ITERATION THEORY (ECIT '04)
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Grazer Math. Ber., ISSN 1016–7692
Bericht Nr. **350** (2006), 96-121

Covariant embeddings of the linear functional equation with respect to an iteration group in the ring of complex formal power series

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Abstract

Let $a(x)$, $b(x)$, $p(x)$ be formal power series in the indeterminate x over $\mathbb{C}$ such that $\operatorname{ord} a(x) = 0$, $\operatorname{ord} p(x) = 1$ and $p(x)$ is embeddable into an iteration group $(\pi(s,x))_{s\in\mathbb{C}}$ in $\mathbb{C}\,[\![x]\!]$. By a covariant embedding of the linear functional equation

$$\varphi(p(x)) = a(x)\varphi(x) + b(x), \tag{L}$$

(for the unknown series $\varphi(x) \in \mathbb{C}\,[\![x]\!]$) with respect to $(\pi(s,x))_{s\in\mathbb{C}}$ we understand families $(\alpha(s,x))_{s\in\mathbb{C}}$ and $(\beta(s,x))_{s\in\mathbb{C}}$ of formal power series which satisfy a system of cocycle equations and boundary conditions such that

$$\varphi(\pi(s,x)) = \alpha(s,x)\varphi(x) + \beta(s,x), \qquad s \in \mathbb{C}, \tag{Ls}$$

holds true for all solutions φ of (L). In this paper we present a complete solution of this problem and we demonstrate how earlier results concerning covariant embeddings with respect to analytic iteration groups can be derived from these more general results.

1 The problem of covariant embeddings

Let $\mathbb{C}\,[\![x]\!]$ be the ring of formal power series in the indeterminate x with complex coefficients. Consider the linear functional equation

$$\varphi(p(x)) = a(x)\varphi(x) + b(x), \tag{L}$$

Mathematics Subject Classification 2000: Primary: 39B12, 39B50; Secondary: 13F25.
Keywords and phrases: Linear functional equation; covariant embedding; cocycle equations; iteration groups without regularity condition; ring of formal power series.

where $p(x), a(x), b(x) \in \mathbb{C}\,[\![x]\!]$ are given formal power series, and $\varphi(x) \in \mathbb{C}\,[\![x]\!]$ should be determined by the functional equation. We always assume that

$$p(x) = \rho x + c_2x^2 + c_3x^3 + \ldots = \rho x + \sum_{n\geq 2} c_nx^n$$

with multiplier $\rho \neq 0$, and

$$a(x) = a_0 + a_1x + a_2x^2 + \ldots = \sum_{n\geq 0} a_nx^n$$

with $a_0 \neq 0$.

L. Reich introduced in [17] the notion of a covariant embedding of (L) with respect to an iteration group.

Let $(\pi(s,x))_{s\in\mathbb{C}}$ *be an iteration group so that* $\pi(t_0,x) = p(x)$ *for some* $t_0 \in \mathbb{C}$*,* $t_0 \neq 0$*. The linear functional equation* (L) *has a covariant embedding with respect to* $(\pi(s,x))_{s\in\mathbb{C}}$*, if there exist families* $(\alpha(s,x))_{s\in\mathbb{C}}$ *and* $(\beta(s,x))_{s\in\mathbb{C}}$ *of formal power series with coefficient functions* α_n *and* β_n *of the form*

$$\alpha(s,x) = \sum_{n\geq 0} \alpha_n(s)x^n, \qquad \beta(s,x) = \sum_{n\geq 0} \beta_n(s)x^n,$$

such that α *and* β *satisfy both the boundary conditions*

$$\alpha(0,x) = 1, \qquad \beta(0,x) = 0, \tag{B1}$$

$$\alpha(t_0,x) = a(x), \qquad \beta(t_0,x) = b(x), \tag{B2}$$

and the cocycle equations

$$\alpha(s+t,x) = \alpha(s,x)\alpha(t,\pi(s,x)), \qquad s,t \in \mathbb{C}, \tag{Co1}$$

$$\beta(s+t,x) = \beta(s,x)\alpha(t,\pi(s,x)) + \beta(t,\pi(s,x)), \qquad s,t \in \mathbb{C}, \tag{Co2}$$

and

$$\varphi(\pi(s,x)) = \alpha(s,x)\varphi(x) + \beta(s,x), \qquad s \in \mathbb{C}, \tag{Ls}$$

holds for all solutions $\varphi(x)$ *of* (L) *in* $\mathbb{C}\,[\![x]\!]$*.*

Such embeddings were studied in a much more general setting by Z. Moszner in [16] and for real-valued functions by G. Guzik in [7] and [9]. The first cocycle equation is also studied in [8] and [10]. It also appears as the triangular equation for instance in [1]. For the theory of linear functional equations we refer the reader to [14] and to [15].

A foundation of the basic calculations with formal power series can be found in [11] or [2]. If $\psi(x) \in \mathbb{C}[\![x]\!]$ is of the form $\psi(x) = \sum_{n\geq k} \psi_n x^n$ with $\psi_k \neq 0$, then k is the order of ψ, which will be indicated as $\operatorname{ord}\psi(x) = k$. Furthermore, the notion of *congruence modulo* x^r will be useful. We write $\varphi \equiv \psi \bmod x^r$ for formal power series $\varphi(x), \psi(x) \in \mathbb{C}[\![x]\!]$ if x^r is a divisor of the difference $\varphi(x) - \psi(x)$. In other words $\varphi(x) - \psi(x) = 0$, or its order is at least r.

In [4] we showed how to motivate covariant embeddings of (L). There we also described the analytic solutions of the problem of covariant embeddings with respect to analytic iteration groups in the generic cases. The non generic cases were studied in [5] based on some results from [3]. In [6] the general solution of the system ((Co1),(Co2),(B1)) where π is an iteration group without any regularity condition was derived. In this paper it was thoroughly discussed how to extend the results of [4]. Finally, in the present paper we completely describe the general solution of the problem of covariant embeddings. We show that all covariant embeddings can be found among the solutions of ((Co1),(Co2),(B1)) by solving suitable linear functional equations which are closely related to the given linear equation (L).

Remark 1. By a simple transformation of the "time parameter" t it is always possible to assume in (B2) that $t_0 = 1$.

How to solve the embedding problem? Remark 2 describes roughly how we will proceed.

Remark 2.

1. If there is no iteration group π such that $\pi(1, x) = p(x)$, then there is no covariant embedding of (L).

   Assume that π is an iteration group with $\pi(1, x) = p(x)$.

2. If there is no solution of the system ((Co1),(Co2),(B1),(B2)), then there is no covariant embedding of (L).

3. If there are solutions (α, β) of ((Co1),(Co2),(B1),(B2)), and no solutions of (L), then each (α, β) is a covariant embedding of (L).

4. Assume that the linear functional equation (L) has a solution φ, and α is a solution of ((Co1),(B1),(B2)). If there is exactly one β which satisfies together with α the cocycle equations and the boundary conditions, then (α, β) is a covariant embedding of (L) (cf. [4, Theorem 4.1]).

5. If (α, β) is a solution of ((Co1),(Co2),(B1),(B2)) and β is not uniquely determined by α, (Co2), (B1), and (B2), then we have to check whether each solution of (L) also satisfies (Ls).

2 Iteration groups

W. Jabłoński and L. Reich succeeded in [12, 13] with the classification of all iteration groups π (without any regularity conditions). In the same way as with analytic iteration groups they distinguished two types of iteration groups:

Iteration groups of type I are of the form

$$\pi(s, x) = \pi_1(s)x + \sum_{\ell \geq 2} P_\ell(\pi_1(s))x^\ell, \qquad s \in \mathbb{C},$$

where $\pi_1 : \mathbb{C} \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ is a generalized exponential function, $\pi_1 \neq 1$, and where $P_\ell(y) \in \mathbb{C}[y]$ is a polynomial of formal degree equal to ℓ. At the moment we do not need the detailed structure, the general form, and the universal character (depending on certain parameters) of these polynomials P_ℓ. Iteration groups of type I have very simple normal forms, since for each π of type I there exists some $S(x) = x + s_2x^2 + \ldots \in \mathbb{C}\,[\![x]\!]$ such that

$$\pi(s, x) = S^{-1}(\pi_1(s)S(x)), \qquad s \in \mathbb{C}.$$

The analytic iteration groups of type I are obtained for $\pi_1(s) = e^{\mu s}$ with $\mu \neq 0$, a regular exponential function. In [4, Theorem 1.3] and [6, Remark 4, Remark 12] it was shown that it is enough to solve the cocycle equations or the embedding problem for these normal forms. Then the solutions can easily be rewritten for general iteration groups of type I which are not in normal form by applying the backwards transformation.

Iteration groups of type II can be described as

$$\pi(s, x) = x + \pi_k(s)x^k + \sum_{\ell > k} P_\ell(\pi_k(s))x^\ell, \qquad s \in \mathbb{C},$$

where $k \geq 2$, $\pi_k : \mathbb{C} \to \mathbb{C}$ is an additive function, $\pi_k \neq 0$, and where $P_\ell(y) \in \mathbb{C}[y]$ is a polynomial of formal degree equal to $\lfloor \frac{\ell-1}{k-1} \rfloor$. Here, too, we do not need the detailed structure, the general form, and the universal character of these polynomials P_ℓ. But the reader should be aware of the fact that even though we use the same letters these polynomials are different from the polynomials occurring in the description of iteration groups of type I. The analytic iteration groups of type II are obtained for $\pi_k(s) = c_k s$ with $c_k \in \mathbb{C}^*$. In comparison with iteration groups of type I, these groups do not have so simple normal forms, and for that reason we do not use normal forms in our approach.

