A non-local conservation law with nonlinear 'radiation' inhomogeneity

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December 21, 2006

Abstract

We study a scalar conservation law with a nonlinear dissipative inhomogeneity, which serves as a simplified model for nonlinear heat radiation effects in high–temperature gases. We establish global existence and uniqueness of weak entropy solutions along with L^1 contraction and monotonicity properties of the solution semigroup. We derive explicit threshold conditions ensuring formation of shocks within finite time. Our main result proves – under further assumptions on the nonlinearity and on the initial datum – large time convergence in L^1 to the self–similar N*–waves* of the homogeneous conservation law.

Key words: nonlinear heat radiation, entropy solutions, large time asymptotics, wave breaking threshold.

AMS subject classification: 35Q53, 35B40, 35L65

1 Introduction

We shall study the following nonlinear and inhomogeneous scalar conservation law

$$\partial_t u + \partial_x f(u) = K * B(u) - B(u)\,. \tag{1.1}$$

Here $f : \mathbb{R} \to \mathbb{R}$ is a convex flux function satisfying $f(0) = f'(0) = 0$. The inhomogeneity $K * B(u) - B(u)$ balances a gain term and a loss term. The latter is defined by the nonlinear function

$$B(u) = u|u|^{m-1} \qquad \text{for some} \;\; m \geq 1\,, \tag{1.2}$$

which the gain term convolutes further with an L^1-normalized, even, and nonnegative kernel $K : \mathbb{R} \to \mathbb{R}$:

$$\|K(x)\|_{L^1(\mathbb{R})} = 1\,, \qquad K \geq 0\,, \qquad K(x) = K(-x). \tag{1.3}$$

Of particular interest is the Green's kernel $K(x) = e^{-|x|}/2$ to the differential operator $-\partial_x^2 + 1$ and the flux function of Burgers' equation $f(u) = u^2/2$, whence equation (1.1) can be rewritten as a hyperbolic–elliptic system :

$$\begin{cases} \partial_t u + u\,\partial_x u = -\partial_x q\,, \\ -\partial_x^2 q + q = -\partial_x B(u)\,. \end{cases} \tag{1.4}$$

This hyperbolic–elliptic system (1.4) can be seen as a simplified model system of the compressible Euler equation for an ideal gas subject to heat radiation phenomena (see [24]), i.e.

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t \left(\rho \left(e + u^2/2 \right) \right) + \partial_x \left(\rho u \left(e + u^2/2 \right) + pu + q \right) = 0, \\ -\partial_x^2 q + aq + b\partial_x(\theta^4) = 0. \end{cases} \tag{1.5}$$

In fact in [8], the model system (1.4) is recovered as a linearization around constant states and by considering a scalar unknown which combines density, velocity and temperature. In particular, the nonlinear temperature term θ^4 is replaced by a linear perturbation of a positive constant state. This leads to the system (1.4) in the case $m = 1$, also called the *Hamer model* of radiating gases. Such a dissipative model has been extensively studied in [20, 17, 14, 11, 10, 9, 15]. For related model systems of dispersive type see e.g. [7, 18].

The nonlocal forcing in (1.1) reflects the global influence of heat sources or gravitation fields and appears in radiative hydrodynamics [24] and in the context of self-gravitating fluids modeled by the Euler-Poisson system [25]. Recently in [4], the system (1.4) with general nondecreasing $B(u)$ has been proposed as a simplified model describing magneto–hydrodynamic phenomena in astrophysics. In this context, the model describes non-local energy transports through radiation. A typical choice is $B(u) = \sigma u^4$ with $\sigma > 0$ which derives from Planck's law of black body radiation. The full mathematical model of radiating plasma flow is given by the compressible Euler equations for macro-quantities such as density, momentum and temperature, coupled with a linear Boltzmann equation for radiation density [24]. We also note that the nonlocal model of this type has also been derived in traffic simulation [22].

We shall analyze the model system (1.4) with $B(u) = u|u|^{m-1}$ when $m > 1$. In the particular case $m = 4$, the system (1.4) can be considered as the simplest model describing the interplay between nonlinear convection and nonlinear radiation. In the gas–dynamical model (1.5) this corresponds to avoid nondegeneracy of the temperature θ.

The scope of this paper spans from i) the existence theory for the equation (1.1) when $m > 1$, over ii) the formation of shocks in smooth solutions in particular for $f(u) = u^2/2$, to iii) the large time asymptotics of the solutions of the system (1.4) to self–similar N–waves of Burger's equation provided smallness of the initial datum.

More precisely, we will establish global well–posedness in the class of weak entropy solutions for (1.1) (see the definition 2.1 below) by taking advantage of the dissipative properties of the inhomogeneous term in a similar fashion to e.g. [14, 20] :

Theorem 1.1 (Existence of a unique entropy solution). *Assume initial data $u_0 \in L^1 \cap L^\infty(\mathbb{R})$. Then, for any $T > 0$ there exists a unique $u \in L^\infty([0, T]; L^1 \cap L^\infty(\mathbb{R}))$ entropy solution to (1.1) with initial datum u_0. In addition, we have for all $t > 0$ and all $p \in [1, +\infty]$*

$$\|u(\cdot, t)\|_{L^p(\mathbb{R})} \leq \|u_0(\cdot)\|_{L^p(\mathbb{R})}. \tag{1.6}$$

Moreover, given u, $\bar{u}$ two entropy solutions to (1.1) with initial data $u_0, \bar{u}_0 \in L^1 \cap L^\infty$, we have for all $t > 0$,

$$\int_{\mathbb{R}} (u(x,t) - \bar{u}(x,t))_+ dx \leq \int_{\mathbb{R}} (u_0(x) - \bar{u}_0(x))_+ dx \,, \tag{1.7}$$

$$\|u(\cdot, t) - \bar{u}(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|u_0(\cdot) - \bar{u}_0(\cdot)\|_{L^1(\mathbb{R})} \,. \tag{1.8}$$

Therefore, $u(x,t) \leq \bar{u}(x,t)$ *almost everywhere if* $u_0(x) \leq \bar{u}_0(x)$ *almost everywhere.*

Remark 1.2. *Theorem 1.1 (with the exception of* (1.6) *for* $p > 1$*), holds readily true for any continuous, nondecreasing function* B*.*

Our second result provides a sufficient condition to predict the formation of shocks within finite time. Assuming the Burger's flux $f(u) = u^2/2$, we prove for nonnegative smooth initial data with a gradient exceeding an explicit negative threshold that the smooth solutions will become discontinuous in finite time :

Theorem 1.3 (Lower regularity threshold). *Assume that* $f(u) = u^2/2$*. Given initial data* $u_0 \in C_b^1(\mathbb{R})$ *with* $u_0 \geq 0$ *bounded and its gradient negative with*

$$\inf_{x\in\mathbb{R}} \partial_x u_0(x) < -\mu^*, \quad \mu^* := \sqrt{[B'(|u_0|_\infty)]^2 + 2K(0)V(B(|u_0|_\infty))}\,],$$

Then, there exists a unique local smooth solution $u \in C_b^1(\mathbb{R} \times [0,T])$ *to (1.1) with a finite life span* $T < \infty$ *and*

$$\lim_{t\to T}(\min_{x\in\mathbb{R}}\{\partial_x u(t,x)\}) = -\infty.$$

Moreover, the following explicit bound on the life span holds

$$T \leq \frac{1}{\mu^*} \min_{x\in\mathbb{R}} \log\left(\frac{\partial_x u_0 - \mu^*}{\partial_x u_0 + \mu^*}\right).$$

To investigate further the wave-breaking phenomenon, we perform numerical experiments. For sub-critical initial data, we observe conditional formation of shocks depending on the exponent m of the nonlinearity of the inhomogeneous term : roughly spoken, the higher m the more the convection term dominates the behavior of the numerical solution.

