], Pólya's Theory of counting in [4]. These are just some suggestions, there exist many other textbooks dealing with these topics as well. The notation in this paper is mainly taken from Kerber's books. Many concepts from mathematical music theory origin from G. Mazzola's book [29]. Similar and further results about applications of combinatorics in music theory can be found for instance in [19, 9, 34, 38, 11, 14, 13, 36, 37, 17, 16].

Using the concept of group actions it is possible to give a mathematical description of an n-scale Z_n consisting of exactly n pitch-classes, which generalizes the concept of 12 tones in one octave to n tones in one octave. Moreover the notion of group actions is used for describing the similarity relation between the objects we are interested in. In general the set of all (symmetry) operations on a set X forms a group called the *symmetry group* of X. Finally it will be demonstrated how to count all the non similar objects and how to construct complete lists of representatives of them.

1 Group actions

A *group action* ${}_GX$ of the (multiplicative) group G on the set X is given by a mapping

$$G \times X \to X, \qquad (g, x) \mapsto gx,$$

which fulfills $1x = x$ and $(g_2g_1)x = g_2(g_1x)$ for all $x \in X$ and $g_1, g_2 \in G$, where 1 is the identity element in G. A group action is called *finite* if both G and X are finite.

A group action ${}_GX$ determines a group homomorphism ϕ from G to the symmetric group S_X by

$$\phi\colon G \to S_X, \qquad g \mapsto \phi(g) := [x \mapsto gx],$$

which is called a *permutation representation* of G on X. Usually we abbreviate $\phi(g)$ by writing $\bar{g}$, which is the permutation of X that maps x to gx. For instance $\bar{1}$ is always the identity on X. Accordingly the image $\phi(G)$ is indicated by $\bar{G}$. It is a *permutation group* on X, i. e. a subgroup of S_X. If X is finite then $\bar{G}$ is finite since it is a subgroup of the symmetric group S_X which is of cardinality $|X|!$. Hence whenever X is finite we can speak of a finite group action.

Before describing some examples of group actions we investigate certain structures which are induced by group actions. A group action ${}_GX$ defines the following equivalence relation on X. $x_1 \sim_G x_2$ if and only if there is some $g \in G$ such that $x_2 = gx_1$. The equivalence classes $G(x)$ with respect to $\sim_G$ are the *orbits* of G on X so that $G(x) = \{gx \mid g \in G\}$. The *set of all orbits* is denoted by $G\backslash\backslash X := \{G(x) \mid x \in X\}$. For each $x \in X$ the *stabilizer* G_x of x which is the set $G_x := \{g \in G \mid gx = x\}$ is a subgroup of G. Finally the *set of all fixed points* of $g \in G$ is denoted by $X_g := \{x \in X \mid gx = x\}$.

Here are some examples of group actions. The first are normal constructions from group theory the latter (which describe actions on the set of functions between two sets X and Y) represent the most important situations in which the methods of combinatorics can be applied.

1.1 Examples.

1. A subgroup U of G acts on G by multiplication from the left

$$U \times G \to G, \qquad (u, g) \mapsto ug.$$

The orbit $U(g)$ is the *right-coset* $Ug := \{ug \mid u \in U\}$ and G is the disjoint union of the different orbits Ug in $U\backslash\backslash G$ which is usually written as $U\backslash G$.

The subgroup U also acts from the right on G,

$$U \times G \to G, \qquad (u, g) \mapsto gu^{-1},$$

the orbits are the *left-cosets* gU and $U\backslash\backslash G = G/U$.

2. A group G acts on itself by *conjugation*

$$G \times G \to G, \qquad (g, h) \mapsto ghg^{-1}.$$

The orbit $G(h) = \{ghg^{-1} \mid g \in G\}$ is the *conjugacy class* of h. The stabilizer of h is the set $\{g \in G \mid gh = hg\}$. The set of fixed points of g is $\{h \in G \mid gh = hg\}$.

3. The direct product of two subgroups U and V of a group G acts on G by

$$(U \times V) \times G \to G, \qquad ((u, v), g) \mapsto ugv^{-1}.$$

The orbits UgV are called *double cosets class*. The set of all double cosets is usually denoted by $U\backslash G/V$.

4. Let ${}_GX$ be a group action, then G acts on Y^X by

1.2 $$G \times Y^X \to Y^X, \qquad (g, f) \mapsto f \circ \bar{g}^{-1},$$

where $\bar{g}$ is the permutation representation of g acting on X. The set of fixed points of g is the set of all functions f which are constant on the cycles (in the cycle decomposition) of $\bar{g}$.

5. Let ${}_HY$ be a group action, then H acts on Y^X by

$$H \times Y^X \to Y^X, \qquad (h, f) \mapsto \bar{h} \circ f,$$

where $(\bar{h} \circ f)(x) = \bar{h}(f(x)) = hf(x)$ for all $x \in X$.

6. Let ${}_GX$ and ${}_HY$ be group actions, then the direct product $H \times G$ acts on Y^X by

$$(H \times G) \times Y^X \to Y^X, \qquad ((h, g), f) \mapsto \bar{h} \circ f \circ \bar{g}^{-1}.$$

The orbits of $f \in Y^X$ defined by the last three group actions are usually called *symmetry types* of mappings. ◇

The main tool for enumeration under group actions is the

1.3. Cauchy-Frobenius-Lemma. *The number of orbits under a finite group action* $_GX$ *is the average number of fixed points.*

$$|G\backslash\backslash X| = \frac{1}{|G|}\sum_{g\in G}|X_g|$$

For instance for the group action on Y^X in the form 1.2 we get

$$\left|G\backslash\backslash Y^X\right| = \frac{1}{|G|}\sum_{g\in G}|Y|^{c(\bar{g})},$$

where $c(\bar{g})$ is the number of cycles (in the cycle decomposition) of $\bar{g}$.

2 The n-scale Z_n

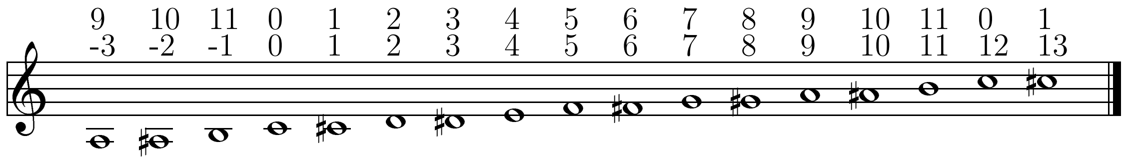
In this section we give a mathematical description of an n-scale (where n is an arbitrary positive integer) which is a generalization of tempered 12-scale that is commonly used in western music. In our model of an n-scale in each octave there are exactly n tones, which are equally distributed over each octave. We label these tones with the integer numbers in $\mathbb{Z}$. For doing this first we choose an arbitrary tone (for instance c^1) and label this tone with 0. Then stepping up from one tone to the next we increase the labels by one, and stepping down we decrease them. This way we get a bijection between the set of all tones and the set of all integer numbers $\mathbb{Z}$. (It is important to mention that there is no difference between tones like *c-sharp* or *d-flat*. They are considered to be the same tones.) When speaking about tones we use their labels for identifying them.

Often for investigations in music theory it is not important which octave a special tone belongs to, for that reason we collect all tones which are any number of octaves apart, into one *pitch-class*, ending up in exactly n pitch-classes in an n-scale. This can be done by introducing a group action on $\mathbb{Z}$. We already know that $(\mathbb{Z},+)$ is a group. The set $n\mathbb{Z} := \{nz \mid z \in \mathbb{Z}\}$ is a subgroup of it. As was described in the first item of 1.1 the subgroup $n\mathbb{Z}$ can be used for defining a group action on $\mathbb{Z}$. The orbits correspond to the cosets of the form $i + n\mathbb{Z} = \{i + nz \mid z \in \mathbb{Z}\}$. Consequently all tones with labels in this set are collected to one class, a pitch-class. These are just the tones which differ from the tone i any number of octaves. Since the action above is not a finite group action we cannot apply the Cauchy-Frobenius-Lemma for determining the number of these orbits. But it is not difficult to deduce that the set of orbits (we introduced the notation $\mathbb{Z}/n\mathbb{Z}$) consists of exactly n elements. To be more precise

2.1 $$\mathbb{Z}/n\mathbb{Z} = \{i + n\mathbb{Z} \mid 0 \le i < n\}$$

and as a standard representative of $z + n\mathbb{Z}$ we choose $i \in z + n\mathbb{Z}$ such that $0 \le i < n$.

The following figure shows a part of the chromatic scale, together with the labelling of the tones in $\mathbb{Z}$ and with the pitch-class numbers.