3 Covariant embeddings with respect to iteration groups of type I

For solving the problem of covariant embeddings we need some results on the linear functional equation. We consider the particular version

$$\varphi(\rho x) = a(x)\varphi(x) + b(x) \tag{Lρ}$$

of (L) for $\rho \neq 1$ and its homogeneous form

$$\varphi(\rho x) = a(x)\varphi(x). \tag{Lρ_{h}}$$

By assumption $a_0 \neq 0$, thus it is possible to divide by a_0. We denote $a(x)/a_0$ by $\hat{a}(x)$ satisfying $\hat{a}(x) \equiv 1 \bmod x$. Using this series instead of $a(x)$ we also consider the functional equations

$$\varphi(\rho x) = \hat{a}(x)\varphi(x) + \hat{b}(x) \tag{$\hat{\mathrm{L}}\rho$}$$

and

$$\varphi(\rho x) = \hat{a}(x)\varphi(x). \tag{$\hat{\mathrm{L}}\rho_{\mathrm{h}}$}$$

Theorem 3. *Consider the linear functional equation* (Lρ).

1. *If $\rho^n \neq a_0$, $n \geq 0$, then there is a unique solution $\varphi(x) = \sum_{n\geq 0} \varphi_n x^n$ of* (Lρ), *where*
$$\varphi_n = \frac{\sum_{r=1}^{n} a_r \varphi_{n-1} + b_n}{\rho^n - a_0}, \qquad n \geq 0. \tag{1}$$

2. *If $\rho^{n_0} = a_0$ and ρ is not a complex root of* 1*, then $\rho^n \neq a_0$ for $n \neq n_0$, and* (Lρ) *has a solution if and only if $b_{n_0} = 0$. In this case, the coefficients φ_n are uniquely determined by* (1) *for $n \neq n_0$, whereas φ_{n_0} remains undetermined. Conversely, if $b_{n_0} = 0$, then any choice of $\varphi_{n_0} \in \mathbb{C}$ yields a solution $\varphi(x)$ of* (Lρ).

3. *If $\rho^{n_0} = a_0$ and ρ is a complex root of* 1 *of order $j_0 > 1$, then let*
$$k_0 := \min\{n \in \mathbb{N}_0 \mid \rho^n = a_0\}$$
$$K := \{n \in \mathbb{N}_0 \mid \rho^n = a_0\} = k_0 + \mathbb{N}_0 j_0.$$
The homogeneous linear functional equation (Lρ_{h}) *has a nontrivial solution if and only if* ($\hat{\mathrm{L}}\rho_{\mathrm{h}}$) *has a nontrivial solution. This is equivalent to*
$$\prod_{\ell=0}^{j_0-1} \hat{a}(\rho^\ell x) = 1. \tag{2}$$

Let

$$A^{(r)}(x) := \sum_{n=0}^{j_0-1} \rho^{-nr} \prod_{\ell=0}^{n-1} a(\rho^\ell x), \qquad 0 \leq r < j_0.$$

Then the general solution of $(\mathrm{L}\rho_\mathrm{h})$ *is given by*

$$\varphi(x) = [A^{(k_0)}(x)]^{-1} j_0 \sum_{t\geq 0} \varphi^*_{tj_0+k_0} x^{tj_0+k_0},$$

where $\sum_{t\geq 0} \varphi^*_{tj_0+k_0} x^{tj_0+k_0} \in \mathbb{C}[\![x]\!]$ *is arbitrary. (Then* $\varphi_{tj_0+k_0} = \varphi^*_{tj_0+k_0}$ *for* $t \geq 0$.*)*

Assume that $(\mathrm{L}\rho_\mathrm{h})$ *has a nontrivial solution and let*

$$B(x) := \sum_{k=0}^{j_0-1} \frac{b(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)}.$$

Then $(\mathrm{L}\rho)$ *has a solution if and only if*

$$B(x) = 0. \tag{3}$$

Under these assumptions the general solution $\varphi(x)$ *of* $(\mathrm{L}\rho)$ *is given by*

$$[A^{(k_0)}(x)]^{-1} \left(j_0 \sum_{t\geq 0} \varphi^*_{tj_0+k_0} x^{tj_0+k_0} - \sum_{n=1}^{j_0-1} \rho^{-nk_0} \prod_{\ell=0}^{n-1} a(\rho^\ell x) \sum_{k=0}^{n-1} \frac{b(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)} \right),$$

where $\sum_{t\geq 0} \varphi^*_{tj_0+k_0} x^{tj_0+k_0} \in \mathbb{C}[\![x]\!]$ *is arbitrary. (Then* $\varphi_{tj_0+k_0} = \varphi^*_{tj_0+k_0}$ *for* $t \geq 0$.*)*

The first two assertions can be verified by easy calculations. For a proof of the third assertion see [3]. The general solution of $(\mathrm{L}\rho_\mathrm{h})$ is given in [3, Theorem 10]. If $a_0 = 1$ then the general solution of $(\mathrm{L}\rho)$ specializes to the formula given in [3, Theorem 15]. If $a_0 = \rho^{k_0}$ with $k_0 > 0$, then it is possible to rewrite the proof of [3, Theorem 15] to verify the formula above.

In [6, Corollary 3] the general solution of ((Co1),(B1)) is given by

$$\alpha(s,x) = \alpha_0(s) \frac{E(\pi(s,x))}{E(x)}, \qquad s \in \mathbb{C}, \tag{4}$$

where α_0 is a generalized exponential function and $E(x) \in \mathbb{C}[\![x]\!]$ with $E(x) \equiv 1 \bmod x$. Dividing both sides by $\alpha_0(s)$ and assuming that the iteration group is in normal form, we obtain

$$\hat{\alpha}(s,x) = \frac{E(\pi_1(s)x)}{E(x)}.$$

Therefore, α satisfies (B2) if and only if $\alpha_0(1) = a_0$ and $\hat{\alpha}$ satisfies

$$\hat{\alpha}(1, x) = \hat{a}(x)$$

which is the same as

$$E(\rho x) = \hat{a}(x)E(x) \tag{5}$$

where $\rho = \pi_1(1) \notin \{0, 1\}$.

Now we analyze how to describe all solutions (α, β) of ((Co1),(Co2),(B1),(B2)).

Case 1: First we assume that ρ is not a complex root of 1. Then ($L\rho_h$) always has a solution. This follows from Theorem 3 since (5) is a homogeneous equation. Hence by (4) we always find a family $(\alpha(s, x))_{s\in\mathbb{C}}$ satisfying ((Co1),(B1),(B2)).

Let $(\alpha(s, x))_{s\in\mathbb{C}}$ be a solution of ((Co1),(B1),(B2)). In order to describe the solutions (α, β) of (((Co1),(Co2),(B1),(B2)) we discuss two cases.

Case 1.1: If $a_0 \neq \rho^n$ for all $n \geq 0$, then according to [6, Theorem 11] the family $(\beta(s, x))_{s\in\mathbb{C}}$ is a solution of (Co2) if and only if

$$\beta(s, x) = \alpha_0(s)E(\pi(s, x))\Big(F(x) - \alpha_0(s)^{-1}F(\pi(s, x))\Big), \qquad s \in \mathbb{C}, \tag{6}$$

where $F(x) \in \mathbb{C}[\![x]\!]$. Therefore, β is a solution of ((Co2),(B1),(B2)) if and only if

$$a_0E(\rho x)\Big(F(x) - a_0^{-1}F(\rho x)\Big) = b(x),$$
$$-E(\rho x)F(\rho x) = -a(x)E(x)F(x) + b(x).$$

Hence, with $G(x) = -E(x)F(x)$ if and only if

$$G(\rho x) = a(x)G(x) + b(x). \tag{7}$$

Using Theorem 3.1 it is always possible to find a solution G, whence we can find some $F(x) \in \mathbb{C}[\![x]\!]$ and a family $(\beta(s, x))_{s\in\mathbb{C}}$. Moreover, G and therefore F and also β are uniquely determined.

Case 1.2: Next we consider $a_0 = \rho^{n_0}$. **Case 1.2.1:** First we assume that $\alpha_0 \neq \pi_1^{n_0}$, then $\alpha_0 \neq \pi_1^n$ for all $n \geq 0$. According to [6, Theorem 11] the family $(\beta(s, x))_{s\in\mathbb{C}}$ is given by (6). The same computations as in Case 1.1 show that β is a solution of ((Co2),(B1),(B2)) if and only if (7) has a solution for $G(x) = -E(x)F(x)$. By Theorem 3.2 this means that $b_{n_0} = 0$. If $b_{n_0} = 0$, then any choice of the coefficient $G_{n_0} \in \mathbb{C}$ yields a solution $G(x) \in \mathbb{C}[\![x]\!]$ which allows to determine all solutions $(\beta(s, x))_{s\in\mathbb{C}}$ of ((Co2),(B1),(B2)).

Case 1.2.2: If $\alpha_0 = \pi_1^{n_0}$, then according to [6, Theorem 11] the general solution of the system ((Co1),(Co2),(B1)) is given by (α, β) with

$$\beta(s, x) = \alpha_0(s)E(\pi(s, x))\Big(A(s)x^{n_0} + F(x) - \alpha_0(s)^{-1}F(\pi(s, x))\Big), \qquad s \in \mathbb{C}, \tag{8}$$

where A is an arbitrary additive function and $F(x) \in \mathbb{C}[\![x]\!]$. The family $(\beta(s,x))_{s\in\mathbb{C}}$ satisfies (B2) if and only if

$$a_0 E(\rho x)\Big(A(1)x^{n_0} + F(x) - a_0^{-1}F(\rho x)\Big) = b(x),$$
$$a_0 E(\rho x)A(1)x^{n_0} + a(x)E(x)F(x) - E(\rho x)F(\rho x) = b(x),$$
$$G(\rho x) = a(x)G(x) + b(x) - a_0 E(\rho x)A(1)x^{n_0}, \qquad (9)$$

where $G(x) = -E(x)F(x)$. Since $\operatorname{ord}(a_0E(\rho x)A(1)x^{n_0}) \geq n_0$, it follows from Theorem 3.2 that by choosing $A(1)$ suitably the linear functional equation for $G(x)$ has a solution. Namely, if $A(1)$ is chosen in such a way that $b_{n_0} = a_0A(1)$, then any choice of G_{n_0} yields a solution β of ((Co2),(B1),(B2)). The family β does not depend on the particular value of F_{n_0} which is uniquely determined by G_{n_0}. However, there exist different additive functions A with $A(1) = b_{n_0}/a_0$.

