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Some applications of functional equations in astronomy

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Abstract

The *mean sun* is an artificial construction in astronomy which is used to define *civil time*. It has the property that local mean time, dependent on the geographical position on the earth's surface, behaves regularly, which is not true for the (true) local time, which depends on the real sun. We present a functional equation, which reflects this regularity property of the mean sun, and show that the solution to this equation essentially is the mean sun. Moreover we discuss the problem of geostationary satellites.

Local solar time is measured by a sundial. When the center of the sun is on an observer's meridian, the observer's local solar time is zero hours (noon). Because the earth moves with varying speed in its orbit at different times of the year and because the plane of the earth's equator is inclined to its orbital plane, the length of the solar day is different depending on the time of year. It is more convenient to define time in terms of the average of local solar time. Such time, called mean solar time, may be thought of as being measured relative to an imaginary sun (the mean sun) that lies in the earth's equatorial plane and about which the earth orbits with constant speed. Every mean solar day is of the same length.[1]

In mathematical terms this means the following: Let $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be the canonical base of $\mathbb{R}^3$ and consider the origin to be positioned at the center of the earth. Then, choosing the initial time and the length of a (mean) solar day appropriately, the law of motion of the mean sun is given by

$$x_m(s) = (-\cos s, \sin s, 0)^\top. \qquad (1)$$

The position of the real or of the mean sun, or rather the vector pointing from the observer's position to the position of the sun then can be viewed as the hand of a sundial

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[1] http://www.infoplease.com/ce6/society/A0845838.html

indicating the local time. More exactly speaking this hand is given by the coordinate vector of this vector when expressed in the observer's local positional frame. So let

$$P(\lambda, \varphi) := r(\cos\varphi\cos\lambda, \cos\varphi\sin\lambda, \sin\varphi)^\top,$$

the observer's position on the surface of the earth (r the radius of the earth, λ the geographical longitude and φ the latitude of that point). Then the local coordinate frame is given by the unit vectors indicating the directions *East*, *North*, and to the *zenith*. The transformation matrix transforming the canonical base $\mathbf{e}$ to the local system, denoted by $\mathbf{e}(\lambda, \varphi)$ is given by

$$M_{\mathbf{e},\mathbf{e}(\lambda,\varphi)} := M(\lambda, \varphi) := \begin{pmatrix} -\sin\lambda & -\sin\varphi\cos\lambda & \cos\varphi\cos\lambda \\ \cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi\sin\lambda \\ 0 & \cos\varphi & \sin\varphi \end{pmatrix}. \tag{2}$$

Thus, if we dispose of the earth's radius (which is very small compared to the distance of the earth's origin to the position of the sun), the local sundial is given by

$$C(\lambda, \varphi)(x_m(s)) := M(\lambda, \varphi)^{-1} x_m(s).$$

We say that it is noon at $P(\lambda, \varphi)$, if the zenith component of $C(\lambda, \varphi)(x_m(s))$ is maximal.

Local mean time (or local civil time) has the convenient property, that,

> if it is noon at Greenwich, it is 1 p.m. at a position 15° east of Greenwich, or more general:
>
> $$C(\lambda + \mu, \varphi)(x_m(s)) = C(\lambda, \varphi)(x_m(\mu + s)).$$

Thus x_m satisfies the functional equation

$$M(\lambda + \mu, \varphi)^{-1} y(s) = M(\lambda, \varphi)^{-1} y(\mu + s), \quad \lambda, \mu, s \in \mathbb{R}, -\pi/2 < \varphi < \pi/2. \tag{3}$$

The solution of (3) is quite simple.

Theorem 1. *Let $y : \mathbb{R} \to \mathbb{R}^3$ be given such that $y(s) \neq 0$ for all real s and such that (3) holds for all $\lambda, \mu, s \in \mathbb{R}$ and* some *$\varphi \in]-\pi/2, \pi/2[$. Let furthermore*

$$U(\mu) := \begin{pmatrix} \cos\mu & -\sin\mu & 0 \\ \sin\mu & \cos\mu & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

Then there is some constant vector $c \in \mathbb{R}^3$, $c \neq 0$, such that

$$y(s) = U(-s)c, \quad s \in \mathbb{R}. \tag{5}$$

Conversely, every y given by (5) satisfies (3).

