38) $b_\sigma(z) = \sum_{\nu \geq 0} \binom{\sigma}{\nu} z^\nu$

Falls $\sigma \in \mathbb{N} \Rightarrow \binom{\sigma}{\sigma+1} = \binom{\sigma}{\sigma} \frac{\sigma-\sigma}{\sigma+1} = 0$

und daher $\binom{\sigma}{\nu} = 0$ für alle $\nu > \sigma$.

$\Rightarrow b_\sigma(z)$ ist ein Polynom $\Rightarrow R = \infty$.

Falls $\sigma \notin \mathbb{N}$: Beh: $R = 1$.

$\binom{\sigma}{\nu} \neq 0$ für alle $\nu \geq 0$. Sei $\nu \geq 1$:

$$\frac{\left|\binom{\sigma}{\nu}\right|}{\left|\binom{\sigma}{\nu+1}\right|} = \frac{\left|\binom{\sigma}{\nu}\right|}{\left|\binom{\sigma}{\nu}\right| \left|\frac{\sigma-\nu}{\nu+1}\right|} = \frac{\nu+1}{|\sigma-\nu|} \quad (*)$$

$\sigma = a + ib$ mit $a, b \in \mathbb{R}$

$$(*) = \frac{\nu+1}{\sqrt{(\nu-a-ib)(\nu-a+ib)}} = \frac{\nu+1}{\sqrt{(\nu-a)^2+b^2}} =$$

$$\frac{\cancel{\nu}\left(1+\frac{1}{\nu}\right)}{\cancel{\nu}\sqrt{1-\frac{2a}{\nu}+\frac{a^2+b^2}{\nu^2}}} \xrightarrow[\nu\to\infty]{} 1.$$

Also ist $R = 1$.

$$b_\sigma'(z) = \sum_{\nu \geq 1} \binom{\sigma}{\nu} \nu z^{\nu-1} \quad (**)$$

$$\nu\binom{\sigma}{\nu} = \nu \frac{\sigma(\sigma-1)\cdots(\sigma-\nu+1)}{\nu\cdot(\nu-1)!} = \sigma \frac{(\sigma-1)\cdots(\sigma+1-\nu)}{(\nu-1)!}$$

$$= \sigma\binom{\sigma-1}{\nu-1}$$

$$(**) = \sum_{\nu \geq 1} \sigma \binom{\sigma-1}{\nu-1} z^{\nu-1} = \sigma \sum_{n \geq 0} \binom{\sigma-1}{n} z^n = \sigma\, b_{\sigma-1}(z).$$

39) G Gebiet $f \in \mathcal{O}(G)$

1) $f(z) = a \exp(bz) \quad z \in G \quad a, b \in \mathbb{C}$

2) $f'(z) = b f(z) \quad z \in G \quad b \in \mathbb{C}$.

1) $\Rightarrow$ 2) $\quad f'(z) = a \exp(bz) \cdot b = b f(z)$.

2) $\Rightarrow$ 1) $\quad h(z) := f(z) \exp(-bz) \quad z \in G$.

$\Rightarrow h \in \mathcal{O}(G)$

$$h'(z) = \underbrace{f'(z)}_{b f(z)} \exp(-bz) + f(z)(-b)\exp(-bz) = 0$$

$\Rightarrow h$ (lokal) konstant $\Rightarrow h$ konstant

Es gibt $a \in \mathbb{C}$ mit $h(z) = a \quad z \in G$

$f(z) \exp(-bz) = a \;\; \forall_{z \in G} \Rightarrow f(z) = a \exp(bz), z \in G$;

40) $f(z) = \sum_{n \geq 1} \frac{z^n}{n^2}$ $\qquad a_n = \frac{1}{n^2} \neq 0$

$$\frac{|a_n|}{|a_{n+1}|} = \frac{1}{n^2} \cdot \frac{n^2+1}{1} = \frac{\cancel{n^2}\left(1+\frac{1}{n^2}\right)}{\cancel{n^2}\cdot 1} = 1 + \frac{1}{n^2} \xrightarrow[n \to \infty]{} 1$$

$R = 1.$

Sei $|z| = 1$. Dann gilt

$$\sum_{n \geq 1} \frac{|z|^n}{n^2} = \sum_{n \geq 1} \frac{1}{n^2} < +\infty \quad \text{konvergent!}$$

d.h. konvergiert für alle $z \in \partial E$.

$$f'(z) = \sum_{n \geq 1} \frac{n z^{n-1}}{n^2} = \sum_{n \geq 1} \frac{z^{n-1}}{n}$$

für $z = 1$ ist die Reihe $\sum_{n \geq 1} \frac{1}{n}$ divergent!

4.2) $\{z \in \mathbb{C} \mid \cos z = 1\}$

$\sin^2 z + \cos^2 z = 1 \Rightarrow \sin^2 z = 0 \Rightarrow \sin z = 0$

$\Rightarrow z \in \{\underset{n}{\cancel{(2n+1)}}\pi \mid n \in \mathbb{Z}\} = \pi\mathbb{Z}.$

$\{z \in \mathbb{C} \mid \tan z = 1\}$

$\frac{\sin z}{\cos z} = 1 \iff \sin z = \cos z$

$\frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{2}(e^{iz} + e^{-iz})$

$e^{iz} - e^{-iz} = i(e^{iz} + e^{-iz})$

$e^{2iz} - 1 = i(e^{2iz} + 1)$

$e^{2iz}(1-i) = 1+i$

$e^{2iz} = \frac{1+i}{1-i}$

$\frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1+2i+i^2}{1-i^2} = \frac{2i}{2} = \underline{i}$

$\cos(2z) + i\sin(2z) = i$

$\cos(2z) = 0$ und $\sin(2z) = 1$

$2z = \frac{\pi}{2} + 2n\pi i$

$z = \frac{\pi}{4} + n\pi i.$

$\{z \in \mathbb{C} \mid \cos z \in \mathbb{R}\}$

$\frac{1}{2}\left(e^{iz} + e^{-iz}\right) \in \mathbb{R}$

$z = x + iy, \quad x, y \in \mathbb{R} \qquad iz = -y + ix \qquad -iz = y - ix$

$e^{iz} = e^{-y+ix} = e^{-y}(\cos x + i \sin x)$

$e^{-iz} = e^{y-ix} = e^{y}(\cos x - i \sin x)$

$\frac{1}{2}\left(e^{iz} + e^{-iz}\right) \in \mathbb{R} \iff$

$e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x) \in \mathbb{R} \iff$

<u>1. Fall</u>: $y = 0 \qquad e^{-y} = e^{y} = 1$

$\cos x + i \sin x + \cos x - i \sin x \cancel{2\cos x} \in \mathbb{R}$

$2\cos x \in \mathbb{R}$ also für alle $x \in \mathbb{R}$

<u>2. Fall</u> $y \neq 0$

$e^{-y}(\cos x + i \sin x + e^{2y}(\cos x - i \sin x)) \in \mathbb{R} \iff$

$\sin x - e^{2y} \sin x = 0 \iff$

$\sin x (1 - e^{2y}) = 0 \iff$

$\sin x = 0$ oder $e^{2y} = 1 \iff$

$x \in \pi\mathbb{Z}$ oder $y = 0$ (unmöglich da 2. Fall) $\Rightarrow$

$x \in \pi\mathbb{Z}$

$\{z \in \mathbb{C} \mid \cos z \in \mathbb{R}\} = \{x + iy \mid y = 0 \text{ oder } x \in \pi\mathbb{Z}\}.$