Case 2: Now we assume that ρ is a complex root of 1 of order j_0. Then α satisfies the boundary condition (B2) if and only if $\alpha(1) = a_0$ and $E(x)$ is a solution of $(\hat{\mathrm{L}}\rho_{\mathrm{h}})$. **Case 2.1:** If $\prod_{\ell=0}^{j_0-1}\hat{a}(\rho^\ell x) \neq 1$, then there is no solution α of ((Co1),(B1),(B2)).

Case 2.2: If $\prod_{\ell=0}^{j_0-1}\hat{a}(\rho^\ell x) = 1$, then there exist nontrivial solutions $(\alpha(s,x))_{s\in\mathbb{C}}$ of the system ((Co1),(B1),(B2)). **Case 2.2.1:** If $a_0 \neq \rho^n$ for all $n \geq 0$, then according to [6, Theorem 11] the family $(\beta(s,x))_{s\in\mathbb{C}}$ is given by (6). It is a solution of ((Co2),(B1),(B2)) if and only if (7) is satisfied for $G(x) := -E(x)F(x)$. Again this linear functional equation has a unique solution by Theorem 3.1.

Case 2.2.2: If $a_0 = \rho^{n_0}$ for some $n_0 \in \mathbb{N}$, then $a_0 = \rho^n$ for all $n \in K$. **Case 2.2.2.1:** If $\alpha_0 \neq \pi_1^n$ for all $n \in K$, then according to [6, Theorem 11] the family $(\beta(s,x))_{s\in\mathbb{C}}$ is given by (6). It satisfies the boundary condition (B2) if and only if (7) holds true for $G(x) := -E(x)F(x)$. Since α satisfies ((Co1),(B1),(B2)), the homogeneous linear functional equation

$$E(\rho x) = \hat{a}(x)E(x)$$

has nontrivial solutions. Therefore, according to Theorem 3.3 the inhomogeneous linear functional (7) equation has a solution if and only if (3) is satisfied.

Case 2.2.2.2: If $\alpha = \pi_1^{n_0}$, then $n_0 \in K$ and n_0 is uniquely determined. In this situation the family $(\beta(s,x))_{s\in\mathbb{C}}$ is given by (8). It satisfies (B2) if and only if (9) has a solution. Putting

$$\tilde{b}(x) := b(x) - a_0E(\rho x)A(1)x^{n_0},$$

we obtain from (3) the condition

$$\sum_{k=0}^{j_0-1}\frac{\tilde{b}(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)} = 0.$$

We have

$$\sum_{k=0}^{j_0-1} \frac{\tilde{b}(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)} = \sum_{k=0}^{j_0-1} \left(\frac{b(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)} - \frac{a_0 E(\rho^{k+1} x) A(1) \rho^{k n_0} x^{n_0}}{\prod_{j=0}^{k} a(\rho^j x)} \right)$$
$$= \sum_{k=0}^{j_0-1} \frac{b(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)} - j_0 E(x) A(1) x^{n_0}.$$

Thus, (9) has a solution if and only if

$$B(x) = j_0 E(x) A(1) x^{n_0}.$$

These cases yield the following description of covariant embeddings of (L) with respect to an iteration group π of type I in normal form, i.e. $\pi(s,x) = \pi_1(s)x$. For $\rho = \pi_1(1)$ the linear functional equation (L) becomes (Lρ):

Theorem 4. *The covariant embeddings of* (L) *with respect to an iteration group* π *of type I in normal form are described by:*

1. *If* $a_0 \neq \rho^n$ *for all* $n \geq 0$, *and if* ρ *is not a complex root of* 1, *then the system* ((Co1),(Co2),(B1),(B2)) *has solutions. They are of the form* (4) *and* (6). *Each solution* (α, β) *is a covariant embedding of* (L) *with respect to* π.

2. *If* $a_0 \neq \rho^n$ *for all* $n \geq 0$, *and if* ρ *is a complex root of* 1, *then there exist solutions* α *of* ((Co1),(B1),(B2)) *if and only if* (2) *holds. They are of the form* (4). *If* (2) *is satisfied, then the system* ((Co1),(Co2),(B1),(B2)) *has solutions* (α, β), *where* β *is of the form* (6). *Moreover, each solution* (α, β) *is a covariant embedding of* (L) *with respect to* π.

3. *If* $\rho^{n_0} = a_0$ *and* $\alpha_0 \neq \pi_1^n$ *for all* $n \geq 0$, *then there is no covariant embedding of* (L).

   (a) *If* ρ *is not a complex root of* 1, *then there exist solutions of the system* ((Co1),(Co2),(B1),(B2)) *if and only if* $b_{n_0} = 0$. *They are of the form* (4) *and* (6).

   (b) *Assume that* ρ *is a complex root of* 1. *If* (2) *is not satisfied, then there exist no solutions of* ((Co1),(B1),(B2)). *If* (2) *is satisfied, then the system* ((Co1),(Co2),(B1),(B2)) *has solutions if and only if* (3) *holds true. They are of the form* (4) *and* (6).

4. *Assume that* $\rho^{n_0} = a_0$, ρ *is not a complex root of* 1, *and* $\alpha_0 = \pi_1^{n_0}$. *The solutions* (α, β) *of* ((Co1),(Co2),(B1),(B2)) *are given by* (4) *and* (8).

(a) *If* (α, β) *is a solution with* $A(1) = 0$*, then* (α, β) *is a covariant embedding of* (L) *if and only if* $A = 0$*.*

(b) *If* (α, β) *is a solution with* $A(1) \neq 0$*, then* (α, β) *is a covariant embedding of* (L)*.*

5. *Assume that* $\rho^{n_0} = a_0$*,* ρ *is a complex root of* 1*, and* $\alpha_0 = \pi_1^{n_0}$*. The solutions* (α, β) *of* ((Co1),(Co2),(B1),(B2)) *are given by* (4) *and* (8)*.*

(a) *If* (α, β) *is a solution with* $A(1) = 0$*, then there exists a family of pairwise different solutions of* (L) *and* (α, β) *is not a covariant embedding of* (L)*.*

(b) *If* (α, β) *is a solution with* $A(1) \neq 0$*, then* (α, β) *is a covariant embedding of* (L) *with respect to* π*.*

Proof. 1. According to Case 1.1, for each α being a solution of ((Co1),(B1),(B2)) there exits exactly one β so that (α, β) is a solution of ((Co1),(Co2),(B1),(B2)). Thus (α, β) is a covariant embedding of (L) by Remark 2.4.
2. According to Case 2.2.1, if α is a solution of ((Co1),(B1),(B2)), then there exits exactly one β so that (α, β) is a solution of ((Co1),(Co2),(B1),(B2)), hence (α, β) is a covariant embedding of (L) by Remark 2.4.
3.(a) According to Case 1.2.1 there exist solutions (α, β) of ((Co1),(Co2),(B1),(B2)) if and only if $b_{n_0} = 0$. If so, from Theorem 3.2 we obtain the existence of a family of pairwise different solutions of (L). Assume that (α, β) is a covariant embedding of (L). Then each solution φ of (L) satisfies (Ls). Thus, we have

$$\begin{aligned}\varphi(\pi_1(s)x) &= \alpha(s,x)\varphi(x) + \beta(s,x) \\ &= \alpha_0(s)\frac{E(\pi_1(s)x)}{E(x)}\varphi(x) + \alpha_0(s)E(\pi_1(s)x)\Big(F(x) - \alpha_0(s)^{-1}F(\pi_1(s)x)\Big) \\ &= \alpha_0(s)E(\pi_1(s)x)\left(\frac{\varphi(x)}{E(x)} + F(x)\right) - E(\pi_1(s)x)F(\pi_1(s)x).\end{aligned}$$

Consequently,

$$\frac{\varphi(\pi_1(s)x) + E(\pi_1(s)x)F(\pi_1(s)x)}{E(\pi_1(s)x)} = \alpha_0(s)\left(\frac{\varphi(x)}{E(x)} + F(x)\right)$$

or

$$H(\pi_1(s)x) = \alpha_0(s)H(x) \tag{10}$$

for

$$H(x) := \frac{\varphi(x)}{E(x)} + F(x). \tag{11}$$

The coefficients of $H(x)$ satisfy $H_n\pi_1(s)^n = \alpha_0(s)H_n$, $n \geq 0$, $s \in \mathbb{C}$. For $n = n_0$ we have $H_{n_0} = 0$ since by assumption $\pi_1^{n_0} \neq \alpha_0$. For $n \neq n_0$ consider $s = 1$. Since $\pi_1(1) = \rho$, $\alpha_0(1) = a_0$ and $\rho^n \neq a_0$ for $n \neq n_0$ it follows that $H_n = 0$. Thus, $H(x) = 0$ and due to (11) $\varphi(x) = -E(x)F(x)$. This, however, is a contradiction since there exist pairwise different solutions of (L). Hence, by Remark 2.5, (α, β) is not a covariant embedding of (L).

3.(b) As was shown in Case 2.2.2.1 there exist solutions (α, β) of ((Co1),(Co2),(B1),(B2)) if and only if (2) and (3) are satisfied. This is equivalent to the fact that (L) has solutions. According to Theorem 3.3 there exists a family of pairwise different solutions of (L). Let (α, β) be a solution of ((Co1),(Co2),(B1),(B2)) and φ a solution of (L). Similarly as in 3.(a) we deduce from φ being a solution of (L) that $H(x)$ defined by (11) satisfies (10) and, therefore, $H(x) = 0$ and $\varphi(x) = -E(x)F(x)$. This is a contradiction, since (L) admits different solutions. As a consequence of Remark 2.5 the pair (α, β) is not a covariant embedding of (L).

