Enumeration of Linear Codes by Applying Methods from Algebraic Combinatorics

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Abstract

It is demonstrated how classes of linear (n, k)-codes can be enumerated using cycle index polynomials and other methods from algebraic combinatorics.

Some results of joined work [9] with Prof. Kerber from the University of Bayreuth on the enumeration of linear codes over $GF(q)$ are presented. Furthermore I will give an introduction to enumeration under finite group actions.

At first let me draw your attention to the enumeration of linear codes. Let p be a prime and let q be a power of p then $GF(q)$ denotes the finite field of q elements. A *linear* (n, k)*-code* over the Galois field $GF(q)$ is a k-dimensional subspace of the vector space $GF(q)^n$. As usual codewords will be written as rows $x = (x_1, \ldots, x_n)$. A $k \times n$-matrix Γ over $GF(q)$ is called *a generator matrix* of the linear (n, k)-code C, if and only if the rows of Γ form a basis of C, so that $C = \{x \cdot \Gamma \mid x \in GF(q)^k\}$. The *Hamming distance* $d(x, y) := |\{i \in \underline{n} \mid x_i \neq y_i\}|$ is a metric on $GF(q)^n$. (The set of integers from 1 to n will be indicated as $\underline{n}$.) The *minimal distance* $d(C)$ of a code C is given by

$$d(C) := \min_{(x,y)\in C^2,\ x\neq y} d(x, y).$$

It can be used to express the quality of a code. A *maximum likelihood decoding algorithm* for instance corrects $(d - 1)/2$ errors and detects $d - 1$ errors, when d is the minimal distance of the code.

In coding theory two linear (n, k)-codes C_1, C_2 are called *equivalent*, if and only if there is an isometry (with respect to the Hamming metric) which maps C_1 onto C_2. This means there is a linear isomorphism $\varphi\colon C_1 \to C_2$ such that $d(x, y) = d(\varphi(x), \varphi(y))$ for all $x, y \in C_1$. It can be shown that such a linear isometry between two linear (n, k)-codes can always be extended to an isometry of $GF(q)^n$. (See [10].) Let's investigate the structure of the group of all linear isometries of $GF(q)^n$. It is enough to consider

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an isometry φ acting on the standard basis $\{e_1,\dots,e_n\}$ where $e_i=(\delta_{i,j})_{j\in\underline{n}}$. Since φ is a homomorphism, $\varphi(e_i)=\sum_{j=1}^n \alpha_{i,j}e_j$, where $\alpha_{i,j}\in GF(q)$. Since φ is an isometry, for each i there is exactly one j such that $\alpha_{i,j}\neq 0$. This defines a function $\pi\colon \underline{n}\to\underline{n}$ such that $\alpha_{i,\pi(i)}\neq 0$ and $\varphi(e_i)=\alpha_{i,\pi(i)}e_{\pi(i)}$. Since φ is an isomorphism, π must be a permutation, i.e. $\pi\in S_{\underline{n}}$. Let's define $\psi(i):=\alpha_{i,\pi(i)}$, then ψ is a mapping from $\underline{n}$ to $GF(q)^*$ (where $GF(q)^*$ denotes the multiplicative group of the Galois field), and φ can be identified with the pair (ψ,π). Then the group of all isometries corresponds to the *wreath product*

$$GF(q)^*\wr S_{\underline{n}}=\left\{(\psi,\pi)\mid \psi\in GF(q)^{*\underline{n}},\pi\in S_{\underline{n}}\right\},$$

which is the semidirect product of $GF(q)^{*\underline{n}}$ and $S_{\underline{n}}$, where the multiplication is given by

$$(\psi,\pi)(\psi',\pi')=(\psi\psi'_\pi,\pi\pi'),$$

where $\psi\psi'_\pi(i)=\psi(i)\psi'_\pi(i)$ and $\psi'_\pi(i)=\psi'(\pi^{-1}i)$. (From now on $GF(q)^n$ will be identified with $GF(q)^{\underline{n}}$, the set of all mappings from $\underline{n}$ to $GF(q)$.) The *complete monomial group* $GF(q)^*\wr S_{\underline{n}}$ of degree n over $GF(q)^*$ acts on $GF(q)^{\underline{n}}$ in the form of the *exponentiation*, i.e.

$$GF(q)^*\wr S_{\underline{n}}\times GF(q)^{\underline{n}}\to GF(q)^{\underline{n}},\quad ((\psi,\pi),(x_1,\dots,x_n))\mapsto(\psi(1)x_{\pi^{-1}1},\dots,\psi(n)x_{\pi^{-1}n}).$$

