Entropy Methods for Reaction-Diffusion Equations with Degenerate Diffusion Arising in Reversible Chemistry*

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An approach entirely based on the entropy dissipation has been used in the past in the study of reaction-diffusion equations arising in reversible chemistry. In particular, it is possible to study the existence (and smoothness) of solutions to these PDEs in many situations, but in general under the requirement that the diffusion be nondegenerate (or at least that all diffusion coefficients are strictly positive a.e.).

We wish to describe here how the entropy approach also enables to treat reaction-diffusion equations which are strongly degenerated, in the sense that one or more of the species do not diffuse at all.

Keywords: reaction-diffusion systems, degenerate diffusion, entropy

1. Introduction

Reaction-Diffusion Systems for Reversible Chemistry

The evolution of a mixture of diffusive species $\mathcal{A}_i, i = 1, 2, \ldots, q$, undergoing a reversible reaction of the type

$$\alpha_1 \mathcal{A}_1 + \cdots + \alpha_q \mathcal{A}_q \rightleftharpoons \beta_1 \mathcal{A}_1 + \cdots + \beta_q \mathcal{A}_q, \qquad \alpha_i, \beta_i \in \mathbb{N}$$

*This proceedings summerizes the contributions of both authors.

is modelled using mass-action kinetics (see [2,5] for a derivation from basic principles) in the following way:

$$\partial_t a_i - d_i\, \Delta_x a_i = (\beta_i - \alpha_i)\bigg(l \prod_{j=1}^{q} a_j^{\alpha_j} - k \prod_{j=1}^{q} a_j^{\beta_j} \bigg)\,, \tag{1}$$

where $a_i := a_i(t,x) \geq 0$ denotes the concentration at time t and point x of the species A_i.

We suppose that $x \in \Omega$, where Ω is a bounded and regular (C^∞) domain of $I\!R^N$ ($N \geq 1$). After a suitable rescaling, it is possible to assume that $l = 1$ (or equally $k = 1$) and that Ω is normalized (i.e. $|\Omega| = 1$).

Moreover, we complement system (1) by homogeneous Neumann boundary conditions:

$$n(x)\cdot\nabla_x a_i(t,x) = 0\,, \qquad \forall t \geq 0, x \in \partial\Omega\,, \tag{2}$$

where $n(x)$ is the outer normal unit vector at point x of $\partial\Omega$.

The particular case $\mathcal{A}_1 + \mathcal{A}_2 \rightleftharpoons \mathcal{A}_3 + \mathcal{A}_4$ (that is, when $q = 4$ with $\alpha_1 = \alpha_2 = 1$, $\beta_3 = \beta_4 = 1$, $\alpha_3 = \alpha_4 = 0$, and $\beta_1 = \beta_2 = 0$). has lately received a lot of attention [8,11]. In order to simplify the notation we further choose $k = 1$ ($k \neq 1$ works analogous with minor modifications) It is proven there that whenever d_1, d_2, d_3, $d_4 > 0$, there exists a global smooth solution for dimensions $N = 1, 2$. Those solutions decay exponentially fast (with explicit rates of decay) when $t \to +\infty$ towards the (unique) equilibrium. In the case $N > 2$, one still can prove [7] the existence of global weak (L^2 and even $L^2\,(\log L)^2$) solutions, but it is not known whether there exist global smooth solutions for general (smooth) initial data, and their large time behavior is not yet studied.

In the present work, we shall first show that exponential convergence (with explicit rates) towards the unique constant equilibrium still holds for any dimension N (see Theorem 2.1 below) when one considers the weak solutions. The proof of Theorem 2.1 is based on an approach where a quantitative entropy-entropy dissipation estimate is established using natural a priori bounds of the system, improving the results of [7].

It is also proven in [8] that weak solutions exist (for all $N \geq 1$) when the diffusions depend on x in such a way that their sum is bounded below (by a strictly positive constant), and that they all are strictly positive a.e.

We shall study in the present paper what subsists of those results when at least one of the d_i-s (assumed here to be constants) is zero.

The paper is organized as follows.

Plan of the paper

We start in Section 2 by presenting the a priori bounds for our system and by overviewing the analytical tools which are available. We show how they can be used to prove Theorem 2.1 about large time behavior.

Then, each following section of this work is devoted to a case in which a certain subset of $\{d_1, d_2, d_3, d_4\}$ is constituted of zeroes. All possible cases are studied (up to symmetries), except the cases when all the d_i-s are strictly positive (already treated in [8,11]) and when all the d_i-s are zero (the system is then only constituted of ODEs). Section 3 is concerned with the case $d_1 = 0$, while the cases $d_1 = d_2 = 0$ and $d_1 = d_3 = 0$ are treated in Section 4. Then, section 5 is devoted to the situation when $d_1 = d_2 = d_3 = 0$. Finally, in a last section we present a few results for the simpler system $\mathcal{A}_1 + \mathcal{A}_2 \rightleftharpoons \mathcal{A}_3$.

2. A priori estimates, analytical tools, and convergence to equilibrium

2.1. *A priori estimates and equilibrium*

The particular case of system (1) that we are concerned with writes

$$\begin{cases} \partial_t a_1 - d_1\,\Delta_x a_1 = a_3\,a_4 - a_1\,a_2, \\ \partial_t a_2 - d_2\,\Delta_x a_2 = a_3\,a_4 - a_1\,a_2, \\ \partial_t a_3 - d_3\,\Delta_x a_3 = a_1\,a_2 - a_3\,a_4, \\ \partial_t a_4 - d_4\,\Delta_x a_4 = a_1\,a_2 - a_3\,a_4, \end{cases} \tag{3}$$

together with the homogeneous Neumann boundary conditions (2).

Two basic a priori estimates, coming naturally out of the thermodynamics, can be written for this system.

(i) Firstly, the conservation of the number of atoms implies (at first for all smooth solutions $(a_i)_{i=1,..,4}$ of (3) with Neumann condition (2)) that for all $t \geq 0$,

$$\begin{cases} \int_\Omega (a_1(t,x) + a_3(t,x))\;dx = \int_\Omega (a_1(0,x) + a_3(0,x))\;dx =: M_{13}\,, \\ \int_\Omega (a_1(t,x) + a_4(t,x))\;dx = \int_\Omega (a_1(0,x) + a_4(0,x))\;dx =: M_{14}\,, \\ \int_\Omega (a_2(t,x) + a_3(t,x))\;dx = \int_\Omega (a_2(0,x) + a_3(0,x))\;dx =: M_{23}\,. \end{cases} \tag{4}$$

(ii) Secondly, introducing the nonnegative entropy (free energy) functional $E(a_i)$ and the entropy dissipation $D((a_i)_{i=1,..,4}) = -\frac{d}{dt}E((a_i))_{i=1,..,4}$

associated to (3) :

$$E(a_i(t,x)_{i=1,..,4}) = \sum_{i=1}^{4} \int_\Omega \Big(a_i(t,x)\log(a_i(t,x)) - a_i(t,x) + 1 \Big)\, dx\,, \quad (5)$$

$$D(a_i(t,x)_{i=1,..,4}) = \sum_{i=1}^{4} \int_\Omega 4\, d_i\, |\nabla_x \sqrt{a_i(t,x)}|^2\, dx \quad (6)$$
$$+ \int_\Omega (a_1\, a_2 - a_3\, a_4) \log\left(\frac{a_1\, a_2}{a_3\, a_4}\right)(t,x)\, dx\,,$$

it is easy to verify that the following a priori estimate holds (still for smooth solutions $(a_i)_{i=1,..,4}$ of (3) with (2)) for all $t \geq 0$

$$E(a_i(t,x)_{i=1,..,4}) + \int_0^t D(a_i(s,x)_{i=1,..,4})\, ds = E(a_i(0,x)_{i=1,..,4})\,. \quad (7)$$

