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SOLUTIONS OF A NON-LOCAL AGGREGATION EQUATION: UNIVERSAL BOUNDS, CONCAVITY CHANGES AND EFFICIENT NUMERICAL SOLUTIONS *, **

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Abstract. We consider a one-dimensional aggregation equation for a non-negative density $\rho(x,t)$ associated with a quartic potential $W(x) = \beta x^2 + \delta x^4$ ($\delta > 0$, $\beta \in \mathbb{R}$). We show that for the case of symmetric initial data [$\rho(x,0) \equiv \rho(-x,0)$] the solution of the aggregation equation can be expressed in terms of an explicit function of x, $L(t)$ and $\Phi(t)$, where the functions $L(t)$ and $\Phi(t)$ are determined by an ordinary differential equation initial value problem, the numerical solution of which is relatively straightforward. The function $L(t)$, which can be interpreted as an upper bound for the radius of the support of $\rho(x,t)$, is finite for all $t > 0$, even if the support of $\rho(x,0)$ is unbounded, while $\Phi(t)$ is a potential related to the time integral of the second spatial moment $\phi(t)$ of $\rho(x,t)$. We develop various bounds on $L(t)$ and $\phi(t)$ and a number of results concerning convexity and monotonicity, that require no knowledge of $\rho(x,0)$ other than its symmetry. We show that many, but not all, of these results persist if the assumption of symmetry is relaxed. Our general results are tested against numerical solutions for two examples of symmetric $\rho(x,0)$. We also exhibit some counterintuitive behaviour of the model, by finding conditions under which $\rho_{xx}(0,t) \to \infty$ as $t \to \infty$, even if we start with $\rho_{xx}(0,0) < 0$ and take $\beta \geqslant 0$, which ensures ultimate collapse to a single Dirac component at the origin.

Résumé. Nous considérons une équation d'agrégation unidimensionnelle pour une densité non négative $\rho(x,t)$ associé à un potentiel quartique $W(x) = \beta x^2 + \delta x^4$ ($\delta > 0$, $\beta \in \mathbb{R}$). Nous montrons que dans le cas de conditions initiales symétriques [$\rho(x,0) \equiv \rho(-x,0)$] la solution de l'équation d'agrégation peut être exprimée sous forme d'une fonction explicite de x, $L(t)$ et $\Phi(t)$, où les fonctions $L(t)$ et $\Phi(t)$ sont déterminées par un problème de valeur initiale, dont la solution est relativement simple. La fonction $L(t)$, qui peut être interprétée comme une borne supérieure du rayon du support de $\rho(x,t)$, est finie pour tout $t > 0$, même si le support de $\rho(x,0)$ est non borné, tandis que $\Phi(t)$ est un potentiel lié à l'intégrale temporelle du deuxième moment spatial $\phi(t)$ de $\rho(x,t)$. Nous développons diverses limites sur $L(t)$ et $\phi(t)$ et un certain nombre de résultats concernant la convexité et la monotonie, qui n'exigent aucune connaissance de $\rho(x,0)$ autre que sa symétrie. Nous montrons que beaucoup de ces résultats (, mais pas tous,) persistent si l'hypothèse de symétrie est relaxée. Nos résultats généraux sont testés via des solutions numériques pour deux exemples de $\rho(x,0)$ symétriques. Nous illustrons également un comportement contre-intuitif du modèle, en recherchant les conditions sous lesquelles $\rho_{xx}(0,t) \to \infty$ lorsque $t \to \infty$, même dans le case où la densité initiale est telle que $\rho_{xx}(0,0) < 0$ avec $\beta \geqslant 0$, ce qui garantit que la solution se réduce à une seule fonction de Dirac centrée à l'origine.

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Introduction

The aggregation equation

$$\frac{\partial \rho}{\partial t} = \nabla_x \cdot \Big[\rho(x,t) \int_{-\infty}^{\infty} \nabla_x W(x-y)\rho(y,t)dy\Big], \tag{1}$$

which describes the evolution of a particle density $\rho(x,t)$ subject to a symmetric interaction potential W is one prominent example of numerous non-local differential equation models for collective behaviour of individuals, which have been studied extensively in recent years.

Examples for such models have many application backgrounds. Simplified inelastic models for granular media are described by convex attractive potentials (see [2,13,34]). Collective behaviour of individuals, such as swarming or flocking gives rise to a variety of continuum models (see [4,5,7–9,14,15,18,22,24,25,31,32]). Equation (1) is used in a model for F-actin filaments in the cellular cytoskeleton (see [21,28]). Finally, related models can be found in opinion dynamics (see [3]) or Lennard-Jones type potentials used in crystallisation (see [33]).