4.(a) Let (α, β) be a solution of ((Co1),(Co2),(B1),(B2)) with $A(1) = 0$. Then $b_{n_0} = 0$ and according to Case 1.2.2 there exist solutions of (L). Theorem 3.2 describes this family of solutions of (L). Assume that (α, β) is a covariant embedding of (L), then each solution φ of (L) satisfies (Ls). Using the particular form (8) of β we obtain

$$\varphi(\pi_1(s)x) = \alpha_0(s)E(\pi_1(s)x)\left(\frac{\varphi(x)}{E(x)} + F(x) + A(s)x^{n_0}\right) - E(\pi_1(s)x)F(\pi_1(s)x).$$

Consequently,

$$\frac{\varphi(\pi_1(s)x)}{E(\pi_1(s)x)} + F(\pi_1(s)x) = \pi_1(s)^{n_0}\left(\frac{\varphi(x)}{E(x)} + F(x) + A(s)x^{n_0}\right),$$

or

$$H(\pi_1(s)x) = \pi_1(s)^{n_0}H(x) + \pi_1(s)^{n_0}A(s)x^{n_0} \tag{12}$$

for $H(x) := \frac{\varphi(x)}{E(x)} + F(x)$. Comparing the coefficients of x^{n_0} we get $H_{n_0}\pi_1(s)^{n_0} = (H_{n_0} + A(s))\pi_1(s)^{n_0}$. Since $\pi_1(s) \neq 0$ for all $s \in \mathbb{C}$ we obtain $A(s) = 0$ for all $s \in \mathbb{C}$, whence $A = 0$.

Conversely assume that $A = 0$. It is easy to prove that $\{c\rho^{n_0}x^{n_0}E(x) \mid c \in \mathbb{C}\}$ is the general solution of (Lρ_{h}). Hence the solutions of (Lρ) are of the form $\varphi(x) = -E(x)F(x) + c\rho^{n_0}x^{n_0}E(x)$. Now we prove that each of these φ satisfies (Ls).

$$\begin{aligned}\alpha(s,x)\varphi(x) + \beta(s,x) = {} & \alpha_0(s)\frac{E(\pi_1(s)x)}{E(x)}\Big(-E(x)F(x) + c\rho^{n_0}x^{n_0}E(x)\Big) \\ & +\alpha_0(s)E(\pi_1(s)x)\Big(F(x) - \alpha_0^{-1}(s)F(\pi_1(s)x)\Big)\end{aligned}$$

$$= c\rho^{n_0}\pi_1(s)^{n_0}x^{n_0}E(\pi_1(s)x) - E(\pi_1(s)x)F(\pi_1(s)x)$$
$$= \varphi(\pi_1(s)x), \qquad s \in \mathbb{C}.$$

By Remark 2.5, (α, β) is a covariant embedding of (L).
4.(b) Let (α, β) be a solution of ((Co1),(Co2),(B1),(B2)) with $A(1) \neq 0$. From Case 1.2.2 and $A(1) \neq 0$ we obtain that (L) does not admit a solution. Hence (α, β) is a covariant embedding of (L) by Remark 2.3.
5.(a) Let (α, β) be a solution of ((Co1),(Co2),(B1),(B2)) with $A(1) = 0$. According to Case 2.2.2.2, the linear functional equation (L) admits solutions. They are described in Theorem 3.3. We suppose that (α, β) is a covariant embedding of (L). Then each solution φ of (L) satisfies (Ls). Similar computations as in 4.(a) yield (12) for $H(x) := \frac{\varphi(x)}{E(x)} + F(x)$. Comparing the coefficients of x^{n_0} we get $H_{n_0}\pi_1(s)^{n_0} = (H_{n_0} + A(s))\pi_1(s)^{n_0}$, $s \in \mathbb{C}$, whence $A = 0$. For $n \neq n_0$ we obtain $H_n\pi_1(s)^{n_0}(\pi_1(s)^{n-n_0} - 1) = 0$, $s \in \mathbb{C}$. Thus, $H_n = 0$ for all $n \neq 0$. If, on the contrary, we suppose that $H_r \neq 0$ for some $r \neq n_0$, then all values $\pi_1(s)$ were complex roots of 1 of order $|r - n_0|$ which is impossible since by assumption $\pi_1 \neq 1$ and, therefore, $\pi_1(s)$ admits infinitely many values.

Consequently, $H(x) = H_{n_0}x^{n_0}$ and $\varphi(x) = E(x)(H_{n_0}x^{n_0} - F(x))$. The set of these series is "smaller" than the set of solutions of (L) described in Theorem 3.3. In the above representation only the coefficient of x^{n_0} in φ is arbitrary, whereas, following Theorem 3.3, the coefficients of infinitely many powers x^ℓ in φ are arbitrary. Therefore, (α, β) is not a covariant embedding of (L) by Remark 2.5.
5.(b) Let (α, β) be a solution of ((Co1),(Co2),(B1),(B2)) with $A(1) \neq 0$. Then we obtain from Case 2.2.2.2 that (L) does not admit a solution. Hence (α, β) is a covariant embedding of (L) by Remark 2.3. □

The analytic covariant embeddings with respect to analytic iteration groups as given in [4] and [5] are described in Remark 5, as consequences of Theorem 4.

Remark 5. If A is an analytic additive function with $A(1) = 0$, then $A = 0$, since all analytic additive functions are of the form $s \mapsto cs$ for some $c \in \mathbb{C}$.

1. If ρ is not a complex root of 1, then the analytic covariant embeddings are described in [5, Theorem 8]. We have $\alpha_0(s) = e^{\mu s}$, $a_0 = e^\mu$, and $\pi_1(s) = e^{\lambda s}$. If $\mu - n\lambda \notin 2\pi i\mathbb{Z}$ for $n \in \mathbb{N}_0$, then the result follows from Theorem 4.1. If $\mu = n_0\lambda$ for some $n_0 \in \mathbb{N}_0$, then $\alpha_0 = \pi_1^{n_0}$, $A(s) = A(1)s$, and the result follows from Theorem 4.4.(a). If $\mu = n_0\lambda + 2\pi i z_0$ for some $n_0 \in \mathbb{N}_0$ and $z_0 \in \mathbb{Z} \setminus \{0\}$, then $a_0 = \rho^{n_0}$ but $\alpha_0 \neq \pi_1^n$ for $n \geq 0$. The result follows from Theorem 4.3.

2. If ρ is a complex root of 1 of order j_0. Then the analytic covariant embeddings are described in [5, Theorem 4]. We have $\alpha_0(s) = e^{\mu s}$, $a_0 = e^\mu$, and $\pi_1(s) = e^{\lambda s}$.

A necessary condition for the existence of covariant embeddings is (2). If $a_0^{j_0} \neq 1$, then $a_0 \neq \rho^n$ for $n \geq 0$ and the result follows from Theorem 4.2. If $a_0^{j_0} = 1$, then $a_0 = \rho^{n_0} = e^{n_0\lambda}$ and $\mu - n_0\lambda \in 2\pi i\mathbb{Z}$. If $\mu - n_0\lambda \neq 0$, then $\alpha_0 \neq \pi_1^n$ for all $n \geq 0$. According to Theorem 4.3.(b) a solution (α, β) of ((Co1),(Co2),(B1),(B2)) with $\alpha_0 \neq \pi_1^n$ for all $n \geq 0$ is not a covariant embedding of (L). Assume that (α, β) is a solution of ((Co1),(Co2),(B1),(B2)) with $\mu = n_0\lambda$, then $\alpha_0 = \pi_1^{n_0}$. If $A = 0$, then (L) admits solutions whence $B(x) = 0$ and the result follows from Theorem 4.5.(a). If (α, β) is a solution with $A \neq 0$, then the result follows from Theorem 4.5.(b).

4 Covariant embeddings with respect to iteration groups of type II

Now we consider the situation that $(\pi(s,x))_{s\in\mathbb{C}}$ is an iteration group of type II. We write

$$\pi(s,x) = x + \pi_k(s)x^k + \sum_{\ell>k} P_\ell(\pi_k(s))x^\ell, \qquad s \in \mathbb{C},$$

where $k \geq 2$, $\pi_k\colon \mathbb{C} \to \mathbb{C}$ is an additive function, $\pi_k \neq 0$, and where each $P_\ell(y) \in \mathbb{C}[y]$ is a polynomial. Since $\pi(1,x) = p(x)$ we have

$$p(x) = x + \pi_k(1)x^k + \sum_{\ell>k} P_\ell(\pi_k(1))x^\ell$$

where $\pi_k(1) \neq 0$. In this context in [6] it was helpful to consider the analytic iteration group $(\pi^*(s,x))_{s\in\mathbb{C}}$ "corresponding" to $(\pi(s,x))_{s\in\mathbb{C}}$. It is given by

$$\pi^*(s,x) = x + \pi_k^*(s)x^k + \sum_{\ell>k} P_\ell(\pi_k^*(s))x^\ell, \qquad s \in \mathbb{C},$$

with the same polynomials P_ℓ as in π and where $\pi_k^*(s) = s$ for $s \in \mathbb{C}$. According to [6, Remark 7], $(\pi^*(s,x))_{s\in\mathbb{C}}$ is indeed an iteration group. The close connection between $(\pi^*(s,x))_{s\in\mathbb{C}}$ and $(\pi(s,x))_{s\in\mathbb{C}}$ is given by

$$\pi(s,x) = \pi^*(\pi_k(s), x), \qquad s \in \mathbb{C}.$$

For solving the problem of covariant embeddings we need again some results on the linear functional equation

$$\varphi(p(x)) = a(x)\varphi(x) + b(x). \tag{L}$$

Substitution of $p(x)$ into $\varphi(x)$ gives

$$\begin{aligned}\varphi(p(x)) &= \sum_{n\geq 0}\varphi_n[p(x)]^n \\ &= \sum_{n=0}^{k-1}\varphi_n x^n + (\varphi_k + \varphi_1\pi_k(1))x^k \\ &\qquad + \sum_{n>k}(\varphi_n + (n-k+1)\varphi_{n-k+1} + \theta_n(\varphi_1,\dots,\varphi_{n-k},\pi_k(1)))x^n\end{aligned}$$

with universal polynomials θ_n, $n > k$, which are linear in φ_j. With $\theta_k = 0$ we obtain

Theorem 6. *1. If $a_0 \notin \{0,1\}$, then* (L) *has a unique solution $\varphi \in \mathbb{C}[\![x]\!]$. The coefficients are recursively given by*

$$\varphi_n = \begin{cases} (1-a_0)^{-1}\left(\displaystyle\sum_{r=1}^{n} a_r\varphi_{n-r} + b_n\right) & 0 \leq n < k, \\ (1-a_0)^{-1}\left(\displaystyle\sum_{r=1}^{n} a_r\varphi_{n-r} + b_n - (n-k+1)\varphi_{n-k+1}\pi_k(1)\right. \\ \qquad\qquad \left. - \theta_n(\varphi_1,\dots,\varphi_{n-k},\pi_k(1))\right) & n \geq k. \end{cases}$$

2. Assume that $a(x) = 1$. If φ is a solution of (L)*, then $b_n = 0$ for $n < k$ and φ_0 is not determined by* (L)*. The other coefficients φ_n, $n \geq 1$, are uniquely determined and recursively given by*

$$\varphi_n = \frac{b_{n+k-1} - \theta_{n+k-1}(\varphi_1,\dots,\varphi_{n-1},\pi_k(1))}{n\pi_k(1)}, \qquad n \geq 1.$$

The coefficients φ_n, $n \geq 1$, do not depend on φ_0.