Finally observe that when all the diffusivity constants d_i are strictly positive (or when at most one of them is equal to 0), there is a unique constant equilibrium state $(a_{i,\infty})_{i=1,..,4}$ (for which the entropy dissipation vanishes). It is defined by the unique positive constants solving $a_{1,\infty}\, a_{2,\infty} = a_{3,\infty}\, a_{4,\infty}$ provided $a_{j,\infty} + a_{k,\infty} = M_{jk}$ for $(j,k) \in (\{1,2\},\{3,4\})$, that is:

$$\begin{cases} a_{1,\infty} = \frac{M_{13}M_{14}}{M}\,, & a_{3,\infty} = M_{13} - \frac{M_{13}M_{14}}{M} = \frac{M_{13}M_{23}}{M}\,, \\ a_{2,\infty} = \frac{M_{23}M_{24}}{M}\,, & a_{4,\infty} = M_{14} - \frac{M_{13}M_{14}}{M} = \frac{M_{14}M_{24}}{M}\,, \end{cases} \quad (8)$$

where M denotes the total initial mass $M = M_{13} + M_{24} = M_{14} + M_{23}$.

2.2. *Analytical tools*

In the sequel, the following tools will systematically be used:

– First, the entropy decay (7) estimate will be exploited as much as possible. Being nonincreasing, the entropy functional will ensure that

$$a_i \in L^\infty([0,+\infty[; L\log L(\Omega)) \qquad \forall i = 1,..,4\,. \quad (9)$$

Considering that the time integral of the entropy dissipation is bounded in (7), its first component will provide the estimate

$$\sqrt{a_i} \in L^2([0,+\infty[; H^1(\Omega)). \qquad \forall i = 1,..,4 \text{ such that } d_i > 0\,, \quad (10)$$

Finally, the second component of the time integral of the entropy dissipation will ensure that, provided $a_3\, a_4 \in L^1_{loc}([0,+\infty[\times\bar{\Omega})$, then $a_1\, a_2 \in L^1_{loc}([0,+\infty[\times\bar{\Omega})$ too. This comes out of the following classical inequality (cf. [10]), which holds for any $\kappa > 1$,

$$a_1\, a_2 \leq \kappa\, a_3\, a_4 + \frac{1}{\log\kappa}\,(a_1\, a_2 - a_3\, a_4)\, \log\left(\frac{a_1\, a_2}{a_3\, a_4}\right). \quad (11)$$

Note that by letting κ be as large as necessary, this inequality also allows to prove that an approximating sequence $a_1^n\, a_2^n$ is (locally in time) weakly compact in L^1 if the sequence $a_3^n\, a_4^n$ is also weakly compact in L^1 (and when estimate (7) holds uniformly with respect to n).

– Secondly, we shall use the duality method as presented, for instance, by Pierre and Schmitt [13]. This method ensures that whenever some quantity $u := u(t,x) \geq 0$ satisfies

$$\begin{cases} \partial_t u - \Delta_x[\phi\, u] \leq 0\,, & \forall t \in [0,T], x \in \Omega\,, \\ n \cdot \nabla_x[\phi\, u] = 0\,, & \forall t \in [0,T], x \in \partial\Omega\,, \\ u(0,\cdot) \in L^2(\Omega)\,, \end{cases} \tag{12}$$

where $0 < \delta_1 \leq \phi(t,x) \leq \delta_2 < +\infty$, then $u(t,x)$ lies in $L^2([0,T] \times \Omega)$.

We shall also use a variant of this estimate adapted to degenerate diffusions (Cf. [8]): Whenever $\alpha_1 := \alpha_1(t,x)$,.., $\alpha_k := \alpha_k(t,x) \geq 0$ satisfy

$$\begin{cases} \partial_t \left(\sum_{i=1}^k \alpha_i\right) - \Delta_x \left(\sum_{i=1}^k d_i\, \alpha_i\right) \leq 0\,, & \forall t \in [0,T], x \in \Omega\,, \\ n \cdot \nabla_x \alpha_i = 0\,, & \forall t \in [0,T], x \in \partial\Omega\,, \\ \alpha_i(0,\cdot) \in L^2(\Omega)\,, \end{cases} \tag{13}$$

for some $d_i \geq 0$, then $(\sum_{i=1}^k \alpha_i)\,(\sum_{i=1}^k d_i\, \alpha_i) \in L^1([0,T] \times \Omega)$.

Still another variant of this estimate is useful (Cf. [14]): If $\alpha_1 := \alpha_1(t,x)$, $\alpha_2 := \alpha_2(t,x) \geq 0$ satisfy

$$\begin{cases} \partial_t \left(\sum_{i=1}^2 \alpha_i\right) - \Delta_x \left(\sum_{i=1}^2 d_i\, \alpha_i\right) \leq 0\,, & \forall t \in [0,T], x \in \Omega\,, \\ n \cdot \nabla_x \alpha_i = 0\,, & \forall t \in [0,T], x \in \partial\Omega\,, \\ \alpha_i(0,\cdot) \in L^p(\Omega) \qquad p < +\infty, \end{cases} \tag{14}$$

for some $d_1, d_2 > 0$, then $\alpha_1 \in L^p([0,T] \times \Omega) \Rightarrow \alpha_2 \in L^p([0,T] \times \Omega)$.

– Then, we shall use the properties of the heat kernel in $[0,T] \times \Omega$. We recall that if $f \in L^p([0,T] \times \Omega)$, then the solution $u := u(t,x)$ of

$$\begin{cases} \partial_t u - \Delta_x u = f & \forall t \in [0,T], x \in \Omega\,, \\ n \cdot \nabla_x u = 0\,, & \forall t \in [0,T], x \in \partial\Omega\,, \\ u(0,\cdot) \in L^\infty(\Omega)\,, \end{cases} \tag{15}$$

lies in $L^q([0,T] \times \Omega)$ for all $q \in [1,+\infty]$ such that

$$\frac{1}{p} + \frac{N}{N+2} < \frac{1}{q} + 1.$$

We shall also use the monotonicity of the heat kernel: $f \geq 0 \Rightarrow u \geq 0$.

Finally, we introduce a lemma which is widely known to hold, but often without reference. We therefore recall an argument of Strook [16], which shows that Sobolev and Poincaré inequality imply the logarithmic Sobolev inequality without confining potential on a bounded domain :