It is by now very well understood that for purely attractive interaction potentials Eq. (1) exhibits aggregation towards measures. This collective aggregation effect caused by the non-local integral term in (1) may indeed be stronger than the dissipative effects of a diffusion term as the famous Patlak–Keller–Segel model of chemotaxis shows for supercritical initial mass (and $W(x) = -(2\pi)^{-1}\log|x|$ in $\mathbb{R}^2$); see e.g. [6,27,30] and the references therein.

Finally, the first-order aggregation model (1) is also linked to second-order swarming models via a vanishing inertia limit (see [19]), which thus share the same stationary states.

In the present paper, we restrict our interest to the one-dimensional aggregation equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\Big[\rho(x,t) \int_{-\infty}^{\infty} W'(x-y)\rho(y,t)dy\Big], \qquad x \in \mathbb{R},\ t \geq 0, \tag{2}$$

which conserves the total mass M (assumed to be finite), and preserves the centre of mass (since W is even), which can thus be normalised to zero without loss of generality, that is,

$$\int_{-\infty}^{\infty} \rho(x,t)dx = M := \int_{-\infty}^{\infty} \rho(x,0)dx < \infty, \qquad \text{and} \qquad \int_{-\infty}^{\infty} x\rho(x,t)dx = 0. \tag{3}$$

The collectively produced velocity field $-\nabla_x W * \rho$ makes the dynamics of Eq. (2) interesting and complicated and implies that the behaviour of solutions depends on initial data and the history of the evolution. In particular, the set of stationary states is often a larger than what can be identified by the two conserved quantities (3). Hence, also equilibria depend via the history of the nonlinear evolution on the initial data and characterising domains of attraction of individual equilibria is typically a hopeless quest.

For analytic potentials W, stationary states of (2) are necessarily finite sums of Dirac measures, while for C^2 potentials, other stationary states might exist, but cannot be linearly stable: see [29]. For double-well potentials W, it has been shown [16,17] that stable stationary states may consist of arbitrarily many Dirac measures (depending on the concavity of the repulsive part) and that the number of Diracs and the details of mass distribution among the Diracs depends sensitively on the initial data and the subsequent collective evolution. In higher space dimensions, the dynamics and sets of stationary states of the aggregation equation (1) are even more complicated, see e.g. [8].

The present paper asks in some sense the opposite to (not) understanding this intricate collective dynamics. *What can be said about solutions when only taking into account a minimal amount of information on the initial data and the interaction potential? Can we derive general bounds, which apply to all the possible trajectories of the collective dynamics?*

As definition of minimal amount on information of the initial data, we focus on the second and third-order moments of the mass distribution

$$M\phi(t) := \int_{-\infty}^{\infty} x^2\rho(x,t)dx, \qquad\qquad M\psi(t) := \int_{-\infty}^{\infty} x^3\rho(x,t)dx. \tag{4}$$

As interaction potential, we will restrict ourselves to a simplest prototype of a smooth potential

$$W(x) = \beta x^2 + \delta x^4, \qquad \beta \in \mathbb{R},\ \delta \in \mathbb{R}_{\geqslant 0}. \tag{5}$$

In an earlier paper [20], we studied equilibria for various cases of the potential $\alpha|x| + \beta x^2 + \gamma|x|^3 + \delta x^4$ when a diffusive component to particle motility was also included. We also discussed some aspects of time evolution in the very special case $\alpha = \gamma = \delta = 0$ using a Fourier transform technique that appears not to extend to the more general potential (5).

For $\beta < 0$ and $\delta > 0$, Eq. (5) is a double-well potential and thus induces a repulsive–attractive interaction. Note that as price of simplicity and in contrast to examples in [16, 17], the double-well potential (5) cannot have stable stationary states with more than two Diracs. It nevertheless features a family of (unstable) three Dirac stationary states (as discussed below), which is a typical complication in understanding the dynamics induced by a potential like (5).

For the potential (5), it follows from the constraints (3) and (4) that

$$\begin{aligned}\int_{-\infty}^{\infty} W'(x-y)\rho(y,t)dy &= \int_{-\infty}^{\infty} [2\beta(x-y) + 4\delta(x-y)^3]\rho(y,t)dy \\ &= 2\beta Mx + 4\delta Mx^3 + 12\delta M\phi(t)x - 4\delta M\psi(t)\end{aligned}$$

and the one-dimensional aggregation equation (2) reduces to

$$\frac{\partial\rho}{\partial t} - \big[2\beta Mx + 4\delta Mx^3 + 12\delta M\phi(t)x - 4\delta M\psi(t)\big]\frac{\partial\rho}{\partial x} = \big[2\beta M + 12\delta Mx^2 + 12\delta M\phi(t)\big]\rho. \tag{6}$$

We can solve this equation in principle via the methods of characteristics treating $\phi(t)$ and $\psi(t)$ as known functions. Once $\rho(x,t)$ has been determined, we can use the integrals (4) to infer an initial value problem for $\phi(t)$ and $\psi(t)$. In practice, the approach is highly effective in the case of symmetric initial data, where only $\phi(t)$ needs to be determined. So that we can check equations that we derive for dimensional consistency, we note that from Eq. (6) we have

$$M\beta = \frac{1}{\text{time}}, \quad M\delta = \frac{1}{\text{length}^2 \times \text{time}}, \quad \phi(t) = \text{length}^2, \quad \frac{\delta\phi(t)}{\beta} = \text{dimensionless}. \tag{7}$$

We shall abuse notation slightly by writing $f(\infty)$ for both an equilibrium value of an observable $f(t)$ in our dynamical system and for $\lim_{t\to\infty} f(t)$. In all cases where the second meaning is implied, the existence of the limit will either be clearly evident or will be substantiated.