For computing the number of *isometry classes* of linear (n,k)-codes we use methods from *algebraic combinatorics*, which shall be described now.
Let me start with the basic concept of a *finite group action*. (More details can be found in [11].) Let G denote a multiplicative finite group and X a nonempty set. A *finite group action* ${}_GX$ of G on X is described by a mapping

$$G\times X\to X,\qquad (g,x)\mapsto gx,$$

such that $g(g'x)=(gg')x$, and $1x=x$. In other words, there is a group homomorphism δ from G into the *symmetric group* S_X on X (i.e. the set of all permutations of X) which is called a *permutation representation of* G *on* X:

$$\delta\colon G\to S_X,\ g\mapsto\delta(g)=:\bar g,\ \text{where } \bar g(x):=gx.$$

A group action ${}_GX$ defines the following equivalence relation on X:

$$x\sim_G x' \text{ iff } \exists g\in G\colon\ x'=gx.$$

The equivalence classes are called *orbits*, and the orbit of $x\in X$ will be indicated as

$$G(x):=\{gx\mid g\in G\}.$$

The set of all orbits will be denoted by

$$G\backslash\backslash X:=\{G(x)\mid x\in X\}.$$

For each $x \in X$ the *stabilizer* G_x of x

$$G_x := \{g \mid gx = x\}$$

is a subgroup of G. The stabilizer of $y = gx$ is given by $G_y = gG_xg^{-1}$, so the stabilizers of all elements in the orbit of x form the conjugacy class of the subgroup G_x of G. The mapping $G(x) \to G/G_x$, $gx \mapsto gG_x$ is a bijection. So we conclude that

$$|G(x)| = |G|/|G_x|.$$

Finally the set of all fixed points of $g \in G$ is denoted by

$$X_g := \{x \mid gx = x\}.$$

The main lemma in the theory of enumeration under finite group actions is the so called *Lemma of* Cauchy Frobenius. It says that the number of orbits of a finite group G acting on a finite set X is equal to the average number of fixed points:

$$|G\backslash\backslash X| = \frac{1}{|G|}\sum_{g\in G}|X_g|.$$

Proof:

$$\sum_{g\in G}|X_g| = \sum_{g\in G}\sum_{x\in X_g}1 = \sum_{x\in X}\sum_{g\in G_x}1 = \sum_{x\in X}|G_x| = \sum_{x\in X}\frac{|G|}{|G(x)|}$$

$$= |G|\sum_{\omega\in G\backslash\backslash X}\sum_{x\in\omega}\frac{1}{|G(x)|} = |G|\sum_{\omega\in G\backslash\backslash X}\sum_{x\in\omega}\frac{1}{|\omega|} = |G|\sum_{\omega\in G\backslash\backslash X}1 = |G|\,|G\backslash\backslash X|\,.$$

Interesting examples for group actions can be found as actions on the set Y^X of all functions from X to Y, where the group action on Y^X is induced from actions on X or Y.

1. Let ${}_GX$ be a finite group action, then G acts on Y^X by the definition

   $$G \times Y^X \to Y^X, \qquad (g, f) \mapsto f \circ g^{-1}, \tag{1}$$

   and the number of G-orbits on Y^X is given by

   $$\left|G\backslash\backslash Y^X\right| = \frac{1}{|G|}\sum_{g\in G}|Y|^{c(\bar{g})}\,,$$

   where $c(\bar{g})$ is the number of cycles of the permutation $\bar{g} = \delta(g) \in S_X$.

   Proof: A function $f \in Y^X$ is a fixed point of g, if and only if $f(g^{-1}x) = f(x)$ for all $x \in X$, i.e. f is constant on each g-orbit (i.e. cycle of $\bar{g}$) on X, and $c(\bar{g})$ is the number of all these cycles.

