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On a functional equation involving group actions

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Summary. During the forty-first ISFE in Noszvaj, Hungary, G. Guzik posed a problem on a functional equation involving group actions which arose in a generalization of Bargman theory occurring in Quantum Mechanics. (Cf. 18. Problem and Remark in "Report of Meeting", Aequationes Math., *67* (2004), 312–313.)

Let $(G, \cdot)$ be a group which is acting on a set X and let $(K, +)$ be an abelian group. Describe all functions $f\colon G \times G \times X \to K$ satisfying

$$f(g_1, g_2, x) + f(g_1g_2, g_3, x) = f(g_2, g_3, g_1^{-1}x) + f(g_1, g_2g_3, x)$$

for all $g_1, g_2, g_3 \in G$ and $x \in X$.

This problem was solved in a particular case by B. Ebanks. (Cf. 19. Remark in "Report of Meeting", Aequationes Math. *67* (2004), p. 313.) We present the general solution of this problem.

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1. Introduction

During the forty-first ISFE in Noszvaj, Hungary, G. Guzik posed a problem on a functional equation involving group actions which arose in a generalization of Bargman theory occurring in Quantum Mechanics (cf. [4]). This problem was solved in a particular case by B. Ebanks [2] during the conference. We present the general solution of this problem.

Problem. Let $(G, \cdot)$ be a group which is acting on a set X and let $(K, +)$ be an abelian group. Describe all functions $f\colon G \times G \times X \to K$ satisfying

$$f(g_1, g_2, x) + f(g_1g_2, g_3, x) = f(g_2, g_3, g_1^{-1}x) + f(g_1, g_2g_3, x) \qquad (1)$$

for all $g_1, g_2, g_3 \in G$ and $x \in X$.

In the functional equation (1) all arguments in the third place of f belong to one orbit only. If f is a solution of (1), then the restriction of f to $G \times G \times G(x_0)$, where $G(x_0)$ is the orbit of an element $x_0 \in X$, is a solution of (1) on $G\times G\times G(x_0)$. If, conversely, we know solutions f_ω of (1) on $G \times G \times \omega$ for all orbits ω of G, then it is obvious how to obtain a solution of (1) on $G \times G \times X$. Hence, without loss of generality, we assume that G acts transitively on X, i.e. $X = G(x_0)$ consists of one orbit only.

Let H be the stabilizer of x_0, i.e.

$$H = \{g \in G \mid gx_0 = x_0\},$$

which is a subgroup of G.

2. Two particular cases

First we consider the two situations $H = \{1\}$ and $H = G$.

If $H = \{1\}$, then the sets $G(x_0)$ and G are one-to-one. A bijection is given by

$$G(x_0) \ni gx_0 \mapsto g \in G.$$

Therefore we rewrite (1) as

$$f(g_1, g_2, g) + f(g_1g_2, g_3, g) = f(g_2, g_3, g_1^{-1}g) + f(g_1, g_2g_3, g) \tag{2}$$

for all $g_1, g_2, g_3, g \in G$. During the forty-first ISFE in Noszvaj, Hungary, B. Ebanks proved the following theorem:

Theorem 1. *The general solution of* (2) *is given by*

$$f(g_1, g_2, g_3) = \varphi(g_3^{-1}, g_1) + \varphi(g_3^{-1}g_1, g_2) - \varphi(g_3^{-1}, g_1g_2), \quad g_1, g_2, g_3 \in G, \tag{3}$$

where φ is an arbitrary mapping from $G \times G$ to K.

Proof. If f is a solution of (2) then by putting $g = 1$ and $\varphi(g_1, g_2) := f(g_1, g_2, 1)$, $g_1, g_2 \in G$, we obtain

$$f(g_2, g_3, g_1^{-1}) = \varphi(g_1, g_2) + \varphi(g_1g_2, g_3) - \varphi(g_1, g_2g_3), \quad g_1, g_2, g_3 \in G.$$

Conversely, direct computations show that each f given by (3) is a solution of (2). □

If $H = G$, then $G(x_0) = \{x_0\}$, whence the third component of f can be omitted. We therefore derive from (1) the *cocycle equation*

$$f(g_1, g_2) + f(g_1g_2, g_3) = f(g_2, g_3) + f(g_1, g_2g_3) \tag{4}$$

for all $g_1, g_2, g_3 \in G$.