From Eq. (6) we see that the equilibrium solutions are given by

$$\frac{d}{dx}\Big\{\big[2\beta Mx + 4\delta Mx^3 + 12\delta M\phi(\infty)x - 4\delta M\psi(\infty)\big]\rho\Big\} = 0.$$

The only possible solutions are Dirac measures (delta functions) located at the roots of the cubic equation

$$x^3 + \Big[3\phi(\infty) + \frac{\beta}{2\delta}\Big]x - \psi(\infty) = 0. \tag{8}$$

If we restrict our analysis (from now until Section 9) to symmetric densities $\rho(x,t)$, then $\psi(\infty) = 0$ and $x = 0$ (which enforces $\phi(\infty) = 0$) is the only real solution to (8) for $\beta \geqslant 0$, while a pair of additional roots $\pm[|\beta|/(2\delta) - 3\phi(\infty)]^{1/2}$ appears for $\beta < 0$ if the condition $|\beta|/(2\delta) - 3\phi(\infty) > 0$ is met. In the second case, since the mass is symmetrically distributed, there is equal mass λM (where $0 \leqslant \lambda \leqslant 1/2$) at each of the two nonzero roots and we have the consistency condition

$$\phi(\infty) = (1 - 2\lambda)\cdot 0 + 2\lambda\cdot[|\beta|/(2\delta) - 3\phi(\infty)],$$

that is,

$$\phi(\infty) = \frac{\lambda|\beta|}{\delta(1+6\lambda)}, \qquad 0 \leqslant \lambda \leqslant 1/2, \tag{9}$$

making the two nonzero locations of masses

$$\pm[|\beta|/(2\delta)-3\phi(\infty)]^{1/2}=\pm\Big[\frac{|\beta|}{2\delta(1+6\lambda)}\Big]^{1/2},$$

the condition $|\beta|/(2\delta)-3\phi(\infty)>0$ being met for all λ in the interval $0\leqslant\lambda\leqslant 1/2$. Writing $\boldsymbol{\delta}(\cdot)$ for the Dirac measure (to avoid a notational conflict with our earlier use of the symbol δ), we have a one-parameter family of symmetric Dirac equilibrium solutions

$$\rho(x,\infty)=(1-2\lambda)M\boldsymbol{\delta}(x)+\lambda M\left[\boldsymbol{\delta}(x-\xi)+\boldsymbol{\delta}(x+\xi)\right],\quad \xi=\Big[\frac{|\beta|}{2\delta(1+6\lambda)}\Big]^{1/2},\quad 0\leqslant\lambda\leqslant 1/2, \tag{10}$$

where $\lambda=0$ reduces the solution to the single Dirac at $x=0$ that we find for $\beta\geq 0$. We note the bounds on the (nonzero) Dirac measure locations and the equilibrium second moment:

$$\left[\frac{|\beta|}{2\delta}\right]^{1/2}\geqslant\xi\geqslant\left[\frac{|\beta|}{8\delta}\right]^{1/2},\quad 0\leqslant\phi(\infty)\leqslant\frac{|\beta|}{8\delta},\qquad 0\leqslant\lambda\leqslant 1/2. \tag{11}$$

1. Outline and results

In this paper, we focus on qualitative statements on solutions to (2), which only involving the initial second-order moment $\phi(0)$ and the two parameters β and δ, which are required by the definition of the equilibria (10). Our goal is to establish results, which are independent of the history dependence of the collective evolution (2) and hold for all possible solutions (with identical $\phi(0)$), including the equilibrium solutions (10).

Despite this huge reduction in complexity compared to the evolution of ρ, the following statements provide a good general picture of the qualitative and large-time behaviour: Theorem 1.1 and its proof will consider symmetric densities $\rho(x,t)$ for the sake of a clear presentation. The second result, Corollary 1.1 is able to generalise almost all results of Theorem 1.1 to general solutions.