2. Let ${}_HY$ be a finite group action, then H acts on Y^X by the definition
$$H \times Y^X \to Y^X, \qquad (h, f) \mapsto h \circ f, \tag{2}$$
and the number of H-orbits on Y^X is given by
$$\left| H\backslash\backslash Y^X \right| = \frac{1}{|H|} \sum_{h \in H} |Y_h|^{|X|}.$$

3. Let ${}_GX$ and ${}_HY$ be finite group actions, then the direct product $G \times H$ acts on Y^X by the definition
$$(G \times H) \times Y^X \to Y^X, \qquad ((g, h), f) \mapsto h \circ f \circ g^{-1}, \tag{3}$$
and the number of $G \times H$-orbits on Y^X is given by
$$\left| G \times H\backslash\backslash Y^X \right| = \frac{1}{|G|\,|H|} \sum_{(g,h) \in G \times H} \prod_{i=1}^{|X|} |Y_{h^i}|^{a_i(\bar{g})},$$
where $(a_1(\bar{g}), \ldots, a_{|X|}(\bar{g}))$ the *cycle type* of the permutation $\bar{g} \in S_X$ is. (I.e. $\bar{g}$ decomposes into a product of $a_i(\bar{g})$ pairwise disjoint cycles of length i for $i = 1, \ldots, |X|$.) Furthermore there is a bijection
$$(G \times H)\backslash\backslash Y^X \to G\backslash\backslash(H\backslash\backslash Y^X), \quad (G \times H)(f) \mapsto G(H(f)),$$
where G acts on $H\backslash\backslash Y^X$ by $g(H(f)) := H(f \circ g^{-1})$. This bijection is due to de Bruijn.

4. Let ${}_GX$ and ${}_GY$ be finite group actions, then G acts on Y^X by the definition
$$G \times Y^X \to Y^X, \qquad (g, f) \mapsto g \circ f \circ g^{-1}, \tag{4}$$
and the number of G-orbits on Y^X is given by
$$\left| G\backslash\backslash Y^X \right| = \frac{1}{|G|} \sum_{g \in G} \prod_{i=1}^{|X|} \left| Y_{g^i} \right|^{a_i(\bar{g})},$$
where $\bar{g}$ is the permutation representation of G on X. The group actions mentioned above can be considered as special cases of this group action. All the group actions mentioned so far can be restricted to group actions on the sets of all injective, surjective or bijective mappings from X to Y.

5. Let ${}_GX$ and ${}_HY$ be finite group actions, then the wreath product $H \wr_X G$ acts in form of the exponentiation on Y^X
$$(H \wr_X G) \times Y^X \to Y^X, \qquad ((\psi, g), f) \mapsto \tilde{f}, \tag{5}$$

where $\tilde{f}(x) = \psi(x) f(\pi^{-1}x)$. There is a bijection due to LEHMANN ([12, 13]) which reduces the action of a wreath product to the action of the group G on the set of all functions from X into the set of all orbits of H on Y:

$$\Phi: H \wr_X G \backslash\backslash Y^X \to G \backslash\backslash (H \backslash\backslash Y)^X, \quad (H \wr_X G(f)) \mapsto G(F),$$

where $F \in (H\backslash\backslash Y)^X$ is given by $F(x) = H(f(x))$, and G acts on $(H\backslash\backslash Y)^X$ by $g(F) := F \circ g^{-1}$. So the number of $H \wr_X G$-orbits on Y^X is given by

$$\left| (H \wr_X G) \backslash\backslash Y^X \right| = \frac{1}{|G|} \sum_{g \in G} |H\backslash\backslash Y|^{c(\bar{g})} .$$

All the group actions above are special cases of this group action.

These enumeration methods can be generalized by introducing weights, which are constant on each orbit. Let $\mathcal{R}$ be a commutative ring such that $\mathbb{Q}$ is a subring of $\mathcal{R}$ and let $w: X \to \mathcal{R}$ be a weight function which is constant on each G-orbit, then the weight of the orbit $G(x)$ can be defined by $W(G(x)) := w(x)$ and from the CAUCHY FROBENIUS Lemma we derive

$$\sum_{\omega \in G\backslash\backslash X} W(\omega) = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in X_g} w(x).$$

Another concept is the enumeration of orbits of given stabilizer type. We have already seen that the stabilizer type of an orbit $G(x)$ is the conjugacy class of the stabilizer G_x. Having detailed information on the *subgroup lattice* of the acting group it is possible to determine the number of orbits of stabilizer type $\tilde{U}$, where $\tilde{U}$ is the conjugacy class of $U \leq G$. In my PhD thesis [7] all the enumeration formulae are given for group actions of the form 1,2,3 and 4 on Y^X.