Conversely, if $b_0 = \ldots = b_{k-1} = 0$, $\varphi_0 \in \mathbb{C}$, and φ_n, $n \geq 1$, as given above, then $\sum_{n\geq 0}\varphi_n x^n$ is a solution of (L)*.*

3. Assume that $a(x) = 1 + a_{n_0}x^{n_0} + \sum_{n>n_0} a_n x^n$, where $n_0 \geq k$ and $a_{n_0} \neq 0$. If φ is a solution of (L)*, then $b_n = 0$ for $n < k$ and φ_0 is not determined by* (L)*. The other coefficients are uniquely determined and recursively given by*

$$\varphi_n = \begin{cases} \dfrac{b_{n+k-1} - \theta_{n+k-1}(\varphi_1,\dots,\varphi_{n-1},\pi_k(1))}{n\pi_k(1)} & 1 \leq n \leq n_0 - k, \\ \dfrac{\sum_{r=0}^{n+k-1-n_0} a_{n+k-1-r}\varphi_r + b_{n+k-1}}{n\pi_k(1)} \\ \qquad - \dfrac{\theta_{n+k-1}(\varphi_1,\dots,\varphi_{n-1},\pi_k(1))}{n\pi_k(1)} & n > n_0 - k. \end{cases}$$

Conversely, if $b_0 = \ldots = b_{k-1} = 0$, $\varphi_0 \in \mathbb{C}$, *and* φ_n, $n \geq 1$, *as given above, then* $\sum_{n\geq 0} \varphi_n x^n$ *is a solution of* (L).

4. *Assume that* $a(x) = 1 + a_{n_0}x^{n_0} + \sum_{n>n_0} a_n x^n$, *where* $n_0 < k-1$ *and* $a_{n_0} \neq 0$. *If* φ *is a solution, then* $b_n = 0$ *for* $n < n_0$ *and the coefficients* φ_n *are uniquely determined and recursively given by*

$$\varphi_{n-n_0} = \begin{cases} \dfrac{-\sum_{r=0}^{n-n_0-1} a_{n-r}\varphi_r - b_n}{a_{n_0}} & n_0 \leq n < k, \\ \dfrac{(n-k+1)\varphi_{n-k+1}\pi_k(1) + \theta_n(\varphi_1, \ldots, \varphi_{n-k}, \pi_k(1))}{a_{n_0}} & \\ \qquad - \dfrac{\sum_{r=0}^{n-n_0-1} a_{n-r}\varphi_r + b_n}{a_{n_0}} & n \geq k. \end{cases}$$

Conversely, if $b_0 = \ldots = b_{n_0-1} = 0$ *and* φ_n, $n \geq 0$, *as given above, then* $\sum_{n\geq 0} \varphi_n x^n$ *is a solution of* (L).

5. *Assume that* $a(x) = 1 + a_{k-1}x^{k-1} + \sum_{n>k-1} a_n x^n$, *where* $a_{k-1} \neq 0$ *and moreover* $a_{k-1}/\pi_k(1) \notin \mathbb{N}$. *If* φ *is a solution, then* $b_n = 0$ *for* $n < k-1$ *and the coefficients* φ_n *are uniquely determined and recursively given by*

$$\varphi_{n-(k-1)} = \begin{cases} \dfrac{-b_{k-1}}{a_{k-1}} & n = k-1, \\ \dfrac{\sum_{r=0}^{n-k} a_{n-r}\varphi_r + b_n - \theta_n(\varphi_1, \ldots, \varphi_{n-k}, \pi_k(1))}{(n-k+1)\pi_k(1) - a_{k-1}} & n \geq k. \end{cases} \tag{13}$$

Conversely, if $b_0 = \ldots = b_{k-2} = 0$ *and* φ_n, $n \geq 0$, *as given above, then* $\sum_{n\geq 0} \varphi_n x^n$ *is a solution of* (L).

6. *Assume that* $a(x) = 1 + a_{k-1}x^{k-1} + \sum_{n>k-1} a_n x^n$, *where* $a_{k-1} \neq 0$ *and moreover* $a_{k-1}/\pi_k(1) = n_1 \in \mathbb{N}$. *If* φ *is a solution, then* $b_n = 0$ *for* $n < k-1$ *and* φ_{n_1} *is not determined by* (L). *For* $n \neq n_1 + k - 1$ *the coefficients* $\varphi_{n-(k-1)}$ *are uniquely determined and recursively given by* (13). *Moreover,*

$$b_{n_1+k-1} = \theta_{n_1+k-1}(\varphi_1, \ldots, \varphi_{n_1-1}, \pi_k(1)) - \sum_{r=0}^{n_1-1} a_{n_1+k-1-r}\varphi_r \tag{14}$$

is satisfied.

Conversely, if $b_0 = \ldots = b_{k-2} = 0$, φ_{n_1} *arbitrarily chosen in* $\mathbb{C}$, φ_n *determined by* (13) *for* $n < n_1$, (14) *satisfied, and* φ_n *determined by* (13) *for* $n > n_1$, *then* φ *is a solution of* (L).

These assertions can be verified by careful computations. The same distinction into different cases occurred already in [4] where we described the covariant embeddings of (L) with respect to an analytic iteration group. If we consider the homogeneous linear equation, then in the last three cases $\operatorname{ord}(\varphi) \geq 1$ for any solution φ, i.e. $\varphi_0 = 0$.

In [6, Theorem 8] the general solution of ((Co1),(B1)) is given by

$$\alpha(s,x) = \alpha_0(s)P(s,x)\frac{E(\pi(s,x))}{E(x)}, \qquad s \in \mathbb{C}, \tag{15}$$

where α_0 is a generalized exponential function, $E(x) \in \mathbb{C}\,[\![x]\!]$ with $E(x) \equiv 1 \bmod x$, and $P(s,x) := P^*(\pi_k(s), x)$ with

$$P^*(\tau,x) = \prod_{n=1}^{k-1} \exp\left(\kappa_n \int_0^\tau [\pi^*(\sigma,x)]^n \, d\sigma\right)$$

with $\kappa_1, \dots, \kappa_{k-1} \in \mathbb{C}$. Dividing both sides of (15) by $\alpha_0(s)$ we obtain

$$\frac{\alpha(s,x)}{\alpha_0(s)} =: \hat{\alpha}(s,x) = P(s,x)\frac{E(\pi(s,x))}{E(x)}.$$

Therefore, α satisfies (B2) if and only if $\alpha_0(1) = a_0$ and $\hat{\alpha}$ satisfies

$$\hat{\alpha}(1,x) = \hat{a}(x)$$

which is the same as

$$E(p(x)) = \frac{\hat{a}(x)}{P(1,x)} E(x). \tag{16}$$

Theorem 7. *For any $\hat{a} \in \mathbb{C}\,[\![x]\!]$ there exists a unique family $(\hat{\alpha}(s,x))_{s\in\mathbb{C}}$ satisfying* (Co1)*, $\hat{\alpha}(0,x) = 1$ and $\hat{\alpha}(1,x) = \hat{a}(x)$.*

Proof. According to the particular representation of solutions α of ((Co1),(B1)) and since

$$\frac{E(\pi(s,x))}{E(x)} \equiv 1 \bmod x^k$$

for every $E(x) \in \mathbb{C}\,[\![x]\!]$ with $E(x) \equiv 1 \bmod x$ and π of type II, we necessarily have to determine $\kappa_1, \dots, \kappa_{k-1}$ so that

$$\frac{\hat{a}(x)}{P(1,x)} \equiv 1 \bmod x. \tag{17}$$

If $\hat{a}(x) \equiv 1 \bmod x^k$, then $\kappa_1 = \ldots = \kappa_{k-1} = 0$, whence $P(s,x) = 1$. Otherwise, applying the same ideas as in [4, Theorem 3.10] it is possible to find uniquely determined coefficients $\kappa_1, \ldots, \kappa_{k-1}$ so that (17) is true. Thus (16) is a homogeneous linear functional equation belonging to case 2 or 3 of Theorem 6. Consequently there exists a unique solution E of (16) with $E(x) \equiv 1 \bmod x$. □

Now we assume that α, given by (15), is a solution of ((Co1),(B1),(B2)) and analyze how to describe all solutions (α, β) of ((Co1),(Co2),(B1),(B2)).