We note that in the case of linear inhomogeneity $m = 1$, both upper threshold for global smooth solution and lower threshold for finite time breakdown are identified in [17]. Here the nonlinear inhomogeneity renders an upper threshold subtle to identify. Nevertheless, the critical threshold phenomenon is indeed generic, and was first observed and studied in [6] for a class of Euler-Poisson equations; and further extended to other problems of various types such as a convolution model for nonlinear conservation laws [17], nonlocal dispersive wave equations [18] as well as relaxation systems in traffic flows [16]. The study of multi-D critical threshold phenomena becomes more challenging, and a new tool of *spectral dynamics* has been first introduced in [19] to estimate the velocity gradient matrix, instead of the velocity slope in 1D problems such as the model studied in this paper.

Our main result concerns with the asymptotic behavior for large times. We prove that the typical L^1–asymptotic state for the model (1.4) with $f(u) = u^2/2$ and $m > 2$ for nonnegative solutions is given by the so called *inviscid N–wave*

$$N(x,t) = \begin{cases} \frac{x}{t} & \text{if } 0 \leq x \leq \sqrt{2\|u_0\|_1 t} \\ 0 & \text{otherwise} \end{cases}.$$

This fact constitutes an essential difference to the Hamer model $m = 1$, where the large time asymptotics is described by diffusive N–waves, i. e. self similar solutions of the viscous Burger's equation (see [9, 15, 5]). Our result on the asymptotic behavior reads as :

Theorem 1.4 (Asymptotic behavior). *Let u be the unique entropy solution to the hyperbolic-elliptic problem (1.4) with $m > 2$ subject to nonnegative initial data $u_0 \in L^1 \cap L^\infty(\mathbb{R})$ satisfying*

$$\|u_0(\cdot)\|_{L^\infty} \leq \left(\frac{1}{m(m-1)}\right)^{\frac{1}{m-2}}. \tag{1.9}$$

Then, the following decay rate holds

$$\|u(\cdot,t)\|_{L^\infty} = O\left(t^{-\frac{1}{2}}\right) \tag{1.10}$$

for large times t. Moreover,

$$\lim_{t\to+\infty} t^{\frac{p-1}{2p}} \|u(\cdot,t) - N(\cdot,t)\|_{L^p(\mathbb{R})} = 0 \tag{1.11}$$

for all $p \in [1, +\infty)$.

The proof of Theorem 1.4 is carried out via a classical rescaling method. We emphasize the techniques used to obtain the compactness needed in passing to the limit in the rescaling. Technical difficulties rise due to the nonlinearity of the inhomogeneous term, which inhibits a method for a one–sided pointwise estimate on the space derivative as available for the linear Hamer model (see [15]).

Instead, we apply a method proposed in [2] for general homogeneous nonlinear semigroups, which detects a uniform bound in BV. Moreover, we overcome regularity issues with a vanishing quadratic nonlinear diffusion on a bounded domain with Dirichlet boundary conditions, while a standard vanishing viscosity approximation fails.

Outline: The three theorems 1.1, 1.3 and 1.4 will be proven in the sections 2, 3 and 4, respectively. We refer to these sections for a more detailed comments and references.

2 Global existence theory

The aim of this section is to prove the global existence and uniqueness stated in the Theorem 1.1 above. Due to the presence of a nonlinear convection term in (1.1), one cannot expect global existence of classical solutions. Since weak solutions are (as well known) in general not unique, we introduce the notion of *weak entropy solution* (see for instance [13, 3]).

Definition 2.1. A bounded measurable function $u : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ is said to be a *weak entropy solution* of (1.1) with initial datum $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ if it verifies the inequality

$$\int_0^T\int_{\mathbb{R}} [\eta(u)\partial_t\psi + q(u)\partial_x\psi]dxdt + \int_{\mathbb{R}} \eta(u_0(x))\psi(x,0)dx \geq -\int_0^T\int_{\mathbb{R}} \eta'(u)L[u]\psi dxdt \tag{2.1}$$

for all convex functions $\eta : \mathbb{R} \to \mathbb{R}$ with q given by

$$q(u) = \int^u f'(s)\eta'(s)ds \tag{2.2}$$

and for all nonnegative Lipschitz continuous test functions $\psi : \mathbb{R} \times [0, T) \to \mathbb{R}$ having compact support intersecting the line $\{t = 0\}$; with $L[u] := K * B(u) - B(u)$.

Existence of a unique weak entropy solutions is shown by a standard vanishing viscosity approximation (see e.g. [3, Chapter 6]). Given $\varepsilon > 0$, we construct classical solutions of

$$\partial_t u + \partial_x f(u) = K * B(u) - B(u) + \varepsilon \partial^2_{xx} u \tag{2.3}$$

and study their limit as $\varepsilon \downarrow 0$. The uniqueness of these weak entropy solutions follows by means of the "variables doubling" technique due to Kružkov [13]. (see also [14] to the problem (1.1) with $B(u) = u$).

The local existence of the approximating solutions of the equation (2.3) is stated in the following proposition.

Proposition 2.2. *Let $\varepsilon > 0$ be fixed and let $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, there exists $T > 0$ such that (2.3) has a unique solution $u \in C([0,T]; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ having u_0 as initial datum. Moreover, the space derivative u_x also belongs to $C([0,T]; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$.*

Proof. The proof follows as in e.g. [14, Theorem A.2], using that B is locally Lipschitz continuous in the inhomogeneous part in the Duhamel type formula for u. Therefore, we shall omit the details. □

The global existence of the approximate solutions u^ϵ in $L^1 \cap L^\infty$ follows in the lines of [3, Theorem 6.2.2] and [14, Theorem 2.2]. For the sake of clarity, we shall shortly reproduce it.

Proposition 2.3. *Let u and $\bar{u}$ be solutions to (2.3) having initial data u_0, $\bar{u}_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ respectively. Then, for all $t > 0$,*

$$\int_{\mathbb{R}} (u(x,t) - \bar{u}(x,t))_+ dx \le \int_{\mathbb{R}} (u_0(x) - \bar{u}_0(x))_+ dx; \tag{2.4}$$

$$\|u(\cdot,t) - \bar{u}(\cdot,t)\|_{L^1(\mathbb{R})} \le \|u_0(\cdot) - \bar{u}_0(\cdot)\|_{L^1(\mathbb{R})}. \tag{2.5}$$

Consequently, if $u_0(x) \le \bar{u}_0(x)$ a.e. on $\mathbb{R}$, then $u(x,t) \le \bar{u}(x,t)$ a.e. on $\mathbb{R} \times [0,T]$. In addition, for all $p \in [1, +\infty]$ we have

$$\|u(\cdot,t)\|_{L^p(\mathbb{R})} \le \|u_0(\cdot)\|_{L^p(\mathbb{R})}. \tag{2.6}$$

Proof. Let $T > 0$ be as given in Proposition 2.2. It is sufficient to prove the above statements for $t \in [0,T]$, as they will extend to all $t > 0$. We consider two local solutions u, $\bar{u} \in C([0,T]; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ with initial data u_0 and $\bar{u}_0$, respectively.