Finally let me introduce the *cycle index* of a finite group action. Let ${}_GX$ be a finite group action, then the cycle index of G acting on X is the following polynomial in the indeterminates $x_1, \ldots, x_{|X|}$ over $\mathbb{Q}$.

$$Z(G, X) := \frac{1}{|G|} \sum_{g \in G} \prod_{i=1}^{|X|} x_i^{a_i(\bar{g})},$$

where $(a_1(\bar{g}), \ldots, a_{|X|}(\bar{g}))$ is the cycle type of the permutation $\bar{g} \in S_X$. Most of the enumerative formulae for group actions on Y^X induced from actions on X or Y can be expressed by using the cycle index notion. Let me just give two examples (see [4]):

1. Let ${}_HY$ be a finite group action, then $S_{\underline{n}} \times H$ acts on $Y^{\underline{n}}$ according to (3) and H-orbits on $Y^{\underline{n}}$ is given by

$$\sum_{n=0}^{\infty} \left| (S_{\underline{n}} \times H) \backslash\backslash Y^{\underline{n}} \right| x^n = Z(H, Y)|_{x_i = \sum_{j=0}^{\infty} x^{ij}} = Z(H, Y)|_{x_i = \frac{1}{1-x^i}}. \quad (6)$$

2. The group action of above can be restricted to the set of all injective mappings from $\underline{n}$ to Y. The corresponding generating function is

$$\sum_{n=0}^{\infty} \left|(S_{\underline{n}} \times H)\backslash\backslash Y^{\underline{n}}_{\mathrm{inj}}\right| x^n = Z(H,Y)|_{x_i=1+x^i}, \tag{7}$$

which is a polynomial.

Returning to the enumeration of linear codes, we translate the equivalence relation for linear (n,k)-codes into an equivalence relation for generator matrices of linear codes, and these generator matrices are considered to be functions $\Gamma\colon \underline{n} \to GF(q)^k \setminus \{0\}$ where $\Gamma(i)$ is the i-th column of the generator matrix Γ. (We exclude 0-columns for obvious reasons.)

Theorem 1: *The matrices corresponding to the two functions Γ_1 and Γ_2 from $\underline{n}$ to $GF(q)^k \setminus \{0\}$ are generator matrices of two equivalent codes, if and only if Γ_1 and Γ_2 lie in the same orbit of the following action of $GL_k(q) \times GF(q)^* \wr S_{\underline{n}}$ as permutation group on $(GF(q)^k \setminus \{0\})^{\underline{n}}$:*

$$(A,(\psi,\pi))(\Gamma) = A\psi(\cdot)\Gamma(\pi^{-1}\cdot),$$

or, more explicitly,

$$(A,(\psi,\pi))(\Gamma)(i) := A\psi(i)\Gamma(\pi^{-1}(i)).$$

Following Slepian [15], we use the following notation:

T_{nkq} := the number of orbits of functions $\Gamma\colon \underline{n} \to GF(q)^k \setminus \{0\}$ under the group action of Theorem 1, i.e. $T_{nkq} = \left|(GL_k(q) \times GF(q)^* \wr S_{\underline{n}})\backslash\backslash(GF(q)^k \setminus \{0\})^n\right|$.

S_{nkq} := the number of equivalence classes of linear (n,k)-codes over $GF(q)$ with no columns of zeros. (A linear (n,k)-code has columns of zeros, if and only if there is some $i \in n$ such that $x_i = 0$ for all codewords x, and so we should exclude such columns.)

The S_{nkq} can be computed from the T_{nkq} by

$$S_{nkq} = T_{nkq} - T_{n,k-1,q}. \tag{8}$$

As initial values we have $S_{n1q} = 1$ for $n \in \mathbb{N}$. It is important to realize that

- T_{nkq} is the number of orbits of functions from n to $GF(q)^k \setminus \{0\}$ without any restrictions on the rank of the induced matrix.
- All matrices which are induced from functions Γ of the same orbit have the same rank.
- The number of orbits of functions Γ which induce matrices of rank less or equal $k-1$ is $T_{n,k-1,q}$.