Lemma 2.1 (Logarithmic Sobolev inequality). *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ *such that the Poincaré (-Wirtinger) and Sobolev inequalities*

$$\|\phi - \textstyle\int_\Omega \phi\, dx\|^2_{L^2(\Omega)} \le P(\Omega)\, \|\nabla_x \phi\|^2_{L^2(\Omega)}\,, \tag{16}$$

$$\|\phi\|^2_{L^q(\Omega)} \le C_1(\Omega)\, \|\nabla_x \phi\|^2_{L^2(\Omega)} + C_2(\Omega)\, \|\phi\|^2_{L^2(\Omega)}\,, \qquad \tfrac{1}{q} = \tfrac{1}{2} - \tfrac{1}{N}\,, \tag{17}$$

hold. Then, the logarithmic Sobolev inequality

$$\int_\Omega \phi^2 \log\left(\frac{\phi^2}{\|\phi\|_2^2}\right) dx \le L(\Omega, N)\, \|\nabla_x \phi\|^2_{L^2(\Omega)} \tag{18}$$

holds (for some constant $L(\Omega, N) > 0$*).*

Proof of Lemma 2.1: Assume firstly that $\|\phi\|_2^2 = 1$. Then, using Jensens inequality for the measure $\phi^2\, dx$, we estimate

$$\begin{aligned}\int_\Omega \phi^2 \log(\phi^2)\, dx &= \frac{2}{q-2}\int_\Omega \log\left(\phi^{q-2}\right)\, (\phi^2 dx) \le \frac{2}{q-2}\log\left(\int_\Omega \phi^q\, dx\right)\\ &= \frac{q}{q-2}\log\left(\|\phi\|_q^2\right) \le \frac{q}{q-2}\left(\|\phi\|_q^2 - 1\right),\end{aligned}$$

using the elementary inequality $\log x \le x - 1$. Hence, we have for general ϕ,

$$\begin{aligned}\int_\Omega \phi^2 \log\left(\frac{\phi^2}{\|\phi\|_2^2}\right) dx &\le \frac{q}{q-2}\ \left(\|\phi\|_q^2 - \|\phi\|_2^2\right)\\ &\le \frac{q}{q-2}\, C_1\, \|\nabla_x \phi\|_2^2 + \frac{q}{q-2}\,(C_2 - 1)\, \|\phi\|_2^2\,,\end{aligned}$$

using the Sobolev inequality (17). Now, in case when $\int_\Omega \phi\, dx = 0$, inequality (18) follows directly from Poincaré inequality (16). Otherwise, considering $\tilde\phi = \phi - \int_\Omega \phi\, dx$, a lengthy calculation [9] shows that

$$\int_\Omega \phi^2\, \log\left(\frac{\phi^2}{\|\phi\|_2^2}\right)\, dx \le \int_\Omega \tilde\phi^2\, \log\left(\frac{\tilde\phi^2}{\|\tilde\phi\|_2^2}\right)\, dx + 2\, \|\tilde\phi\|_2^2\,,$$

and the inequaltiy (18) follows from Poincaré inequality (16).

Remark 2.1. On convex domains Ω, an alternative proof of (18) consists in building a limiting procedure with a sequence of logarithmic Sobolev inequalitites on $\mathbb{R}^N$ (see e.g. [1,4]) with a convex confining potential, which is made constant inside the bounded domain (by using the Holley-Strook perturbation lemma [12]) and tends to infinity outside of the bounded domain.

2.3. *Convergence to equilibrium*

We end this section by a proof of exponential decay towards equilibrium (with explicit rates) for the weak solutions of system (3) when there is no degeneracy (all d_i are strictly positive).

Theorem 2.1. *Let Ω be a bounded domain such that Lemma 2.1 holds. Let $(d_i)_{i=1,..,4} > 0$ be strictly positive diffusion rates. Finally, let the initial data $(a_{i,0})_{i=1,..,4}$ be nonnegative functions of $L^2\,(\log L)^2(\Omega)$ with strictly positive masses $(M_{jk})_{(j,k)\in(\{1,2\},\{3,4\})} > 0$ (see (4)). Then, the global weak solution a_i of (3), (2) (as shown to exist in [8]) decay exponentially towards the positive equilibrium state $(a_{i,\infty})_{i=1,..,4} > 0$ defined by (8) :*

$$\sum_{i=1}^{4} \|a_i(t,\cdot) - a_{i,\infty}\|^2_{L^1(\Omega)} \le C_1 \left(E((a_{i,0})_{i=1,..,4}) - E((a_{i,\infty})_{i=1,..,4}) \right) e^{-C_2\,t} \,,$$

where the constants C_1 and C_2 can be computed explicitly.

Remark 2.2. The above Theorem generalizes to all space dimensions the convergence result obtained in [7]. It avoids a slowly growing L^∞-bound (available only in 1D and maybe 2D) by using at some step the logarithmic Sobolev inequality (18) to control the relative entropy of the concentrations a_i with their averages $\overline{a_i} = \int_\Omega a_i\,dx$ (recall that $|\Omega| = 1$). The last (and main) part of the proof follows then [7]: It uses the global bounds of the averages $\overline{a_i}$ in all dimensions, which are obtained thanks to the conservation laws (4).

Note also that exponential decay towards equilibrium in $L^p(\Omega)$ with $1 < p < 2$ follows by interpolation with a slowly (polynomially) growing $L^2(\Omega)$-bound [8].

Proof of Theorem 2.1. : The proof is based on an entropy approach, where the entropy dissipation $D((a_i)_{i=1,..,4}) = -\frac{d}{dt}E((a_i)_{i=1,..,4}) = -\frac{d}{dt}(E((a_i)_{i=1,..,4}) - E((a_{i,\infty})_{i=1,..,4}))$ is controlled from below in terms of the relative entropy with respect to equilibrium. That is, we look for an estimate like

$$\begin{aligned} D((a_i)_{i=1,..,4}) &\geq C\, E((a_i)_{i=1,..,4}|(a_{i,\infty})_{i=1,..,4}) \qquad (19) \\ &= C \sum_{i=1}^{4} \int_\Omega \left[a_i \log\left(\frac{a_i}{a_{i,\infty}}\right) - (a_i - a_{i,\infty}) \right] dx\,, \end{aligned}$$

with a constant C provided that all the conservation laws (4) are observed. Then, a simple Gronwall lemma yields exponential convergence in relative entropy to the equilibrium $(a_{i,\infty})_{i=1,..,4}$. Furthermore, convergence in L^1 as stated in Theorem 2.1 follows from a Cziszar-Kullback type inequality [7, Proposition 4.1].

In order to establish the entropy-entropy dissipation estimate (19), we firstly use an additivity property of the relative entropy

$$\begin{aligned} E((a_i)_{i=1,..,4}|(a_{i,\infty})_{i=1,..,4}) &= E((a_i)_{i=1,..,4}|(\overline{a_i})_{i=1,..,4}) \\ &\quad + E((\overline{a_i})_{i=1,..,4}|(a_{i,\infty})_{i=1,..,4})\,, \end{aligned}$$

to – roughly speaking – separate in relative entropy how far away the concentrations a_i are from their averages $\overline{a_i}$ and how much the averages $\overline{a_i}$ differ from the equilibrium constants $a_{i,\infty}$.

The first term can be estimated thanks to the logarithmic Sobolev inequality (18) (recall the conservation laws (4)) by

$$E((a_i)_{i=1,..,4}|(\overline{a_i})_{i=1,..,4}) = \sum_{i=1}^{4} \int_\Omega a_i \log\left(\frac{a_i}{\overline{a_i}}\right) dx \leq L(\Omega) \sum_{i=1}^{4} \int_\Omega |\nabla_x \sqrt{a_i}|^2\, dx\,,$$

which is clearly bounded by the entropy dissipation $D((a_i)_{i=1,..,4})$.

