

# Enumeration of non-isomorphic canons

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## Abstract

In this note we describe how to apply group actions in order to enumerate the isomorphism classes of rhythmic canons.

The notion of a rhythmic canon was originally formalized by O. Messiaen. He proposed to study canonical musical forms in which the imitation between the different voices concerns only rhythmical aspects, independent from melody or harmony. The first mathematical investigation of rhythmic canons goes back to Vuza's papers [7, 9, 8, 10] in which he actually introduced even much more complicated families of canons, the "Regular Complementary Canons of Maximal Category" (in short RCMC-canons). Their definition will be given below.

Before describing rhythmic canons in cyclic time, we view them as they may appear within a musical composition, in free linear time which has no cyclicity. Like in a melodic canon, one has several voices that may enter one after the other until all voices are present. As in the case of a melodic canon, all voices are just copies of a ground voice that is suitably translated in the time axis. For simplicity we suppose here that all voices are extended in both directions of the time axis ad infinitum. We further suppose that the ground voice is a periodic rhythm that we will call the *inner rhythm*. Following Vuza's definition, a *periodic rhythm* is an infinite subset  $R$  of the rationals  $\mathbb{Q}$  (marking the attack times, or onsets) with  $R = R + d$  for a suitable period  $d$ . Furthermore,  $R$  is supposed to be locally finite (i.e. the intersection of  $R$  with every time segment  $[a, b]$  (of finite length  $b - a \geq 0$ ) is finite). The *period* of a periodic rhythm  $R$  is the smallest positive rational number  $d = d(R)$  satisfying  $R = R + d$ . We also mention another important characteristic of a periodic rhythm, its *pulsation*  $p = p(R)$ . It is defined as the  $\max \{q \in \mathbb{Q}_+ \mid \forall r \in R - R \exists z \in \mathbb{Z} : r = zq\}$ , in other words, it is the biggest rational  $q$  such that all distances between the attack points of  $R$  are integer multiples of  $q$ . Obviously, the period  $d$  is an integer multiple of the pulsation  $p$ .

Let  $A$  be the set of all rational numbers  $a$  such that the attack times of the voices of the canon can be expressed as  $R + a$ . Then  $A$  itself is a periodic rhythm, called the *outer rhythm* of the canon, with period  $d(A)$ , where  $d(R)$  is an integer multiple of  $d(A)$ . Note that  $R$  and  $A$  may have different pulsations  $p(R)$  and  $p(A)$ . Hence, the pulsation of a canon is defined as the greatest rational number  $p$  such that both  $p(R)$  and  $p(A)$  are integer multiples of  $p$ .

In order to switch from linear time to circular time the fraction  $n = d(R)/p$  must be computed. Let  $r$  denote a fixed attack time within the inner rhythm  $R$ . Then each

attack time in any of the voices is of the form  $r + tp$  for a suitable integer  $t$ , i.e. the whole canon is contained in  $r + p\mathbb{Z} \subset \mathbb{Q}$ . Because everything is periodic with period  $d(R)$ , we can restrict to the factor space  $(r + p\mathbb{Z})/d(R)\mathbb{Z}$  which may be identified with  $\mathbb{Z}/n\mathbb{Z}$ , the residue class ring of  $\mathbb{Z}$  modulo  $n\mathbb{Z}$ .

From now on, we consider the whole canon within  $Z_n := \mathbb{Z}/n\mathbb{Z}$  and assume that  $R$ ,  $A$ , and  $V_a$  denote the projections of the inner rhythm, the outer rhythm, and the voices of a canon.

The concept of a canon used in the present paper is described by G. Mazzola in [4] and was presented by him to the author in the following way: A *canon* is a subset  $K \subseteq Z_n$  together with a covering of  $K$  by pairwise different subsets, the voices,  $V_i \neq \emptyset$  for  $1 \leq i \leq t$ , where  $t \geq 1$  is the number of voices of the canon,

$$K = \bigcup_{i=1}^t V_i,$$

such that for all  $i, j \in \{1, \dots, t\}$

1. the set  $V_i$  can be obtained from  $V_j$  by a translation of  $Z_n$ ,
2. there is only the identity translation which maps  $V_i$  to  $V_i$ ,
3. the set of differences in  $K$  generates  $Z_n$ , i.e.  $\langle K - K \rangle := \langle k - l \mid k, l \in K \rangle = Z_n$ .