To show the estimate (2.4) we consider a suitable regularization of the positive part function $z \mapsto z_+ = \max\{0, z\}$, for instance

$$\alpha_\eta(z) = \begin{cases} 0 & \text{if } z \le 0 \\ \frac{z^2}{4\eta} & \text{if } 0 < z \le 2\eta \\ z - \eta & \text{if } z > 2\eta. \end{cases} \qquad \text{for} \quad \eta > 0$$

and compute

$$\frac{d}{dt} \int_{\mathbb{R}} \alpha_\eta(u - \bar{u}) dx = - \int_{\mathbb{R}} \alpha'_\eta(u - \bar{u})(f(u) - f(\bar{u}))_x dx \ + \int_{\mathbb{R}} \alpha'_\eta(u - \bar{u})(L[u] - L[\bar{u}]) dx$$
$$+ \varepsilon \int_{\mathbb{R}} \alpha'_\eta(u - \bar{u})(u - \bar{u})_{xx} dx\,.$$

Then, integration by parts shows the last term nonpositive ($\alpha_\eta'' \geq 0$), while the first term vanishes in the limit $\eta \downarrow 0$ due to the regularity of u and $\bar{u}$ stated in Proposition 2.2

$$\alpha_\eta''(u-\bar{u})(f(u)-f(\bar{u}))(u_x-\bar{u}_x) \to 0 \quad \text{as } \eta \downarrow 0 \,.$$

Thus, with $\alpha_\eta' \to H$ the Heaviside characteristic function of $[0,+\infty)$ it remains to estimate the convolution term (using that $0 \leq H \leq 1$, $0 \leq K$, and that B is nondecreasing)

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} (u(x,t)-\bar{u}(x,t))_+ dx \quad &\leq \quad \int_{\mathbb{R}} H(u-\bar{u})[K*(B(u)-B(\bar{u}))-(B(u)-B(\bar{u}))](x,s)dx\,,\\ &\leq \quad \int_{\mathbb{R}}\int_{\mathbb{R}} K(x-y)(B(u(y))-B(\bar{u}(y)))_+\,dydx\\ &\qquad -\int_{\mathbb{R}} H(B(u)-B(\bar{u}))(B(u)-B(\bar{u}))\,dx = 0\,,\end{aligned}$$

since we recall that $\|K\|_1 = 1$. This proves (2.4) after integration over the time interval $[0,t]$. Moreover, the inequality (2.5) easily follows by interchanging the roles of u and $\bar{u}$, while the monotonicity statement is a direct consequence of (2.4).

In order to prove (2.6), we use a regularization of the modulus $\beta_\eta(z) \to |z|$ as $\eta \to 0$ with that same properties as α_η above. For $p \in [1,+\infty)$, after integration by parts we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \beta_\eta(u)^p dx = &-p\int_{\mathbb{R}} \beta_\eta(u)^{p-1}\beta_\eta'(u)f(u)_x\,dx + p\int_{\mathbb{R}} \beta_\eta(u)^{p-1}\beta_\eta'(u)L[u]\,dx\\ &-p\,\varepsilon\int_{\mathbb{R}}[(p-1)\beta_\eta(u)^{p-2}(\beta_\eta'(u))^2+\beta_\eta^{p-1}\beta_\eta''(u)]u_x^2\,dx.\end{aligned}$$

We observe that the last term above is nonpositive and that for the first term

$$\int_{\mathbb{R}} \beta_\eta(u)^{p-1}\beta_\eta'(u)f(u)_x dx = \int_{\mathbb{R}} \partial_x\left(\int^u \beta_\eta(z)^{p-1}\beta_\eta'(z)f'(z)dz\right)dx = 0\,.$$

Thus, recalling that $B(u) = u|u|^{m-1}$ we have in the limit $\eta \downarrow 0$

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} |u(t)|^p dx &\leq \int_{\mathbb{R}} |u|^{p-1}\mathrm{sign}(u)K*(u|u|^{m-1})\,dx - \int_{\mathbb{R}} |u|^{p-1+m}dx\\ &\leq \|u\|^{p-1}_{L^{p+m-1}(\mathbb{R})}\|K*|u|^m\|_{L^{\frac{p+m-1}{m}}(\mathbb{R})} - \|u\|^{p-1+m}_{L^{p+m-1}(\mathbb{R})} \leq 0\end{aligned}$$

using Young's inequality for the convolution with $\|K\|_{L^1} = 1$. Finally, the statement (2.6) for $p = +\infty$ follows by sending $p \to +\infty$. □

The next step provides compactness (in a suitable sense) of the solutions to (2.3) with respect to ε (see also [3, 14]) :

Lemma 2.4. *For a given $\varepsilon > 0$, let u^ε be the solution to (2.3) with initial datum $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, there exists a function $\omega : [0,+\infty)$ with $\omega(r) \downarrow 0$ as $r \downarrow 0$, ω depending on $\|u_0\|_{L^\infty}$ and on $\|u_0\|_{L^1}$, such that*

$$\int_{\mathbb{R}} |u^\varepsilon(x+h,t+k)-u^\varepsilon(x,t)|dx \leq \omega(|h|)+\omega(k^{\frac{1}{3}})\,,$$

for all $h \in \mathbb{R}$ and $k > 0$.

Proof. For simplicity we replace u^ε by u throughout the proof. First, for any fixed $t > 0$, we use (2.5) with $\bar{u}(x,t) := u(x+h,t)$, i.e.

$$\int_{\mathbb{R}} |u(x+h,t) - u(x,t)|dx \leq \int_{\mathbb{R}} |u_0(x+h) - u_0(x)|dx \leq \omega(|h|)\,, \tag{2.7}$$

for a function ω as specified in the Lemma thanks to the absolute continuity of the measure $u_0 dx$ with respect to the Lebesgue measure.

Secondly, it remains to show that

$$\int_{\mathbb{R}} |u(x,t+k) - u(x,t)|dx \leq \omega_1(k^{\frac{1}{3}})\,, \quad \text{for } k > 0\,, \tag{2.8}$$

for a function ω_1 having the same properties as ω before. This is done as in [14, Lemma 2.3], where (with ω is as in (2.7))

$$\int_{\mathbb{R}} |u(x,t+k) - u(x,t)|dx - \int_{\mathbb{R}} \phi(x)[u(x,t+k) - u(x,t)]dx \leq 4\,\omega(k^{\frac{1}{3}}), \tag{2.9}$$

and ϕ is a Friedrich's regularization of $\operatorname{sign}(u(x,t+k) - u(x))$ with 'step' $k^{1/3}$, i.e.

$$\phi(x) = \int_{\mathbb{R}} k^{-\frac{1}{3}} \rho(k^{-\frac{1}{3}}(x-\xi))\, \operatorname{sign}(u(\xi,t+k) - u(\xi,t)) d\xi\,,$$

where $\rho \geq 0$ with $\|\rho\|_1 = 1$ is a smooth mollifier with compact support in $[-1,1]$. Thus, we estimate

$$\begin{aligned}
&\int_{\mathbb{R}} \phi(x)[u(x,t+k) - u(x,t)]dx = \int_t^{t+k}\!\!\int_{\mathbb{R}} \phi(x) u_\tau(x,\tau) dx d\tau \\
&= \int_t^{t+k}\!\!\int_{\mathbb{R}} \big(\phi'(x) f(u(x,\tau)) + \mu\phi''(x) u(x,\tau) + \phi(x)[K * B(u) - B(u)](x,\tau)\big)\, dx d\tau \\
&\leq C k^{\frac{2}{3}} \|u_0\|_{L^1} \sup_{0 \leq u \leq \|u_0\|_{L^\infty}} f'(u) + \mu C k^{\frac{1}{3}} \|u_0\|_{L^1} + 2k\|u_0\|_{L^1} \sup_{0 \leq u \leq \|u_0\|_{L^\infty}} B'(u),
\end{aligned}$$

where the constant C only depends on the mollifier ρ. This completes the proof of (2.8) and the proof of the Lemma. □

As a consequence of Lemma 2.4 and of Proposition 2.3 follows the

Proof of Theorem 1.1. Let $u_0 \in L^1 \cap L^\infty(\mathbb{R})$. Then, for any $T > 0$ the family $\{u^\varepsilon\}_{\varepsilon>0}$ of solutions to (2.3) with u_0 as initial datum converges (up to subsequences) strongly in $L^p_{loc}(\mathbb{R} \times [0,T])$ for all $p \in [1,+\infty)$ to $u \in L^1 \cap L^\infty(\mathbb{R} \times [0,T])$, as a consequence of the Riesz–Frechet–Kolmogorov compactness Theorem and of Lemma 2.4. Moreover, by extracting a subsequence converging almost everywhere, it is easy to verify that the limit u is an entropy solution to (1.1) with u_0 as initial datum. The uniqueness of entropy solutions to (1.1) can be proven in the same way as in [3, Theorem 6.2.2] or as in [14, Theorem 2.5] (based on the 'variables doubling' method by Kružkov [13]) using the dissipative nature of the source term $K * B(u) - B(u)$ as in Proposition 2.3 above. Therefore, we omit the details. □

3 Critical thresholds and numerical experiments

Throughout this section we consider (1.1) in particular for the Burger's flux $f(u) = u^2/2$, i.e.

$$\partial_t u + u\,\partial_x u = K * B(u) - B(u)\,, \tag{3.1}$$

subject to nonnegative initial data

$$u(x,0) = u_0(x) \geq 0\,, \quad x \in \mathbb{R}\,. \tag{3.2}$$

The monotonicity property obtained in last section ensures the estimate

$$0 \leq u(t,x) \leq \max_{x\in R} u_0(x).$$

when $u_0 \in L^\infty$. This bound leads to the following estimates, which will be used in figuring out our threshold conditions.