Theorem 1.1. *Let $\beta\in\mathbb{R}$ and $\delta>0$. Suppose symmetric, integrable, non-negative initial data $\rho(x,0)$ with finite (conserved) mass M and finite second-order moment $\phi(0)$. Consider the corresponding symmetric solutions to the one-dimensional aggregation equation* (2) *subject to the interaction potential* (5) *in terms of the second-order moment $\phi(t)$.*

Then, the interval of equilibrium values $\phi(\infty)$ in (11) *is globally and monotonic attractive for all solutions with initial moments $\phi(0)>|\beta|/(8\delta)$ and invariant for all solutions with initial moments $\phi(0)\leqslant|\beta|/(8\delta)$ (in the sense of Lemma 4.1). Moreover, $\phi(t)$ is a convex function as long as $\phi(t)>|\beta|/(8\delta)$ for $t>0$, see Section 6.*

An alternative viewpoint onto the qualitative behaviour of solutions to (2) *is provided by the properties of the function*

$$\Phi(t)=\int_0^t 4M\big[\beta+6\delta\phi(\tau)\big]d\tau.$$

Assuming knowledge of $\Phi(t)$, it is possible to deduce support estimates for $\rho(x,t)$ (see Lemma 2.1), to distinguish the parameter cases $\beta>0$, $\beta=0$ and $\beta<0$ and the corresponding qualitative behaviour and equilibrium values of $\phi(t)$ (see Lemmas 3.1 and 5.1). Finally, $\Phi(t)$ is given by formula (55) *independently of $\phi(t)$ and can thus be interpreted as potential function of the second-order moment $\phi(t)$.*

Our findings of Theorem 1.1 are illustrated by numerical simulations for particular initial densities $\rho(x,0)$ in Section 8. We note that our method of characteristics analysis ensures that $\phi(t)$ is continuous, so $\Phi'(t)=4M\beta+24M\delta\phi(t)$ and $\lim_{t\to\infty}\phi(t)$ exists if and only if $\lim_{t\to\infty}\Phi'(t)$ exists.

Corollary 1.1. *Consider general asymmetric solutions to the one-dimensional aggregation equation* (2) *under the otherwise same assumption as in Theorem 1.1.*

Then, the range of equilibrium values $\phi(\infty)$ remains $0\leqslant\phi(\infty)\leqslant|\beta|/(8\delta)$ as in (9)*. Moreover, all of the statements of Theorem 1.1, which are derived in Sections 3–5 hold true. Only the convexity of $\phi(t)>|\beta|/(8\delta)$ as shown in Section 6 holds no longer for asymmetric solutions; see Section 9.*

Remark 1.1 (Existence of solutions).
Note that the existence theory for the one-dimensional aggregation equation (2) *is well established for initial data in the space of probability measures with bounded second moment and subject to a large class of interaction potentials* (*including* (5)), *see e.g.* [10–12] *and the references therein.*

For the results of the present paper, we only require solutions $\rho(x,t)$ such that $\phi(t) \in C[0,\infty) \cap C^1(0,\infty)$. Without relying on the previous existence theory results [10–12], *the required continuity and smoothmess of $\phi(t)$ follows from the results of Sections 2 and 3 for non-negative probability densities $\rho(x,0) \in L^1_+(1+x^2, dx)$ via standard fixed-point type arguments. More precisely, the method of characteristics* (12)*–*(13) *is globally well-posed thanks to* (21) *for continuous $\phi(t)$, which follows in return from formula* (27) *and the global bound* (34). *By the same argument, we see that we can allow for $\rho(x,0)$ to contain also a finite sum of Dirac measures, which then move along their characteristics. This also includes the equilibria* (10) *as admissible initial data.*

Remark 1.2 (Applications of Theorem 1.1 and Corollary 1.1).
Our interest in deriving estimates on solutions to the aggregation equation (2) *is motivated by a project with an interdisciplinary cooperation of mathematics and arts. More precisely, sound artists have long used swarming-type models like* (1) *to spatialise music within multi-speaker sound systems. In our collaboration, we are aiming for an interactive way of manipulating complicated collective behaviour by a comparatively simple trigger variable such as the function $\Phi(t)$.*

Remark 1.3 (Restrictions assumptions and future research).
Restricting ourselves to the one-dimensional aggregation equation and the smooth potential (5) *are limitations, which we hope to be less severe than they might look at first glance.*

Following particle trajectories like the method of characteristics is equally possible in all space dimensions. In fact, it has been shown [11] *that smooth densities can be approximated by atomic measures and the dynamics of* (1) *by the corresponding evolution of a cloud of particles. However, it is not hard to check that in higher space dimensions, there is no single differential inequality for the second order moment ϕ* (*see, e.g.* (44) *or* (46)). *We would need to analyse a system for the vector of second order moments with respect to all space directions. This would introduce another level of complexity into our present study and is left for future work.*

For the interaction potential, we conjecture the polynomial potential (5) *to serve as a kind of normal form for smooth (double-well) potentials and that their dynamics (at least near critical points) would be isomorphic. However, we have no proof for such a statement, which remains an interesting question for future research.*