Case 1: If $\alpha_0 \neq 1$, then according to [6, Theorem 16] β is of the form

$$\beta(s,x) = \alpha_0(s)P(s,x)E(\pi(s,x))\Big(F(x) - \alpha_0(s)^{-1}\frac{F(\pi(s,x))}{P(s,x)}\Big), \qquad s \in \mathbb{C}, \tag{18}$$

for some $F(x) \in \mathbb{C}\,[\![x]\!]$. **Case 1.1:** If $a_0 \neq 1$, then β satisfies (B2) if and only if

$$a_0 P(1,x)E(p(x))\Big(F(x) - a_0^{-1}\frac{F(x)}{P(1,x)}\Big) = b(x)$$

which is equivalent to

$$\begin{aligned} a_0 P(1,x)\frac{\hat{a}(x)}{P(1,x)}E(x)F(x) - E(p(x))F(p(x)) &= b(x) \\ -E(p(x))F(p(x)) &= -a(x)E(x)F(x) + b(x), \\ G(p(x)) &= a(x)G(x) + b(x) \end{aligned} \tag{19}$$

for $G(x) := -E(x)F(x)$. According to Theorem 6.1 there exists a unique solution G, which means that F and also β are uniquely determined.

Case 1.2: If $a_0 = 1$, then $a(x) = \hat{a}(x)$ and, depending on $a(x)$, we have to consider different situations. **Case 1.2.1:** We assume that $a(x) = 1$, whence $P(s,x) = 1$. Then (19) is simply

$$G(p(x)) = G(x) + b(x).$$

According to Theorem 6.2, there exists a solution G if and only if $b_n = 0$ for $n < k$. If $b_n = 0$, $n < k$, then G_0 is not determined by the equation, whereas G_n does not depend on G_0 and is uniquely determined for $n > 1$. Therefore, if $b_n = 0$, $n < k$, then F_n is uniquely determined for $n > 0$. In conclusion, there exist solutions β if and only if $b_n = 0$, $n < k$. If so, then β is not uniquely determined by α and ((Co2),(B1),(B2)). **Case 1.2.2:** We assume that $a(x) = 1 + a_{n_0}x^{n_0} + \ldots$ with $a_{n_0} \neq 0$ and $n_0 \geq k$. Again we have $P(s,x) = 1$. According to Theorem 6.3, there exists a solution G of (19) if and only if $b_n = 0$ for $n < k$. If $b_n = 0$, $n < k$, then G_0

is not determined by the equation, whereas G_n is uniquely determined for $n > 1$. Now the coefficients G_n also depend on G_0. Similarly as above, there exist solutions β if and only if $b_n = 0$, $n < k$. If so, then β is not uniquely determined by α and ((Co2),(B1),(B2)). **Case 1.2.3:** We assume that $a(x) = 1 + a_{n_0}x^{n_0} + \dots$ with $a_{n_0} \neq 0$ and $n_0 < k-1$. Then according to Theorem 6.4, there exists a solution G of (19) if and only if $b_n = 0$ for $n < n_0$. If $b_n = 0$, $n < n_0$, then G is uniquely determined, whence β is uniquely determined by α and ((Co2),(B1),(B2)). **Case 1.2.4:** We assume that $a(x) = 1 + a_{n_0}x^{n_0} + \dots$ with $a_{n_0} \neq 0$ and $n_0 = k-1$. **Case 1.2.4.1:** If $a_{k-1}/\pi_k(1) \notin \mathbb{N}$, then according to Theorem 6.5, there exists a solution G of (19) if and only if $b_n = 0$ for $n < k-1$. If so, then G is uniquely determined, whence β is uniquely determined by α and ((Co2),(B1),(B2)). **Case 1.2.4.2:** If $a_{k-1}/\pi_k(1) = n_1 \in \mathbb{N}$, then according to Theorem 6.6, there exist solutions G of (19) if and only if (L) has a solution. To be more precise, there are solutions of (L) if and only if $\operatorname{ord}(b(x)) \geq k-1$ and (14) is satisfied. However, G and therefore β is not uniquely determined by α and ((Co2),(B1),(B2)).

Case 2: Assume that $\alpha_0 = 1$, then necessarily $a_0 = 1$ and $a = \hat{a}$. **Case 2.1:** Consider

$$P(s,x) = 1 + \kappa_r \pi_k(s) x^r + \dots, \qquad s \in \mathbb{C},$$

with $\kappa_r \neq 0$, and either $r < k-1$ or $r = k-1$ and $\kappa_{k-1} \notin \mathbb{N}_0$. Due to the particular form (15) of α and $\alpha(1,x) = a(x)$ we have $a(x) = 1 + a_r x^r + \dots$ where $a_r = \pi_k(1)\kappa_r$. Using the notation of Theorem 6, $n_0 = r$ and either $n_0 < k-1$ or $n_0 = k$ and $a_{n_0}/\pi_k(1) \notin \mathbb{N}$. According to [6, Theorem 17] β is of the form

$$\beta(s,x) = P(s,x)E(\pi(s,x))\Big(F(x) - \frac{F(\pi(s,x))}{P(s,x)} + Q(s,x)\Big), \qquad s \in \mathbb{C}, \tag{20}$$

where $F(x) \in \mathbb{C}\llbracket x \rrbracket$ and

$$Q(s,x) = \sum_{n=0}^{n_0-1} \int_0^{\tau} \frac{\ell_n[\pi^*(\sigma,x)]^n}{P^*(\sigma,x)E(\pi^*(\sigma,x))}\, d\sigma \Big|_{\tau=\pi_k(s)},$$

with $\ell_0, \dots, \ell_{n_0-1} \in \mathbb{C}$. Then β satisfies (B2) if and only if

$$P(1,x)E(p(x))\Big(F(x) - \frac{F(p(x))}{P(1,x)} + Q(1,x)\Big) = b(x)$$

$$a(x)E(x)F(x) - E(p(x))F(p(x)) + a(x)E(x)Q(1,x) = b(x),$$

$$G(p(x)) = a(x)G(x) + \tilde{b}(x) \tag{21}$$

for $G(x) = -E(x)F(x)$ and $\tilde{b}(x) = b(x) - a(x)E(x)Q(1,x)$. According to Theorem 6.4 or Theorem 6.5 there exists a solution G of (21) if and only if $\operatorname{ord}(\tilde{b}(x)) \geq n_0$.

By construction, for $0 \leq n < n_0$ the n-th summand of $Q(s,x)$ is of the form $\ell_n x^n + \ldots$, whence it is possible to determine ℓ_n, $0 \leq n < n_0$, so that $\operatorname{ord}(b(x) - a(x)E(x)Q(1,x)) \geq n_0$. By doing this $Q(s,x)$ is uniquely determined. Using this Q there exists a unique solution G of (21) and consequently a unique series $F(x)$. In conclusion β is uniquely determined by α and ((Co2),(B1),(B2)).

Case 2.2: Assume that $P(s,x) = 1$, then either $a(x) = 1$ or $a(x) = 1 + a_{n_0}x^{n_0} + \ldots$ with $a_{n_0} \neq 0$ and $n_0 \geq k$. According to [6, first part of Theorem 18] β is of the form

$$\beta(s,x) = E(\pi(s,x))\Big(A(s) + F(x) - F(\pi(s,x)) + Q(s,x)\Big), \qquad s \in \mathbb{C}, \tag{22}$$

where $A\colon \mathbb{C} \to \mathbb{C}$ is additive, $F(x) \in \mathbb{C}[\![x]\!]$, and

$$Q(s,x) = \sum_{n=0}^{k-1} \int_0^\tau \frac{\ell_n[\pi^*(\sigma,x)]^n}{E(\pi^*(\sigma,x))}\, d\sigma\Big|_{\tau=\pi_k(s)},$$

with $\ell_0, \ldots, \ell_{k-1} \in \mathbb{C}$. Then β satisfies (B2) if and only if

$$a(x)E(x)F(x) - E(p(x))F(p(x)) + a(x)E(x)(A(1) + Q(1,x)) = b(x),$$
$$G(p(x)) = a(x)G(x) + \tilde{b}(x) \tag{23}$$

for $G(x) = -E(x)F(x)$ and $\tilde{b}(x) = b(x) - a(x)E(x)(A(1) + Q(1,x))$. According to Theorem 6.2 or Theorem 6.3 there exist solutions of (23) if and only if $\operatorname{ord}(\tilde{b}(x)) \geq k$. For any choice of $A(1)$ there exist unique $\ell_0, \ldots, \ell_{k-1}$ so that $\operatorname{ord}(b(x) - a(x)E(x)(A(1) + Q(1,x))) \geq k$. Using this Q there exist solutions G of (23). The coefficient G_0 can be chosen arbitrarily, the remaining G_n are uniquely determined. In conclusion there exist solutions β of ((Co2),(B1),(B2)), but β is not uniquely determined by α and this system of equations.

Case 2.3: Assume that

$$P(s,x) = 1 + \kappa_{k-1}\pi_k(s)x^{k-1} + \ldots, \qquad s \in \mathbb{C},$$

with $\kappa_{k-1} = n_1 \in \mathbb{N}$. Then $a(x) = 1 + a_{k-1}x^{k-1} + \ldots$, with $a_{k-1} = n_1\pi_k(1)$. According to [6, second part of Theorem 18] β is of the form

$$\beta(s,x) = P(s,x)E(\pi(s,x))\Big(A(s)\delta(x) + F(x) - \frac{F(\pi(s,x))}{P(s,x)} + Q(s,x)\Big), \qquad s \in \mathbb{C}, \tag{24}$$

where $A\colon \mathbb{C} \to \mathbb{C}$ is additive, $\delta(x) = x^{n_1} + \sum_{n>n_1} \delta_n x^n$, so that $A(s)\delta(x)$ is a solution of

$$\Delta(s+t,x) = \Delta(s,x) + \frac{\Delta(t,\pi(s,x))}{P(s,x)}, \qquad s,t \in \mathbb{C},$$

$F(x) \in \mathbb{C}\,[\![x]\!]$, and

$$Q(s,x)=\left(\sum_{n=0}^{k-2}\int_0^\tau \frac{\ell_n[\pi^*(\sigma,x)]^n}{P^*(\sigma,x)E(\pi^*(\sigma,x))}\,d\sigma+\int_0^\tau \frac{\ell_{n_1+k-1}[\pi^*(\sigma,x)]^{n_1+k-1}}{P^*(\sigma,x)}\,d\sigma\right)\Big|_{\tau=\pi_k(s)},$$

with $\ell_0,\dots,\ell_{k-2},\ell_{n_1+k-1}\in\mathbb{C}$. Then β satisfies (B2) if and only if

$$a(x)E(x)F(x)-E(p(x))F(p(x))+a(x)E(x)(A(1)\delta(x)+Q(1,x))=b(x),$$
$$G(p(x))=a(x)G(x)+\tilde{b}(x) \quad (25)$$

for $G(x)=-E(x)F(x)$ and $\tilde{b}(x)=b(x)-a(x)E(x)(A(1)\delta(x)+Q(1,x))$. According to Theorem 6.6 there exist solutions of (25) if and only if $\operatorname{ord}(\tilde{b}(x))\geq k-1$ and (14) is satisfied. For any choice of $A(1)$ there exist unique $\ell_0,\dots,\ell_{k-2}$ and ℓ_{n_1+k-1} so that these conditions are satisfied. Using this Q there exist solutions G of (25). The coefficient G_{n_1} can be chosen arbitrarily, the remaining G_n are uniquely determined. In conclusion there exist solutions β of ((Co2),(B1),(B2)) but β is not uniquely determined by α and this system of equations.