On the other hand, estimating the second contribution can be done in the following way: We define

$$\phi(x,y) = \frac{x\,\ln(x/y) - (x-y)}{(\sqrt{x}-\sqrt{y})^2} = \phi(\frac{x}{y}, 1).$$

Note that thanks to the conservation laws (4), $\phi(\overline{a_i}/a_{i,\infty}, 1) \leq C(M_{jk})$. Then, we can write

$$\begin{aligned} E((\overline{a_i})_{i=1,..,4}|(a_{i,\infty})_{i=1,..,4}) &= \sum_{i=1}^{4} \left[\overline{a_i} \log\left(\frac{\overline{a_i}}{a_{i,\infty}}\right) - (\overline{a_i} - a_{i,\infty}) \right] \\ \leq \sum_{i=1}^{4} \phi(\overline{a_i}, a_{i,\infty}) \left|\sqrt{\overline{a_i}} - \sqrt{a_{i,\infty}}\right|^2 &\leq C(M_{jk}) \sum_{i=1}^{4} \left|\sqrt{\overline{a_i}} - \sqrt{a_{i,\infty}}\right|^2\,. \end{aligned}$$

Finally, the expression $\sum_{i=1}^{4} \left|\sqrt{\overline{a_i}} - \sqrt{a_{i,\infty}}\right|^2$ is bounded in terms of equation (47) in [7, Lemma 3.2]. This Lemma entails as a consequence that it can be bounded in terms of the entropy dissipation $D((a_i)_{i=1,..,4})$. We recall that the proof can then be concluded thanks to the Cziszar-Kullback type inequality [7, Proposition 4.1]. □

3. Existence and smoothness, one diffusion coefficient missing

This section is devoted to the study of system (3), (2) in the case when $d_1 = 0$, d_2, d_3, $d_4 > 0$.

Theorem 3.1. *Let $\Omega \subset I\!R^N$ be a bounded and regular (C^∞) domain, and $d_1 = 0$, d_2, d_3, $d_4 > 0$. Then there exists a weak ($L^1_{loc}([0,+\infty] \times \bar{\Omega}) \times (L^2(\log L)^2_{loc}([0,+\infty] \times \bar{\Omega})^3$) solution to system (3), (2) for all smooth (and compatible with the Neumann boundary condition) nonnegative initial data.*

Moreover, this solution is smooth ($C^\infty([0,+\infty[\times\bar{\Omega})^4$) when $N = 1$.

Proof of Theorem 3.1. We consider a (strong) solution of system (3), (2) and observe that

$$\partial_t\left(\sum_{i=1}^{4}(a_i\,\log a_i - a_i + 1)\right) - \Delta_x\left(\sum_{i=1}^{4} d_i\,(a_i\,\log a_i - a_i + 1)\right) \le 0. \quad (20)$$

Using the method of duality, we get the estimate

$$\left[\sum_{i=2}^{4} d_i\,(a_i\,\log a_i - a_i + 1)\right]\left[\sum_{i=1}^{4}(a_i\,\log a_i - a_i + 1)\right] \in L^1_{loc}([0,+\infty[\times\bar{\Omega}).$$

As a consequence and since d_2, d_3, $d_4 > 0$, we see that a_2, a_3, $a_4 \in L^2(\log L)^2_{loc}([0,+\infty[\times\bar{\Omega})$. In particular $a_3\,a_4 \in L\log L_{loc}([0,+\infty[\times\bar{\Omega})$. Then, one uses inequality (11) in order to see that $a_1\,a_2 \in L^1_{loc}([0,+\infty[\times\bar{\Omega})$.

Since these bounds enable to define all the terms in system (3), (2), existence can then be proven using the following approximating system:

$$\begin{cases} \partial_t a_1^n - d_1\,\Delta_x a_1^n = \frac{a_3^n\,a_4^n - a_1^n\,a_2^n}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n}, \\ \partial_t a_2^n - d_2\,\Delta_x a_2^n = \frac{a_3^n\,a_4^n - a_1^n\,a_2^n}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n}, \\ \partial_t a_3^n - d_3\,\Delta_x a_3^n = \frac{a_1^n\,a_2^n - a_3^n\,a_4^n}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n}, \\ \partial_t a_4^n - d_4\,\Delta_x a_4^n = \frac{a_1^n\,a_2^n - a_3^n\,a_4^n}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n}, \end{cases} \quad \forall t \in I\!R_+, x \in \Omega \quad (21)$$

$$n\cdot\nabla_x a_i^n = 0\,, \qquad \forall t \in I\!R_+, x \in \partial\Omega\,, \quad (22)$$

$$a_i^n(0,x) = a_i(0,x)\,, \qquad \forall x \in \Omega\,. \quad (23)$$

Existence of a smooth solution for system (21), (22), (23) (for a given $n \in I\!N^*$ and for $d_i \geq 0$) is a consequence of the existence of smooth solutions $U : I\!R_+ \times \bar\Omega \to I\!R^d$ of systems like

$$\partial_t U - D\,\Delta_x U = f(U),$$

where D is a diagonal constant matrix whose entries are all nonnegative, and f is globally Lipschitz-continuous. Those solutions are limits of the sequence U_k, where

$$\partial_t U_{k+1} - D\,\Delta_x U_{k+1} = f(U_k).$$

For details we refer for example to [6].

Then, we show that it is possible to pass to the limit in (21), (22), (23) in order to recover a (weak) solution to (3), (2). This is done by verifying that the a priori estimates proven above for the solutions of (3), (2) still hold (uniformly with respect to n) for the approximated system (21), (22), (23). Note first that the method of duality ensures that

$$\left[\sum_{i=2}^4 d_i\,(a_i^n\,\log a_i^n - a_i^n + 1)\right]\left[\sum_{i=1}^4 (a_i^n\,\log a_i^n - a_i^n + 1)\right]$$

is bounded in $L^1_{loc}([0,+\infty[\times\bar\Omega)$. As a consequence, $a_3^n\,a_4^n$ is bounded in $L\log L_{loc}([0,+\infty[\times\bar\Omega)$. Then, the entropy dissipation estimate writes (for the approximated system):

$$\sup_{t\in[0,T]}\int_\Omega \sum_{i=1}^4 (a_i^n \log a_i^n - a_i^n + 1)\,dx + \int_0^T\!\!\int_\Omega \left[\sum_{i=1}^4 d_i \frac{|\nabla_x a_i^n(s,x)|^2}{a_i^n(s,x)}\right] dxds$$
$$+ \int_0^T\!\!\int_\Omega \frac{a_1^n(s,x)\,a_2^n(s,x) - a_3^n(s,x)\,a_4^n(s,x)}{1+\frac{1}{n}\,\sum_{i=1}^4 a_i^n(s,x)}\,\log\left(\frac{a_1^n(s,x)\,a_2^n(s,x)}{a_3^n(s,x)\,a_4^n(s,x)}\right) dxds \leq C,$$

for a constant C independent of n. The first term in this estimate ensures that the $(a_i^n)_{i=1,..,4}$ are weakly compact in $L^1_{loc}([0,+\infty[\times\bar\Omega)$. Up to extraction, they converge therefore to some limit a_i.