We prefer to write a canon  $K$  as a set of its subsets  $V_i$ . Two canons  $K = \{V_1, \dots, V_t\}$  and  $L = \{W_1, \dots, W_s\}$  are called *isomorphic* if  $s = t$  and if there exists a translation  $T$  of  $Z_n$  and a permutation  $\pi$  in the symmetric group  $S_t$  such that  $T(V_i) = W_{\pi(i)}$  for  $1 \leq i \leq t$ . Then obviously  $T(K) = L$ .

The aim of this note is to determine the number of non-isomorphic canons for given  $n$ . First we present the definition of canons in more details and give a short description of group actions, which are the standard tool to describe isomorphism classes of different objects (cf.[2, 3]). Since we are interested in the attack times modulo  $n$ , we assume that  $K$  is a subset of  $Z_n$ . The set of translations of  $Z_n$  is the cyclic group  $C_n$  generated by the permutation  $\sigma_n := (0, 1, \dots, n-1)$  which maps  $i$  to  $i+1 \pmod n$ .  $C_n$  acts in a natural way both on  $Z_n$  and on the set of all subsets of  $Z_n$ :

$$\sigma_n(i) = i + 1, \quad i \in Z_n, \quad \text{and} \quad \sigma_n(A) = \{\sigma_n(i) \mid i \in A\}, \quad A \subseteq Z_n.$$

As usual we identify a subset  $A$  of  $Z_n$  with its *characteristic function*  $\chi_A : Z_n \rightarrow \{0, 1\}$  given by

$$\chi_A(i) = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{otherwise.} \end{cases}$$

The set of all functions from a set  $X$  to a set  $Y$  will be indicated as  $Y^X$ . For arbitrary  $f \in \{0, 1\}^{Z_n}$  let  $\bar{f}$  denote the subset  $f^{-1}(\{1\})$  of  $Z_n$ , whence  $f = \chi_{\bar{f}}$ .

Sometimes it is convenient to write functions  $f$  from  $Z_n$  to  $\{0, 1\}$  as vectors of the form  $(f(0), f(1), \dots, f(n-1))$  using the natural order of the elements of  $Z_n$ , because

then the set of all these functions is totally ordered by the *lexicographic order*. For functions  $f, g \in \{0, 1\}^{Z_n}$  we say  $f < g$  if there exists an  $i \in Z_n$  such that  $f(j) = g(j)$  for  $0 \leq j < i$  and  $f(i) < g(i)$ . The group action of  $C_n$  on the set of all subsets of  $Z_n$  described as an action on the set of all characteristic function is the following

$$C_n \times \{0, 1\}^{Z_n} \rightarrow \{0, 1\}^{Z_n} \quad (\sigma_n^j, f) \mapsto f \circ \sigma_n^{-j}.$$

As the *canonical representative* of the orbit  $C_n(f) = \{f \circ \sigma_n^j \mid 0 \leq j \leq n - 1\}$  we choose the function  $f_0 \in C_n(f)$  such that  $f_0 \leq g$  for all  $g \in C_n(f)$ . Moreover, we choose  $\bar{f}_0$  as the canonical representative of the orbit  $C_n(\bar{f}) = \{\sigma_n^j(\bar{f}) \mid 0 \leq j \leq n - 1\}$ . In general, representatives of orbits  $C_n(f)$  of functions  $f$  under the action of the cyclic group  $C_n$  are called *necklaces*.

**Lemma 1.** *If  $f \neq 0$  denotes a function from  $Z_n$  to  $\{0, 1\}$ , then the canonical representative  $f_0$  of the orbit  $C_n(f)$  fulfills  $f_0(n - 1) = 1$ .*

A function  $f \in \{0, 1\}^{Z_n}$  (or the corresponding vector and the set  $\bar{f}$ ) is called *acyclic* if  $C_n(f)$  consists of  $n$  different objects. The canonical representative of the orbit of an acyclic function is usually called a *Lyndon word*. If  $f$  is acyclic, then all elements in the orbit  $C_n(f)$  are acyclic as well, and we call  $C_n(f)$  an *acyclic orbit*.