Lemma 3.1. *Let u be a smooth nonnegative solution in $[0,T]$. Then it holds*

$$0 \leq K * B(u) \leq \max_{x\in\mathbb{R}} B(u_0), \tag{3.3}$$

$$-K(0)V(B(u_0)) \leq K * \partial_x B(u) \leq K(0)V(B(u_0)), \tag{3.4}$$

where $V(B(u_0)) := -\min_{x\in\mathbb{R}} B(u_0(x)) + \max_{x\in\mathbb{R}} B(u_0(x))$.

Proof. The first inequality follows from the fact $K * 1 = 1$ and the L^∞ bound $0 \leq u(t,\cdot) \leq \max_{x\in\mathbb{R}} u_0(x)$. We shall prove the second inequality as follows :

$$\begin{aligned}
K * (\partial_x B(u)) &= \int_{\mathbb{R}} K(x-y)\,\partial_y B(u)(t,y)\,dy = \int_{\mathbb{R}} \partial_x K(x-y)\,B(u)(t,y)\,dy \\
&= \left[\int_{-\infty}^{x} \partial_x K(x-y)\,B(u)(t,y)\,dy + \int_{x}^{+\infty} \partial_x K(x-y)\,B(u)(t,y)\,dy\right] \\
&\leq \min_{x\in\mathbb{R}} B(u_0(x)) \int_{-\infty}^{x} \partial_x K(x-y)\,dy + \max_{x\in\mathbb{R}} B(u_0(x)) \int_{x}^{+\infty} \partial_x K(x-y)\,dy \\
&\leq K(0)\left[-\min_{x\in\mathbb{R}} B(u_0(x)) + \max_{x\in\mathbb{R}} B(u_0(x))\right] = K(0)V(B(u_0)).
\end{aligned}$$

The lower bound $-K(0)V(B(u_0))$ is clear from the above estimate. □

The existence of T is ensured by the local existence theorem stated in the following

Lemma 3.2. *Consider the Cauchy problem (3.1)-(3.2) with non-negative initial data $u_0 \in C_b^1(\mathbb{R})$. Then there exists a positive constant T, depending only on $\|u_0\|_{C_b^1(\mathbb{R})}$ such that (3.1)-(3.2) has a unique smooth solution in $C_b^1(\mathbb{R}\times[0,T])$.*

The proof of this local existence may be obtained by a standard iteration scheme, the details are omitted.

Equipped with the above preliminary facts, we turn to a discussion of wave breaking criterion.

Lemma 3.3. *Consider the Cauchy problem (3.1)-(3.2). The maximal existence time T is finite if and only if the gradient of the solution becomes unbounded from below in finite time.*

Proof. ¿From the local existence in Lemma 3.2 it follows that if the gradient of the solution becomes unbounded in finite time, then $T < \infty$. Let the life span $T < \infty$ and assume that for some constant $M > 0$ we have

$$\partial_x u(t,x) \geq -M, \quad (t,x) \in [0,T) \times \mathbb{R}. \tag{3.5}$$

Then the slope $\mu := \partial_x u$ satisfies

$$\partial_t \mu + u\,\partial_x \mu + \mu^2 = K * \partial_x B(u) - B'(u)\mu. \tag{3.6}$$

Using the Lemma 3.1 and $\mu \geq -M$ we obtain

$$\frac{d}{dt}\mu \leq K(0)V(B(u_0)) + \max_{x\in\mathbb{R}} B'(u_0)M =: C,$$

along the characteristics $dx/dt = u, x(0) = \alpha$. From this it follows that

$$\mu \leq \mu_0 + CT < \infty.$$

Therefore the standard continuation argument enables us to extend solution to $[0, T+\delta)$ with $\delta > 0$, and thereby one must have $T = \infty$. This contradiction ensures that

$$\lim_{t\to T-} (\min_{x\in\mathbb{R}} \partial_x u(t,x)) = -\infty.$$

□

We are ready now to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. The smoothness of u before the breakdown time ensures that there exists a smooth curve $x(t,\alpha)$ satisfying

$$\frac{d}{dt}x(t,\alpha) = u(t, x(t,\alpha)), \quad x(0,\alpha) = \alpha, \quad \alpha \in \mathbb{R}.$$

Evaluating the above μ-equation (3.6) along $x(t,\alpha)$ and using $K * (\partial_x B(u)) \leq A$ with $A := K(0)V(B(u_0))$ as stated in Lemma 3.1, we have (denoting $' := \partial_t + u\,\partial_x$)

$$\mu' + \mu^2 = K * \partial_x B(u)(t, x(t,\alpha)) - B'(u)\mu \leq A - B'(u)\mu, \quad ' := \partial_t + u\,\partial_x$$

for $t \in (0,T)$. This leads to

$$\mu' \leq A + \frac{1}{2}|\max_{x\in\mathbb{R}} B'(u_0)|^2 - \frac{1}{2}\mu^2. \tag{3.7}$$

Solving this differential inequality we obtain

$$\mu(t, x(t,\alpha)) \leq \mu^* \frac{(\mu_0 + \mu^*) + (\mu_0 - \mu^*)e^{-\mu^* t}}{(\mu_0 + \mu^*) - (\mu_0 - \mu^*)e^{-\mu^* t}},$$

where $\mu^* = \sqrt{2A + |\max_{x\in\mathbb{R}} B'(u_0)|^2}$. From this if follows that if $\mu_0 < -\mu^*$, then $\mu(t) \to -\infty$ before t reaches

$$T^* = \frac{1}{\mu^*} \min_{x\in\mathbb{R}} \log\left(\frac{\mu_0 - \mu^*}{\mu_0 + \mu^*}\right).$$

This proves that the solution breaks down in finite time once $\partial_x u_0 \geq -\mu^*$ fails. □

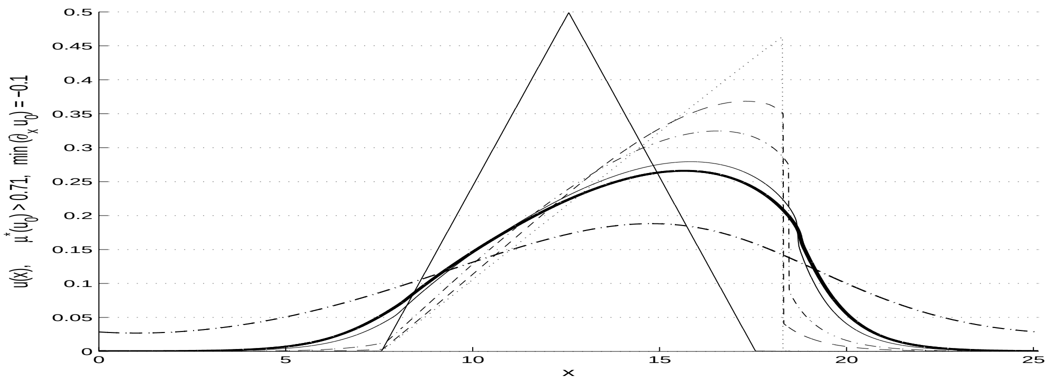

Figure 3.1: Wave Breaking for Undercritical Initial Data : Dependence on the nonlinearity. Initial triangular peak and numerical solutions at time $t = 13$ for $m = 10$, 4, 3, 2.2, 2, 1, in the order : dotted ($m = 10$), dashed ($m = 4$), dash-dotted ($m = 3$), solid ($m = 2.2$), bold ($m = 2$), dashed ($m = 1$) on top of dotted ($m = 1$, spectral method).