Using the following variant of inequality (11):

$$\frac{a_1^n\,a_2^n}{1+\frac{1}{n}\,\sum_{i=1}^4 a_i^n} \leq \kappa\,\frac{a_3^n\,a_4^n}{1+\frac{1}{n}\,\sum_{i=1}^4 a_i^n} + \frac{1}{\log\kappa}\,\frac{(a_1^n\,a_2^n - a_3^n\,a_4^n)\,\log\left(\frac{a_1^n\,a_2^n}{a_3^n\,a_4^n}\right)}{1+\frac{1}{n}\,\sum_{i=1}^4 a_i^n},$$

we see (thanks to Dunford-Pettis' criterion, cf. [3] for example) that the quantity

$$\frac{a_1^n\,a_2^n}{1+\frac{1}{n}\,\sum_{i=1}^4 a_i^n}$$

lies in a weak compact set of $L^1_{loc}([0,+\infty[\times\bar\Omega)$. As a consequence, $\partial_t a_i^n - d_i\,\Delta_x a_i^n$ lies in a weak compact set of $L^1_{loc}([0,+\infty[\times\bar\Omega)$, and the properties of the heat kernel ensure that the sequences $(a_i^n)_{i=2,3,4}$ converge a.e. towards a_i. We then rewrite the first equation of (21) as

$$a_1^n(t,x) = a_1^n(0,x)\, e^{-\int_0^t \frac{a_2^n(\sigma,x)}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n(\sigma,x)}d\sigma} + \int_0^t \frac{a_3^n(s,x)\,a_4^n(s,x)}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n(s,x)}\, e^{-\int_s^t \frac{a_2^n(\sigma,x)}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n(\sigma,x)}d\sigma}\, ds.$$

Then, $\frac{1}{n}\sum_{i=1}^4 a_i^n$ converges a.e. to 0 so that

$$\frac{a_2^n}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n} \to a_2 \quad a.e. \quad \text{and} \quad \frac{a_3^n\,a_4^n}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n} \to a_3\,a_4 \quad a.e.$$

Thanks to the bounds on $(a_i^n)_{i=2,3,4}$ in $L^2(\log L)^2_{loc}([0,+\infty[\times\bar\Omega)$, we see that a_1^n converges a.e. to a_1. Finally, the weak compactness in $L^1_{loc}([0,+\infty[\times\bar\Omega)$ of $\frac{a_1^n\,a_2^n}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n}$ and $\frac{a_3^n\,a_4^n}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n}$ enables to pass to the limit in (the weak formulation) of (21), (22), (23).

In dimension 1, it is in fact possible to show that these solutions are strong. This follows by observing that a_2, a_3, a_4 satisfy a heat equation with a r.h.s. in $L^1_{loc}([0,+\infty[\times\bar\Omega)$. As a consequence, they lie in $L^{3-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$ for all $\varepsilon > 0$. Then, a_2 satisfies a heat equation with a r.h.s. in $L^{3/2-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$, and it lies therefore in $L^p_{loc}([0,+\infty[\times\bar\Omega)$ for all $p \in [1,+\infty[$. We observe that (for all $\varepsilon > 0$)

$$\partial_t a_1 \le a_3\,a_4 \in L^{3/2-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega), \tag{24}$$

so that $a_1\,a_2 \in L^{3/2-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$, and $a_3\,a_4$ satisfies a heat equation with a r.h.s. in $L^{3/2-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$, and lies as a consequence in $L^p_{loc}([0,+\infty[\times\bar\Omega)$ for all $p \in [1,+\infty[$. Using again (24), this also holds for a_1. A last application of the properties of the heat kernel and of (24) ensures that all a_i are bounded, and the smoothness is an easy consequence of general properties of the heat operator and the regularity with respect to a parameter of solutions of ODEs. This ends the proof of Theorem 3.1. □

4. Existence and smoothness, two diffusion coefficients missing

4.1. *First case*

Theorem 4.1. *Let $\Omega \subset I\!R^N$ be a bounded and regular (C^∞) domain, and $d_1 = d_3 = 0$, d_2, $d_4 > 0$. Then there exists a smooth ($(C^\infty([0,+\infty[\times\bar\Omega))^4$)*

solution to system (3), (2) for all smooth initial nonnegative data (compatible with the Neumann boundary condition).

Proof of Theorem 4.1. We consider a solution to system (3), (2). Then, $\partial_t(a_1 + a_3) = 0$, so that $a_1, a_3 \in L^\infty_{loc}([0,+\infty[\times\bar\Omega)$. Moreover, $a_2, a_4 \in L^\infty([0,+\infty[; L^1(\Omega))$ thanks to the conservation laws, so that $a_1\, a_2$ and $a_3\, a_4 \in L^1_{loc}([0,+\infty[\times\bar\Omega)$.

Using the properties of the heat kernel, we see that $a_2, a_4 \in L^{(N+2)/N-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$ for all $\varepsilon > 0$, and the same is true for $a_1\, a_2$, $a_3\, a_4$. An immediate induction shows that $a_2, a_4 \in L^{p_s-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$ for $(p_s)_{s\in I\!N^*}$ such that

$$p_1 = \frac{N}{N+2}; \qquad \frac{1}{p_s} + \frac{N}{N+2} = \frac{1}{p_{s+1}} - 1.$$

Finally, $a_i \in L^\infty_{loc}([0,+\infty[\times\bar\Omega)$ for all $i = 1,..,4$, and the smoothness can easily be recovered.

Existence is obtained through the use of the approximated system (21), (22), (23). The proof used above (when applied to a_i^n instead of a_i) shows that the sequences a_i^n are bounded in $L^\infty_{loc}([0,+\infty[\times\bar\Omega)$. Then, a_2^n and a_4^n converge a.e. to some limit a_2, a_4 thanks to the properties of the heat kernel. Denoting $\phi(x) := a_1^n(t,x) + a_3^n(t,x) = a_1(0,x) + a_3(0,x)$, we see that

$$a_1^n(t,x) = a_1^n(0,x)\, e^{-\int_0^t \frac{a_2^n(\sigma,x)+a_4^n(\sigma,x)}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n(\sigma,x)}d\sigma} + \int_0^t \frac{\phi(x)\, a_4^n(s,x)}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n(s,x)}\, e^{-\int_s^t \frac{a_2^n(\sigma,x)+a_4^n(\sigma,x)}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n(\sigma,x)}d\sigma}\, ds.$$

Then, $\frac{1}{n}\sum_{i=1}^4 a_i^n$ converges a.e. to 0 so that a_1^n converges a.e to some limit a_1. The same holds for a_3^n. As a consequence, it is possible to pass to the limit in (21), (22), (23) and this ends the proof of Theorem 4.1. □

4.2. *Second case*

Theorem 4.2. *Let $\Omega \subset I\!R^N$ be a bounded, regular (C^∞) domain, and $d_1 = d_2 = 0$, d_3, $d_4 > 0$. Then there exists a weak $((L^1_{loc}([0,+\infty[\times\bar\Omega))^2 \times (L^2(\log L)^2_{loc}([0,+\infty[\times\bar\Omega))^2)$ solution to system (3), (2) for all smooth initial nonnegative data (compatible with the Neumann boundary condition).*

Moreover this solution is smooth ($C^\infty([0,+\infty[\times\bar\Omega)^4$) when $N = 1$.

Proof of Theorem 4.2. We consider a solution to system (3), (2). We notice that thanks to the method of duality, inequality (20) leads to the estimate

$$(d_3\,a_3 + d_4\,a_4)\,(a_1 + a_2 + a_3 + a_4) \in L\log L_{loc}([0,+\infty[\times\bar{\Omega}), \tag{25}$$

so that $a_3, a_4 \in L^2(\log L)^2_{loc}([0,+\infty[\times\bar{\Omega})$.

Using now the second term of entropy dissipation integral (7) (that is, inequality (11)), we see that

$$a_3\,a_4 \in L^1_{loc}([0,+\infty[\times\bar{\Omega}) \qquad \Rightarrow \qquad a_1\,a_2 \in L^1_{loc}([0,+\infty[\times\bar{\Omega})\,,$$

so that all terms in system (3), (2) can be defined. In order to prove existence, we consider the approximated system (21), (22), (23) and verify that the arguments above can be used (uniformly w.r.t. n). Estimate (25) still holds uniformly w.r.t. n, and leads to the boundedness in $L\log L_{loc}([0,+\infty[\times\bar{\Omega})$ of the sequence $a_3^n\,a_4^n$. Inequality (11) shows then that $a_1^n\,a_2^n$ is weakly compact in $L^1_{loc}([0,+\infty[\times\bar{\Omega})$. Finally, thanks to the properties of the heat kernel, we know that a_3^n and a_4^n converge a.e. to some a_3, a_4.