From the first two properties of a canon we derive that all voices  $V_i$  belong to the same orbit  $C_n(V_1)$  and that each voice is acyclic. For that reason, we can describe a canon  $K = \{V_1, \dots, V_t\}$  as a pair  $(L, A)$  where  $L$  is the Lyndon word, which is the canonical representative of the orbit  $C_n(\chi_{V_1})$ , and  $A = \{a_1, \dots, a_t\}$  is a  $t$ -subset of  $Z_n$  such that  $V_i = \sigma_n^{a_i}(L)$ . In other words,  $L$  describes the inner rhythm and  $A$  the outer rhythm of  $K$ .

Now we analyse in which situations a pair  $(L, A)$ , as described above, is not a canon, i.e. when it fails to fulfill the third property. Assume that  $(L, A)$  is not a canon, then the differences  $K - K$  generate a proper subgroup of  $Z_n$ . This subgroup is isomorphic to  $Z_{n/d}$  where  $d > 1$  is a divisor of  $n$ . In other words,  $d$  is a divisor of all differences  $k - l$  for all  $k, l \in K$ . We write  $d \mid r$  in order to express that the integer  $d$  is a divisor of the integer  $r$ . (It is not connected with division in the ring  $Z_n$ .) It is easy to prove the following

**Lemma 2.** *Let  $L \neq 0$  be a Lyndon word of length  $n$ , and let  $A$  be a  $t$ -subset of  $Z_n$ . The pair  $(L, A)$  does not describe a canon in  $Z_n$  if and only if there exists an integer  $d > 1$  such that  $d \mid n$ ,  $d \mid k - l$  for all  $k, l \in \bar{L}$ , and  $d \mid k - l$  for all  $k, l \in A$ .*

Consider the set  $S$  of all pairs  $(L, A)$  where  $L$  is a Lyndon word and  $A$  is a  $t$ -subset of  $Z_n$ . The group of translations of  $Z_n$  acts on  $S$  according to

$$C_n \times S \rightarrow S \quad (\sigma_n^j, (L, A)) \mapsto (L, \sigma_n^j(A)).$$

If  $(L, A)$  is a canon, then the orbit  $C_n(L, A) = (L, C_n(A))$  describes the isomorphism class of  $(L, A)$ . In general, the canonical representative of  $C_n(L, A)$  is  $(L, A_0)$  where  $A_0$  is the canonical representative of  $C_n(A)$ . From Lemma 1 we already know that  $\chi_{A_0}(n - 1) = 1$  and  $L(n - 1) = 1$  if  $L \neq 0$ . This together with Lemma 2 proves

**Lemma 3.** *Let  $L \neq 0$  be a Lyndon word of length  $n$ , and let  $A$  be a  $t$ -subset of  $Z_n$ . The pair  $(L, A)$  does not describe a canon in  $Z_n$  if and only if there exists a divisor  $d > 1$  of  $n$  such that  $L(i) = 1$  implies  $i \equiv d - 1 \pmod{d}$ , and  $\chi_{A_0}(i) = 1$  implies  $i \equiv d - 1 \pmod{d}$ , where  $A_0$  is the canonical representative of  $C_n(A)$ .*

In the next part we show that functions  $f \in \{0, 1\}^{Z_n}$  with the property  $f(i) = 1$  implies  $i \equiv -1 \pmod{d}$  (for  $d \mid n, d > 1$ ) can be constructed from functions in  $\{0, 1\}^{Z_{n/d}}$ . Let  $\psi_d$  be a function defined on  $\{0, 1\}$ , such that  $\psi_d(0)$  is the vector  $(0, 0, \dots, 0)$  consisting of  $d$  entries of 0, and  $\psi_d(1) = (0, \dots, 0, 1)$  is a vector consisting of  $d - 1$  entries of 0 and 1 in the last position. We write the values of  $\psi_d$  in the form

$$\psi_d(0) = 0^d, \quad \psi_d(1) = 0^{d-1}1.$$

If we apply  $\psi_d$  to each component of a vector  $f \in \{0, 1\}^{Z_r}$  by replacing each component 0 in  $f$  by  $0^d$  and each component 1 by  $0^{d-1}1$ , we get a vector  $\psi_d(f) \in \{0, 1\}^{Z_{rd}}$  and

$$\psi_d(\{0, 1\}^{Z_r}) = \left\{ f \in \{0, 1\}^{Z_{rd}} \mid f(i) = 1 \text{ implies } i \equiv -1 \pmod{d} \right\}.$$