Wave Breaking for Undercritical Initial Data : Numerical Experiments

The critical breaking threshold of Theorem 1.3 is not sharp. Figure 3.1 shows numerical solutions for undercritical initial value problems for several values of the exponent $m = 10$, 4, 3, 2.2, 2, 1 ($K(x) = 0.5 \exp(-|x|)$, $\min \partial_x u_0(x) = -0.1$ while $0.71 < \mu^*$ for all considered m). Wave breaking is observed approximately for $m > 2$. In the strongest nonlinear case $m = 10$ the solution is quickly close to the asymptotic nonnegative N-wave (see section 4 below).

The numerical solutions are calculated by a splitting scheme : For the hyperbolic transport we use up-winding with timesteps chosen according to the CFL-condition; for $K * (u^m) - u^m$, instead of calculating the convolution $K * u^m$, we solve (taking u^m at the old time) the elliptic equation $-(K * u^m)'' + K * u^m = u^m$ by a mass-preserving centered-difference scheme. In the linear case $m = 1$, the numerical solution is alternatively calculated using an implicit spectral method (cf. [1, Part II, Chapter 8]) with no visible difference (see Figure 3.1).

4 Asymptotic behavior

This section is devoted to the proof of Theorem 1.4. Recalling the assumptions, we consider nonnegative initial data $u_0(x) \geq 0$ and thus nonnegative solutions $u(t, x) \geq 0$ for the particular flux function $f(u) = u^2/2$ and the nonlinearity $B(u) = u^m$ for $m > 2$. Moreover, we regard especially the Green's kernel $K(x) = e^{-|x|}/2$ to the differential operator $-\partial_x^2 + 1$. Thus, alltogether the equation (1.1) can be rephrased as the hyperbolic–elliptic system (1.4).

These choices of f and B are related to a simplified model for radiating gases described in [24, 8]. In [8], linearization around constant states yields the exponent $m = 1$ in the elliptic equation of the model system (1.4). (see also the related works [10, 11, 9]).

Here we don't perform any linearization: we model the convective motion of the compressible Euler equations by the scalar convection of Burger's equation (compare e.g. [7] for a related simplification of two species Euler-Poisson systems), whereas we leave the nonlinearity in the terms describing the radiation phenomena unchanged. In particular, $m = 4$ is a physically significant case arising from the stationary radiative transfer equation in the original model (see also [4]).

In this section we investigate the large time behavior in L^1 of the solutions to (1.4). We shall use the classical scaling method described, for instance, in [23] (see also [15, 14]).

4.1 Rescaling

Given a positive parameter $\lambda > 0$, we introduce the rescaled quantities

$$v^\lambda(y,\tau) := \lambda u(\lambda y, \lambda^2 \tau)\,, \qquad q^\lambda(y,\tau) := \lambda^2 q(\lambda y, \lambda^2\tau)\,,$$

and the corresponding rescaled kernel

$$K^\lambda(y) := \lambda K(\lambda y)\,. \tag{4.1}$$

Considering the system (1.4), it is easy to verify that v^λ and q^λ satisfy the rescaled system

$$\begin{cases} \partial_\tau v^\lambda + v^\lambda\, \partial_y v^\lambda = -\partial_y q^\lambda \\ -\lambda^{-2}\partial_y^2 q^\lambda + q^\lambda = -\lambda^{1-m}\partial_y (v^\lambda)^m, \end{cases} \tag{4.2}$$

or, equivalently, the rescaled equation

$$\partial_\tau v^\lambda + v^\lambda\, \partial_y v^\lambda = \lambda^{3-m}[K^\lambda * (v^\lambda)^m - (v^\lambda)^m] = \lambda^{1-m}\partial_y[(\partial_y K^\lambda) * (v^\lambda)^m]\,. \tag{4.3}$$

where the last equality follows from the Green's function properites of K^λ. Finally, we note that the recsaled initial datum v_0 satisfies

$$v_0(y) = \lambda u_0(\lambda y)\,, \tag{4.4}$$

for nonegative $u_0 > 0$ is taken in $L^1 \cap L^\infty(\mathbb{R})$.

By the rescaling $t = \lambda^2\tau$, we analyse the large time behavior of u in terms of the limit of v^λ when $\lambda \to \infty$. Formally, since the inhomogeneous part in (4.3) or (4.2) clearly vanishes as $\lambda \to \infty$, we expect the limiting behavior of v^λ to be described in terms of the inviscid Burgers' equation

$$\partial_\tau v^\infty + v^\infty\, \partial_y v^\infty = 0\,. \tag{4.5}$$

This is a different asymptotic regime compared to the linear case $m = 1$, in which the large time behavior is governed by the viscous Burger's equation as was proven by means of the same rescaling as above in [14].

To prove the convergence of the family $\{v^\lambda\}$ for $\lambda \to \infty$, we require compactness properites of the solution semigroup. A result in this direction has been obtained in [15, 20] in the linear case $m = 1$ in terms of a one–sided estimate of the spatial derivative u_x, which implies a time decay estimate of the L^∞ norm of u. Consequently, the L^1 norm of u_x and of u_t are controlled uniformly for large times when the initial datum is purely in L^1. Here, unfortunately, in the nonlinear case $B(u) = u^m$ for $m > 1$, the technique to derive the one–sided estimate of u_x seems to fail.

Instead, we succeeded to adapt an approach of Crandall and Pierre for conservation laws [2], which shows an L^1 estimate of the time derivative of the solution (compare Lemma 4.2 below). This approach relies on the regularizing properties of the convective part, expressed by the quadratic flux function $f(u) = u^2/2$. The inhomogeneous part is treated as a perturbation, which entails both the assumptions of having overquadratic nonlinearity ($m > 2$) and of the smallness of the L^∞ norm.

As already mentioned in [2], the method requires a suitable regularization precedure for the solutions, which were Yosida-type approximations in case of the pure conservation law.

Here, in presence of the nonlinear inhomogeneous part, a quadratic nonlinear diffusion approximation on bounded intervals with Dirichlet boundary conditions fits this purpose. This is due to the fact that the leading operator in the generator of the semigroup must be homogeneous in order to apply the Crandall–Pierre technique. It is therefore that a standard vanishing viscosity argument as in section 2 does not apply.

In the proof of the compactness properties, we shall work directly with the rescaled equation (4.3). For the sake of simplicity, we shall drop the index λ and replace v^λ by v.

4.2 Approximation via quadratic diffusion

For fixed $\epsilon > 0$, $\lambda > 0$ and $n \in \mathbb{N}$, we introduce the nonlinear diffusive approximation

$$\begin{cases} \partial_\tau v + v\,\partial_y v = -\partial_y q + \epsilon\,\partial_y^2 v^2\,, \\ -\lambda^{-2}\,\partial_y^2 q + q = -\lambda^{1-m}\,\partial_y v^m\,, \end{cases} \tag{4.6}$$

or, equivalently, the approximating equation

$$\partial_\tau v + v\,\partial_y v = L^\lambda[v^m] + \epsilon\,\partial_y^2 v^2\,, \tag{4.7}$$

on the strip $(y,\tau) \in [-n,n] \times [0,+\infty)$, with Dirichlet boundary conditions

$$v(\pm n,\tau) \equiv \frac{1}{n}, \qquad q(\pm n,\tau) \equiv 0. \tag{4.8}$$

We impose the following initial condition

$$v(y,0) = v^\epsilon_{0,n}(y) := \frac{1}{n} + v_0(y), \qquad y \in [-n,n], \tag{4.9}$$

where v_0 is as in (4.4). The solution q to the elliptic equation in approximating system (4.6) can be expressed as

$$q(x,t) = -\lambda^{1-m}\int_{-n}^{n} K^{\lambda,n}(x;y)(v^m(y))_y dy\,,$$

given the approximating Kernel $K^{\lambda,n}$

$$K^{\lambda,n}(x;y) = \tfrac{\lambda e^{4\lambda n}}{2(e^{4\lambda n}-1)}\left[e^{-\lambda|x-y|} - e^{-2\lambda n}\left(e^{\lambda(x+y)} + e^{-\lambda(x+y)}\right) + e^{-4\lambda n + \lambda|x-y|}\right]\,, \tag{4.10}$$

$$\textstyle\int_{-n}^{n} K^{\lambda,n}(x;y)dy = 1 - \frac{\lambda e^{\lambda n}}{e^{2\lambda n}+1}(e^{-\lambda x} + e^{\lambda x})\,, \qquad K^{\lambda,n} \geq 0\,, \qquad K^{\lambda,n}(x;y) = K^{\lambda,n}(y;x)\,.$$