On the set $\mathbb{Z}/n\mathbb{Z}$ we can define two inner composition $+$ and $\cdot$ by setting $(i+n\mathbb{Z})+(j+n\mathbb{Z}):=(i+j)+n\mathbb{Z}$ and $(i+n\mathbb{Z})\cdot(j+n\mathbb{Z}):=ij+n\mathbb{Z}$.

Instead of $\mathbb{Z}/n\mathbb{Z}$ we will usually use the notation Z_n. So we end up that the n-scale is the set Z_n together with the two inner compositions $+$ and $\cdot$, which is a commutative ring with 1. It is the *residue-class-ring* of $\mathbb{Z}$ modulo $n\mathbb{Z}$. The set of units in Z_n is given by

$$Z_n^* = \{a \in Z_n \mid \gcd(a,n) = 1\}.$$

From the unique composition of the integer $n > 1$ as a product of primes

$$n = \prod_{i=1}^{r} p_i^{a_i}$$

it is possible to prove that the mapping ϕ, which is given below, defines a ring isomorphism from Z_n to the product of the rings $Z_{p_i^{a_i}}$.

2.2
$$\phi\colon Z_n \to Z_{p_1^{a_1}} \times Z_{p_2^{a_2}} \times \ldots \times Z_{p_r^{a_r}} =: \bigtimes_{i=1}^{r} Z_{p_i^{a_i}}$$

$$j \mapsto \phi(j) := (j \bmod p_1^{a_1}, j \bmod p_2^{a_2}, \ldots, j \bmod p_r^{a_r})$$

The rings Z_p for a prime number p are the *finite fields* of order p. From the prime number decomposition of 12 we get that

$$Z_{12} = Z_3 \times Z_4,$$

where Z_3 is a field, but Z_4 is not a field.

Now we are coming back to some objects from music theory when describing how the musical operators *transposing* and *inversion* can be defined on an n-scale. First we realize that transposing by k tones up or down means replacing each tone by the tone which is exactly k tones higher or lower. Since we agreed to speak about tones by using their labels, the operation of transposing by k tones is described by exchanging all labels $z \in \mathbb{Z}$ by $z+k$ or $z-k$ respectively. Furthermore we realize that transposing by k tones is the same as transposing k times by one tone. For that reason we introduce the operator T *transposing by one tone* as the following bijection

$$T\colon \mathbb{Z} \to \mathbb{Z}, \qquad z \mapsto T(z) := z+1$$

on $\mathbb{Z}$, the set of all labels of tones. Applying this operator to the elements of a pitch-class $i+n\mathbb{Z}$ we get all the elements in the pitch-class $(i+1)+n\mathbb{Z}$. In other words, $T(i+n\mathbb{Z}) = (i+1)+n\mathbb{Z}$. For that reason it is possible to introduce the operator *transposing by one pitch-class* on the set Z_n. We also denote it by T, then T is the following mapping

$$T\colon Z_n \to Z_n, \qquad i \mapsto T(i) := i + 1.$$

This mapping is a bijection on Z_n, in other words it is a permutation of Z_n and its standard cycle decomposition is of the form $(0, 1, \ldots, n-1)$. Hence T is a cycle of length n.

The musical operator inversion means that tone-steps (in a motive, in a melody or in a tone-row) or just intervals of given size up or down, are exchanged by steps or intervals of the same size but exchanged direction, i. e. down or up. Here we must distinguish two cases: Either there exists a reference tone which is not changed, tones higher than this tone are exchanged into tones lower than this tone and vice versa. Or there are two adjacent tones which are exchanged, and simultaneously tones higher than these two tones are exchanged into tones lower than these two tones and vice versa. Let z_0 be the label of the reference tone, which is not changed, or let z_0 and $z_0 + 1$ be the labels of the two adjacent tones which are exchanged. In the first case let $r := z_0$ in the second case let $r := z_0 + \frac{1}{2}$. Then we can define the operator *inversion with respect to* r, as the following bijection

$$I_r\colon \mathbb{Z} \to \mathbb{Z}, \qquad z \mapsto I_r(z) := r - (z - r) = 2r - z$$

on $\mathbb{Z}$, the set of all labels of tones. Applying I_r to all elements of the pitch-class $(2r - i) + n\mathbb{Z}$ we get all the elements in the pitch-class $i + n\mathbb{Z}$. In other words, $I_r((2r - i) + n\mathbb{Z}) = i + n\mathbb{Z}$. Especially we are interested in $r = z_0 = 0$ then we define the operator I, *inversion at pitch-class 0,* on the set Z_n by

$$I\colon Z_n \to Z_n, \qquad i \mapsto I(i) := -i.$$

This mapping is a bijection on Z_n, hence it is a permutation of Z_n and its standard cycle decomposition is of the form $(0)(1, n-1)(2, n-2)\ldots$. Depending on n the permutation I decomposes into 2 fixed points and $\frac{n-2}{2}$ transpositions if n is even, or 1 fixed point and $\frac{n-1}{2}$ transpositions if n is odd.

With these two operators we define two symmetry groups on the n-scale Z_n. First consider the permutation group $\langle T \rangle$. It consists of all powers T^i for $1 \leq i \leq n$, the operator T^n is the identity element. Since T stands for transposing by one pitch-class, T^i stands for transposing by i pitch-classes. Thus the group $\langle T \rangle$ describes all the possibilities to transpose in an n-scale. It is a cyclic group of order n which later will be denoted as C_n.

The group $\langle T, I \rangle$ consists of all possibilities to combine powers of T with the inversion operator I. It is the *dihedral group* D_n of order $2n$ for $n \geq 3$. It is easy to check that $I \circ T = T^{-1} \circ I$ and that all elements of $\langle T, I \rangle$ can be written as $T^k \circ I^j$ such that $k \in \{0, 1, \ldots, n-1\}$ and $j \in \{0, 1\}$.

Sometimes a symmetry of another type is applied in Z_{12}, it is the so called *quart-circle symmetry* Q, which is defined by

$$Q\colon Z_{12} \to Z_{12}, \qquad i \mapsto Q(i) := 5i.$$

Since 5 is a unit element in Z_{12} this mapping is a permutation of Z_{12} and its standard cycle decomposition is of the form $(0)(1,5)(2,10)(3)(4,8)(6)(7,11)(9)$. Applying Q to the chromatic scale we get the sequence of all quarts 0, 5, 10, 3, ... Together with Q usually the other operators T and I are taken into consideration as well, such that we end up with the permutation group $\langle T, I, Q\rangle$ acting on Z_{12}. The product $I \circ Q$ equals $Q \circ I$ which is called the *quint-circle symmetry*. Since $Q \circ T = T^5 \circ Q$ all the elements of this group can be expressed as $T^k \circ I^j \circ Q^l$ such that $k \in \{0, 1, \dots, 11\}$ and $j, l \in \{0, 1\}$. Therefore the group $\langle T, I, Q\rangle$ consists of 48 permutations. This group is the set of all mappings $\{i \mapsto a \cdot i + b \mid a \in Z_{12}^*,\ b \in Z_{12}\}$.

We could generalize this approach in order to define a symmetry group on the n-scale Z_n for arbitrary n. For $a \in Z_n^*$ and $b \in Z_n$ the mapping

2.3 $$\pi_{a,b}\colon Z_n \to Z_n, \qquad i \mapsto \pi_{a,b}(i) := ai + b$$

is a bijection on Z_n and the set $\mathrm{Aff}_1(Z_n) := \{\pi_{a,b} \mid a \in Z_n^*,\ b \in Z_n\}$ is a permutation group on Z_n. It is called the group of all *affine mappings* from Z_n to Z_n.

3 Pólya's Theorem, cycle indices and applications

In this section we come back to enumeration under group actions. In order to get some more information about the G-orbits on X we introduce *weight functions* on X. Then we can count the number of G-orbits of given weight. This approach will be applied for the enumeration of orbits of k-chords in an n-scale and later for the enumeration of k-motives and other objects from music theory.

A weight function is a function $w\colon X \to R$ where R is a commutative ring, such that $\mathbb{Q}$ is a sub-ring of R. The function w must be constant on each G-orbit on X, i. e. $w(x) = w(gx)$ for all $g \in G$ and $x \in X$. Then it makes sense to define the weight of an orbit $G(x)$ as the weight of an arbitrary element, say x, of $G(x)$.

3.1 $$w(G(x)) := w(x)$$

When summing up weights of G-orbits instead of counting them we derive the *weighted version* of the *Cauchy-Frobenius-Lemma*.

3.2 Theorem. *The sum of weights of G-orbits defined by the weight function 3.1 under a finite group action ${}_GX$ is given by*

$$\sum_{\omega \in G\backslash\backslash X} w(\omega) = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in X_g} w(x).$$

We get the original version 1.3 of this theorem by setting $w(x) = 1$ for all $x \in X$.