From Lemma 3 we conclude that  $(L, A)$  does not describe a canon if and only if there exists a  $d > 1$  which divides  $n$  such that both  $L$  and  $\chi_{A_0}$  are elements of  $\psi_d(\{0, 1\}^{Z_{n/d}})$ .

Some properties of  $\psi_d$  are collected in the next

**Lemma 4.** *Let  $f, g$  be functions from  $Z_r$  to  $\{0, 1\}$ . Then*

1.  $\psi_d(f) = \psi_d(g)$  if and only if  $f = g$ .
2.  $\psi_d(f \circ \sigma_r) = \psi_d(f) \circ \sigma_{rd}^d$ .
3.  $\psi_d(C_r(f)) \subset C_{rd}(\psi_d(f))$ .
4.  $f < g$  if and only if  $\psi_d(f) < \psi_d(g)$ .
5.  $f_0$  is the canonical representative of  $C_r(f)$  if and only if  $\psi_d(f_0)$  is the canonical representative of  $C_{rd}(\psi_d(f))$ .
6.  $\psi_d$  describes a bijection between the  $C_r$ -orbits on  $\{0, 1\}^{Z_r}$  and the  $C_{rd}$ -orbits on  $\{0, 1\}^{Z_{rd}}$  which have non-empty intersection with  $\psi_d(\{0, 1\}^{Z_r})$ , thus

$$\begin{aligned} & |\{C_{rd}(A) \mid A \subseteq Z_{rd}, A \neq \emptyset, d \mid k - l \forall k, l \in A\}| = \\ & |\{C_r(A) \mid A \subseteq Z_r, A \neq \emptyset\}| =: \alpha(r). \end{aligned}$$

7.  $f \neq 0$  is acyclic if and only if  $\psi_d(f)$  is acyclic.
8.  $\psi_d$  describes a bijection between the acyclic  $C_r$ -orbits on  $\{0, 1\}^{Z_r}$  and the acyclic  $C_{rd}$ -orbits on  $\{0, 1\}^{Z_{rd}}$  which have non-empty intersection with  $\psi_d(\{0, 1\}^{Z_r})$ , thus

$$\begin{aligned} & |\{C_{rd}(A) \mid A \subseteq Z_{rd}, A \neq \emptyset, d \mid k - l \forall k, l \in A, A \text{ acyclic}\}| = \\ & |\{C_r(A) \mid A \subseteq Z_r, A \neq \emptyset, A \text{ acyclic}\}| =: \lambda(r). \end{aligned}$$

9. Both  $f$  and  $\psi_d(f)$  have the same number of components which are 1.

**Proof.** Since  $\psi_d$  is an injective mapping from each component of  $f$  into the set  $\{0^d, 0^{d-1}1\}$ , the first statement is clear.

In order to show that the second item is true, assume that  $\psi_d(f \circ \sigma_r)(j) = 1$  for some  $j \in Z_{rd}$ . According to the definition of  $\psi_d$ , this is equivalent to  $j = sd - 1$  and  $(f \circ \sigma_r)(s - 1) = 1$  for some  $s \in \{1, \dots, r\}$ . In other words,  $j = sd - 1$  and  $f(s) = 1$ , which is equivalent to  $j = sd - 1$  and  $\psi_d(f)((s + 1)d - 1) = 1$ . This, however, is the same as  $\psi_d(f)(j + d) = 1$ , whence  $(\psi_d(f) \circ \sigma_{rd}^d)(j) = 1$ . If  $f = 0$  the assertion is always true.

The third part is an immediate consequence of the second.