The kernel $K^{\lambda,n}$ converges to K^λ given in (4.1) as we have that

$$K^{\lambda,n}(x,y) \to K^\lambda(x,y) \quad \text{for all } (x,y)\,.$$

The existence of local smooth solutions to the approximating system can be easily proven via Schauder's fixed point theorem. Moreover, the estimates of Proposition 2.3 hold thanks to the dissipative nature of the diffusion term, as summarized in the following

Proposition 4.1. *For fixed $\epsilon > 0$, $\lambda > 0$ and $n \in \mathbb{N}$, there exists a unique smooth solution v to approximating system (4.6)–(4.8) subject to the initial data (4.9). Moreover, v satisfies the statements of Proposition 2.2 and Proposition 2.3, in particular the solution is global and bounded below*

$$\|v(\cdot,\tau)\|_{L^p} \leq \|v_0(\cdot)\|_{L^p} \qquad p \in [1,+\infty]\,,$$
$$v(y,\tau) \geq \frac{1}{n} \qquad \text{for all } y \in [-n,n],\ \tau \geq 0\,,$$

where the last property ensures that v is globally smooth as a solution of a non–degenerate parabolic equation.

Proof. The proof follows the proof of Proposition 2.3. The nonlinear diffusion term $\epsilon\,\partial_y^2 v^2$ is dissipative and it can be treated similarly to the linear diffusion term in the approximation (2.3). The estimates of the nonlocal inhomogeneous term follow – after extending v, $\bar{v}$, and $K^{\lambda,n}$ by zero outside the interval $[-n,n]$ – as in Proposition 2.3 since $\|K^{\lambda,n}\|_{L^1(\mathbb{R})} \leq 1$. □

The next lemma will be useful in the sequel.

Lemma 4.2. *Let $f \in L^1([-n,n])$ and introduce the approximating convolution product $*_n$*

$$(\partial_y K^{\lambda,n}) *_n (f) := \int_{-n}^{n} \partial_x K^{\lambda,n}(x;y) f(y) dy\,.$$

Then, for n and λ large enough (e.g. $n\lambda > 1$) we have

$$\|(\partial_y K^{\lambda,n}) *_n (f)\|_{L^1([-n,n])} \leq M_n \lambda \|f\|_{L^1([-n,n])},$$

for some constant $M_n \geq 1$ such that $M_n \searrow 1$ as $n \to +\infty$.

Proof. A simple but tedious calculation yields

$$\begin{aligned}\|(\partial_y K^{\lambda,n}) *_n (f)\|_{L^1} \leq \frac{\lambda^2 e^{4\lambda n}}{2(e^{4\lambda n}-1)}\Bigg[\int_{-n}^{n}\int_{-n}^{n} e^{-\lambda|x-y|}|f(x)|dxdy \\ -e^{-4\lambda n}\int_{-n}^{n}\int_{-n}^{n} e^{\lambda|x-y|}|f(x)|dxdy + e^{-2\lambda n}\int_{-n}^{n}\int_{-n}^{y}[e^{-\lambda x-\lambda y} - e^{\lambda x+\lambda y}]|f(x)|dxdy \\ +e^{-2\lambda n}\int_{-n}^{n}\int_{y}^{n}[-e^{-\lambda x-\lambda y} + e^{\lambda x+\lambda y}]|f(x)|dxdy\Bigg]\,.\end{aligned}$$

By computing the above integrals and by discarding nonpositive terms we obtain the desired estimate. □

The following two Lemmata will provide the compactness of v as $\lambda \to \infty$.

Lemma 4.3. *Assume u_0 satisfying $\|u_0\|_{L^\infty_x}^{m-2} < \frac{1}{m}$. Then, the solution v of (4.6)-(4.8)-(4.9)-(4.4) satisfies the following estimate :*

$$\|(v^2/2)_y\|_{L^1([-n,n])} \leq \frac{1}{1 - M_n m \|u_0\|_{L^\infty_x}^{m-2}} \|v_\tau\|_{L^1([-n,n])}\,, \tag{4.11}$$

where M_n is the same constant as in Lemma 4.2 for n large enough.

Proof. We multiply the right-hand side of (4.7) with a smoothed version of $\text{sign}(v_y)$ and use the result in Lemma 4.2 to obtain the following estimate

$$\begin{aligned}\|(v^2/2)_y\|_{L^1([-n,n])} &\leq \|v_\tau\|_{L^1([-n,n])} + m\lambda^{1-m}\|v\|^{m-2}_{L^\infty([-n,n])}\int_{-n}^{n}(\partial_y K^{\lambda,n}) *_n \left|(v^2/2)_y\right| dy \\ &\leq \|v_\tau\|_{L^1([-n,n])} + M_n m\|u_0\|^{m-2}_{L^\infty_x}\|(v^2/2)_y\|_{L^1([-n,n])}\,,\end{aligned}$$

where we have used (4.4) and

$$\epsilon\int_{-n}^{n}\text{sign}(v_y)(v^2)_{yy}dy = 4\epsilon\int_{-n}^{n} v_y\int_0^{v_y} s\delta(s)\,dx\,dy \to 0$$

in the approximation of the sign function together with the results of Proposition 4.1. □

Lemma 4.4 (Estimate of the time derivative). *Assume initial data satisfying*

$$\|u_0\|^{m-2}_{L^\infty_x} < \frac{1}{m(m-1)}.$$

Then, the solution v of (4.6)-(4.8)-(4.9)-(4.4) satisfies the following estimate :

$$\|v_\tau(\tau)\|_{L^1([-n,n])} \leq \frac{1}{\tau}C(m, \|u_0\|_{L^1_x}, \|u_0\|_{L^\infty_x}) \qquad \tau \geq 0 \tag{4.12}$$

for a constant C depending only on m, $\|u_0\|_{L^1_x}$ and $\|u_0\|^{m-2}_{L^\infty_x} < \frac{1}{m(m-1)}$ and for n large enough.

Proof. We follow the approach of [2], which considers for $\alpha > 0$ the function

$$w = \tau v_\tau + \alpha v \qquad w_\tau = \tau v_{\tau\tau} + (1+\alpha)v_\tau\,,$$

satisfying the equation

$$\begin{aligned} w_\tau &= \tau v_{\tau\tau} + (1+\alpha)v_\tau \\ &= -(vw)_y + \alpha(v^2)_y - \tau q_{\tau y} + 2\epsilon(wv)_{yy} - 2\epsilon\alpha(v^2)_{yy} + (1+\alpha)v_\tau \\ &= -(vw)_y + 2\epsilon(wv)_{yy} - 2\alpha q_y - \tau q_{\tau y} + (1-\alpha)v_\tau\,. \end{aligned}$$

Thus, we calculate readily that

$$\begin{aligned}\frac{d}{dt}\left[\tau\|w\|_1\right] &= \int_{-n}^{n}\text{sign}(w)\,(\tau w_\tau + w)\,dy \\ &= \int_{-n}^{n}\text{sign}(w)\,[-\tau(vw)_y + 2\tau\epsilon(wv)_{yy}]\,dy + \int_{-n}^{n}\text{sign}(w)\,[(2-\alpha)w - \alpha(1-\alpha)v]\,dy \\ &\quad + \int_{-n}^{n}\text{sign}(w)\left(-\tau^2 q_{\tau y}\right)dy + \int_{-n}^{n}\text{sign}(w)\,(-2\alpha\tau q_y)\,dy =: \sum_{k=1}^{4} I_k.\end{aligned}$$