A group action ${}_GX$ induces an action of G an the set Y^X as was described in 1.2. Let R be a commutative ring such that $\mathbb{Q}$ is a sub-ring of R and let $W : Y \to R$ be an arbitrary function. Then the function $w\colon Y^X \to R$ defined by

3.3 $$w(f) := \prod_{x \in X} W(f(x))$$

is a weight function. It is constant on each G-orbit on Y^X, since multiplication in R is commutative, and applying any group element g to a function f just leads to a reordering of the terms in the product caused by the permutation $\bar{g}$ on X. For any $g \in G$ we have

$$w(gf) = \prod_{x \in X} W(f(\bar{g}^{-1}x)) = \prod_{x \in X} W(f(x)) = w(f).$$

For this situation we rewrite 3.2 in order to derive the famous theorem by G. Pólya [33, 32].

3.4. Pólya's Theorem. *The sum of weights of G-orbits on Y^X induced by a finite group action ${}_GX$ is given by*

$$\sum_{\omega \in G \backslash\backslash Y^X} w(\omega) = \frac{1}{|G|} \sum_{g \in G} \prod_{i=1}^{|X|} \left(\sum_{y \in Y} W(y)^i \right)^{a_i(\bar{g})},$$

where $(a_1(\bar{g}), \ldots, a_{|X|}(\bar{g}))$ is the cycle type of the permutation $\bar{g}$.

The result above motivates the following definition of the *cycle index* of an action ${}_GX$. It is a polynomial over $\mathbb{Q}$ in indeterminates $z_1, z_2, \ldots, z_{|X|}$ which collects for all $g \in G$ the information about the cycle types of the induced permutations $\bar{g}$ on X in a useful way. It is given by

$$C(G, X) := \frac{1}{|G|} \sum_{g \in G} \prod_{i=1}^{|X|} z_i^{a_i(\bar{g})}.$$

Comparing this definition with the formula in 3.4 we realize that the sum of weights of G orbits on Y^X can be computed from $C(G, X)$ by replacing each variable z_i by $\sum_{y \in Y} W(y)^i$. We indicate this substitution in the following way:

$$C(G, X; z_i := \sum_{y \in Y} W(y)^i).$$

Using this notation we can reformulate Pólya's Theorem again. For a given finite group action ${}_GX$, a finite set Y, a function $W: Y \to R$, where R is a commutative ring, such that $\mathbb{Q}$ is a sub-ring of R, and for a weight function $w: Y^X \to R$ defined by 3.3, the sum of weights of G orbits for the action defined by 1.2 on Y^X is

$$\sum_{\omega \in G \backslash\backslash Y^X} w(\omega) = C(G, X; z_i := \sum_{y \in Y} W(y)^i).$$

In order to apply Pólya's Theorem we have to know the cycle indices of the acting groups. Actually there is no general routine for computing cycle indices, it depends on the group and the action of the group which method to use. For groups of small order it is possible to compute the cycle type of all the induced permutations. Often it is not necessary to compute the cycle type of $\bar{g}$ for each element $g \in G$. In the second item of 1.1 we introduced an action of G on G in form of the conjugation. The orbits under this action are the conjugacy classes in G. Since conjugate elements are of the same cycle type it is possible to compute the cycle index in the form

$$C(G,X) = \frac{1}{|G|}\sum_{C\in\mathcal{C}}|C|\prod_{i=1}^{|X|} z_i^{a_i(\bar{g}_C)},$$

where $\mathcal{C}$ is the set of all conjugacy classes of elements of G and g_C is an arbitrary element of the conjugacy class $C \in \mathcal{C}$.

In order to apply Pólya's theory to the enumeration problems arising in music theory we have to know the cycle indices of the cyclic groups C_n, the dihedral groups D_n, the symmetric groups $S_{\underline{n}}$ and of the affine groups $\mathrm{Aff}_1(Z_n)$. The symbol $\underline{n}$ is used to represent the set $\{1,\ldots,n\}$.

3.5 Lemma. *The cycle indices of the permutation groups C_n and D_n are given by*

$$C(C_n, Z_n) = \frac{1}{n}\sum_{d|n}\varphi(n/d)z_{n/d}^d,$$

$$C(D_n,Z_n) = \frac{1}{2}C(C_n,Z_n) + \begin{cases} \frac{1}{2}z_1 z_2^{(n-1)/2} & \textit{if } n \textit{ is odd,} \\ \frac{1}{4}(z_1^2 z_2^{(n-2)/2} + z_2^{n/2}) & \textit{if } n \textit{ is even,} \end{cases}$$

where φ is the Euler function. The cycle index of the natural action of the symmetric group on $\underline{n}$ can be computed as

$$C(S_{\underline{n}},\underline{n}) = \sum_{a\dashv\dashv n}\prod_k \frac{1}{a_k!}\left(\frac{z_k}{k}\right)^{a_k},$$

where the sum is taken over all $a = (a_1,\ldots,a_n)$ such that $\sum_{i=1}^n ia_i = n$.

Before presenting the cycle indices of the affine groups we investigate a product of two group actions. Let ${}_GX$ and ${}_HY$ be two group actions then a natural action of the direct product $G \times H$ on $X \times Y$ can be described by

$$(G\times H)\times(X\times Y)\to X\times Y \qquad ((g,h),(x,y))\mapsto(gx,hy).$$

In [22] the following product operator was defined. Let A and B be polynomials in indeterminates $z_1, z_2, \ldots$ over $\mathbb{Q}$ given by

$$A(z_1,z_2,\ldots,z_n) = \sum_{(j)} a_{(j)}\prod_{i=1}^n z_i^{j_i}, \quad B(z_1,z_2,\ldots,z_m) = \sum_{(k)} b_{(k)}\prod_{i=1}^m z_i^{k_i},$$

where the first sum is taken over finitely many n-tuples $(j) = (j_1,\ldots,j_n) \in \mathbb{N}_0^n$ and the second sum over finitely many m-tuples $(k) = (k_1,\ldots,k_m) \in \mathbb{N}_0^m$. Then

$$A(z_1,\ldots,z_n) \times B(z_1,\ldots,z_m) := \sum_{(j)}\sum_{(k)} a_{(j)}b_{(k)}\left(\prod_{i=1}^n z_i^{j_i}\right)\times\left(\prod_{i=1}^m z_i^{k_i}\right),$$

where

$$\left(\prod_{i=1}^{n} z_i^{j_i}\right) \times \left(\prod_{i=1}^{m} z_i^{k_i}\right) := \prod_{i=1}^{n}\prod_{l=1}^{m}(z_i^{j_i} \times z_l^{k_l})$$

and

$$z_i^{j_i} \times z_l^{k_l} := z_{\operatorname{lcm}(i,l)}^{j_i k_l \gcd(i,l)}.$$

It can be used for computing the cycle index of the action above.

3.6 Lemma. *The cycle index of the natural action of $G \times H$ on $X \times Y$ induced by two finite action ${}_GX$ and ${}_HY$ can be expressed as*

$$C(G \times H, X \times Y) = C(G, X) \times C(H, Y).$$

Applying the ring isomorphism ϕ between Z_n and $\times_{i=1}^{r} Z_{p_i^{a_i}}$ from 2.2 to the action of $\mathrm{Aff}_1(Z_n)$ on Z_n we realize that $\phi(\mathrm{Aff}_1(Z_n)) = \times_{i=1}^{r} \mathrm{Aff}_1(Z_{p_i^{a_i}})$ acts on $\times_{i=1}^{r} Z_{p_i^{a_i}}$ where each $\mathrm{Aff}_1(Z_{p_i^{a_i}})$ acts in a natural way on $Z_{p_i^{a_i}}$ for $i = 1, 2, \ldots, r$. A consequence of 3.6 is

$$C(\mathrm{Aff}_1(Z_n), Z_n) = \bigtimes_{i=1}^{r} C(\mathrm{Aff}_1(Z_{p_i^{a_i}}), Z_{p_i^{a_i}})$$

In [43] the following formulae are given for these cycle indices.