Remark 1.4 (Energy functional and gradient flow structure).
The aggregation equation (1) *also satisfies a very nice and useful gradient flow structure on the space of probability measures and the Wasserstein distance, see e.g.* [11] *and the references therein. However, in view of only assuming the minimal amount of information on solutions as in Theorem 1.1 and Corollary 1.1, the energy and its decay seem much less suitable since they involve/combine higher-order moments, that is,*

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x,t)W(x-y)\rho(y,t)\,dxdy = M^2\phi(t)[\beta + 3\delta\phi(t)] + M^2\delta\mu_4(t),$$

where

$$\mu_4(t) = M^{-1}\int_{-\infty}^{\infty} x^4\rho(x,t)\,dx$$

denotes the fourth-order moment. Moreover,

$$\begin{aligned}\frac{d}{dt}E(t) &= -\int_{-\infty}^{\infty}\rho(x)\left(W' * \rho\right)^2\,dx \\ &= -4M^3[\beta+6\delta\phi(t)]\{\phi(t)[\beta+6\delta\phi(t)] + 4\delta\mu_4(t)\} - 16\delta^2M^3[\mu_6(t) - \psi^2(t)] \le 0,\end{aligned}$$

where

$$\mu_6(t) = M^{-1}\int_{-\infty}^{\infty} x^6\rho(x,t)\,dx$$

is the sixth-order moment. We note that $\mu_6(t) \geqslant \psi^2(t)$ from Jensen's inequality.

2. Method of characteristics analysis for a symmetric initial distribution

Under the assumption that $\rho(x,t)$ is symmetric (that is, that $\rho(x,0)$ is symmetric and sufficiently regular such that that symmetry is preserved under the flow of (2)), the method of characteristics converts the partial differential equation (6) with a prescribed initial density $\rho(x,0)$ to the ordinary differential equation initial value problem

$$\frac{dx}{dt} = -2M\big[\beta + 6\delta\phi(t)\big]x - 4\delta M x^3 = -4\delta M x\Big[\frac{\beta+6\delta\phi(t)}{2\delta} + x^2\Big], \qquad x\Big|_{t=0} = X, \tag{12}$$

$$\frac{d\rho}{dt} = 2M\big[\beta + 6\delta\phi(t)\big]\rho + 12\delta M x^2\rho, \qquad \rho\Big|_{t=0} = \rho(X,0). \tag{13}$$

First, we remark that Eq. (12) for the position variable x has the form of a (non-autonomous) pitchfork bifurcation at $\beta + 6\delta\phi(t) = 0$, which only occurs if $\beta < 0$ (since $\phi(t) \geq 0$). Interestingly, the condition $\beta + 6\delta\phi(t) = 0$ appeared already in our earlier paper ([20], p. 32), where we considered equilibrium solutions to a (2) *with an additional diffusion term present*. In [20], it was shown that the equilibrium density was unimodal for $\beta + 6\phi\delta \geqslant 0$ and bimodal for $\beta + 6\phi\delta < 0$. In Eq. (2) without diffusion, the role of the condition $\beta + 6\delta\phi(t) = 0$ in the subsequent analysis is different, but still points towards a mechanism of the formation of a single versus two aggregates: see Remark 4.3.

Secondly, before solving system (12)–(13), we observe that Eq. (12) is invariant under the change of variable $x \leftrightarrow -x$ and that we need consider only the characteristics starting from $(x,t) = (X,0)$ with $X \geqslant 0$. The initial velocity on the characteristics is continuous, so there are no shocks present at time $t = 0$. In the special case $X = 0$, the system (12)–(13) clearly has the solution $x = 0$, $\rho = \rho(0,0)\exp[\Phi(t)/2]$, where

$$\Phi(t) = \int_0^t 4M\big[\beta + 6\delta\phi(\tau)\big]d\tau \tag{14}$$

is the function introduced in Theorem 1.1. We may therefore assume for the following analysis that $X > 0$. By applying the Bernoulli ansatz $z = x^{-2}$, we obtain

$$\frac{dz}{dt} = -\frac{2}{x^3}\frac{dx}{dt} = \frac{4M\big[\beta + 6\delta\phi(t)\big]x + 8\delta M x^3}{x^3} = 4M\big[\beta + 6\delta\phi(t)\big]z + 8\delta M = \Phi'(t)z + 8\delta M.$$

By applying the integrating factor $\exp[-\Phi(t)]$, we find that

$$z\exp[-\Phi(t)] - z\big|_{t=0} = 8\delta M\int_0^t \exp[-\Phi(\tau)]d\tau.$$

With $z\big|_{t=0} = X^{-2}$ and after multiplying with $\exp[\Phi(t)]$ we have

$$\frac{1}{x^2} = \frac{\exp[\Phi(t)]}{X^2} + \frac{1}{L^2(t)}, \tag{15}$$

where we have introduced the positive function $L(t)$ defined for all $t > 0$ by

$$\frac{1}{L^2(t)} := 8\delta M\int_0^t \exp[\Phi(t) - \Phi(\tau)]d\tau. \tag{16}$$