The term I_1 can be proven to be nonpositive by choosing a suitable smoothed version of the sign function and by taking the limit in the smoothing parameter, similarly to Proposition 2.3. The term I_2 is controlled by

$$I_2 \leq (2-\alpha)\|w\|_{L^1} + \alpha|1-\alpha|\|v\|_{L^1}\,.$$

In order to estimate the term I_3, we introduce the notations

$$-\lambda^{-2}Q_{yy} + Q = \lambda^{3-m}v^m \qquad -\lambda^{-2}P_{yy} + P = m\lambda^{3-m}v^{m-1}w\,,$$

and write

$$-\tau q_{y\tau} = \tau\left[P - \alpha m Q - m\lambda^{3-m}v^{m-1}w + \alpha\lambda^{3-m}mv^m\right].$$

Therefore, we have

$$\begin{aligned} I_3 &= \tau\int_{-n}^{n}\text{sign}(w)\left(P - m\lambda^{3-m}v^{m-1}w\right)dy + \tau\alpha m\int_{-n}^{n}\text{sign}(w)\left(\lambda^{3-m}v^m - Q\right)dy \\ &\le \tau\left(\|P\|_{L^1} - m\lambda^{3-m}\|v^{m-1}w\|_{L^1}\right) + \tau\alpha m\int_{-n}^{n}\text{sign}(w)q_y\,dy\,. \end{aligned}$$

Similarly to the proof of Proposition 2.3, we have that $\|P\|_{L^1} \le m\lambda^{3-m}\|v^{m-1}w\|_{L^1}$ since $\|K^{\lambda,n}\|_1 \le 1$. Therefore, by using Lemma 4.2 and the result of Proposition 4.1 we obtain

$$\begin{aligned} I_3 + I_4 &\le \alpha\tau(m-2)\int_{\mathbb{R}}\text{sign}(w)q_y dy \le \alpha\tau(m-2)\|q_y\|_{L^1} \\ &\le M_n\alpha\tau m(m-2)\|u_0\|_{L^\infty_x}^{m-2}\|(v^2/2)_y\|_{L^1_y}. \end{aligned}$$

All together, we obtain that

$$\frac{d}{dt}\left[\tau\|w\|_1\right] \le M_n\alpha m(m-2)\|u_0\|_\infty^{m-2}\,\tau\,\|(v^2/2)_y\|_1 + (2-\alpha)\|w\|_1 + \alpha|\alpha-1|\|v\|_1\,,$$

and further using lemma 4.3 and $\tau\|v_\tau\|_1 \le \|w\|_1 + \alpha\|v\|_1$

$$\begin{aligned} \frac{d}{dt}\left[\tau\|w\|_1\right] \le &\left(\frac{M_n\alpha m(m-2)\|u_0\|_\infty^{m-2}}{1 - M_n m\|u_0\|_\infty^{m-2}} + 2 - \alpha\right)\|w\|_1 \\ &+ \left(\frac{M_n\alpha^2 m(m-2)\|u_0\|_\infty^{m-2}}{1 - M_n m\|u_0\|_\infty^{m-2}} + \alpha|\alpha-1|\right)\|u_0\|_1\,. \end{aligned}$$

It is now easy to verify that the round bracket of the first term $(\cdot)\|w\|_1$ equals zero if

$$\|u_0\|_\infty^{m-2} \le \frac{1}{mM_n(m-1)}\,, \qquad \text{and} \qquad \alpha = \frac{2\left(mM_n\|u_0\|_\infty^{m-2} - 1\right)}{1 - mM_n(m-1)\|u_0\|_\infty^{m-2}}\,,$$

and thus, integration with respect to time yields

$$\|w(\tau)\|_1 \le C\left(m, \|u_0\|_\infty, \|u_0\|_1\right)$$

for some constant $C > 0$. By recalling the definition of w we obtain the desired estimate. □

We use the previous estimates to establish suitable compactness of the family $\{v^\lambda\}_\lambda$.

Proposition 4.5 (Compactness). *Under the assumption*

$$\|u_0(\cdot)\|_{L^\infty_x}^{m-2} < \frac{1}{m(m-1)}, \tag{4.13}$$

the family of solutions v^λ is relatively compact in $\mathcal{C}([\tau_1,\tau_2];L^1(\mathbb{R}))$ for $0 < \tau_1 < \tau_2$.

Proof. Take a sequence $\epsilon_n \to 0$ as $n \to \infty$. Due to the results in Lemmas 4.3 and 4.4, the solution $v^\lambda_{\epsilon_n,n}$ to the problem (4.6)-(4.8)-(4.9)-(4.4) (extended to zero outside the interval $[-n,n]$) with $\epsilon = \epsilon_n$ satisfies

$$\|v^\lambda_{\epsilon_n,n}\|_{L^1_y} + \|\partial_\tau(v^\lambda_{\epsilon_n,n})\|_{L^1_y} + \|\partial_y[(v^\lambda_{\epsilon_n,n})^2]\|_{L^1_y} \leq \frac{1}{\tau}C(m,\|u_0\|_{L^1_x},\|u_0\|_{L^\infty_x}), \tag{4.14}$$

which implies

$$\|v^\lambda_{\epsilon_n,n}(\tau)\|_{L^\infty_y} \leq \frac{1}{\tau^{1/2}}[C(m,\|u_0\|_{L^1_x},\|u_0\|_{L^\infty_x})]^{1/2}. \tag{4.15}$$

Therefore, $\{[v^\lambda_{\epsilon_n,n}]^2\}_n$ is an equi–bounded family in $BV([-R,R]\times[\tau_1,\tau_2])$ for fixed $R>0$ with equi–bounded time derivative $\frac{d}{d\tau}\|[v^\lambda_{\epsilon_n,n}]^2\|_1$. Hence, by Helly's selection theorem and Ascoli–Arzelà's theorem (see [12]), there exists a subsequence $v^\lambda_{\epsilon_{n_k},n_k}$ converging almost everywhere and in $\mathcal{C}([\tau_1,\tau_2];L^1([-R,R]))$. By using the estimate (4.15) and the definition of $K^{\lambda,n}$, one can repeat the same computation as in [15, Lemma 5] and prove that

$$\int_{|y|\geq 2R}|v^\lambda_{\epsilon_{n_k},n_k}(y,\tau)|dy \leq \int_{|y|\geq R}|u_0(x)|dx + C_1\left(\frac{\tau^{1/2}}{R}+\frac{\tau}{R}\right)+R(k),$$

with $R(k)$ decaying to zero as $k \to \infty$. These assertions imply that $v^\lambda_{\epsilon_{n_k},n_k}$ is strongly convergent in $\mathcal{C}([\tau_1,\tau_2];L^1(\mathbb{R}))$. Now, by considering the weak formulation of the problem (4.6), we can easily prove that $v^\lambda_{\epsilon_{n_k},n_k}$ converges almost everywhere on $[\tau_1,\tau_2]\times\mathbb{R}$ (eventually by extracting a further subsequence) to the unique entropy solution v^λ of the rescaled problem (4.3). Moreover, it is not difficult to pass to the limit (as $n\to\infty$) in the above estimates (e.g. by weak lower semicontinuity) and obtain the same estimates (4.14) for v^λ. Since the constant in (4.14) does not depend on λ, one can repeat the whole compactness argument above applied to the family $\{v^\lambda\}_\lambda$ and the proof is complete. □

4.3 Limit as $\lambda \to \infty$

¿From the result in Proposition 4.5 one deduces the convergence up to subsequences of the family v^λ in $\mathcal{C}([\tau_1,\tau_2];L^1(\mathbb{R}))$. We shall prove that the limit is the same for all subsequences. Thus the whole family v^λ converges to a v^∞ as $\lambda\to\infty$, which we identify as the unique nonnegative entropy solution of the homogeneous Burger's equation

$$v_\tau + v\,v_y = 0 \tag{4.16}$$

with initial datum the Dirac delta measure, i. e. the N–wave solution

$$N(y,\tau) = \begin{cases} \frac{y}{\tau} & \text{if } 0\leq y\leq \sqrt{2\|u_0\|_1\tau} \\ 0 & \text{otherwise} \end{cases}. \tag{4.17}$$

We proceed in the following two propositions.