The equation on a_1^n can then be rewritten as

$$\partial_t a_1^n = \gamma^n - \delta^n\,a_1^n\,(a_1^n - \phi)\,, \tag{26}$$

where $\phi(x) := a_1^n(t,x) - a_2^n(t,x)$ is constant for all n, t. Moreover, $\delta^n(t,x) = (1 + \frac{1}{n}\sum_{i=1}^4 a_i^n)^{-1}$ tends to 1 a.e. and lives in $[0,1]$, and $\gamma_n = \delta^n a_3^n\,a_4^n$ converges in $L^1([0,T]\times\Omega)$. We first observe that (up to extraction of a subsequence), the quantities $\int_0^T |\delta^n(t,x) - 1|\,dt$ and $\int_0^T |\gamma^n(t,x) - a_3(t,x)\,a_4(t,x)|\,dt$ converge towards 0 for a.e. $x \in \Omega$. Then, for a given $x \in \Omega$, we introduce the sequence of functions of time $\alpha_p^n := \alpha_p^n(t)$ approximating the ODE (26) in the following way:

$$\alpha_{p+1}^n(t) = a_1^n(0)\,e^{-\int_0^t \delta^n(s)\,(\alpha_p^n(s)-\phi)\,ds} + \int_0^t \gamma^n(s)\;e^{-\int_s^t \delta^n(\sigma)\,(\alpha_p^n(\sigma)-\phi)\,d\sigma}\,ds.$$

It is easy to see that (still for a given x), $\sup_{t\in[0,T]}\sup_{p,n\in I\!N}|\alpha_p^n(t)| < \infty$ and $\lim_{p\to\infty}\sup_{n\in I\!N}\sup_{t\in[0,T]}|\alpha_p^n(t) - \alpha^n(t)| = 0$. Then, it is possible to prove by induction on p that (for all p) $\lim_{n\to\infty}|\alpha_p^n(t) - \alpha_p(t)| = 0$, where

$$\alpha_{p+1}(t) = a_1^n(0)\,e^{-\int_0^t(\alpha_p(s)-\phi)\,ds} + \int_0^t a_3(s)\,a_4(s)\,e^{-\int_s^t(\alpha_p(\sigma)-\phi)\,d\sigma}\,ds.$$

It is therefore possible to prove that the sequence $\alpha_p(t)$ is a Cauchy sequence in $L^\infty([0,T])$, and consequently that there exists (still for a given x) a function $a_1 := a_1(s)$ such that $\lim_{p\to\infty}\sup_{t\in[0,T]}|\alpha_p(t) - a_1(t)| = 0$. Thanks to the previous construction, we see that a_1^n converges to a_1 uniformly on $[0,T]$ (for a given x). Therefore, we obtain the convergence a.e. of a_1^n to a_1, and the same holds of course for a_2^n. This is enough to pass to the limit in the system.

Moreover, when $N = 1$, the properties of the heat kernel imply that (for all $\varepsilon > 0$) $a_3, a_4 \in L^{3-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$, so that $a_3\, a_4 \in L^{3/2-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$. Then, we observe that (with $\phi(x) := a_1(t,x) - a_2(t,x)$)

$$\partial_t(a_2^{2-2\varepsilon}/(2-2\varepsilon)) \le a_2^{1-2\varepsilon}\, a_3\, a_4 - a_2^{3-2\varepsilon} - a_2^{2-2\varepsilon}\,\phi$$
$$\le a_2^{3-2\varepsilon}(1 - \frac{2}{3-2\varepsilon}) + \frac{(a_3 a_4)^{3/2-\varepsilon}}{3/2-\varepsilon} - a_2^{3-2\varepsilon} - a_2^{2-2\varepsilon}\phi\,,$$

due to Young's inequality. As a consequence, $a_2 \in L^{3-2\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$. Then the properties of the heat kernel imply that $a_3\, a_4 \in L^p_{loc}([0,+\infty[\times\bar\Omega)$ for all $p \in [1,+\infty[$. A last iteration ensures that all a_i lie in $L^\infty_{loc}([0,+\infty[\times\bar\Omega)$, which implies smoothness of the solution without difficulty. This ends the proof of Theorem 4.2 □

5. Existence and smoothness, three diffusion coefficients missing

Theorem 5.1. *Let $\Omega \subset I\!R^N$ be a bounded and regular (C^∞) domain, and $d_1 = d_2 = d_3 = 0$, $d_4 > 0$. Then there exists a smooth ($(C^\infty([0,+\infty[\times\bar\Omega))^4$) solution to system (3), (2) for all smooth initial nonnegative data (compatible with the Neumann boundary condition).*

Proof of Theorem 5.1. We consider a solution to system (3), (2). We first observe that

$$a_1(t,x) + a_3(t,x) = a_1(0,x) + a_3(0,x) := \phi(x) \le C\,, \qquad \forall t \ge 0, x \in \Omega\,,$$
$$a_2(t,x) + a_3(t,x) = a_2(0,x) + a_3(0,x) := \psi(x) \le C\,, \qquad \forall t \ge 0, x \in \Omega\,,$$

where $C \ge 0$ is defined as

$$C = \sup\left\{ ||a_1(0,\cdot)||_{L^\infty} + ||a_3(0,\cdot)||_{L^\infty}, ||a_2(0,\cdot)||_{L^\infty} + ||a_3(0,\cdot)||_{L^\infty} \right\}.$$

Then, the system can be rewritten as

$$\partial_t a_3 = (\phi - a_3)\,(\psi - a_4) - a_3\, a_4 \le C^2,$$
$$\partial_t a_4 - d_4\,\Delta_x a_4 = (\phi - a_3)\,(\psi - a_4) - a_3\, a_4 \le C^2\,.$$

Since the left-hand side is bounded (when a_3, a_4 are nonnegative), it is possible to prove existence of a bounded solution by a simple continuation argument (Cf. [6]). Note that we do not need to use an approximated system here. Smoothness follows then easily. This ends the proof of Theorem 5.1□

6. Existence and smoothness: the case of a three-species reaction

In this section we present brief proofs of theorems of existence and smoothness in the case of the simpler chemical reaction $\mathcal{A} + \mathcal{B} \rightleftharpoons \mathcal{C}$. The corresponding system writes

$$\begin{cases} \partial_t a - d_1 \, \Delta_x a = c - a\, b, \\ \partial_t b - d_2 \, \Delta_x b = c - a\, b, \\ \partial_t c - d_3 \, \Delta_x c = a\, b - c, \end{cases} \tag{27}$$

together with the boundary condition

$$n\cdot\nabla_x a = 0, \quad n\cdot\nabla_x b = 0, \quad n\cdot\nabla_x c = 0\,, \qquad \forall t \geq 0,\ x \in \partial\Omega\,. \tag{28}$$

In order to keep the proofs short, only the a priori estimates are presented. Approximation arguments similar to those used in Sections 3 to 5 can then be used in order to build up the solutions.