Applying LEHMANN's bijection for the action of the wreath product $GF(q)^* \wr S_{\underline{n}}$ we realize that T_{nkq} is the number of orbits under the following action of $S_{\underline{n}} \times GL_k(q)$ on the set of all functions $\bar{\Gamma}$ from $\underline{n}$ to $GF(q)^* \backslash\backslash GF(q)^k \setminus \{0\}$:

$$(\pi, A)(\bar{\Gamma}) = A\bar{\Gamma}\pi^{-1},$$

where $\Gamma \in (GF(q)^k \setminus \{0\})^{\underline{n}}$ defines $\bar{\Gamma}$ by $\bar{\Gamma}(i) = GF(q)^*(\Gamma(i))$ and $S_{\underline{n}}$ acts on the set of the functions $\left(GF(q)^* \backslash\backslash (GF(q)^k \setminus \{0\})\right)^{\underline{n}}$ by $\pi(\bar{\Gamma}) = \bar{\Gamma} \circ \pi^{-1}$. Furthermore $GL_k(q)$ acts on $GF(q)^* \backslash\backslash (GF(q)^k \setminus \{0\})$ by $A(GF(q)^*(v)) = GF(q)^*(Av)$. The set of the $GF(q)^*$-orbits $GF(q)^* \backslash\backslash (GF(q)^k \setminus \{0\})$ is the $(k-1)$-dimensional projective space

$$GF(q)^* \backslash\backslash GF(q)^k \setminus \{0\} =: PG_{k-1}(q)$$

and the representation of $GL_k(q)$ as a permutation group is the projective linear group $PGL_k(q)$. This proves in fact the following to be true:

Theorem 2: *The isometry classes of linear (n, k)-codes over $GF(q)$ are the orbits of $GL_k(q) \times S_{\underline{n}}$ on the set of mappings $PG_{k-1}(q)^{\underline{n}}$. This set of orbits is equal to the set of orbits of $GL_k(q)$ on the set $S_{\underline{n}} \backslash\backslash PG_{k-1}(q)^{\underline{n}}$, which can be represented by a complete set of mappings of different content, if the content of $f \in PG_{k-1}(q)^{\underline{n}}$ is defined to be the sequence of orders of inverse images $|f^{-1}(x)|$.*
Thus the set of isometry classes of linear (n, k)-codes over $GF(q)$ is equal to the set of orbits of $GL_k(q)$ on the set of mappings $f \in PG_{k-1}(q)^{\underline{n}}$ of different content that form $k \times n$-matrices of rank k.
The particular classes of elements with orders of inverse images $|f^{-1}(x)| \leq 1$ are the classes consisting of Hamming codes.

Knowing the cycle index of $PGL_k(q)$ acting on $PG_{k-1}(q)$ equation (6) can be applied for computing T_{nkq}. When restricting the group action of $PGL_k(q) \times S_{\underline{n}}$ to an action on the set of all injective functions $\bar{\Gamma}: \underline{n} \to GF(q)^* \backslash\backslash (GF(q)^k \setminus \{0\})$ one can derive the number of classes of *injective codes*, which are codes without proportional columns.
In [15] SLEPIAN explained how the cycle index of $GL_k(2)$ can be computed using results of ELSPAS [5]. In [6] the author generalized this concept for computing the cycle indices of $GL_k(q)$ and $PGL_k(q)$ acting on $GF(q)^k$ or $PG_{k-1}(q)$ respectively. These cycle indices are now available in the computer algebra package SYMMETRICA [16]. (See [8] for more details on the implementation.) They were applied for computing the tables of linear codes over $GF(9)$, which can be found on the next pages.
In order to minimize the number of orbits that must be enumerated or represented, and following SLEPIAN again, we can restrict attention to *indecomposable* linear (n, k)-codes. Let C_1 be a linear (n_1, k_1)-code over $GF(q)$ with generator matrix Γ_1 and let C_2 be a linear (n_2, k_2)-code over $GF(q)$ with generator matrix Γ_2, then the code C with generator matrix

$$\Gamma := \left(\begin{array}{c|c} \Gamma_1 & 0 \\ \hline 0 & \Gamma_2 \end{array}\right)$$

Table 1: Isometry classes of linear (n,k)-codes over $GF(9)$

$n\backslash k$	1	2	3	4
1	1	0	0	0
2	1	1	0	0
3	1	2	1	0
4	1	5	3	1
5	1	8	12	4
6	1	17	62	28
7	1	27	430	475
8	1	54	4150	28856
9	1	91	42401	2.417364
10	1	168	413259	197.609449
11	1	275	3.762158	14874.498092
12	1	477	31.881605	1.029632.967128
13	1	764	252.307220	65.891528.575554
14	1	1247	1873.439094	3920.491447.510867
15	1	1937	13111.695528	217978.744960.455289

is called the *direct sum* of the codes C_1 and C_2, and it will be denoted by $C = C_1 \oplus C_2$. A code C is called *decomposable*, if and only if it is equivalent to a code which is the direct sum of two or more linear codes. Otherwise it is called *indecomposable*. The number of all indecomposable linear (n,k)-codes over $GF(q)$ will be denoted by R_{nkq}.
In [15] SLEPIAN proves that every decomposable linear (n,k)-code is equivalent to a direct sum of indecomposable codes, and that this decomposition is unique up to equivalence and order of the summands. SLEPIAN used a generating function scheme for computing the numbers R_{nk2}. However after constructing these codes the author realized that in some situations this formula doesn't work correctly. For that reason we are giving another formula to determine R_{nkq}. For the rest of this article let $n \geq 2$.