If G is abelian and K is a divisible abelian group, which is uniquely divisible by 2, then the general solution of (4) is given by

$$f(g_1, g_2) = \psi(g_1) + \psi(g_2) - \psi(g_1g_2) + \Psi(g_1, g_2), \qquad g_1, g_2 \in G, \tag{5}$$

where ψ is an arbitrary mapping from G to K and Ψ is a mapping from $G\times G \to K$ so that $\Psi(g_1, g_2) = -\Psi(g_2, g_1)$ and $\Psi(g_1g_1', g_2) = \Psi(g_1, g_2) + \Psi(g_1', g_2)$ for all $g_1, g_1', g_2 \in G$. (See e.g. [7], [3], [5], [1], [6].)

If G is an arbitrary group, then each f of the form (5) is a solution of the cocycle equation.

3. The general case

Assume that the stabilizer H of x_0 is an arbitrary subgroup of G. Let f be a solution of (1), then for all $g_1,\, g_2,\, g_3 \in G$ we have

$$f(g_2,\, g_3,\, g_1^{-1}x_0) = f(g_1,\, g_2,\, x_0) + f(g_1g_2,\, g_3,\, x_0) - f(g_1,\, g_2g_3,\, x_0).$$

Define the function $\varphi\colon G \times G \to K$ by $\varphi(g_1,\, g_2) := f(g_1,\, g_2,\, x_0)$. Thus we obtain

$$f(g_2,\, g_3,\, g_1^{-1}x_0) = \varphi(g_1,\, g_2) + \varphi(g_1g_2,\, g_3) - \varphi(g_1,\, g_2g_3), \qquad g_1,\, g_2,\, g_3 \in G. \quad (6)$$

If $g_1 \in H$, then $g_1^{-1}x_0 = x_0$ and consequently

$$\varphi(g_2,\, g_3) = \varphi(h_1,\, g_2) + \varphi(h_1g_2,\, g_3) - \varphi(h_1,\, g_2g_3), \qquad h_1 \in H,\ g_2,\, g_3 \in G. \quad (7)$$

In particular

$$\varphi(h_2,\, h_3) = \varphi(h_1,\, h_2) + \varphi(h_1h_2,\, h_3) - \varphi(h_1,\, h_2h_3), \qquad h_1,\, h_2,\, h_3 \in H,$$

which is the cocycle equation on H. Therefore the restriction of φ to $H \times H$ is a solution of the cocycle equation on H. Putting $h_1 := h$ and $h_2 := h_3 := 1$ we obtain that $\varphi(h, 1) = \varphi(1, 1)$ for all $h \in H$ and all solutions φ of the cocycle equation.

Let $(\gamma_i)_{i\in I}$ with $\gamma_0 = 1$ be a complete system of representatives of the right cosets of H in G. Consequently

$$G = \dot{\bigcup_{i\in I}} H\gamma_i.$$

In (7) we set $g_2 := h_2 \in H$ and $g_3 := \gamma_i$ for some $i \in I$. Then we get

$$\varphi(h_1,\, h_2\gamma_i) = \varphi(h_1,\, h_2) + \varphi(h_1h_2,\, \gamma_i) - \varphi(h_2,\, \gamma_i).$$

For $i \in I$ define $\varphi_i\colon H \to K$ by $\varphi_i(h) := \varphi(h,\, \gamma_i)$. Then we derive

$$\varphi(h_1,\, h_2\gamma_i) = \varphi(h_1,\, h_2) + \varphi_i(h_1h_2) - \varphi_i(h_2), \qquad h_1,\, h_2 \in H,\ i \in I.$$

Especially $\varphi_0(h) = \varphi(h,\, 1) = \varphi(1, 1)$ for $h \in H$.

Lemma 2. *The function $\varphi\colon H \times G \to K$ is a solution of*

$$\varphi(h_2,\, g_3) = \varphi(h_1,\, h_2) + \varphi(h_1h_2,\, g_3) - \varphi(h_1,\, h_2g_3), \qquad h_1,\, h_2 \in H,\ g_3 \in G, \quad (7')$$

if and only if there exists a solution $\phi\colon H \times H \to K$ of the cocycle equation on H and functions $\varphi_i\colon H \to K$ which are arbitrary for $i \in I \setminus \{0\}$ and $\varphi_0(h) = \phi(1, 1)$, $h \in H$, so that

$$\varphi(h_1,\, h_2\gamma_i) = \phi(h_1,\, h_2) + \varphi_i(h_1h_2) - \varphi_i(h_2), \qquad h_1, h_2 \in H,\ i \in I. \quad (8)$$

Therefore, φ is an extension of ϕ.