3.7 Lemma. *Let p be a prime and $a \in \mathbb{N}$. Then the cycle index of the natural action of $\mathrm{Aff}_1(Z_{p^a})$ on Z_{p^a} is of the following form: If $p = 2$ then*

$$C(\mathrm{Aff}_1(Z_2), Z_2) = \frac{1}{2}(z_1^2 + z_2)$$

$$C(\mathrm{Aff}_1(Z_4), Z_4) = \frac{1}{8}(z_1^4 + 2z_1^2 z_2 + 3z_2^2 + 2z_4).$$

and for $a \geq 3$

$$C(\mathrm{Aff}_1(Z_{2^a}), Z_{2^a}) = \frac{1}{2^{2a-1}}\Bigg(2^{2(a-1)-1} z_{2^a} + \sum_{i=1}^{a-1}\left(2^{2(i-1)} + \varphi(2^{i-1})2^{a-1}\right) z_{2^i}^{2^{a-i}} +$$

$$+ \sum_{i=0}^{a-2} \varphi(2^i)\left(2^i z_1^{2^{a-i}} + 2^{a-1} z_1^2 z_2^{2^{a-i-1}-1}\right)\left(\prod_{j=1}^{i} z_{2^j}\right)^{2^{a-i-1}}\Bigg).$$

If p is a prime number different from 2 then for $a \geq 1$

$$C(\mathrm{Aff}_1(Z_{p^a}), Z_{p^a}) = \frac{1}{p^{2a-1}(p-1)}\Bigg(\sum_{i=1}^{a} p^{2(i-1)}(p-1) z_{p^i}^{p^{a-i}} +$$

$$+ \sum_{i=0}^{a-1}\sum_{d|p-1} p^{i+\delta(d)(a-i)} \varphi(p^i d) z_1 z_d^{(p^{a-i-1}-1)/d}\left(\prod_{j=0}^{i} z_{p^j d}\right)^{p^{a-i-1}(p-1)/d}\Bigg),$$

where

$$\delta(d) = \begin{cases} 1 & \textit{if } d > 1 \\ 0 & \textit{if } d = 1. \end{cases}$$

For our next example it is important to mention that when a group G acts on X then G acts in a natural way on $\mathcal{P}(X) := \{S \mid S \subseteq X\}$, which is the set of all subsets of X. For $g \in G$ and $S \in \mathcal{P}(X)$ the subset gS of X is defined as $gS := \{gx \mid x \in S\}$. It is possible to identify each $S \subseteq X$ with its *characteristic function* χ_S. This is a function from X to $\{0, 1\}$ defined by

$$\chi_S(x) := \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

Moreover each function $f\colon X \to \{0, 1\}$ can be interpreted as the characteristic function of the subset $S = f^{-1}(\{1\})$ of X. If we define the weight function W on $Y = \{0, 1\}$ by $W(0) = 1$ and $W(1) = z$ where z is an indeterminate over $\mathbb{Q}$ then a function $f\colon X \to \{0, 1\}$ is the characteristic function of a k-subset of X if and only if the weight $w(f)$ defined by 3.3 is equal to z^k. We call such a function f a function of weight k.

The group action of G on $\mathcal{P}(X)$ can be translated into a group action of G on $\{0, 1\}^X$ in the form of 1.2 since $\chi_{gS}(x) = 1$ if and only if $x \in gS$ or equivalently $g^{-1}x \in S$ which means that $\chi_S(g^{-1}x) = 1$.

3.8 Example. (k-chords) Any k-subset of Z_n is called a *k-chord* in Z_n. Especially 2-chords are called *intervals.* As was shown in section 2 the action of the permutation groups $\langle T\rangle$, $\langle T, I\rangle$ or $\mathrm{Aff}_1(Z_n)$ can be motivated from music theory. Therefore it makes sense to apply the elements of these groups to k-chords. Let G be one of these groups, then the G orbit $G(S)$ of a k-chord $S \subseteq Z_n$ is the collection of all k-chords which are G-equivalent to S. Consequently the number of different k-chords is the number of G-orbits on the set of all k-subsets of Z_n which is the coefficient of z^k in

$$C(G, Z_n; z_i := 1 + z^i).$$

As a matter of fact for different choices of G we get different classes of G-equivalent chords, consequently different numbers of G-orbits. The following numbers of different G-orbits of k-chords in Z_{12} for $k = 1, 2, \ldots, 12$ and G being one of the groups C_{12}, D_{12} or $\mathrm{Aff}_1(Z_{12})$ can also be found in [38, 11, 37].

$G\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12
C_{12}	1	6	19	43	66	80	66	43	19	6	1	1
D_{12}	1	6	12	29	38	50	38	29	12	6	1	1
$\mathrm{Aff}_1(Z_{12})$	1	5	9	21	25	34	25	21	9	5	1	1

$\diamond$

Similar methods can be applied for enumerating the number of symmetry patterns under the other group actions introduced in 1.1 (cf. [4, 27, 28]). Here we want to deal with another problem. The action of $H\times G$ induces the following action of H on $G\backslash\backslash Y^X$, the set of all G-orbits on Y^X by

$$H\times (G\backslash\backslash Y^X)\to G\backslash\backslash Y^X,\qquad (h,G(f))\mapsto G(\bar h\circ f).$$

For $h\in H$ the G-orbit $G(f)$ is called *h-invariant* if and only if $G(f)=G(\bar h\circ f)$, in other words, $G(f)$ is h-invariant if $G(f)$ is a fixed point of h under the action above. Then the number of h-invariant orbits is (compare this result with [5])

$$\frac{1}{|G|}\sum_{g\in G}\prod_{i=1}^{|X|}|Y_{h^i}|^{a_i(\bar g)}=C(G,X;z_i:=\sum_{j|i}j\cdot a_j(\bar h)).$$

3.9 Example. (self-complementary chords) The *complement* of a k-chord S in Z_n is the the $n-k$-chord $Z_n\setminus S$. If S is given by its characteristic function χ_S then the characteristic function of the complement of S is $1-\chi_S$ which means that each 1 in the characteristic function of S must be replaced by 0 and each 0 must be replaced by 1. This operation can also be described by multiplying χ_S from the left with the transposition $(0,1)\in S_{\{0,1\}}$. This permutation is of cycle type $(0,1)$.

If n is an even positive integer, then the $n/2$-chord S is called *self-complementary* if and only if it is G-equivalent to its complement $Z_n\backslash S$, where G is a (musically motivated) group acting on Z_n. In mathematical terms the orbits of self-complementary $n/2$-chords are just the $(0,1)$-invariant orbits of $n/2$-chords. Their number is equal to

$$C(G,Z_n;z_{2k}:=2,\ z_{2k+1}:=0)$$

which means that the indeterminates z_i in $C(G,Z_n)$ must be replaced by 0 if i is odd, and by 2 if i is even. Especially, for G being one of the groups C_{12}, D_{12} or $\mathrm{Aff}_1(Z_{12})$ the numbers of self-complementary 6-chords in Z_{12} are 8, 20 and 18 respectively. $\diamond$

3.10 Example. (interval structure of a chord) In this example we assume that the acting group G is a subgroup of D_n. The set of all possible intervals between two different tones in the n-scale Z_n is $Z_n\setminus\{0\}$. Considering on Z_n the natural linear order $0<1<2<\ldots<n-1$ we assume that the chord $S=\{i_1,i_2,\ldots,i_k\}$ is given in the form such that $i_1<i_2<\ldots<i_k$. Then the *interval structure* of S is defined to be the G-orbit $G(f_S)$ of the function f_S given by

$$f_S\colon\{1,2,\ldots,k\}\to Z_n\setminus\{0\}$$

$$f_S(j):=i_{j+1}-i_j \text{ for } j<k \text{ and } f_S(k):=i_1-i_k.$$

In [11] it is shown that two chords belong to the same G-orbit if and only if they have the same interval structure.

It is obvious that the orbit $G(f)$ of an arbitrary function $f\colon\{1,2,\ldots,k\}\to Z_n\setminus\{0\}$ is the interval structure of a k-chord, if and only if

$$\sum_{i=1}^{k}f(i)=n,$$

which must be read as a sum of positive integers.

Let $x, y_1, y_2, \ldots, y_n$ be indeterminates over $\mathbb{Q}$ and let R be the ring $\mathbb{Q}[x, y_1, y_2, \ldots, y_n]$. After defining the following weight W on $Z_n \setminus \{0\}$

$$i \mapsto W(i) := x^i y_i$$

it is easy to show that $G(f)$ is the interval structure of a k-chord, if and only if

$$w(f) = x^n \prod_{i=1}^{k} y_{f(i)},$$

where w is defined by 3.3.

3.11 Theorem. *The weights of all interval structures of k-chords in Z_n are the summands in the expansion of*

$$C(G, Z_n; z_j = \sum_{i=1}^{n-1} x^{ji} y_i^j)$$

containing the term x^n.

For instance the interval structures of 3-chords under D_{12} symmetry are given by ${y_1}^2 y_{10} + y_1(y_2 y_9 + y_3 y_8 + y_4 y_7 + y_5 y_6) + {y_2}^2 y_8 + y_2(y_3 y_7 + y_4 y_6 + {y_5}^2) + {y_3}^2 y_6 + y_3 y_4 y_5 + {y_4}^3$.