Theorem 3: *The number R_{nkq} is equal to*

$$S_{nkq} - \sum_a \sum_b \prod_{\substack{j=1\\ a_j \neq 0}}^{n-1} \left(\sum_{\substack{c=(c_1,\ldots,c_{a_j}) \in \mathbb{N}^{a_j} \\ j \geq c_1 \geq \ldots \geq c_{a_j} \geq 1,\ \sum c_i = b_j}} U(j,a,c) \right),$$

where

$$U(j,a,c) = \prod_{i=1}^{j} Z(S_{\underline{\nu(i,a_j,c)}}, \underline{\nu(i,a_j,c)})|_{x_\ell = R_{jiq}}, \quad \nu(i,a_j,c) = |\{1 \leq l \leq a_j \mid c_l = i\}|,$$

Table 2: Isometry classes of injective linear (n, k)-codes over $GF(9)$

$n\backslash k$	1	2	3	4
1	1	0	0	0
2	0	1	0	0
3	0	1	1	0
4	0	2	2	1
5	0	2	7	3
6	0	2	38	21
7	0	1	250	409
8	0	1	2178	26436
9	0	1	19067	2.206042
10	0	1	153106	176.938649
11	0	0	1119490	13005.200885
12	0	0	7.444639	876508.494146
13	0	0	45.193018	54.475570.780948
14	0	0	251.681833	3140.099483.972559
15	0	0	1291.732944	168727.736939.110698

and where the first sum is taken over the cycle types $a = (a_1, \ldots, a_{n-1})$ *of* n, *(which means that* $a_i \in \mathbb{N}_0$ *and* $\sum ia_i = n$*) such that* $\sum a_i \leq k$, *while the second sum is over the* $(n-1)$*-tuples* $b = (b_1, \ldots, b_{n-1}) \in \mathbb{N}_0^{n-1}$, *for which* $a_i \leq b_i \leq ia_i$, *and* $\sum b_i = k$. *The numerical results show that for fixed* q *and* n *the sequence of* R_{nkq} *is unimodal and symmetric. (It is easy to prove that this sequence must be symmetric, but the proof of the unimodality is still open.)*

The numbers of classes of indecomposable linear codes over $GF(9)$ are given in the next two tables.
Using double coset methods and the combinatorial method of *orderly generation* [14, 3] we got a complete overview of indecomposable linear (n, k)-codes over $GF(q)$ for quite a number of parameter triples (n, k, q) by constructing lists of representatives of the isometry classes of indecomposable linear codes. (See for instance [1, 2, 17].)

Table 3: Isometry classes of indecomposable linear (n, k)-codes over $GF(9)$

$n\backslash k$	1	2	3	4
1	1	0	0	0
2	1	0	0	0
3	1	1	0	0
4	1	3	1	0
5	1	6	6	1
6	1	14	49	14
7	1	24	402	402
8	1	50	4097	28353
9	1	87	42296	2412712
10	1	163	413066	197.562376
11	1	270	3.761800	14874.037714
12	1	471	31.880975	1.029628.744436
13	1	758	252.306117	65.891492.470922
14	1	1240	1873.437231	3920.491159.098191
15	1	1930	13111.692422	217978.742798.601836

Table 4: Isometry classes of indecomposable injective linear (n, k)-codes over $GF(9)$

$n\backslash k$	1	2	3	4
1	1	0	0	0
2	0	0	0	0
3	0	1	0	0
4	0	2	1	0
5	0	2	5	1
6	0	2	36	13
7	0	1	248	369
8	0	1	2177	26181
9	0	1	19066	2203858
10	0	1	153105	176.919574
11	0	0	1119489	13005.047772
12	0	0	7.444639	876507.374648
13	0	0	45.193018	54.475563.336302
14	0	0	251.681833	3140.099438.779534
15	0	0	1291.732944	168727.736687.428860

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