Proof. If φ satisfies (7′), then we have just shown that φ has the desired representation.

If φ is given by (8), then φ is an extension of ϕ,

$$\begin{aligned}\varphi(h_1,\, h_2) &= \varphi(h_1,\, h_2\gamma_0)\\ &= \phi(h_1,\, h_2) + \varphi_0(h_1h_2) - \varphi_0(h_2)\\ &= \phi(h_1,\, h_2) + \phi(1,\, 1) - \phi(1,\, 1)\\ &= \phi(h_1,\, h_2), \qquad h_1,\, h_2 \in H.\end{aligned}$$

Now we prove that φ satisfies (7′). Consider $h_1,\, h_2 \in H$ and $g_3 \in G$. Then there exists some $i \in I$ and $h_3 \in H$ so that $g_3 = h_3\gamma_i$ and

$$\begin{aligned}&\varphi(h_1,\, h_2) + \varphi(h_1h_2,\, h_3\gamma_i) - \varphi(h_1,\, h_2h_3\gamma_i)\\ &\overset{(8)}{=} \phi(h_1,\, h_2) + \phi(h_1h_2,\, h_3) + \varphi_i(h_1h_2h_3) - \varphi_i(h_3) - \phi(h_1,\, h_2h_3) - \varphi_i(h_1h_2h_3)\\ &\quad + \varphi_i(h_2h_3)\\ &= \phi(h_1,\, h_2) + \phi(h_1h_2,\, h_3) - \phi(h_1,\, h_2h_3) + \varphi_i(h_2h_3) - \varphi_i(h_3)\\ &\overset{(4)}{=} \phi(h_2,\, h_3) + \varphi_i(h_2h_3) - \varphi_i(h_3)\\ &\overset{(8)}{=} \varphi(h_2,\, h_3\gamma_i)\end{aligned}$$

which finishes the proof. □

Again we assume that φ satisfies (7). In (7) we set $g_2 := \gamma_i$ for some $i \in I$. Then we get

$$\varphi(h_1\gamma_i,\, g_3) = \varphi(\gamma_i,\, g_3) + \varphi(h_1,\, \gamma_i g_3) - \varphi(h_1,\, \gamma_i).$$

For $i \in I$ define $\theta_i \colon G \to K$ by $\theta_i(g) := \varphi(\gamma_i,\, g)$. Then we derive

$$\varphi(h_1\gamma_i,\, g_3) = \theta_i(g_3) + \varphi(h_1,\, \gamma_i g_3) - \varphi(h_1,\, \gamma_i), \qquad h_1 \in H,\ i \in I,\ g_3 \in G.$$

Especially $\theta_0(g) = \varphi(1,\, g) = \varphi(1,1)$ for $g \in G$ (which follows from (7) by setting $h_1 := g_2 := 1$ and $g_3 := g$).

Lemma 3. *The function $\varphi \colon G \times G \to K$ is a solution of* (7) *if and only if there exists a solution $\phi \colon H \times G \to K$ of* (7′) *and functions $\theta_i \colon G \to K$ which are arbitrary for $i \in I \setminus \{0\}$ and $\theta_0(g) = \phi(1,1)$, $g \in G$, so that*

$$\varphi(h_1\gamma_i,\, g_3) = \theta_i(g_3) + \phi(h_1,\, \gamma_i g_3) - \phi(h_1,\, \gamma_i), \qquad h_1 \in H,\ i \in I,\ g_3 \in G. \tag{9}$$

Therefore, φ is an extension of ϕ.

Proof. If φ satisfies (7), then we have just shown that φ has the desired representation.