◇

3.12 Example. (k-rhythms) The easiest form of describing a rhythm is to find a very dense subdivision of the rhythm into equidistant beats such that all rhythmical events coincide with some of these beats. Collecting $m \in \mathbb{N}$ beats into a bar we are speaking of an *m-bar*. In order to make investigations easier we restrict our attention just to one bar. And if for some reason a given rhythm exceeds one bar then we continue this rhythm starting at the beginning of this bar again. In other words, an m-bar has a cyclic structure, hence we can use the set Z_m as a model of an m-bar. The same way as we introduced symmetry operations for pitch-classes we can introduce symmetry operations on an m-bar. For instance as standard operators we could introduce S the *cyclic shift by one beat* and R the *retrograde inversion* which reverses a given rhythm. They can be defined as the following permutations on Z_m.

$$S: Z_m \to Z_m, \quad i \mapsto S(i) := i + 1 \qquad R: Z_m \to Z_m, \quad i \mapsto R(i) := -i$$

Again we realize that the permutation groups $\langle S \rangle$ and $\langle S, R \rangle$ and their actions on Z_m are isomorphic to C_m and D_m and to their actions on $\underline{m}$. A *k-rhythm* in an m-bar (for $1 \leq k \leq m$) is defined as a k-subset of Z_m. The actions of the groups $G = \langle S \rangle$ or $G = \langle S, R \rangle$ are motivated by music theory. Applying the same methods as in 3.8 it is possible to determine the number of non-G-equivalent k-rhythms. ◇

3.13 Example. (k-motives) In order to describe both tonal and rhythmical aspects of music G. Mazzola introduced in [29] a mathematical notion of *motives*. When speaking about motives we first have to find all possible combinations of beats in an m-bar Z_m and pitch-classes in an n-scale Z_n. The set of all these combinations is the product $Z_m \times Z_n$. Then for $1 \leq k \leq mn$ each k-subset S of $Z_m \times Z_n$ is a *k-motive*. When $(i, j) \in Z_m \times Z_n$ belongs to the motive S it means that a tone of pitch-class j occurs at the beat i in the motive S. So the first parameter describes the rhythmical aspects the second the tonal aspects. Usually when drawing pairs (i, j) on a sheet of paper the first component describes the position on a horizontal axis the second component the position on a vertical axis. This point of view coincides with the musical notation where rhythmical aspects are described horizontally and tonal aspects vertically.

In the case $m = n$ G. Mazzola suggested to investigate the following group G acting on Z_n^2.

$$G := \langle \{T, \phi_A \mid A \in \{U, P, D_l \mid l \in Z_n^*\}\} \rangle,$$

where

$$T\colon Z_n^2 \to Z_n^2, \qquad (i,j) \mapsto T(i,j) := (i, j+1)$$

$$\phi_A\colon Z_n^2 \to Z_n^2, \qquad (i,j) \mapsto \phi_A(i,j) = A \begin{pmatrix} i \\ j \end{pmatrix}$$

and

$$U := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \; P := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; D_l := \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}.$$

T stands for transposing by one pitch-class, the matrix U describes an *arpeggio* since for $k \leq n$ the k-chord $\{(x, y_1), (x, y_2), \ldots, (x, y_k)\}$, which is played at beat x is transformed into

$$\{(x + y_1, y_1), (x + y_2, y_2), \ldots, (x + y_k, y_k)\}.$$

The matrices D_l define an *augmentation* since the rhythm $\{(x_1, y), (x_2, y), \ldots, (x_k, y)\}$, which is played at pitch-class y is replaced by the rhythm

$$\{(lx_1, y), (lx_2, y), \ldots, (lx_k, y)\}.$$

The matrix P describes the exchange of rhythmical and tonal properties. Moreover the product $\phi_P \circ T \circ \phi_P$ is the cyclic shift S by one beat, $S(i,j) = (i+1, j)$.

3.14 Lemma. *The group G of symmetry operations of motives introduced above is the group $\mathrm{Aff}_2(Z_n)$ of all affine mappings from Z_n^2 to Z_n^2.*

Generalizing the approach above the group of all symmetries of $Z_m \times Z_n$ could be described as the set of all mappings of the form

$$Z_m \times Z_n \ni (i,j) \mapsto \pi_{A,b}(i,j) := A \begin{pmatrix} i \\ j \end{pmatrix} + b := \begin{pmatrix} a_{11}i + a_{12}j + b_1 \\ a_{21}i + a_{22}j + b_2 \end{pmatrix} \in Z_m \times Z_n,$$

where A is a matrix representing a group automorphism of $Z_m \times Z_n$ and b is an arbitrary element of $Z_m \times Z_n$.

In order to determine the number of non G-equivalent k-motives by applying Pólya's Theorem 3.4 we have to compute the cycle index of the acting group G first which is not a trivial task any more. In the general case $n \neq m$ we do not have enough information in order to get nice results. But for $n = m = \prod_{i=1}^r p_i^{a_i}$ the action of $\mathrm{Aff}_2(Z_n)$ on Z_n^2 can be replaced by the action of the direct product $\bigtimes_{i=1}^r \mathrm{Aff}_2(Z_{p_i^{a_i}})$ on $\bigtimes_{i=1}^r Z^2_{p_i^{a_i}}$ and the cycle index can be computed as

$$C(\mathrm{Aff}_2(Z_n), Z_n^2) = \bigtimes_{i=1}^r C(\mathrm{Aff}_2(Z_{p_i^{a_i}}), Z^2_{p_i^{a_i}}).$$

In the case $a_i = 1$ the residue-class-ring Z_{p_i} is a field and we can apply a lot of theory about fields, polynomials over fields etc. in order to compute the cycle indices of the natural actions of $\mathrm{GL}_2(Z_{p_i})$ or $\mathrm{Aff}_2(Z_{p_i})$ on $Z^2_{p_i}$. Going into details would carry us too far. See for instance [23, 15]. With this decomposition method the cycle index of $\mathrm{Aff}_2(Z_{12})$ can be computed as $C(\mathrm{Aff}_2(Z_3), Z_3^2) \times C(\mathrm{Aff}_2(Z_4), Z_4^2)$ which is the following polynomial

$\frac{1}{663552}(z_1^{144} + 18z_1^{72}z_2^{36} + 36z_1^{48}z_2^{48} + 24z_1^{48}z_3^{32} + 72z_1^{36}z_2^{54} + 48z_1^{36}z_2^{18}z_4^{18} + 648z_1^{24}z_2^{60} + 432z_1^{24}z_2^{12}z_3^{16}z_6^{8} + 192z_1^{18}z_2^{9}z_4^{27} + 9z_1^{16}z_2^{64} + 72z_1^{16}z_2^{16}z_6^{16} + 54z_1^{16}z_4^{32} + 108z_1^{16}z_8^{16} + 2592z_1^{12}z_2^{66} + 1728z_1^{12}z_2^{30}z_4^{18} + 1728z_1^{12}z_2^{18}z_3^{8}z_6^{12} + 1152z_1^{12}z_2^{6}z_3^{8}z_4^{6}z_6^{4}z_{12}^{4} + 128z_1^{9}z_3^{45} + 384z_1^{9}z_3^{9}z_6^{18} + 162z_1^{8}z_2^{68} + 1296z_1^{8}z_2^{20}z_6^{16} + 972z_1^{8}z_2^{4}z_4^{32} + 1944z_1^{8}z_2^{4}z_8^{16} + 6912z_1^{6}z_2^{15}z_4^{27} + 4608z_1^{6}z_2^{3}z_3^{4}z_4^{9}z_6^{2}z_{12}^{6} + 648z_1^{4}z_2^{70} + 432z_1^{4}z_2^{34}z_4^{18} + 5184z_1^{4}z_2^{22}z_6^{16} + 3456z_1^{4}z_2^{10}z_4^{6}z_6^{8}z_{12}^{4} + 3888z_1^{4}z_2^{6}z_4^{32} + 7776z_1^{4}z_2^{6}z_8^{16} + 2592z_1^{4}z_2^{2}z_4^{34} + 5184z_1^{4}z_2^{2}z_4^{2}z_8^{16} + 4608z_1^{3}z_2^{3}z_3^{15}z_6^{15} + 13824z_1^{3}z_2^{3}z_3^{3}z_6^{21} + 3072z_1^{3}z_3^{47} + 9216z_1^{3}z_3^{11}z_6^{18} + 1728z_1^{2}z_2^{17}z_4^{27} + 13824z_1^{2}z_2^{5}z_4^{9}z_6^{4}z_{12}^{6} + 10368z_1^{2}z_2z_4^{35} + 20736z_1^{2}z_2z_4^{3}z_8^{16} + 1152z_1z_2^{4}z_3^{5}z_6^{20} + 3456z_1z_2^{4}z_3z_6^{22} + 9216z_1z_2z_3^{5}z_6^{21} + 27648z_1z_2z_3z_6^{23} + 6912z_1z_3^{5}z_4^{2}z_{12}^{10} + 13824z_1z_3^{5}z_8z_{24}^{5} + 20736z_1z_3z_4^{2}z_6^{2}z_{12}^{10} + 41472z_1z_3z_6^{2}z_8z_{24}^{5} + 3174z_2^{72} + 2208z_2^{36}z_4^{18} + 6624z_2^{24}z_6^{16} + 4608z_2^{12}z_4^{6}z_6^{8}z_{12}^{4} + 3726z_2^{8}z_4^{32} + 7452z_2^{8}z_8^{16} + 2592z_2^{4}z_4^{34} + 5184z_2^{4}z_4^{2}z_8^{16} + 7224z_3^{48} + 1008z_3^{24}z_6^{12} + 9288z_3^{16}z_6^{16} + 25536z_3^{12}z_6^{18} + 2688z_3^{12}z_6^{6}z_{12}^{6} + 1296z_3^{8}z_6^{20} + 10752z_3^{6}z_6^{3}z_{12}^{9} + 32832z_3^{4}z_6^{22} + 3456z_3^{4}z_6^{10}z_{12}^{6} + 13824z_3^{2}z_6^{5}z_{12}^{9} + 38400z_4^{36} + 36864z_4^{12}z_{12}^{8} + 41472z_4^{4}z_8^{16} + 8832z_6^{24} + 6144z_6^{12}z_{12}^{6} + 39936z_8^{18} + 18432z_8^{6}z_{24}^{4} + 49152z_{12}^{12} + 24576z_{24}^{6})$.