Proof. (3) is equivalent to

$$M(\lambda,\varphi)M(\lambda+\mu,\varphi)^{-1}y(s)=y(s+\mu),\quad s,\lambda,\mu\in\mathbb{R}.$$

But one easily can see that

$$M(\lambda,\varphi)M(\lambda+\mu,\varphi)^{-1}=M(\lambda,\varphi)M(\lambda+\mu,\varphi)^{\top}=U(-\mu).$$

Thus (3) is equivalent to

$$U(-\mu)y(s)=y(s+\mu),\quad s,\mu\in\mathbb{R},$$

which for $s=0$ and $c=y(0)$ gives (after replacing μ by s) the desired result. The remaining part is obvious. □

Remark 1. The result of this theorem says, that every object behaving like the mean sun with respect to (3) is given by a clockwise rotation with center at some point of the earth's axis and rotational velocity of one full rotation per day.

Now we come to a problem for which it is important to take the earth's radius into account. Satellite dishes point to positions given by geostationary satellites, and thus have the desired property of fixed positions invariant with respect to time. Thus, if $y(s)$ is the position of the satellite at time s, we require the following: the direction vector belonging to $\overrightarrow{P(\lambda,\varphi)y(s)}$, i.e., the unit vector corresponding to this vector, when expressed in local coordinates, should be independent of s:

$$M(\lambda,\varphi)^{-1}\frac{y(s)-P(\lambda,\varphi)}{\|y(s)-P(\lambda,\varphi)\|}=\text{const.}=c(\lambda,\varphi),\quad s\in\mathbb{R}. \tag{6}$$

Of course we may suppose that r, the radius of the earth, equals 1. Then we get the following.

Theorem 2. *If for $y:\mathbb{R}\to\mathbb{R}^3$, $\|y(s)\|\neq 1$ for all $s\in\mathbb{R}$, the expression*

$$M(\lambda,\varphi)^{-1}\frac{y(s)-P(\lambda,\varphi)}{\|y(s)-P(\lambda,\varphi)\|}$$

is independent of s for at least three different points $P(\lambda_i,\mu_i)$, $i=1,2,3$, then y itself is constant. Moreover every constant vector of length different from 1 satisfies (6).

Proof. (6) implies

$$y(s)\in P(\lambda,\varphi)+\mathbb{R}\,u(\lambda,\varphi),$$

for some unit vector $u(\lambda,\varphi)$. Thus

$$y(s)\in g_i:=P(\lambda_i,\varphi_i)+\mathbb{R}\,u(\lambda_i,\varphi_i)$$

for $i=1,2,3$, where the g_i are straight lines in $\mathbb{R}^3$.

If, say, $u(\lambda_i, \varphi_i)$ and $u(\lambda_j, \varphi_j)$ are linearly independent, the intersection of g_i and g_j contains at most one point. Since $y(s) \in g_i \cap g_j$ this implies that $y(s)$ equals the point of intersection of those two lines for all s. Thus y has to be constant.

The (only) other possibility left means that the two vectors $u(\lambda_1, \varphi_1)$, $u(\lambda_2, \varphi_2)$ as well as the vectors $u(\lambda_1, \varphi_1)$ and $u(\lambda_3, \varphi_3)$ are linearly dependent. But then all three lines g_1, g_2, g_3 coincide, since they have a point in common and since they have proportional direction vectors. Thus the three different points $P(\lambda_1, \varphi_1)$, $P(\lambda_2, \varphi_2)$, $P(\lambda_3, \varphi_3)$ of the same length are collinear, which is impossible. □

Now, as we have been successful with taking into account the radius of the earth in solving this problem, we want to reconsider, in the same spirit, the original problem. We again suppose this radius r to be of length 1 and we suppose for y that $\|y(s)\| \neq 1$ for all s. The functional equation then reads as

$$M(\lambda+\mu, \varphi)^{-1} \frac{y(s) - P(\lambda+\mu, \varphi)}{\|y(s) - P(\lambda+\mu, \varphi)\|} = M(\lambda, \varphi)^{-1} \frac{y(s+\mu) - P(\lambda, \varphi)}{\|y(s+\mu) - P(\lambda, \varphi)\|}, \tag{7}$$

for all $\lambda, \mu, s \in \mathbb{R}$ and all $-\pi/2 < \varphi < \pi/2$.