For later reference we note that

$$\frac{1}{L^2(t)} \sim 8\delta M t \quad \text{as } t \to 0^+. \tag{17}$$

We observe that $\exp[\Phi(t)]/X^2 > 0$ in (15) implies that $x^2 < L^2(t)$ for all $X \in \mathbb{R}$, that is, the support of $\rho(x,t)$ is bounded within the interval

$$-L(t) < x < L(t), \qquad \text{for all} \quad t > 0. \tag{18}$$

We shall refer to $L(t)$ as the *maximum support radius*: $L(t)$ is finite for all positive times, no matter how small, but because $L(t) \to \infty$ as $t \to 0^+$, our solution is valid for an unbounded initial support.

After inverting (15), we calculate by using $\frac{d}{dt}\big(L^{-2}(t)\exp[-\Phi(t)]\big) = 8\delta M \exp[-\Phi(t)]$ that

$$\begin{aligned} x^2 &= \frac{1}{X^{-2}\exp[\Phi(t)] + L^{-2}(t)} = \frac{X^2\exp[-\Phi(t)]}{1+\exp[-\Phi(t)]X^2/L^2(t)} \\ &= \frac{1}{8\delta M}\frac{d/dt\big\{\exp[-\Phi(t)]X^2/L^2(t)\big\}}{1+\exp[-\Phi(t)]X^2/L^2(t)} = \frac{1}{8\delta M}\frac{d}{dt}\log\left\{1+\exp[-\Phi(t)]\frac{X^2}{L^2(t)}\right\}. \end{aligned} \tag{19}$$

Observe now that Eq. (13) can be written as

$$\frac{d}{dt}\log\rho = \frac{\Phi'(t)}{2} + \frac{3}{2}\frac{d}{dt}\log\left\{1+\exp[-\Phi(t)]\frac{X^2}{L^2(t)}\right\}$$

and so

$$\rho(x,t) = \rho(X,0)\exp\Big[\frac{\Phi(t)}{2}\Big]\Big\{1+\exp[-\Phi(t)]\frac{X^2}{L^2(t)}\Big\}^{3/2}. \tag{20}$$

Equations (19) and (20) both contain the starting coordinate X of the characteristics. We record, for avoidance of doubt, two observations about the solutions we have exhibited.

(i) The density ρ remains bounded along each characteristic.

(ii) At fixed t and for all $X > 0$, the relation $x = \big(X^{-2}\exp[\Phi(t)] + L^{-2}(t)\big)^{-1/2}$ implies

$$\frac{\partial x}{\partial X}\Big|_{t \text{ fixed}} = \frac{X^{-3}\exp[\Phi(t)]}{\big(X^{-2}\exp[\Phi(t)]+L^{-2}(t)\big)^{3/2}} = \frac{\exp[-\Phi(t)/2]}{\big(1+\exp[-\Phi(t)]X^2/L^2(t)\big)^{3/2}} > 0, \tag{21}$$

so the characteristics never cross. Therefore shocks never occur in finite time.

We therefore know that we can eliminate X in favour of x and rewrite the solution at position x at time t explicitly as

$$\rho(x,t) = \rho\Big(\frac{x\exp[\Phi(t)/2]}{\sqrt{1-x^2/L(t)^2}}, 0\Big)\exp\Big[\frac{\Phi(t)}{2}\Big]\Big[1-\frac{x^2}{L(t)^2}\Big]^{-3/2}. \tag{22}$$

We see that for all $t > 0$, the solution $\rho(x,t)$ retains the same degree of spatial smoothness as the initial profile $\rho(x,0)$. Although we have arrived at the solution (22) by assuming that $X > 0$, this form of the solution agrees with our earlier special solution for $X = 0$, and as we have assumed symmetry in $\rho(x,0)$, Eq. (22) is valid for all $x \in \mathbb{R}$.

Remark 2.1 (A simple special case: the pure quadratic potential ($\delta = 0$))**.**
For the sake of completeness, we remark the method of characteristics yields for the quadratic potential $W = \beta x^2$ the initial value problem

$$\frac{dx}{dt} = -2\beta M x, \qquad x\Big|_{t=0} = X; \qquad \frac{d\rho}{dt} = 2\beta M\rho, \qquad \rho\Big|_{t=0} = \rho(X,0). \tag{23}$$

and we arrive at $x = Xe^{-2\beta Mt}$ and $\rho = \rho(X,0)e^{2\beta t}$ giving the closed-form solution

$$\rho(x,t) = e^{2\beta Mt}\rho(xe^{2\beta Mt}, 0). \tag{24}$$

We find that our exact solution has the form of a "similarity solution": see Figure 1. Note that the only restriction in this case is finiteness of total mass.