If  $f < g$ , then there exists  $i \in Z_r$  such that  $f(j) = g(j)$  for all  $j < i$  and  $f(i) < g(i)$ , whence  $f(i) = 0$  and  $g(i) = 1$ . For that reason,  $\psi_d(f(i)) = 0^d$  and  $\psi_d(g(i)) = 0^{d-1}1$ . Hence,  $\psi_d(f)(j) = \psi_d(g)(j)$  for  $j < id - 1$  and  $\psi_d(f)(id - 1) = 0 < 1 = \psi_d(g)(id - 1)$ , and consequently  $\psi_d(f) < \psi_d(g)$ . If, conversely,  $\psi_d(f) < \psi_d(g)$ , then there exists an  $i \in Z_{rd}$  such that  $\psi_d(f)(j) = \psi_d(g)(j)$  for all  $j < i$  and  $\psi_d(f)(i) < \psi_d(g)(i)$ , whence  $\psi_d(g)(i) = 1$  and  $i \equiv -1 \pmod{d}$ . Assume that  $i$  is of the form  $sd - 1$ . Then  $\psi_d(f(j)) = \psi_d(g(j))$  for  $j < s$ , and  $\psi_d(f(s)) = 0^d$  whereas  $\psi_d(g(s)) = 0^{d-1}1$ . Since  $\psi_d$  is an injective mapping,  $f(j) = g(j)$  for  $j < s$  and  $f(s) = 0 < 1 = g(s)$ , which implies that  $f < g$ .

From the definition of the canonical representative of an orbit and from the items 4. and 2. it follows immediately that  $\psi_d(f_0) < \psi_d(f_0 \circ \sigma_r^j) = \psi_d(f_0) \circ \sigma_{rd}^{dj}$  for all  $j \in \{1, \dots, r - 1\}$ . Moreover, if  $n \not\equiv 0 \pmod{d}$ , then  $(\psi_d(f_0) \circ \sigma_{rd}^n)(rd - 1) = \psi_d(f_0)(n - 1) \neq 1$ , since  $n - 1 \not\equiv -1 \pmod{d}$ . According to Lemma 2, the representative  $\psi_d(f_0) \circ \sigma_{rd}^n$  cannot be the canonical representative of the orbit  $C_{rd}(\psi_d(f_0))$ . Hence, the canonical representative is  $\psi_d(f_0)$ .

Then 6. is a trivial consequence of 5. Similar arguments can be applied for proving 7. and 9. Item 8. follows immediately from 7.  $\square$

In order to compute the number of all  $C_r$ -orbits on  $\{0, 1\}^{Z_r}$  we apply *Pólya's theory* cf. [2, 3, 5, 6]. In the present situation we have to compute the cycle index of the group  $C_r$  acting on  $Z_r$  which is a polynomial in  $z_1, \dots, z_r$  over  $\mathbb{Q}$  given by

$$C(C_r, Z_r) = \frac{1}{r} \sum_{s|r} \varphi(s) z_s^{r/s},$$

where  $\varphi$  is the *Euler totient function*. Replacing each indeterminate  $z_i$  in  $C(C_r, Z_r)$  by  $|\{0, 1\}|$ , which is equal to 2, we compute the number of all  $C_r$ -orbits on  $\{0, 1\}^{Z_r}$  as

$$C(C_r, Z_r, z_i := 2) = \frac{1}{r} \sum_{s|r} \varphi(s) 2^{r/s}.$$

If we replace  $z_i$  by  $1 + z^i$ , where  $z$  is an indeterminate over  $\mathbb{Q}$ , then the number of  $C_r$ -orbits of  $k$ -sets  $A \subseteq Z_r$  is the coefficient of  $z^k$  in

$$C(C_r, Z_r, z_i := 1 + z^i) = \frac{1}{r} \sum_{s|r} \varphi(s) (1 + z^i)^{r/s}.$$