If φ is given by (9), then φ is an extension of ϕ, since

$$\begin{aligned}\varphi(h,\, g) &= \varphi(h\gamma_0,\, g)\\ &= \theta_0(g) + \phi(h,\, g) - \phi(h,\, \gamma_0)\\ &= \phi(1,\, 1) + \phi(h,\, g) - \phi(h,\, 1)\\ &= \phi(h,\, g), \qquad h \in H,\ g \in G.\end{aligned}$$

Now we prove that φ satisfies (7). Consider $h_1 \in H$ and $g_2, g_3 \in G$. Then there exist $i, j \in I$ and $h_2, h_3 \in H$ so that $g_2 = h_2\gamma_i$ and $g_3 = h_3\gamma_j$. Moreover $\gamma_i h_3 \gamma_j = h_4\gamma_k$ for some $h_4 \in H$ and $k \in I$. We obtain

$$\begin{aligned}&\varphi(h_1,\, h_2\gamma_i) + \varphi(h_1h_2\gamma_i,\, h_3\gamma_j) - \varphi(h_1,\, h_2\gamma_i h_3\gamma_j)\\ &\overset{(9)}{=} \phi(h_1,\, h_2\gamma_i) + \theta_i(h_3\gamma_j) + \phi(h_1h_2,\, \gamma_i h_3\gamma_j) - \phi(h_1h_2,\, \gamma_i) - \phi(h_1,\, h_2h_4\gamma_k)\\ &\overset{(8)}{=} \phi(h_1,\, h_2) + \varphi_i(h_1h_2) - \varphi_i(h_2) + \theta_i(h_3\gamma_j) + \phi(h_1h_2,\, h_4) + \varphi_k(h_1h_2h_4)\\ &\quad - \varphi_k(h_4) - \phi(h_1h_2,\, 1) - \varphi_i(h_1h_2) + \varphi_i(1) - \phi(h_1, h_2h_4) - \varphi_k(h_1h_2h_4)\\ &\quad + \varphi_k(h_2h_4)\\ &= \phi(h_1,\, h_2) + \phi(h_1h_2,\, h_4) - \phi(h_1, h_2h_4) + \theta_i(h_3\gamma_j) + \varphi_k(h_2h_4) - \varphi_k(h_4)\\ &\quad - \varphi_i(h_2) - \phi(1,\, 1) + \varphi_i(1)\\ &\overset{(4)}{=} \phi(h_2,\, h_4) + \theta_i(h_3\gamma_j) + \varphi_k(h_2h_4) - \varphi_k(h_4) - \varphi_i(h_2) - \phi(1,\, 1) + \varphi_i(1).\end{aligned}$$

And

$$\begin{aligned}&\varphi(g_2,\, g_3)\\ &= \varphi(h_2\gamma_i,\, h_3\gamma_j)\\ &\overset{(9)}{=} \theta_i(h_3\gamma_j) + \phi(h_2,\, \gamma_i h_3\gamma_j) - \phi(h_2,\, \gamma_i)\\ &\overset{(8)}{=} \theta_i(h_3\gamma_j) + \phi(h_2,\, h_4) + \varphi_k(h_2h_4) - \varphi_k(h_4) - \phi(h_2,\, 1) - \varphi_i(h_2) + \varphi_i(1)\\ &= \phi(h_2,\, h_4) + \theta_i(h_3\gamma_j) + \varphi_k(h_2h_4) - \varphi_k(h_4) - \varphi_i(h_2) - \phi(1,\, 1) + \varphi_i(1)\end{aligned}$$

which finishes the proof. □

Corollary 4. *Let $(G, \cdot)$ be a group, H a subgroup of G, $(\gamma_i)_{i\in I}$ with $\gamma_0 = 1$ a complete system of representatives of the right cosets of H in G, and $(K, +)$ an abelian group. The function $\varphi\colon G \times G \to K$ is a solution of (7) if and only if φ restricted to $H \times H$ satisfies the cocycle equation and*

$$\begin{aligned}\varphi(h_1,\, h_2\gamma_i) &= \varphi(h_1,\, h_2) + \varphi_i(h_1h_2) - \varphi_i(h_2) \qquad h_1, h_2 \in H,\ i \in I,\\ \varphi(h_1\gamma_i,\, g_3) &= \theta_i(g_3) + \varphi(h_1,\, \gamma_i g_3) - \varphi(h_1,\, \gamma_i) \quad h_1 \in H,\ i \in I,\ g_3 \in G.\end{aligned}$$

where $\varphi_i\colon H \to K$ and $\theta_i\colon G \to K$, $i \in I$, are arbitrary functions with exception of $\varphi_0 = \theta_0 = \varphi(1, 1)$.