6.1. *One diffusion missing, first case*

We consider the system (27) in the case when $d_1, d_3 > 0$, and $d_2 = 0$. We prove the

Theorem 6.1. *Let $\Omega \subset I\!R^N$ be a bounded and regular (C^∞) open set of $I\!R^N$, and $d_1,\ d_3 > 0$. Then there exists a smooth ($(C^\infty([0,+\infty[\times\bar\Omega))^4$) solution to system (27) – (28) for all smooth initial nonnegative data (compatible with the Neumann boundary condition).*

Proof of Theorem 6.1. We consider a smooth solution to system (27), (28). We observe that

$$\partial_t\bigg((a\,\log a - a + 1) + (b\,\log b - b + 1) + (c\,\log c - c + 1)\bigg)$$
$$-\Delta_x\bigg(d_1\,(a\,\log a - a + 1) + d_2\,(b\,\log b - b + 1) + d_3\,(c\,\log c - c + 1)\bigg) \leq 0.$$

The method of duality implies then that

$$\bigg[d_1\,(a\,\log a - a + 1) + d_2\,(b\,\log b - b + 1) + d_3\,(c\,\log c - c + 1)\bigg] \tag{29}$$
$$\times \bigg[(a\,\log a - a + 1) + (b\,\log b - b + 1) + (c\,\log c - c + 1)\bigg] \in L^1_{loc}([0,+\infty[\times\bar\Omega),$$

so that $a, c \in L^2(\log L)^2_{loc}([0,+\infty[\times\bar\Omega)$.

Then, $\partial_t b \leq c$ implies also that $b \in L^2(\log L)^2_{loc}([0,+\infty[\times\bar\Omega)$. This is enough to define weak solutions to the system. We now present three methods which work under less and less stringent conditions on the dimension N.

We first suppose that $N \leq 5$. Thanks to the properties of the heat kernel, we know that (for all $\varepsilon > 0$) $a \in L^{2(N+2)/(N-2)-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$ (the number $\frac{2\,(N+2)}{N-2}$ is replaced by ∞ for $N=1$ and by any $p < +\infty$ when $N=2$). As a consequence, $a\,b \in L^{(N+2)/N-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$, and thanks again to the properties of the heat kernel, $c \in L^{(N+2)/(N-2)-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$.

We now present the bootstrap argument only when $N=5$ (the cases $N=1,2,3,4$ being easier). We know that $a \in L^{14/3-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$, $c \in L^{7/3-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$, and $\partial_t b \leq c$, so that $b \in L^{7/3-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$. Using the heat kernel first for a, we get $a \in L^{7-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$, so that $a\,b \in L^{7/4-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$, and using it then for c, we see that $c \in L^{7/2-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$. As a consequence, $b \in L^{7/2-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$. Still thanks to the heat kernel properties, $a \in L^{p}_{loc}([0,+\infty[\times\bar\Omega)$ for all $p \in [1,+\infty[$, so that $a\,b \in L^{7/2-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$, and $c \in L^{p}_{loc}([0,+\infty[\times\bar\Omega)$ for all $p \in [1,+\infty[$. So does b and it remains to use a last time the heat kernel and $\partial_t b \leq c$ to get the boundedness of a,b,c. Smoothness is then easy to prove.

We introduce a different type of arguments in the case when $5 < N \leq 12$. Let us assume that $c \in L^{q}_{loc}([0,+\infty[\times\bar\Omega)$. Then, the same holds for b, and thanks to the properties of the heat kernel (for all $\varepsilon > 0$), $a \in L^{\frac{q\,(N+2)}{N+2-2q}-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$. As a consequence, $c\,b^{q-1} \in L^{1}_{loc}([0,+\infty[\times\bar\Omega)$ and $c\,a^{\frac{q-1}{1-(2q)/(N+2)}-\varepsilon} \in L^{1}_{loc}([0,+\infty[\times\bar\Omega)$. Then, we notice that for all $p \in [1,+\infty[$,

$$\partial_t(b^p/p) + a\,b^p = b^{p-1}\,c\,,$$
$$\partial_t(a^p/p) + b\,a^p + d_1\,(p-1)a^{p-2}|\nabla_x a|^2 - d_1\,\nabla_x\cdot(a^{p-1}\nabla_x a) = a^{p-1}\,c\,.$$

Since a and b are nonnegative, this ensures for all $T>0$

$$\int_0^T\!\!\int_\Omega a\,b^p\,dxdt \leq \int_0^T\!\!\int_\Omega c\,b^{p-1}\,dxdt, + \int_\Omega b^p(t=0)/p\,dx\,,$$
$$\int_0^T\!\!\int_\Omega a^p\,b\,dxdt \leq \int_0^T\!\!\int_\Omega c\,a^{p-1}\,dxdt + \int_\Omega a^p(t=0)/p\,dx\,,$$

so that (using first $p=q$ and then $p = \frac{q\,(N+2)}{N+2-2q} - \varepsilon$) $a\,b^q \in L^1_{loc}([0,+\infty[\times\bar\Omega)$ and $a^{1+\frac{q-1}{1-(2q)/(N+2)}-\varepsilon}\,b \in L^1_{loc}([0,+\infty[\times\bar\Omega)$. By interpolation, we end up with $a\,b \in L^{1+\frac{q-1}{2\,(1-q/(N+2))}-\varepsilon}$. Finally, thanks to the properties of the heat kernel, $c \in L^r_{loc}([0,+\infty[\times\bar\Omega)$, with

$$\frac{1}{r} > \frac{2\,(1-q/(N+2))}{1+q\,(1-2/(N+2))} - \frac{2}{N+2}.$$

This leads to the following induction formula: $c \in L^{q_n-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$, where

$$q_0 = 2, \qquad \frac{1}{q_{n+1}} = \frac{2\,(1-q_n/(N+2))}{1+q_n\,(1-2/(N+2))} - \frac{2}{N+2}.$$

It can be verified than when $N \le 12$, the sequence q_n is increasing (after a finite number of steps, the estimates on the heat kernel imply that $c \in L^\infty$). Smoothness is then easy to prove. Note that the method above applied to the non degenerate case ($d_1, d_2, d_3 > 0$) leads to the existence of smooth solutions when $N \le 16$.

We finally introduce the proof which holds for any dimension N. According to the variant (14, see also section 2.2) of the method of duality, since

$$\partial_t(a+c) - \Delta_x(d_1\,a + d_3\,c) = 0,$$

we know that any estimate in L^p_{loc} for a can be transfered to c. Thanks to the properties of the heat kernel used for a, we see that $a \in L^p_{loc} \Rightarrow c \in L^p_{loc} \Rightarrow a \in L^q_{loc}$, with $1/q > 1/p - 2/(N+2)$. A simple induction leads to a L^∞_{loc} estimate for a, c. The same can be obtained for b since $\partial_t b \le c$. This ends the proof of Theorem 6.1 □

6.2. *One diffusion missing, second case*

Theorem 6.2. *Let $\Omega \subset I\!R^N$ be a bounded and regular (C^∞) domain, and d_1, $d_2 > 0$, $d_3 = 0$. Then there exists a weak ($(L^2_{loc}([0,+\infty[\times\bar\Omega))^4$) solution to system (27) – (28) for all smooth initial nonnegative data (compatible with the Neumann boundary condition).*

Moreover, if $N \le 3$, then this solution is smooth ($(C^\infty([0,+\infty[\times\bar\Omega))^4$).

Proof of Theorem 6.2. We consider a solution to system (27), (28). We observe that (29) still holds, so that $a, b \in L^2(\log L)^2_{loc}([0,+\infty[\times\bar\Omega)$.

Then, $\partial_t c \le a\,b$ also implies that $c \in L\,\log L_{loc}([0,+\infty[\times\bar\Omega)$. This is enough to define weak solutions of the system.