In conclusion (for the definition of  $\alpha(r)$  see 6. in Lemma 4)

$$\alpha(r) = C(C_r, Z_r, z_i := 2) - 1,$$

since we don't count the orbit of the empty set. Similar methods can be applied to enumerate the number of acyclic  $C_r$ -orbits on  $\{0, 1\}^{Z_r}$  or, what is equivalent to this, to enumerate all Lyndon words of length  $r$  over the alphabet  $\{0, 1\}$ . For  $r > 1$ , this number is given as (for the definition of  $\lambda(r)$  see 8. in Lemma 4)

$$\lambda(r) = \frac{1}{r} \sum_{s|r} \mu(s) 2^{r/s},$$

where  $\mu$  is the *classical Moebius function*. Moreover, the number of Lyndon words of length  $r$  over  $\{0, 1\}$  having  $r_1$  components 1 and  $r - r_1$  components 0 is

$$\frac{1}{r} \sum_{s|\gcd(r-r_1, r_1)} \mu(s) \binom{r/s}{r_1/s}.$$

And the generating function of Lyndon words of length  $r$  over the alphabet  $\{0, 1\}$  with  $r_1$  components 1 is of the form

$$\frac{1}{r} \sum_{r_1=1}^{r-1} \left( \sum_{s|\gcd(r-r_1, r_1)} \mu(s) \binom{r/s}{r_1/s} \right) y^{r_1}.$$

In the case  $r = 1$  there are two Lyndon words, namely  $f_0 = (0)$  and  $f_1 = (1)$ . The first one is the characteristic function of the empty set in  $Z_1$  so we don't want to count it. For that reason, we must set  $\lambda(1) = 1$ . Moreover, it should be mentioned that  $f_0$  is the unique Lyndon word such that  $\psi_d(f_0)$  is not a Lyndon word for  $d > 1$ .

Finally, we have to combine all these results for enumerating the isomorphism classes of canons. Let  $n \geq 1$  be an integer. For any divisor  $d \geq 1$  of  $n$ , let  $M_{n,d}$  be the set of all pairs  $(L, C_n(A))$ , where  $L$  is a Lyndon word of length  $n$  over  $\{0, 1\}$ , (in the case  $n = 1$  different from 0) such that  $L(i) = 1$  implies  $i \equiv -1 \pmod{d}$ , and  $A$  is a non-empty subset of  $Z_n$ , such that  $d \mid k - l$  for all  $k, l \in A$ . Hence

$$M_{n,d} = \left\{ (L, C_n(A)) \mid \begin{array}{l} L \text{ Lyndon word, } L \neq 0, L(i) = 1 \Rightarrow i \equiv -1 \pmod{d} \\ A \subseteq Z_n, A \neq \emptyset, d \mid k - l \forall k, l \in A \end{array} \right\}.$$

From Lemma 4 we deduce that  $\psi_d$  describes a bijection between  $M_{n,d}$  and  $M_{n/d,1}$ , thus

$$|M_{n,d}| = |M_{n/d,1}| = \lambda(n/d) \alpha(n/d),$$

which is the number of possibilities to choose  $L$  and  $C_n(A)$  according to the desired properties of  $M_{n,d}$ . Finally, let  $\kappa_n$  be the set of isomorphism classes of canons in  $Z_n$ ,

$$\kappa_n = \{(L, C_n(A)) \in M_{n,1} \mid (L, A) \text{ describes a canon}\}.$$

From Lemma 3 we deduce that

$$\kappa_n = M_{n,1} \setminus \bigcup_{\substack{d>1 \\ d|n}} M_{n,d}.$$

**Theorem.** *The number of isomorphism classes of canons in  $Z_n$  is*

$$|\kappa_n| = \sum_{d|n} \mu(d) \lambda(n/d) \alpha(n/d).$$

**Proof.** First we prove that the set  $M_{n,1}$  is the disjoint union

$$M_{n,1} = \dot{\bigcup}_{d|n} \psi_d(\kappa_{n/d}),$$

where

$$\psi_d(\kappa_{n/d}) = \{ \psi_d(L, C_{n/d}(A)) \mid (L, C_{n/d}(A)) \in \kappa_{n/d} \}$$

and

$$\psi_d(L, C_{n/d}(A)) = (\psi_d(L), C_n(\overline{\psi_d(\chi_A)})).$$

It is clear that  $M_{n,1}$  contains this union. Moreover, this union is disjoint, since for a canon  $K = (L, C_{n/d}(A)) \in \kappa_{n/d}$  we have