Lemma 5. *Let G act transitively on X, take $x_0 \in X$ fixed, let H be the stabilizer of x_0 and assume that for $x \in X$*

$$x = g_1^{-1} x_0 = \tilde{g}_1^{-1} x_0$$

hence $\tilde{g}_1 = hg_1$ for some $h \in H$. Let φ be a solution of (7)*, then*

$$\varphi(g_1,\, g_2) + \varphi(g_1 g_2,\, g_3) - \varphi(g_1,\, g_2 g_3) = \varphi(\tilde{g}_1,\, g_2) + \varphi(\tilde{g}_1 g_2,\, g_3) - \varphi(\tilde{g}_1,\, g_2 g_3).$$

Proof. We prove that

$$\varphi(g_1,\, g_2) + \varphi(g_1 g_2,\, g_3) - \varphi(g_1,\, g_2 g_3) = \varphi(hg_1,\, g_2) + \varphi(hg_1 g_2,\, g_3) - \varphi(hg_1,\, g_2 g_3)$$

for all $g_1,\, g_2,\, g_3 \in G$ and $h \in H$. This will be done by showing that the right hand side does not depend on h, so we can take $h = 1$ and obtain the left hand side.

Let $(\gamma_i)_{i \in I}$ be a complete system of representatives of the right cosets of H in G with $\gamma_0 = 1$. Then there exist $h_1, \ldots, h_6 \in H$ and $\gamma_{(1)}, \gamma_{(2)}, \gamma_{(3)}, \gamma_{(12)}, \gamma_{(23)}, \gamma_{(123)}$ elements of $(\gamma_i)_{i \in I}$ so that $g_1 = h_1 \gamma_{(1)}$, $g_2 = h_2 \gamma_{(2)}$, $g_3 = h_3 \gamma_{(3)}$ and

$$\begin{aligned}
\gamma_{(1)} h_2 \gamma_{(2)} &= h_4 \gamma_{(12)} \\
\gamma_{(2)} h_3 \gamma_{(3)} &= h_5 \gamma_{(23)} \\
\gamma_{(1)} h_2 \gamma_{(2)} h_3 \gamma_{(3)} &= h_6 \gamma_{(123)} \\
\gamma_{(12)} h_3 \gamma_{(3)} &= h_4^{-1} \gamma_{(1)} h_2 \gamma_{(2)} h_3 \gamma_{(3)} \\
&= h_4^{-1} h_6 \gamma_{(123)} \\
\gamma_{(1)} h_2 h_5 \gamma_{(23)} &= \gamma_{(1)} h_2 \gamma_{(2)} h_3 \gamma_{(3)} \\
&= h_6 \gamma_{(123)}.
\end{aligned}$$

We obtain:

$$\begin{aligned}
&\varphi(hg_1,\, g_2) + \varphi(hg_1 g_2,\, g_3) - \varphi(hg_1,\, g_2 g_3) \\
&= \varphi(hh_1 \gamma_{(1)},\, h_2 \gamma_{(2)}) + \varphi(hh_1 \gamma_{(1)} h_2 \gamma_{(2)},\, h_3 \gamma_{(3)}) - \varphi(hh_1 \gamma_{(1)},\, h_2 \gamma_{(2)} h_3 \gamma_{(3)}) \\
&= \varphi(hh_1 \gamma_{(1)},\, h_2 \gamma_{(2)}) + \varphi(hh_1 h_4 \gamma_{(12)},\, h_3 \gamma_{(3)}) - \varphi(hh_1 \gamma_{(1)},\, h_2 h_5 \gamma_{(23)}) \\
&\overset{(9)}{=} \theta_{(1)}(h_2 \gamma_{(2)}) + \varphi(hh_1,\, \gamma_{(1)} h_2 \gamma_{(2)}) - \varphi(hh_1,\, \gamma_{(1)}) + \theta_{(12)}(h_3 \gamma_{(3)}) \\
&\quad + \varphi(hh_1 h_4,\, \gamma_{(12)} h_3 \gamma_{(3)}) - \varphi(hh_1 h_4,\, \gamma_{(12)}) - \theta_{(1)}(h_2 h_5 \gamma_{(23)}) \\
&\quad - \varphi(hh_1,\, \gamma_{(1)} h_2 h_5 \gamma_{(23)}) + \varphi(hh_1,\, \gamma_{(1)}) \\
&= \theta_{(1)}(h_2 \gamma_{(2)}) + \theta_{(12)}(h_3 \gamma_{(3)}) - \theta_{(1)}(h_2 h_5 \gamma_{(23)}) + \varphi(hh_1,\, h_4 \gamma_{(12)}) \\
&\quad - \varphi(hh_1 h_4,\, \gamma_{(12)}) + \varphi(hh_1 h_4,\, h_4^{-1} h_6 \gamma_{(123)}) - \varphi(hh_1,\, h_6 \gamma_{(123)}) \\
&\overset{(*)}{=} \theta_{(1)}(h_2 \gamma_{(2)}) + \theta_{(12)}(h_3 \gamma_{(3)}) - \theta_{(1)}(h_2 h_5 \gamma_{(23)}) + \varphi(hh_1,\, h_4) - \varphi(h_4,\, \gamma_{(12)}) \\
&\quad + \varphi(hh_1 h_4,\, h_4^{-1}) - \varphi(h_4^{-1},\, h_6 \gamma_{(123)}) \\
&\overset{(\circ)}{=} \theta_{(1)}(h_2 \gamma_{(2)}) + \theta_{(12)}(h_3 \gamma_{(3)}) - \theta_{(1)}(h_2 h_5 \gamma_{(23)}) - \varphi(h_4,\, \gamma_{(12)}) \\
&\quad - \varphi(h_4^{-1},\, h_6 \gamma_{(123)}) + \varphi(h_4,\, h_4^{-1}) + \varphi(hh_1,\, 1)
\end{aligned}$$

which is independent of h since $\varphi(hh_1, 1) = \varphi(1, 1)$. Applications of (7) yield $(*)$ and $(\circ)$. For $(*)$ we applied (7) twice, namely for $(h_1, g_2, g_3) = (hh_1, h_4, \gamma_{(12)})$ and for $(h_1, g_2, g_3) = (hh_1h_4, h_4^{-1}, h_6\gamma_{(123)})$. For $(\circ)$ we used $(h_1, g_2, g_3) = (hh_1, h_4, h_4^{-1})$. (Actually in the last case (7) is the cocycle equation on H.) □

In Corollary 4 we have described the general solution of (7) under the assumption that we know all solutions of the cocycle equation on H. Now we are able to prove the main theorem. It gives the desired answer in terms of solutions of the cocycle equation on H.

Theorem 6. *Consider a transitive action of a group $(G, \cdot)$ on a set X and let $(K, +)$ be an abelian group. Let $x_0 \in X$ and let H denote the stabilizer of x_0. The general solution $f\colon G \times G \times X \to K$ of* (1) *is given by* (6) *where $\varphi\colon G \times G \to K$ is a solution of* (7).

Proof. If f satisfies (1), then we have already shown that f has the desired representation.

Assume that f is given by (6) where φ, described in Corollary 4, is a solution of (7). According to Lemma 5, f is well defined. We still have to prove that f satisfies (1). Consider some $x \in X$, then there exists some $g \in G$ so that $x = g^{-1}x_0$. Simple calculations show that

$$\begin{aligned} &f(g_1, g_2, x) + f(g_1g_2, g_3, x) \\ &= f(g_1, g_2, g^{-1}x_0) + f(g_1g_2, g_3, g^{-1}x_0) \\ &= \varphi(g, g_1) + \varphi(gg_1, g_2) + \varphi(gg_1g_2, g_3) - \varphi(g, g_1g_2g_3) \end{aligned}$$

and also

$$\begin{aligned} &f(g_2, g_3, g_1^{-1}x) + f(g_1, g_2g_3, x) \\ &= f(g_2, g_3, g_1^{-1}g^{-1}x_0) + f(g_1, g_2g_3, g^{-1}x_0) \\ &= \varphi(gg_1, g_2) + \varphi(gg_1g_2, g_3) + \varphi(g, g_1) - \varphi(g, g_1g_2g_3). \end{aligned}$$

□

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