Replacing the indeterminate z_i in $C(\mathrm{Aff}_2(Z_{12}), Z_{12}^2)$ by $1+z^i$ we see that the coefficient of z^k is the number of different k-motives in $Z_{12} \times Z_{12}$. Here are these numbers of k-motives for small values of k. $1 + z + 5z^2 + 26z^3 + 216z^4 + 2024z^5 + 27806z^6 + 417209z^7 + 6345735z^8 + 90590713z^9 + 1190322956z^{10} + \ldots$. This polynomial must be read in the following way: There is (are) exactly one 1-motive, five 2-motives, twenty six 3-motives etc. The complete list of numbers can be found in [10]. $\diamond$

Another possibility to combine tonal and rhythmical aspects will be described in the next

3.15 Example. (tone-rows) For $n \geq 3$ a *tone-row* in Z_n is a bijective mapping $f\colon Z_n \to Z_n$ where $f(i)$ is the tone which occurs in i-th position in the tone-row.

Usually two tone-rows f_1, f_2 are considered to be similar if f_1 can be constructed by transposing, inversion and retrograde inversion of f_2. Thus the similarity classes of tone-rows are the $\langle R\rangle \times D_n$ orbits on the set of all bijections on Z_n. The number of these orbits can be computed using a special version of the Cauchy-Frobenius-Lemma, special cycle index methods or the Redfield $\cap$-product (cf. [4, 27, 28]). According to this particular choice of the acting groups we can derive the following formulae:

3.16 Theorem. *For $n \geq 3$ the number of similarity classes of tone-rows in Z_n is*

$$\begin{cases} \frac{1}{4}\Big((n-1)! + (n-1)!!\Big) & \text{if } n \equiv 1 \bmod 2 \\ \frac{1}{4}\Big((n-1)! + (n-2)!!(\frac{n}{2}+1)\Big) & \text{if } n \equiv 0 \bmod 2, \end{cases}$$

where $n!!$ for $n \in \mathbb{N}$ is defined by

$$n!! = \begin{cases} n \cdot (n-2) \cdot \ldots \cdot 2 & \text{if } n \equiv 0 \bmod 2 \\ n \cdot (n-2) \cdot \ldots \cdot 1 & \text{if } n \equiv 1 \bmod 2. \end{cases}$$

Especially there are 9985920 classes of tone-rows in 12-tone music.

In [37] the author investigates $D_n \times D_n$ orbits of tone-rows as well.

A special kind of tone-rows are the *all interval-rows*. A tone-row $f\colon Z_n \to Z_n$ is an all interval-row if each possible interval occurs as the interval between two consecutive tones in the tone-row, i. e. if $\{f(i) - f(i-1) \mid i \in Z_n \setminus \{0\}\} = Z_n \setminus \{0\}$. This property is too complicated in order to be described by a weight function. But it is possible to list all these tone-rows for small values of n by using a backtrack algorithm. In [8, 11, 12] all interval-rows are counted and representatives of the similarity classes are listed. $\diamond$

So far we were investigating subsets, Cartesian products and permutations of Z_n. There is still another mathematical construction which is considered in music theory.

3.17 Example. (mosaics) A *partition* π of Z_n is a collection of subsets of Z_n, such that the empty set is not an element of π and such that for each $i \in Z_n$ there is exactly one $P \in \pi$ with $i \in P$. If π consists of exactly k subsets, then π is called a *partition of size* k. Let Π_n denote the set of all partitions of Z_n, and let $\Pi_{n,k}$ be the set of all partitions of Z_n of size k. A permutation group G of Z_n induces the following group action of G on Π_n:

$$G \times \Pi_n \to \Pi_n, \qquad (g, \pi) \mapsto g\pi := \{gP \mid P \in \pi\},$$

where $gP := \{gi \mid i \in P\}$. This action can be restricted to an action of G on $\Pi_{n,k}$. The G*-orbits* on Π_n are called G*-mosaics*. Correspondingly the G-orbits on $\Pi_{n,k}$ are called G*-mosaics of size* k. (See chapters 2 and 3 of [1] or [26].)

In [4, 6] it is explained how G-orbits of partitions can be identified with $G \times S_{\underline{n}}$-orbits on the set of all functions from Z_n to $\underline{n}$. From this construction one can easily deduce that G-mosaics of size k correspond to $G \times S_{\underline{k}}$-orbits on the set of all *surjective* functions from Z_n to $\underline{k}$. There are again several approaches how to enumerate these orbits (cf. [27, 28, 4]).

3.18 Theorem. *Let M_k be the number of $S_{\underline{k}} \times G$-orbits on $\underline{k}^{Z_n}$, i. e.*

$$M_k = \frac{1}{|G|\,|S_{\underline{k}}|} \sum_{(g,\sigma)\in G\times S_{\underline{k}}} \prod_{i=1}^{n} a_1(\sigma^i)^{a_i(g)},$$

where $a_i(g)$ or $a_i(\sigma)$ are the numbers of i-cycles in the cycle decomposition of g or σ respectively. Then the number of G-mosaics in Z_n is given by M_n, and the number of G-mosaics of size k is given by $M_k - M_{k-1}$, where $M_0 := 0$.

Here are the numbers of G-mosaics in Z_{12}:

$G\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12
C_{12}	1	179	7254	51075	115100	110462	52376	13299	1873	147	6	1
D_{12}	1	121	3838	26148	58400	56079	26696	6907	1014	96	6	1
$\mathrm{Aff}_1(Z_{12})$	1	87	2155	13730	30121	28867	13835	3667	571	63	5	1

In conclusion there are 351773 C_{12}-mosaics, 179307 D_{12}-mosaics and 93103 $\mathrm{Aff}_1(Z_{12})$-mosaics in twelve tone music.

If $\pi \in \Pi_n$ consists of λ_i blocks of size i for $i \in \underline{n}$ then π is said to be of *block-type* $\lambda = (\lambda_1, \ldots, \lambda_n)$. Now we determine the number of G-mosaics of type λ (i. e. G-patterns of partitions of block-type λ). For doing that let $\bar{\lambda}$ be any partition of type λ. (For instance $\bar{\lambda}$ can be defined such that the blocks of $\bar{\lambda}$ of size 1 are given by $\{1\}$, $\{2\}$, …, $\{\lambda_1\}$, the blocks of $\bar{\lambda}$ of size 2 are given by $\{\lambda_1+1, \lambda_1+2\}$ $\{\lambda_1+3, \lambda_1+4\}$, …, $\{\lambda_1+2\lambda_2-1, \lambda_1+2\lambda_2\}$, and so on.) According to [27] the *stabilizer* H_λ of $\bar{\lambda}$ in the symmetric group $S_{\underline{n}}$ (H_λ is the set of all permutations $\sigma \in S_{\underline{n}}$ such that $\sigma\bar{\lambda} = \bar{\lambda}$) is *similar* to the *direct sum* $\bigoplus_{i=1}^{n} S_{\underline{\lambda_i}}[S_{\underline{i}}]$ of *compositions* of symmetric groups, which is a permutation representation of the *direct product* of *wreath products* of symmetric groups. For more details see [27, 28, 17]. In other words H_λ is the set of all permutations $\sigma \in S_{\underline{n}}$, which map each block of the partition $\bar{\lambda}$ again onto a block (of the same size) of the partition. It is well known that the cycle index of the composition of two groups can be determined from the cycle indices of the two groups ([27]), so we know how to compute the cycle index of H_λ. Either from Ruch's lemma (cf. [28]) or from a direct proof (cf. [17]) it follows that the G-orbits of mosaics of type λ correspond to the double cosets $G\backslash S_{\underline{n}}/H_\lambda$ or to the mapping patterns under the group action

$$(G \times H_\lambda) \times \underline{n}_{\mathrm{bij}}^{Z_n} \to \underline{n}_{\mathrm{bij}}^{Z_n}, \qquad ((g,\sigma), f) \mapsto g \circ f \circ \sigma^{-1}.$$