$$\langle K - K \rangle = Z_{n/d} \cong \langle \psi_d(K) - \psi_d(K) \rangle.$$

Finally, we have to show that each element of  $M_{n,1}$  belongs to this union. If  $(L, C_n(A)) \in M_{n,1}$ , then choose the biggest  $d$  such that  $(L, C_n(A))$  belongs to  $M_{n,d}$ . (I.e.  $(L, C_n(A))$  does not belong to  $M_{n,d'}$  for  $d' > d$ .) It is always possible to find such  $d$ , since, if  $(L, C_n(A))$  belongs both to  $M_{n,d_1}$  and  $M_{n,d_2}$ , then it also belongs to  $M_{n,\text{lcm}(d_1,d_2)}$ . Then there exists  $(L', C_{n/d}(A')) \in \kappa_{n/d}$  such that  $L = \psi_d(L')$  and  $\chi_{A_0} = \psi_d(\chi_{A'_0})$  for the canonical representatives  $A_0$  and  $A'_0$  of  $C_n(A)$  and  $C_{n/d}(A')$ . Moreover,  $(L', C_{n/d}(A')) \in \kappa_{n/d}$  because otherwise there would be some  $d' > 1$  which is a divisor of  $n/d$  such that  $(L', C_{n/d}(A')) \in M_{n/d,d'}$ . But then  $(L, C_n(A))$  also belongs to  $M_{n,dd'}$ , which is a contradiction to the choice of  $d$ .

Hence,

$$|M_{n,1}| = \sum_{d|n} |\kappa_{n/d}| = \sum_{d|n} |\kappa_d|,$$

and by Moebius inversion we get

$$|\kappa_n| = \sum_{d|n} \mu(n/d) |M_{d,1}| = \sum_{d|n} \mu(d) |M_{n/d,1}| = \sum_{d|n} \mu(d) |M_{n,d}| = \sum_{d|n} \mu(d) \lambda(n/d) \alpha(n/d),$$

which finishes the proof.  $\square$

If  $\lambda$  and  $\alpha$  are replaced by the corresponding generating functions

$$\bar{\lambda}(r) = \begin{cases} \frac{1}{r} \sum_{r_1=1}^{r-1} \left( \sum_{s|\gcd(r-r_1, r_1)} \mu(s) \binom{r/s}{r_1/s} \right) y^{r_1} & \text{if } r > 1 \\ y & \text{if } r = 1 \end{cases}$$

and

$$\bar{\alpha}(r) = C(C_r, Z_r, z_i := 1 + z^i) - 1,$$

then the coefficient of  $y^s z^t$  in

$$\sum_{d|n} \mu(d) \bar{\lambda}(n/d) \bar{\alpha}(n/d)$$

gives the number of non-isomorphic canons in  $Z_n$  which consist of  $t$  voices where each voice contains exactly  $s$  attack times.

For example the numbers of isomorphism classes of canons in  $Z_n$  for  $1 \leq n \leq 10$  are:

$n$	1	2	3	4	5	6	7	8	9	10
$ \kappa_n $	1	1	5	13	41	110	341	1035	3298	10550

Finally, we have a closer look at the 13 isomorphism classes of canons in  $Z_4$ , which can be classified according to their number of voices and attack times per voice by  $z^4 y^3 + z^4 y^2 + z^4 y + z^3 y^3 + z^3 y^2 + z^3 y + 2z^2 y^3 + 2z^2 y^2 + z^2 y + z y^3 + z y^2$ . From this list we can read, for instance, that there are exactly three classes of canons with 4 voices which have 1, 2, or 3 attack times per voice. Or we see that there are five classes of canons with 2 voices, two of them having 3 attack times per voice, two of them having 2 attack times per voice, and only one having 1 attack time in each of its two voices.

We can even compute the canonical representative of each isomorphism class. For instance, there are exactly three Lyndon words  $L_1, L_2, L_3$  of length 4 over  $\{0, 1\}$  and five representatives  $f_1, \dots, f_5$  of the  $C_4$ -orbits of non empty subsets of  $Z_4$ :

$$\begin{array}{ll} L_1 = (0, 0, 0, 1) & f_1 = (0, 0, 0, 1) \\ L_2 = (0, 0, 1, 1) & f_2 = (0, 0, 1, 1) \\ L_3 = (0, 1, 1, 1) & f_3 = (0, 1, 0, 1) \\ & f_4 = (0, 1, 1, 1) \\ & f_5 = (1, 1, 1, 1) \end{array}$$