With the change of variable of the methods of characteristics being justified by the strict inequality (21) and thus with Eq. (20) providing a solution formula to (2) for the potential (5) in the case of symmetric densities $\rho(x,t)$, we begin out analysis by the following support estimates.

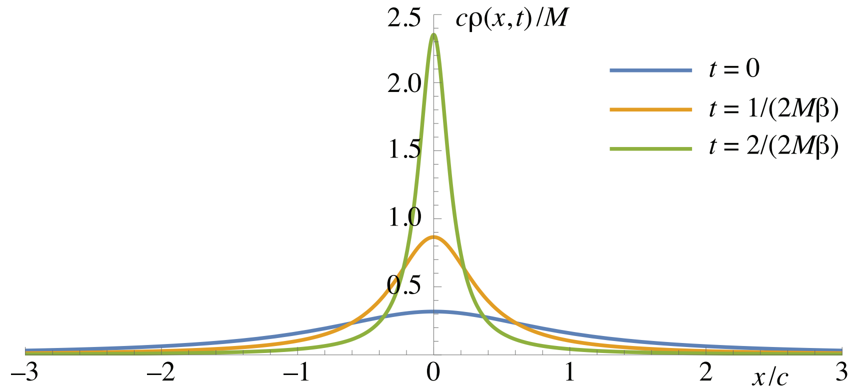

FIGURE 1. Collapse of the initial mass distribution $\rho(x,0) = Mc/[\pi(x^2 + c^2)]$ under the attractive potential $W(x) = \beta x^2$. Horizontal axis is x/c; vertical axis is $c\rho(x,t)/M$. Times are $t = 0$ (blue—the curve with the lowest peak), $t = 1/(2M\beta)$ (brown) and $t = 2/(2M\beta)$ (green—the curve with the highest peak).

Lemma 2.1 (Upper bounds on the maximum support radius $L(t)$ for symmetric ρ)**.**
The positive function $L(t)$ defined for all $t > 0$ by Eq. (16) satisfies

$$L(t) \leqslant \begin{cases} \left(\frac{|\beta|}{2\delta}\right)^{1/2}[1 - \exp(-4M|\beta|t)]^{-1/2} \xrightarrow{t\to\infty} \left(\frac{|\beta|}{2\delta}\right)^{1/2} & \text{if} \quad \beta < 0, \\ (8M\delta t)^{-1/2} \xrightarrow{t\to\infty} 0 & \text{if} \quad \beta = 0, \\ \left(\frac{\beta}{2\delta}\right)^{1/2}[\exp(4M\beta t) - 1]^{-1/2} \sim \left(\frac{\beta}{2\delta}\right)^{1/2}\exp(-2M\beta t) \xrightarrow{t\to\infty} 0 & \text{if} \quad \beta > 0. \end{cases} \tag{25}$$

Proof. From the Mean Value Theorem,

$$\Phi(t) - \Phi(\tau) = \Phi'(\theta(\tau))(t - \tau) \qquad \text{for some } \theta(\tau) \in (\tau, t)$$

and with $\Phi'(t) = 4M\beta + 24M\delta\phi(t) \geqslant 4M\beta$ for all $t > 0$, we discover the bound

$$L^{-2}(t) \geqslant 8\delta M \int_0^t \exp[4M\beta(t-\tau)]d\tau = \begin{cases} \frac{2\delta}{|\beta|}[1 - \exp(-4M|\beta|t)] & \text{if} \quad \beta < 0, \\ 8M\delta t & \text{if} \quad \beta = 0, \\ \frac{2\delta}{\beta}[\exp(4M\beta t) - 1] & \text{if} \quad \beta > 0, \end{cases}$$

which proves (25). □

Corollary 2.1 (Global aggregation for $\beta \geqslant 0$)**.**
In the cases that $\beta \geq 0$, the support estimates (25) *imply global aggregation in the sense that*

$$\lim_{t\to\infty} \phi(t) = 0 = \begin{cases} O(t^{-1}) & \beta = 0, \\ O(\exp(-4M\beta t)) & \beta > 0, \end{cases} \qquad \textit{and} \qquad \lim_{t\to\infty} \rho(x,t) = M\boldsymbol{\delta}(x) \quad \textit{for } \beta \geqslant 0.$$

Proof. The support estimate $\operatorname{supp}\rho(\cdot,t) \in [-L(t), L(t)]$ yields from Eq. (4) that

$$\phi(t) \leqslant L(t)^2, \tag{26}$$

and Lemma 2.1 yields directly the ϕ-related statements of Corollary 2.1.