3.19 Theorem. *The number of G-mosaics of type λ can be derived as*

$$\frac{1}{|G|\,|H_\lambda|} \sum_{\substack{(g,\sigma)\in G\times H_\lambda \\ z(g)=z(\sigma)}} \prod_{i=1}^{n} a_i(\sigma)!\, i^{a_i(g)},$$

where $z(g)$ and $z(\sigma)$ are the cycle types of the permutations induced by the actions of g on the set Z_n and of σ on $\underline{n}$ respectively, given in the form $(a_i(g))_{i\in\underline{n}}$ or $(a_i(\sigma))_{i\in\underline{n}}$. In other words we are summing over those pairs (g,σ) such that g and σ determine permutations of the same cycle type.

A table of the numbers of D_{12}-mosaics of type λ can be found in [17].

If n is even then a mosaic consisting of two blocks of size $n/2$ corresponds to a *trope* introduced by J. M. Hauer in [24, 25]. By applying the *power group enumeration theorem* ([21]) an explicit formula for the number of all orbits of tropes under a given group action was determined in [11]. ◇

3.20 Example. (two voices) Let ϵ be an indeterminate over Z_n. In [29] G. Mazzola suggested to use the ring $R := Z_n[\epsilon]/(\epsilon^2)$ as a model for compositions in two voices. The elements of R are all of the form $a + b\epsilon$ which should represent a pitch-class a in the first voice and a pitch-class b in the second voice. Then he motivates that the symmetry group G of R consists of all permutations of R of the form $\xi \mapsto \alpha\xi + \beta$, where α is a unit element in R and β is an arbitrary element of R. In other words it is again an affine group. All the units $\alpha \in R$ are of the form $\alpha = a + b\epsilon$ such that $a \in Z_n^*$, $b \in Z_n$. Again using the decomposition of n into powers of primes it is a very fast process to determine the cycle index of this group action. For $n = 12$ it is given by

$$C(G,R) = \frac{1}{6912}(640x_{24}^6 + 640x_{12}^{12} + 768x_8^6x_{24}^4 + 320x_8^{18} + 320x_6^{12}x_{12}^6 + 420x_6^{24} + 768x_4^{12}x_{12}^8 + 320x_4^{36} + 320x_3^{12}x_6^6x_{12}^6 + 160x_3^{12}x_6^{18} + 40x_3^{24}x_6^{12} + 20x_3^{48} + 384x_2^{12}x_4^6x_6^8x_{12}^4 + 504x_2^{24}x_6^{16} + 160x_2^{36}x_4^{18} + 210x_2^{72} + 288x_1^4x_2^{10}x_4^6x_6^8x_{12}^4 + 144x_1^4x_2^{22}x_6^{16} + 144x_1^4x_2^{34}x_4^{18} + 72x_1^4x_2^{70} + 36x_1^8x_2^{20}x_6^{16} + 18x_1^8x_2^{68} + 96x_1^{12}x_2^6x_3^8x_4^6x_6^4x_{12}^4 + 48x_1^{12}x_2^{18}x_3^8x_6^{12} + 18x_1^{16}x_2^{16}x_6^{16} + 9x_1^{16}x_2^{64} + 12x_1^{24}x_2^{12}x_3^{16}x_6^8 + 16x_1^{36}x_2^{18}x_4^{18} + 8x_1^{36}x_2^{54} + 6x_1^{48}x_3^{32} + 2x_1^{72}x_2^{36} + 1x_1^{144})$$

A two voice composition of length n could be represented as a function from $\underline{n}$ to R. When introducing a dihedral symmetry on $\underline{n}$ then the symmetry classes of these compositions are described as symmetry types of mappings. Here are the numbers of them for small values of n

n	1	2	3	4	5	6	7
$\lvert G \times D_n \backslash\backslash R^{\underline{n}} \rvert$	1	18	179	11177	999882	112.506578	13532.154277

◇

4 Construction of orbit representatives

In the last part of this article we describe a method which allows to compute complete lists of orbit -representatives which are usually called a *transversals*. We describe this method again in a more general setting. Consider a finite group action ${}_GX$ which induces a group action by 1.2 on the set of all mappings from X to a finite set Y. Without loss of generality we assume that there is a total order defined both on X and Y such that X and Y can be identified with $\underline{n}$ and $\underline{m}$ respectively and a function $f \in Y^X = \underline{m}^{\underline{n}}$ can be written as an n-tuple in the form $f = (f(1), f(2), \ldots f(n))$. There is a total order on the set of all n-tuples over $\underline{m}$ given by the *lexicographic order*. With respect to this order a function f is smaller than a function $h \in \underline{m}^{\underline{n}}$ if and only if there is an index $i_0 \in \underline{n}$ such that $f(i) = h(i)$ for all $i < i_0$ and $f(i_0) < h(i_0)$. As the *canonic representative* of the orbit $G(f)$ we choose that function $h \in G(f)$ such that $h \geq gf = f \circ \bar{g}^{-1}$ for all $g \in G$. In other words, the canonic representative h is given as

$$h = \max\{gf \mid g \in G\}.$$

If S is a subset of Y^X and $f \in Y^X$ then when writing $f \geq S$ we mean $f \geq h$ for all $h \in S$. In order to decide whether a given function $f \in \underline{m}^{\underline{n}}$ is the canonic representative of the orbit $G(f)$ we have to test whether $f \geq G(f)$, i. e. $f \geq gf$ for all $g \in G$. We call this procedure the *maximum-test* of f. For applying this test we first have to find a method to generate all elements of the acting group. Therefore we introduce the Sims-chain of a permutation group, which allows to generate all group elements as a product of the strong generators. Moreover we have to analyze the maximum-test in order to make it faster. Testing whether $f \geq gf$ for all $g \in G$ depends heavily on the order of G. Often it is possible to find certain shortcuts during this test. We will see that in many situations it is not necessary to take all group elements into account, often we can do certain jumps as will be described later. Proceeding this way, we generate all functions $f \in \underline{m}^{\underline{n}}$ starting from the biggest one and stepping from one function f to its successor, which is the function h that fulfills $f > h$ such that there exists no function k with the property $f > k > h$. Each of the functions $f \in \underline{m}^{\underline{n}}$ is tested whether it is maximal in its orbit. If it is maximal then it is a canonic representative, if not then we know that we already listed the canonic representative of the orbit $G(f)$ since we are listing the functions f according to the lexicographic order in decreasing way. Finally we will see that it is not necessary to test each function $f \in \underline{m}^{\underline{n}}$ for maximality. In certain situations it will be possible to decide from previous results that f cannot be a canonic representative. In general we are usually not interested in a complete list of all orbit representatives. For instance there are 33.608135.013344.714280.178360.727460.692224 representatives of motives in $Z_{12} \times Z_{12}$. For that reason we usually fix our attention to orbits of given weight.