The Lyndon words describe the distribution of the attack times in one voice (i.e. the inner rhythm of the canon), the necklaces describe the distribution of the voices over the complete period, here in this example over  $Z_4$ , which is the outer rhythm. Only  $L_1$  can be constructed as  $\psi_4((1))$  or  $\psi_2((0, 1))$  from shorter Lyndon words, so all pairs  $(L_i, f_j)$  for  $i \in \{2, 3\}$  and  $j \in \{1, \dots, 5\}$  describe canons in  $Z_4$ . Since  $f_1 = \psi_4((1)) = \psi_2((0, 1))$  and  $f_3 = \psi_2((1, 1))$ , and  $(1), (0, 1)$ , and  $(1, 1)$  are necklaces of length 1 or 2 over  $\{0, 1\}$ , the pairs  $(L_1, C_4(f_1)) = \psi_4((1), C_1(1))$  and  $(L_1, C_4(f_3)) = \psi_2((0, 1), C_2(1, 1))$  do not belong to  $\kappa_4$ . But the pairs  $(L_1, f_j)$  for  $j \in \{2, 4, 5\}$  describe again canons in  $Z_4$ .



From this example we also see how to compute complete lists of isomorphism classes of canons in  $Z_n$  for arbitrary  $n$ . There exist fast algorithms for computing all Lyndon words and all necklaces of length  $n$  over  $\{0, 1\}$ . Then each pair  $(L_i, f_j)$  must be tested whether there exists some  $d > 1$  such that  $(L_i, f_j) = (\psi_d(L'), \psi_d(f'))$  for a Lyndon word  $L'$  and a necklace  $f'$  of length  $n/d$ .

There exist more complicated definitions of canons. A pair  $(R, A)$  of inner and outer rhythm defines a *rhythmic tiling canon* with voices  $V_a$  for  $a \in A$  in  $Z_n$  if and only if

1. the voices  $V_a$  cover entirely the cyclic group  $Z_n$ ,
2. the voices  $V_a$  are pairwise disjoint.

It is obvious that periods and pulsations can be determined in cyclic time as well. Rhythmic tiling canons with the additional property,

3. the periods  $d(R)$  and  $d(A)$  coincide,

are called *regular complementary canons of maximal category*. So far we were not dealing with that kind of canons.

**Acknowledgment:** The author wants to express his thanks to Guerino Mazzola for pointing his attention to the problem of classification of canons. He is also grateful to Moreno Andreatta, who provided very useful historic and background information about different definitions of canons (cf [1]).

## References

- [1] M. Andreatta, Th. Noll, C. Agon, and G. Assayag. The geometrical groove: rhythmic canons between theory, implementation and musical experiment. In *Les Actes des 8e Journées d'Informatique Musicale, Bourges 7-9 juin 2001*, pages 93–97, 2001.
- [2] A. Kerber. *Algebraic Combinatorics via Finite Group Actions*. B.I. Wissenschaftsverlag, Mannheim, Wien, Zürich, 1991. ISBN 3-411-14521-8.
- [3] A. Kerber. *Applied Finite Group Actions*, volume 19 of *Algorithms and Combinatorics*. Springer, Berlin, Heidelberg, New York, 1999. ISBN 3-540-65941-2.
- [4] G. Mazzola. Topos of music. To appear.
- [5] G. Pólya. Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen. *Acta Mathematica*, 68:145 – 254, 1937.
- [6] G. Pólya and R.C. Read. *Combinatorial Enumeration of Groups, Graphs and Chemical Compounds*. Springer Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, 1987. ISBN 0-387-96413-4 or ISBN 3-540-96413-4.

- [7] D.T. Vuza. Supplementary sets and regular complementary unending canons (part one). *Perspectives of New Music*, 29(2):22–49, 1991.
- [8] D.T. Vuza. Supplementary sets and regular complementary unending canons (part three). *Perspectives of New Music*, 30(2):102–125, 1992.
- [9] D.T. Vuza. Supplementary sets and regular complementary unending canons (part two). *Perspectives of New Music*, 30(1):184–207, 1992.
- [10] D.T. Vuza. Supplementary sets and regular complementary unending canons (part four). *Perspectives of New Music*, 31(1):270–305, 1993.

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