Theorem 3. *Assume that (7) holds for all $\lambda, \mu, s \in \mathbb{R}$ and* some $-\pi/2 < \varphi < \pi/2$, *where $y : \mathbb{R} \to \mathbb{R}^3$ satisfies $\|y(s)\| \neq 1$ for all s. Then there is some constant vector $c \in \mathbb{R}^3$, $\|c\| \neq 1$, such that*

$$y(s) = U(-s)c, \quad s \in \mathbb{R},$$

where U is given by (4). *Moreover any y of the above form solves (7) for all $\lambda, \mu, s \in \mathbb{R}$ and* all $-\pi/2 < \varphi < \pi/2$.

Proof. Fixing $\varphi \in]-\pi/2, \pi/2[$ we put $c(s, \lambda) := \|y(s) - P(\lambda, \varphi)\|$ and $u(s, \lambda) := (y(s) - P(\lambda, \varphi))/c(s, \lambda)$. This means

$$y(s) = P(\lambda, \varphi) + c(s, \lambda)u(s, \lambda), \tag{8}$$

where $c(s, \lambda) > 0$ and $u(s, \lambda)$ is a unit vector. Then (7) and the relation at the beginning of the proof of the first theorem imply

$$u(s, \lambda+\mu) = U(\mu)u(s+\mu, \lambda). \tag{9}$$

Using (9) with $\lambda = 0$ gives

$$u(s, \mu) = U(\mu)u(s+\mu, 0). \tag{10}$$

Moreover $P(\lambda, \varphi) = U(\lambda)P(0, \varphi)$ and $U(\lambda+\mu) = U(\lambda)U(\mu)$. Thus (8) reads as

$$y(s) = U(\lambda)\left[P(0, \varphi) + c(s, \lambda)u(s+\lambda, 0)\right]. \tag{11}$$

We also have, by (8) for $\lambda = 0$

$$y(s) = P(0, \varphi) + c(s, 0)u(s, 0). \tag{12}$$

For $\lambda \in \mathbb{R}$ we consider the straight line ℓ_λ determined by $y(0)$ and $P(\lambda, \varphi)$.

$$\ell_\lambda := y(0) + \mathbb{R} \cdot (P(\lambda, \varphi) - y(0)) = P(\lambda, \varphi) + \mathbb{R} \cdot u(0, \lambda).$$

Note that any ℓ_λ may contain at most two points of unit length, one of them being $P(\lambda, \varphi)$.

Thus for any fixed real t there is some λ such that $P(\lambda, \varphi) \notin \ell_t$. Then obviously $y(0) \in \ell_\lambda \cap \ell_t$; and it is the *only* point in that intersection of the two lines ℓ_λ and ℓ_t, since $P(\lambda, \varphi) \notin \ell_t$:

$$\ell_\lambda \cap \ell_t = \{y(0)\}. \tag{13}$$

On the other hand we have by (8) with λ replaced by $\lambda - t$ and s replaced by t

$$\begin{aligned} y(t) &= P(\lambda - t, \varphi) + c(t, \lambda - t)u(t, \lambda - t) \in \\ &\in U(-t)\,[P(\lambda, \varphi) + \mathbb{R}\, U(t)u(t, \lambda - t)] = \\ &= U(-t)\,[P(\lambda, \varphi) + \mathbb{R}\, u(0, \lambda)] = U(-t)\ell_\lambda, \end{aligned}$$

where we also used (9) with μ replaced by t, s by 0 and λ by $\lambda - t$. We also have

$$\begin{aligned} y(t) &\in P(0, \varphi) + \mathbb{R}\ (y(t) - P(0, \varphi)) = P(0, \varphi) + \mathbb{R}\, u(t, 0) = \\ &= U(-t)\,[U(t)\,[P(0, \varphi) + \mathbb{R}\, u(t, 0)]] = U(-t)\,[P(t, \varphi) + \mathbb{R}\, U(t)u(t, 0)] = \\ &= U(-t)\ell_t, \end{aligned}$$

where we used (10) with $s = 0$ and $\mu = t$.

Thus

$$U(t)y(t) \in \ell_\lambda \cap \ell_t = \{y(0)\} \quad \text{or} \quad y(t) = U(-t)y(0),$$

as desired. (The second part of the theorem is obvious.) □

Remark 2. A good description on the relation between the real and the mean sun may be found in [1], one may also consult [4]. Generalizations of the questions treated here, especially connected with the results of the first theorem, may be found in [2] and [3].

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