Theorem 8. *The covariant embeddings of* (L) *with respect to an iteration group of type II are described by:*

1. *If $a_0\neq 1$, then the system* ((Co1),(Co2),(B1),(B2)) *has solutions (α,β). They are of the form* (15) *and* (18)*. Each solution is a covariant embedding of* (L).
2. *Assume that $a(x)=1+a_{n_0}x^{n_0}+\dots$ with $a_{n_0}\neq 0$ and either $n_0<k-1$ or $n_0=k-1$ and $a_{n_0}\neq n\pi_k(1)$ for $n\in\mathbb{N}$.*
   - (a) *There exist solutions (α,β) of* ((Co1),(Co2),(B1),(B2)) *with $\alpha_0\neq 1$ if and only if* (L) *has solutions. They are of the form* (15) *and* (18)*. If (α,β) is a solution with $\alpha_0\neq 1$, then it is a covariant embedding of* (L).
   - (b) *There exists a unique solution (α,β) of* ((Co1),(Co2),(B1),(B2)) *with $\alpha_0=1$. It is of the form* (15) *and* (20)*, and it is a covariant embedding of* (L).
3. *Assume that either $a(x)=1$ or $a(x)=1+a_{n_0}x^{n_0}+\dots$ with $a_{n_0}\neq 0$ and $n_0\geq k$.*
   - (a) *There exist solutions (α,β) of* ((Co1),(Co2),(B1),(B2)) *with $\alpha_0\neq 1$ if and only if* (L) *has solutions. They are of the form* (15) *and* (18) *with $P(s,x)=1$. If (α,β) is a solution with $\alpha_0\neq 1$, then it is not a covariant embedding of* (L).

(b) *There exist solutions* (α, β) *of* ((Co1),(Co2),(B1),(B2)) *with* $\alpha_0 = 1$ *if and only if the linear functional equation* (23) *has solutions. They are of the form* (15) *and* (22). *Assume that* (α, β) *is a solution with* $\alpha_0 = 1$ *and an additive function* A.

   i. *If there is no solution of* (L), *i.e. there exists some* $n < k$ *so that* $b_n \neq 0$, *then* (α, β) *is a covariant embedding of* (L).

   ii. *Assume that* $b_n = 0$ *for* $n < k$, *i.e. there exist solutions of* (L). *Then* (α, β) *is a covariant embedding of* (L) *if and only if* $A = c\pi_k$ *for some* $c \in \mathbb{C}$.

4. *Assume that* $a(x) = 1 + a_{k-1}x^{k-1} + \dots$ *with* $a_{k-1} = n_1\pi_k(1)$ *for some* $n_1 \in \mathbb{N}$.

   (a) *There exist solutions* (α, β) *of* ((Co1),(Co2),(B1),(B2)) *with* $\alpha_0 \neq 1$ *if and only if* (L) *has solutions. They are of the form* (15) *and* (18). *If* (α, β) *is a solution with* $\alpha_0 \neq 1$, *then it is not a covariant embedding of* (L).

   (b) *There exist solutions* (α, β) *of* ((Co1),(Co2),(B1),(B2)) *with* $\alpha_0 = 1$ *if and only if the linear functional equation* (25) *has solutions. They are of the form* (15) *and* (24). *Assume that* (α, β) *is a solution with* $\alpha_0 = 1$ *and additive function* A.

      i. *If there is no solution of* (L), *then* (α, β) *is a covariant embedding of* (L).

      ii. *Assume that there exist solutions of* (L). *Then* (α, β) *is a covariant embedding of* (L) *if and only if* $A = c\pi_k$ *for some* $c \in \mathbb{C}$.

Proof. 1. If $a_0 \neq 1$ and $\alpha_0(1) = a_0$, then $\alpha_0 \neq 1$. According to Case 1.1, for each α being a solution of ((Co1),(B1),(B2)) there exits exactly one β so that (α, β) is a solution of ((Co1),(Co2),(B1),(B2)). Thus (α, β) is a covariant embedding of (L) by Remark 2.4.
2.(a) According to Case 1.2.3 for $n_0 < k - 1$ or Case 1.2.4.1 for $n_0 = k - 1$ and $a_{n_0} \neq n\pi_k(1)$, $n \in \mathbb{N}$, we have: If α is a solution of ((Co1),(B1),(B2)) with $\alpha_0 \neq 1$, then there exists a solution (α, β) of ((Co1),(Co2),(B1),(B2)) if and only if (L) has a solution. In this case β is uniquely determined by α and ((Co2),(B1),(B2)), hence (α, β) is a covariant embedding of (L) by Remark 2.4.
2.(b) According to Theorem 7 and Case 2.1 there exists a unique solution α of ((Co1),(B1),(B2)) with $\alpha_0 = 1$ and there exists a unique β so that (α, β) is a solution of ((Co2),(B1),(B2)). Hence (α, β) is a covariant embedding of (L) by Remark 2.4.
3.(a) Assume that α is a solution of ((Co1),(B1),(B2)) with $\alpha_0 \neq 1$. According to Case 1.2.1 or Case 1.2.2 there exist solutions (α, β) of ((Co1),(Co2),(B1),(B2)) if and only if (L) has a solution. Supposing that (α, β) is a covariant embedding of

(L) we deduce that each solution φ of (L) satisfies (Ls). According to Theorem 6.3 for each φ_0 there exists exactly one solution φ of (L) with $\varphi \equiv \varphi_0 \bmod x$. From (15) and (18) we deduce that each of these φ satisfies

$$\begin{aligned}\varphi(\pi(s,x)) &= \alpha_0(s)\frac{E(\pi(s,x))}{E(x)}\varphi(x)\\ &\quad +\alpha_0(s)E(\pi(s,x))(F(x)-\alpha_0(s)^{-1}F(\pi(s,x))),\\ \frac{\varphi(\pi(s,x))}{E(\pi(s,x))}+F(\pi(s,x)) &= \alpha_0(s)\left(\frac{\varphi(x)}{E(x)}+F(x)\right),\\ H(\pi(s,x)) &= \alpha_0(s)H(x) \qquad (26)\end{aligned}$$

for all $s \in \mathbb{C}$ where $H(x) = \frac{\varphi(x)}{E(x)} + F(x)$. Choosing some $s \in \mathbb{C}$ so that $\alpha_0(s) \neq 1$ we obtain from Theorem 6.1 that $H(x) = 0$. Whence $\varphi(x) = -E(x)F(x)$ which is a contradiction to the arbitrary choice of $\varphi_0 \in \mathbb{C}$. Hence, by Remark 2.5, (α, β) is not a covariant embedding of (L).

3.(b) The assertion follows from Case 2.2.

3.(b)i. If (α, β) is a solution of ((Co1),(Co2),(B1),(B2)), then the assertion is obviously true by Remark 2.3.

3.(b)ii. Assume that (α, β) is a solution of ((Co1),(Co2),(B1),(B2)) with $\alpha_0 = 1$. In [6, proof of Theorem 18] we have shown that when all the coefficient functions of β are polynomials in $\pi_k(s)$, then it is possible to assume that $A = 0$. Therefore, it is possible to assume that $A(1) = 0$. Since, if $A(1) \neq 0$, then we express the additive function A as a sum of two additive functions $A = A_1 + A_2$. The first one is a nonzero multiple of π_k,

$$A_1(s) := \frac{A(1)}{\pi_k(1)}\pi_k(s)$$

and the second one

$$A_2(s) := \left(A(s) - \frac{A(1)}{\pi_k(1)}\pi_k(s)\right)$$

takes the value 0 for $s = 1$. Then $\beta(s,x) - A_2(s)E(\pi(s,x))$ can be expressed as

$$E(\pi(s,x))\Big(\tilde{F}(x) - \tilde{F}(\pi(s,x)) + \tilde{Q}(s,x)\Big),$$

thus

$$\beta(s,x) = E(\pi(s,x))\Big(A_2(s) + \tilde{F}(x) - \tilde{F}(\pi(s,x)) + \tilde{Q}(s,x)\Big), \qquad s \in \mathbb{C},$$

with suitable $\tilde{F}$ and $\tilde{Q}$.