Proposition 4.6. *v^∞ is an entropy solution to (4.16) on the set $\mathbb{R}\times(0,+\infty)$.*

Proof. In the rescaled variables for $0<\tau_1<\tau_2$, we recover the entropy formulation of (4.3) as

$$\int_{\tau_1}^{\tau_2}\int_{\mathbb{R}}[\eta(v)\psi_\tau + q(v)\psi_y]dyd\tau \geq -\lambda^{3-m}\int_{\tau_1}^{\tau_2}\int_{\mathbb{R}}\eta'(v)\,[K^\lambda * v^m - v^m]\psi\,dyd\tau := I$$

for all convex functions $\eta : \mathbb{R} \to \mathbb{R}$ with $q(v) = \int^v s\eta'(s)ds$ and for all nonnegative Lipschitz continuous test functions $\psi : \mathbb{R} \times [\tau_1, \tau_2] \to \mathbb{R}$ having compact support. By a change of variables the right-hand-side I equals to

$$\begin{aligned} I &= \lambda^{3-m} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \eta'(v(y)) K^\lambda(z) [v^m(y) - v^m(y-z)] \psi(y) \, dydzd\tau \\ &= -\lambda^{3-m} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} v^m(y) K^\lambda(z) [\eta'(v(y))\psi(y) - \eta'(v(y-z))\psi(y-z)] \, dydzd\tau \,. \end{aligned}$$

By spiltting the square brackets like

$$[\eta'(v(y))\psi(y) - \eta'(v(y-z))\psi(y-z)] = \eta'(v(y))[\psi(y) - \psi(y-z)] + [\eta'(v(y)) - \eta'(v(y-z))]\psi(y) \,,$$

we have thus for the right-hand-side

$$\begin{aligned} I = I_1 + I_2 &= -\lambda^{3-m} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} v^m(y) K^\lambda(z) \eta'(v(y)) [\psi(y) - \psi(y-z)] \, dydzd\tau \\ &\quad -\lambda^{3-m} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} v^m(y) K^\lambda(z) [\eta'(v(y)) - \eta'(v(y-z))] \psi(y) \, dydzd\tau \,. \end{aligned}$$

For I_1, a rescaling $z \to \frac{z}{\lambda}$ and a Taylor expansion

$$[\psi(y) - \psi(y - \frac{z}{\lambda})] = \psi'(y)\frac{z}{\lambda} + O\left(\frac{z^2}{2}\right)$$

shows that I_1 tends to zero similarly to [14, Theorem 3.13]. For I_2, we change once more variables and rescale $z \to \frac{z}{\lambda}$ to obtain

$$I_2 = -\lambda^{3-m} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} K(z) \eta'(v(y)) [v^m(y)\psi(y) - v^m(y - \frac{z}{\lambda})\psi(y - \frac{z}{\lambda})] \, dydzd\tau \,.$$

Similar to the above, we add and subtract $-v^m(y - \frac{z}{\lambda})\psi(y)$ to the square bracket and estimate then the first difference by the Lipschitz continuity of $v \mapsto v^{m/2}$ for $m > 2$

$$\begin{aligned} I_3 &= \lambda^{3-m} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} K(z) \eta'(v(y)) \psi(y) [v^m(y) - v^m(y - \frac{z}{\lambda})] \, dydzd\tau \\ &\leq C\lambda^{3-m} \int_0^T \int_{\mathbb{R}} K(z) \int_{\mathbb{R}} |v^2(y) - v^2(y - \frac{z}{\lambda})| \, dydzd\tau \leq C\lambda^{2-m} \int_0^T K(z)|z| dz \,, \end{aligned}$$

since $v^2(y) \in BV(\mathbb{R})$ (uniformly in λ) by Lemma 4.4, and therefore (see e.g. [21])

$$\left\| v^2(y) - v^2(y - \frac{z}{\lambda}) \right\|_1 \leq C \left| \frac{z}{\lambda} \right| .$$

Thus, the I_3 tends to zero as $\lambda \to \infty$, and the same is true for second difference since the testfunction ψ is Lipschitz $\left\| \psi(y) - \psi(y - \frac{z}{\lambda}) \right\|_1 \leq C \left| \frac{z}{\lambda} \right|$. □

Finally, we follow [15] to identify the initial datum of v^∞.

Proposition 4.7. *For all test functions $\phi \in C_0^\infty(\mathbb{R})$, the function $v^\infty(y,\tau)$ satisfies*

$$\lim_{\tau \searrow 0} \left| \int_{\mathbb{R}} v(y,\tau)\phi(y)dy - \|u_0\|_{L^1_x} \langle \delta, \phi \rangle \right| = 0,$$

where $\|u_0\|_{L^1_x}$ is the initial mass and δ is Dirac's delta distribution.

Proof. For a test function $\phi \in C_0^\infty$ multiplied with (4.3), we integrate by parts the convection term and change variables in the convolution to calculate (using $\|v\|_1 \leq \|u_0\|_1$)

$$\begin{aligned} &\left| \int_{\mathbb{R}} v(y,\tau)\phi(y)dy - \int_{\mathbb{R}} u_0(y)\phi(y/\lambda)dy \right| = \left| \int_0^\tau \int_{\mathbb{R}} v_\tau \, \phi(y)dyd\tau \right| \\ &\leq \|\phi\|_{W^{1,\infty}} \frac{\|u_0\|_1}{2} \int_0^\tau \|v\|_\infty(\tau')\, d\tau' + \lambda^{3-m} \|K^\lambda * \phi - \phi\|_\infty \int_0^\tau \|v\|_m^m(\tau')\, d\tau' \, , \end{aligned}$$

where we have as in [15] that

$$\|K^\lambda * \phi - \phi\|_\infty \leq \lambda^{-2} \|\phi''\|_\infty + \lambda^{-3} \frac{1}{6} \|\phi'''\|_\infty \int_{\mathbb{R}} |z|^3 K(z) dz \, .$$

Thus, by the decay estimates (4.15) and $\|v\|_m^m \leq \lambda^{m-1}\|u_0\|_m^m$, there are constants C_1 and C_2 such that

$$\left| \int_{\mathbb{R}} v(y,\tau)\phi(y)dy - \int_{\mathbb{R}} u_0(y)\phi(y/\lambda)dy \right| \leq C_1 \tau^{1/2} + C_2 \tau \, .$$

Moreover, in the limit $\lambda \to \infty$,

$$\left| \int_{\mathbb{R}} v_\infty(y,\tau)\phi(y)dy - \|u_0\|_{L^1_x} \phi(0) \right| \leq C_1 \tau^{1/2} + C_2 \tau \, ,$$

and the limit $\tau \to 0$ shows the desired assertion. □

In order to complete the proof of Theorem 1.4, we choose $\tau = 1$ and obtain

$$\int |u(x,\lambda^2)dx - N(x,\lambda^2)|dx = \int |v^\lambda(y,1) - v^\infty(y,1)|dy \to 0 \qquad \text{as } \lambda \to +\infty.$$

Moreover, the uniform L^∞ bound for v^λ trivially implies the decay rate (1.10). The L^p estimates easily follow by interpolation.

Acknowledgments

Liu's research was supported by the USA National Science Foundation under Grant DMS05-05975. Marco Di Francesco acknowledges support from the Wolfgang Pauli Institute (WPI) of Vienna. KF was partially supported by the Vienna Science and Technology Fund (WWTF) project "How do cells move?"

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