Let ${}_G\underline{n}$ be a group action, and let $\bar{G}$ be the induced permutation representation of G on $\underline{n}$. The *pointwise stabilizer* or the *centralizer* of $\underline{k} \subseteq \underline{n}$ is the subgroup

$$C_{\bar{G}}(\underline{k}) := \left\{\bar{g} \in \bar{G} \mid \bar{g}(i) = i \text{ for all } i \in \underline{k}\right\}$$

of $\bar{G}$. These pointwise stabilizers form a subgroup chain of $\bar{G}$ of the form

$$\{\mathrm{id}\} = C_{\bar{G}}(\underline{n}) \leq C_{\bar{G}}(\underline{n-1}) \leq \ldots \leq C_{\bar{G}}(\underline{1}) \leq C_{\bar{G}}(\emptyset) = \bar{G}$$

which is called the *Sims-chain* of $\bar{G}$ (cf. [39, 40]). Let b the smallest element in $\underline{n}$ such that $\{\mathrm{id}\} = C_{\bar{G}}(\underline{b})$ then b is the *length* of the Sims-chain

$$\{\mathrm{id}\} = C_{\bar{G}}(\underline{b}) \leq \ldots \leq C_{\bar{G}}(\emptyset) = \bar{G}.$$

For each $i \in \underline{b}$ we have to find a complete set of left-coset representatives $\pi_j^{(i)} \in C_{\bar{G}}(\underline{i-1})$ (cf. the first item of 1.1) of $C_{\bar{G}}(\underline{i-1})/C_{\bar{G}}(\underline{i})$ which implies

$$C_{\bar{G}}(\underline{i-1}) = \bigcup_{j=1}^{r(i)} \pi_j^{(i)} C_{\bar{G}}(\underline{i}).$$

These $\pi_j^{(i)}$ are called *strong generators* of $\bar{G}$. Without loss of generality we can assume that $\pi_1^{(i)} = \mathrm{id} \in C_{\bar{G}}(\underline{i-1})$. Then each element of $\bar{G}$ can uniquely be expressed as a product of strong generators in the form

$$\pi_{j_1}^{(1)} \circ \pi_{j_2}^{(2)} \circ \ldots \circ \pi_{j_b}^{(b)}$$

for some $1 \le j_k \le r(k)$, $1 \le k \le b$.

Since $\pi_j^{(i)}(k) = k$ for $i > k$ the orbit $G(k)$ equals

$$G(k) = \left\{ \pi_{j_1}^{(1)} \circ \pi_{j_2}^{(2)} \circ \ldots \circ \pi_{j_k}^{(k)}(k) \mid j_1 \in \underline{r(1)}, \ldots, j_k \in \underline{r(k)} \right\}.$$

The elements of $\bar{G}$ can be expressed as the leaves (which are the final nodes) of the following tree.

$$\begin{array}{ccccccccccccc}
 & & \pi_1^{(1)} & & \ldots & & \ldots & & \ldots & & \pi_{r(1)}^{(1)} & & \\
 & \swarrow & & \ldots & & \searrow & \ldots & \swarrow & & & \ldots & & \searrow \\
\pi_1^{(1)} \circ \pi_1^{(2)} & & \ldots & & \pi_1^{(1)} \circ \pi_{r(2)}^{(2)} & & \ldots & & \pi_{r(1)}^{(1)} \circ \pi_1^{(2)} & & \ldots & & \pi_{r(1)}^{(1)} \circ \pi_{r(2)}^{(2)} \\
\swarrow \ldots \searrow & & \ldots & & \swarrow \ldots \searrow & & \ldots & & \swarrow \ldots \searrow & & \ldots & & \swarrow \ldots \searrow
\end{array}$$

For testing whether a function $f \in \underline{m}^{\underline{n}}$ is maximal in its orbit or not we have to compute $gf = f \circ \bar{g}^{-1}$ for all g which are the leaves of this tree. But sometimes it is possible to cut certain branches of it which makes the maximum-test much faster.

4.1 Lemma. *Assume that* $f > f \circ \pi_j^{(i)}$ *and* $f(i) > (f \circ \pi_j^{(i)})(i)$, *then*

$$f > C_{\bar{G}}(\underline{i})(f \circ \pi_j^{(i)}).$$

In general the functions $f \in \underline{m}^{\underline{n}}$ are not injective, so there exist non-trivial permutations $\pi, \sigma \in \bar{G}$ such that $f \circ \pi = f \circ \sigma$. This fact can also be used for making the maximum-test faster.

4.2 Lemma. *Let* $f \ge C_{\bar{G}}(\underline{i})(f)$ *and assume that there exists a permutation* $\sigma \in C_{\bar{G}}(\underline{i})$ *such that* $f \circ \pi_j^{(i)} \circ \sigma = f$, *then*

$$C_{\bar{G}}(\underline{i})(f \circ \pi_j^{(i)}) \le f.$$

This way R.C. Read's method of *orderly generation* can be described in the following way. (Cf. [2, 3].)

1. Determine the biggest function (of given weight) $f \in \underline{m}^{\underline{n}}$ with respect to the lexicographic order.

2. Using the maximum-test determine whether f is maximal in its orbit or not. In the case f is maximal add f to the list of canonic representatives.

3. If it is possible determine the successor of f with respect to the lexicographic order and jump to 2. Otherwise return the complete list of all representatives.

Finally we describe an iterative method for the construction of a transversal of orbits of weight $k+1$ from a transversal of orbits of weight k. We had agreed to describe the objects to be listed as characteristic functions of k-subsets. Knowing the characteristic functions of all k-subsets, a possible way of constructing all characteristic functions of $k+1$-subsets is the following augmentation: Let f be a characteristic function of a k-subset. Find the maximal i such that $f(i) = 1$. The *augmentation* $\mathcal{A}(f)$ consists of all those characteristic functions h such that $h(j) = f(j)$ for $1 \leq j \leq i$ and there is exactly one $j > i$ such that $h(j) = 1$.

The next theorem is a special case of *Read's recursion algorithm* (cf. [35]). In this context it is interesting to mention that during the maximum-test we are learning both from positive and from negative results. (For more details see [20].)

4.3 Theorem. *If f is not the canonic representative of the G-orbit $G(f)$ then no h in the augmentation $\mathcal{A}(f)$ is the canonic representative of its orbit $G(h)$.*

4.4 Example. (motive representatives) The following list is a transversal of motives in $Z_3 \times Z_3$. It is computed by using Read's recursion. For *labelling* the elements of $Z_m \times Z_n$ the bijection $Z_m \times Z_n \ni (i,j) \mapsto i + m \cdot j + 1 \in \underline{mn}$ was used. We first test characteristic functions of weight 1 for maximality. Whenever we find a canonic representative we apply the maximum-test to all characteristic functions in its augmentation. Since we list all characteristic functions of given weight in decreasing way the characteristic function $(1,0,0,0,0,0,0,0,0)$ is tested first. It is the biggest characteristic function of weight 1 and it is the first representative m_1 in our list. The next lines contain elements m_i of the augmentation $\mathcal{A}(m_{i-1})$ for $2 \leq i \leq 9$. There exist no functions in the augmentation of m_9 and no more functions in the augmentation of m_8. All further functions in the augmentations of m_7 and m_6 are not canonic, so finally we find the next canonic representative m_{10} as an element in the augmentation of m_5, hence it is the characteristic function of a 6-motive. The last three canonic representatives turn out to be characteristic functions of 5-, 3- and 4-motives.

$$
\begin{array}{rlrl}
m_1 := & (1,0,0,0,0,0,0,0,0) & m_2 := & (1,1,0,0,0,0,0,0,0) \\
m_3 := & (1,1,1,0,0,0,0,0,0) & m_4 := & (1,1,1,1,0,0,0,0,0) \\
m_5 := & (1,1,1,1,1,0,0,0,0) & m_6 := & (1,1,1,1,1,1,0,0,0) \\
m_7 := & (1,1,1,1,1,1,1,0,0) & m_8 := & (1,1,1,1,1,1,1,1,0) \\
m_9 := & (1,1,1,1,1,1,1,1,1) & m_{10} := & (1,1,1,1,1,0,1,0,0) \\
m_{11} := & (1,1,1,1,0,0,1,0,0) & m_{12} := & (1,1,0,1,0,0,0,0,0) \\
m_{13} := & (1,1,0,1,1,0,0,0,0) & &
\end{array}
$$

After realizing that only m_3 and m_{12} are canonic representatives in the augmentation of m_2 and that there are no canonic representatives different from m_2 in the augmentation of m_1 we have to test the next characteristic functions of weight 1 for maximality. They all turn out not to be maximal in their orbits.

Of course the empty set is always the canonic representative of the unique 0-motive. ◇

More details about the construction of representatives of motives can be found in [14, 13, 16].

In situations when the list of orbit representatives is too long, or when the order of the acting group is big such that the computation of a transversal takes too long time, then it is useful and it makes sense to apply probabilistic methods for generating orbit representatives uniformly at random. I. e., for any given orbit the probability that a generated representative belongs to this orbit does not depend on the special choice of the orbit. In other words, for all orbits this probability is the same and it is given as the fraction 1 divided by the number of different orbits. This way it is possible to generate in very short time huge lists of unprejudiced representatives. The method of this random generation is known as the *Dixon-Wilf-algorithm* (cf. [7, 27, 28]) which was originally designed for the random generation of linear graphs. It can also be applied for uniform generation of objects from music theory.

Most of these algorithms for computing cycle indices, counting G-orbits, for the construction of transversals and for the random generation of orbit representatives are implemented in the computer algebra system SYMMETRICA [41]. Moreover special C-code can be loaded from the author's home page [18].

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