We assume that $A(1) = 0$, whence $F = \tilde{F}$ and $Q = \tilde{Q}$. Since by assumption $b(x) \equiv 0 \bmod x^k$, we derive that $Q(s,x) = 0$. Therefore, the linear functional equation (23) becomes (L). The solutions of (L) are described in Theorem 6.3. In the present situation the set of all solutions of (L) is

$$\{-E(x)F(x) + cE(x) \mid c \in \mathbb{C}\}.$$

Simple calculations show that $-E(x)F(x) + cE(x)$ is a solution of (L). For $\varphi_0 \in \mathbb{C}$ let $c := \varphi_0 + F_0$, then $-E(x)F(x) + cE(x) \equiv -F_0 + (\varphi_0 + F_0) = \varphi_0 \bmod x$ which shows that each solution of (L) can be represented in this way. Now we analyze when (Ls) is satisfied. For $\varphi(x) = -E(x)F(x) + cE(x)$ we obtain

$$\begin{aligned}\varphi(\pi(s,x)) &- \alpha(s,x)\varphi(x) - \beta(s,x) = \\ &- E(\pi(s,x))F(\pi(s,x)) + cE(\pi(s,x)) + E(\pi(s,x))F(x) - cE(\pi(s,x)) \\ &\qquad - E(\pi(s,x))A(s) - E(\pi(s,x))F(x) + E(\pi(s,x))F(\pi(s,x)) = \\ &- E(\pi(s,x))A(s).\end{aligned}$$

Thus (Ls) is satisfied if and only if $A = 0$, whence $A = 0\pi_k$.
4.(a) Assume that α is a solution of ((Co1),(B1),(B2)) with $\alpha_0 \neq 1$. According to Case 1.2.4.2 there exist solutions (α, β) of ((Co1),(Co2),(B1),(B2)) if and only if (L) has a solution. Supposing that (α, β) is a covariant embedding of (L) we deduce that each solution φ of (L) satisfies (Ls). According to Theorem 6.6 for each choice of φ_{n_1} in $\mathbb{C}$ there exists exactly one solution φ of (L) where φ_{n_1} is the coefficient of x^{n_1} in φ. From (15) and (18) we deduce that all these φ satisfy

$$\begin{aligned}\varphi(\pi(s,x)) &= \alpha_0(s)P(s,x)\frac{E(\pi(s,x))}{E(x)}\varphi(x) \\ &\qquad + \alpha_0(s)P(s,x)E(\pi(s,x))\Big(F(x) - \alpha_0(s)^{-1}\frac{F(\pi(s,x))}{P(s,x)}\Big), \\ \frac{\varphi(\pi(s,x))}{E(\pi(s,x))} + F(\pi(s,x)) &= \alpha_0(s)P(s,x)\left(\frac{\varphi(x)}{E(x)} + F(x)\right), \\ H(\pi(s,x)) &= \alpha_0(s)P(s,x)H(x) \qquad (27)\end{aligned}$$

for all $s \in \mathbb{C}$ where $H(x) = \frac{\varphi(x)}{E(x)} + F(x)$. Choosing some $s \in \mathbb{C}$ so that $\alpha_0(s) \neq 1$ we obtain from Theorem 6.1 that $H(x) = 0$. Whence $\varphi(x) = -E(x)F(x)$ which is a contradiction to the arbitrary choice of $\varphi_{n_1} \in \mathbb{C}$. Hence, by Remark 2.5, (α, β) is not a covariant embedding of (L).
4.(b) The assertion follows from Case 2.3.
4.(b)i. If (α, β) is a solution of ((Co1),(Co2),(B1),(B2)), then the assertion is obviously true by Remark 2.3.

4.(b)ii. Assume that (α,β) is a solution of ((Co1),(Co2),(B1),(B2)) with $\alpha_0 = 1$ where β is of the form (24). The same arguments as in 3.(b)ii. show that it is possible to assume that $A(1) = 0$.

Since by assumption $b(x) \equiv 0 \bmod x^k$, we derive that $Q(s,x) = 0$. Therefore, the linear functional equation (25) becomes (L). The solutions of (L) are described in Theorem 6.6. In [6, Theorem 20] we have shown that

$$P(s,x) = \frac{\delta(\pi(s,x))}{\delta(x)},$$

whence $\delta(p(x))E(p(x)) = a(x)\delta(x)E(x)$. In other words, δE is a solution of the homogeneous linear functional equation and consequently, the set of all solutions of (L) is

$$\{-E(x)F(x) + c\delta(x)E(x) \mid c \in \mathbb{C}\}.$$

Simple calculations show that $-E(x)F(x) + c\delta(x)E(x)$ is a solution of (L) and that each solution of (L) can be represented in this way. Now we analyze when (Ls) is satisfied. For $\varphi(x) = -E(x)F(x) + c\delta(x)E(x)$ we obtain

$$\begin{aligned}\varphi(\pi(s,x)) &- \alpha(s,x)\varphi(x) - \beta(s,x) = \\ &- E(\pi(s,x))F(\pi(s,x)) + c\delta(\pi(s,x))E(\pi(s,x)) + P(s,x)E(\pi(s,x))F(x) \\ &\qquad - c\delta(x)P(s,x)E(\pi(s,x)) - P(s,x)E(\pi(s,x))A(s)\delta(x) \\ &\qquad - P(s,x)E(\pi(s,x))F(x) + E(\pi(s,x))F(\pi(s,x)) = \\ &- P(s,x)E(\pi(s,x))A(s)\delta(x).\end{aligned}$$

Thus (Ls) is satisfied if and only if $A = 0$, whence $A = 0\pi_k$. □

Remark 9. For analytic iteration groups of type II we have $\pi_k(s) = c_k s$ for some $c_k \in \mathbb{C} \setminus \{0\}$. If A is an arbitrary analytic additive function with $A(1) = 0$, then $A = 0$, since all analytic additive functions are of the form $s \mapsto cs$ for some $c \in \mathbb{C}$. Thus, any nontrivial analytic additive function is a multiple of π_k.

1. If $a_0 \neq 1$, then the analytic covariant embeddings are described in [4, Corollary 4.2] and [5, Theorem 19]. The result follows from Theorem 8.1.

2. Assume that $a(x) = 1 + a_{n_0}x^{n_0} + \dots$ with $a_{n_0} \neq 0$ and either $n_0 < k-1$ or $n_0 = k-1$ and $a_{n_0} \neq nc_k$ for $n \in \mathbb{N}$. The analytic covariant embeddings of (L) are described in [4, Corollary 4.2] and [5, Theorem 19]. We have $\alpha_0(s) = e^{\mu s}$. The situations $\mu \neq 0$ or $\mu = 0$ are described by Theorem 8.2.(a) or Theorem 8.2.(b) respectively.

3. Consider $a(x) = 1$ or $a(x) = 1 + a_{n_0}x^{n_0} + \dots$ with $a_{n_0} \neq 0$ and $n_0 \geq k$. We have $\alpha_0(s) = e^{\mu s}$. If $\mu \neq 0$ then according to [5, Theorem 15] there are no covariant embeddings, which follows from Theorem 8.3.(a). The case $\mu = 0$ is described in [5, Theorem 19] and follows from Theorem 8.3.(b). In the analytic case the additive function A does not appear, since A is of the form $c\pi_k$ and, therefore, it can be included in $Q(s, x)$.

4. The analytic covariant embeddings in the situation $a(x) = 1 + a_{k-1}x^{k-1} + \dots$ with $a_{k-1} = n_1\pi_k(1)$ for some $n_1 \in \mathbb{N}$ are described in [5, Theorem 12]. We have $\alpha_0(s) = e^{\mu s}$. For $\mu \neq 0$ or $\mu = 0$ the results follow from Theorem 8.4.(a) and Theorem 8.4.(b). In the analytic case the additive function A does not appear, since A is of the form $c\pi_k$ and, therefore, it can be included in $Q(s, x)$.

References

[1] J. Aczél, B. Forte, and C.T. Ng. L'équation fonctionnelle triangulaire; son application á une géneralisation de l'équation de Cauchy. *C. R. Acad. Sci. Paris Sér. A-B*, 275:A605–A607, 1972.

[2] H. Cartan. *Elementary theory of analytic functions of one or several complex variables.* Addison- Wesley Publishing Company, Reading (Mass.), Palo Alto, London, 1963.

[3] H. Fripertinger and L. Reich. On a linear functional equation for formal power series. *Sitzungsberichte der Österr. Akademie der Wissenschaften, Abt. II*, 210:85–134, 2001.

[4] H. Fripertinger and L. Reich. On covariant embeddings of a linear functional equation with respect to an analytic iteration group. *International Journal of Bifurcation and Chaos*, 13(7):1853–1875, 2003.

[5] H. Fripertinger and L. Reich. On covariant embeddings of a linear functional equation with respect to an analytic iteration group in some non-generic cases. *Aequationes Math.*, 68:60–87, 2004.

[6] H. Fripertinger and L. Reich. On the general solution of the system of cocycle equations without regularity conditions. *Aequationes Math.*, 68:200–229, 2004.

[7] G. Guzik. On embedding of a linear functional equation. *Rocznik Nauk.-Dydakt. WSP w Krakowie. Prace Matematyczne*, 16:23–33, 1999.

[8] G. Guzik. On continuity of measurable cocycles. *Journal of Applied Analysis*, 6(2):295–302, 2000.

[9] G. Guzik. On embeddability of a linear functional equation in the class of differentiable functions. *Grazer Math. Ber.*, 344:31–42, 2001.

[10] G. Guzik, W. Jarczyk, and J. Matkowski. Cocycles of continuous iteration semigroups. *Bull. Polish Acad. Sci. Math.*, 51(2):185–197, 2003.

[11] P. Henrici. *Applied and computational complex analysis. Vol. I: Power series, integration, conformal mapping, location of zeros.* John Wiley & Sons, New York etc., 1974.

[12] W. Jabłoński and L. Reich. On the solutions of the translation equation I. To appear in Abh. Math. Sem. Univ. Hamburg.

[13] W. Jabłoński and L. Reich. On the solutions of the translation equation II. In preparation.

[14] M. Kuczma. *Functional equations in a single variable*, volume 46 of *Monografie Mat.* Polish Scientific Publishers, Warsaw, 1968.

[15] M. Kuczma, B. Choczewski, and R. Ger. *Iterative Functional Equations*, volume 32 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1990.

[16] Z. Moszner. Sur le prolongement covariant d'une équation linéaire par rapport au groupe d'itération. *Sitzungsberichte ÖAW, Math.-nat. Kl. Abt. II*, 207:173–182, 1999.

[17] L. Reich. 24. Remark in The Thirty-fifth International Symposium on Functional Equations, September 7-14, 1997, Graz-Mariatrost, Austria. *Aequationes Math.*, 55:311–312, 1998.

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