Moreover, if $N = 1$, then thanks to the properties of the heat kernel (for all $\varepsilon > 0$), $a, b \in L^{3-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$, so that $c \in L^{3-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$. Then, still thanks to the properties of the heat kernel, $a, b \in L^p_{loc}([0,+\infty[\times\bar\Omega)$ for all $p \in [1,+\infty[$, so that $c \in L^p_{loc}([0,+\infty[\times\bar\Omega)$ for all $p \in [1,+\infty[$. A last iteration shows that $a, b, c \in L^\infty_{loc}([0,+\infty[\times\bar\Omega)$.

We now propose an alternative method when $N = 2$ or $N = 3$. We present it only in the case $N = 3$ since the case $N = 2$ is easier. We observe that

$$(\partial_t - d_1\,\Delta_x)(a^2/2) = a\,(c - a\,b) - d_1\,|\nabla_x a|^2 \le a\,c.$$

Using the variant (13) of the method of duality, we see that $a\,c \in L^1_{loc}([0,+\infty[\times\bar\Omega)$. The same holds for $b\,c$. Using the properties of the heat kernel, we end up with $a^2, b^2 \in L^{5/3-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$ for all $\varepsilon > 0$. But $\partial_t c \le a\,b$, so that $c \in L^{5/3-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$. We then proceed by induction using the properties of the heat kernel. We obtain first $a, b \in L^{5-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$, so that $c \in L^{5/2-\varepsilon}_{loc}([0,+\infty[\times\bar\Omega)$, and then $a, b \in L^p_{loc}([0,+\infty[\times\bar\Omega)$ for all $p < +\infty$. A last iteration enables to conclude the proof of Theorem 6.2 □

6.3. *Two diffusions missing, first case*

Theorem 6.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded and regular (C^∞) domain, and $d_1 > 0$, $d_2 = d_3 = 0$. Then there exists a smooth ($(C^\infty([0,+\infty[\times\bar\Omega))^4$) solution to system (27) – (28) for all smooth initial nonnegative data (compatible with the Neumann boundary condition).*

Proof of Theorem 6.3. It is enough to notice that $\partial_t(b + c) = 0$ so that

$$b(t,x) + c(t,x) = b(0,x) + c(0,x) := \phi(x).$$

Then, the system can be rewritten

$$\partial_t a - d_1\,\Delta_x a = \phi - b - a\,b,$$
$$\partial_t b = \phi - b - a\,b.$$

Since the right-hand-side of the system is bounded, it is easy to prove existence of smooth solutions. This ends the proof of Theorem 6.3. □

6.4. *Two diffusions missing, second case*

Theorem 6.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded and regular (C^∞) domain, and $d_3 > 0$, $d_1 = d_2 = 0$. Then there exists a smooth ($(C^\infty([0,+\infty[\times\bar\Omega))^4$) solution to system (27) – (28) for all smooth initial nonnegative data (compatible with the Neumann boundary condition).*

Proof of Theorem 6.4. We notice first that $\partial_t(a-b)=0$ and, thus (for all times $t \geq 0$)

$$a(t,x)-b(t,x)=a(0,x)-b(0,x):=-\phi(x).$$

Then, the system can be rewritten as

$$\partial_t a = c - a\,(a+\phi),$$
$$\partial_t c - d_3\,\Delta_x c = a\,(a+\phi) - c.$$

Thanks to the duality method, we already know that $c \in L^2_{loc}([0,+\infty[\times\bar{\Omega})$, and hence (thanks to the first equation) that $a \in L^2_{loc}([0,+\infty[\times\bar{\Omega})$. Then, the first equation also implies that

$$\partial_t(a^3/3) \leq c\,a^2 - a^4 - a^3\,\phi \leq c^2/2 - a^4/2 - a^3\phi\,.$$

As a consequence (integrating w.r.t. time) we see that $a \in L^4_{loc}([0,+\infty[\times\bar{\Omega})$. Then, using the properties of the heat kernel, $c \in L^{(N+2)/(N-2)-\varepsilon}$ for all $\varepsilon > 0$ (the exponent is replaced by ∞ when $N=1$ and by any $p \in [1,+\infty[$ if $N=2$).

In fact, if we know that $c \in L^p_{loc}([0,+\infty[\times\bar{\Omega})$, then

$$\begin{aligned}\partial_t(a^{2p-1}/(2p-1)) + a^{2p} &\leq c\,a^{2p-2} - \phi\,a^{2p-1}\\ &\leq c^p/p + a^{2p}\,(1-1/p) - \phi\,a^{2p-1}\,,\end{aligned}$$

using Young's inequality. As a consequence, $a \in L^{2p}_{loc}([0,+\infty[\times\bar{\Omega})$ and using the properties of the heat kernel, $c \in L^q_{loc}([0,+\infty[\times\bar{\Omega})$ with $1/q > 1/p - 2/(N+2)$. An immediate induction shows that $a, c \in L^\infty_{loc}([0,+\infty[\times\bar{\Omega})$, and this is enough to guarantee the smoothness of the solution. This ends the proof of Theorem 6.4 □

References

1. A. Arnold, P. Markowich, G. Toscani, A. Unterreiter *On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations* Comm. Partial Diff. Equ. **26** (2001), no. 1-2, 43–100.
2. M. Bisi and L. Desvillettes. *From reactive Boltzmann equations to reaction-diffusion systems* J. Stat. Phys. **125** (2006), no. 1, pp. 249–280.
3. H. Brézis. *Analyse Fonctionnelle.* Masson, Paris, 1983.
4. J. Carrillo, A. Jüngel, P. Markowich, G. Toscani, A. Unterreiter. *Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities.* Monatsh. Math. *133* no. 1 (2001), pp. 1–82.
5. A. De Masi, E. Presutti. *Mathematical methods for hydrodynamic limits.* Springer–Verlag, Berlin, 1991.
6. L. Desvillettes. *Relation between kinetic and Reaction-Diffusion Equations*, to appear in Rivista dell'Universita' di Parma.

7. L. Desvillettes, K. Fellner. *Entropy methods for reaction-diffusion equations: slowly growing a priori bounds* to appear in Revista Matemática Iberoamericana
8. L. Desvillettes, K. Fellner, M. Pierre and J. Vovelle *About global existence for quadratic systems of reaction-diffusion* J. Advanced Nonlinear Studies **7** (2007) no 3., 491–11.
9. J.-D. Deuschel, D. W. Stroock *Large deviations*, Pure and Applied Mathematics, **137**, Academic Press Inc. Boston, (1989)
10. R. J. Di Perna, P. L. Lions *On the Cauchy problem for Boltzmann equations: Global existence and weak stability* Annals of Mathematics, **130** (1989) pp. 321–366.
11. T. Goudon, A. Vasseur *Regularity analysis for systems of reaction-diffusion equations* preprint.
12. R. Holley, D. Stroock, Daniel *Logarithmic Sobolev inequalities and stochastic Ising models* J. Statist. Phys. **46** (1987), no. 5-6, 1159–1194.
13. M. Pierre, D. Schmitt *Blowup in reaction-diffusion systems with dissipation of mass* SIAM Review, **42** (2000) no. 1, pp. 93–106.
14. P. Quittner, P. Souplet *Superlinear Parabolic Problems, blow-up, global existence and steady states*, Birkhauser advanced texts, Basel, 2007.
15. F. Rothe. *Global solutions of reaction-diffusion systems.* Lecture Notes in Mathematics, Springer, Berlin, 1984.
16. D. Stroock *Logarithmic Sobolev inequalities for gibbs states* Lecture Notes in Mathematics **1563** (1993) pp. 194–228.