In order to prove convergence of ρ to a single Dirac measure at the origin, we recall that

$$\int_{-L(t)}^{L(t)} \rho(x,t)dx = M,$$

and that $L(t) \to 0$ as $t \to \infty$ for $\beta \geqslant 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We have

$$\Big|\int_{-\infty}^{\infty} \rho(x,t)f(x)dx - Mf(0)\Big| = \Big|\int_{-L(t)}^{L(t)} \rho(x,t)[f(x)-f(0)]dx\Big| \leqslant \int_{-L(t)}^{L(t)} \rho(x,t)\big|f(x)-f(0)\big|dx.$$

Let $\epsilon > 0$ be given. From the continuity of f, there exists $\Delta > 0$ such that $\big|f(x) - f(0)\big| < \epsilon/M$ whenever $|x| < \Delta$. Thus if $L(t) < \Delta$ we have

$$\Big|\int_{-\infty}^{\infty} \rho(x,t)f(x)dx - Mf(0)\Big| < \int_{-L(t)}^{L(t)} \rho(x,t)\frac{\epsilon}{M}dx = \epsilon.$$

Because $L(t) \to 0$ as $t \to \infty$ we have shown that given any $\epsilon > 0$, for all sufficiently large t we have

$$\Big|\int_{-\infty}^{\infty} \rho(x,t)f(x)dx - Mf(0)\Big| < \epsilon.$$

Thus $\lim_{t\to\infty} \rho(x,t)f(x)dx = Mf(0)$ for arbitrary continuous f and so [23] $\rho(x,t) \to M\boldsymbol{\delta}(x)$ as $t \to \infty$ whenever $\beta \geqslant 0$. $\square$

The second-order moment $\phi(t) = \int_{-\infty}^{\infty} x^2\rho(x,t)dx$ plays a crucial role in the definition of $\Phi(t)$ in (14) and, hence, in formula (20). The following Lemma provides a formula for $\phi(t)$ in terms of the method of characteristics.

Lemma 2.2 (Formula for the second-order moment $\phi(t)$)**.**
The second-order moment $\phi(t)$ satisfies

$$M\phi(t) = \int_{-\infty}^{\infty} \frac{\rho(X,0)dX}{L(t)^{-2} + X^{-2}\exp[\Phi(t)]} = 2\int_0^{\infty} \frac{L(t)^2X^2\rho(X,0)dX}{X^2 + L(t)^2\exp[\Phi(t)]}, \qquad \textit{for all} \quad t \geq 0. \tag{27}$$

Proof. From (20) and (21) we have

$$\rho(x,t) = \rho(X,0)\frac{\partial X}{\partial x}\Big|_{t \text{ fixed}}. \tag{28}$$

Using (19), we find that for $t > 0$

$$M\phi(t) = \int_{-L(t)}^{L(t)} x^2\rho(x,t)\,dx = \int_{-L(t)}^{L(t)} \frac{\rho(X,0)\frac{\partial X}{\partial x}\Big|_{t \text{ fixed}}dx}{X^{-2}\exp[\Phi(t)] + L^{-2}(t)} = \int_{-\infty}^{\infty} \frac{\rho(X,0)\,dX}{X^{-2}\exp[\Phi(t)] + L^{-2}(t)}.$$

Note that at $t = 0$, we have $L^{-2}(0) = 0 = \Phi(0)$ and thus, $M\phi(0) = \int_{-\infty}^{\infty} X^2\rho(X,0)dX$, which proves that (27) holds also at $t = 0$. $\square$

Lemma 2.2 allows to write a closed evolution problem for the quantity $\Phi(t)$ by recalling that $\Phi'(t) = 4M\beta + 24M\delta\phi(t)$.

Corollary 2.2. *The variable $\Phi(t)$ satisfies the following system:*

$$\Phi'(t) = 4M\beta + 48\delta\int_0^{\infty} \frac{L(t)^2X^2\rho(X,0)dX}{X^2 + L(t)^2\exp[\Phi(t)]}, \tag{29}$$

$$L(t) = \Big\{8\delta M\int_0^t \exp[\Phi(t) - \Phi(\tau)]d\tau\Big\}^{-1/2}. \tag{30}$$

If we are prepared to solve two coupled first-order ordinary differential equations, we can replace Eq. (30) *with a differential equation easily obtained by differentiating* $L(t)^{-2}$ *and using* (30):

$$L'(t) = -\frac{1}{2}\Phi'(t)L(t) - 4\delta M L(t)^3. \tag{31}$$

Remark 2.2. *A possibly better numerical scheme sets* $S(t) = 1/L(t)^2$*, so* $S(0) = 0$*, leading to*

$$\Phi'(t) = 4M\beta + 48\delta \int_0^\infty \frac{X^2\rho(X,0)dX}{X^2 S(t) + \exp[\Phi(t)]}, \tag{32}$$

$$S'(t) = \Phi'(t)S(t) + 8\delta M. \tag{33}$$

3. Global estimates for $\phi(t)$ and $\Phi(t)$

The support estimates (25) provide a good insight into the long time behaviour of Eq. (2), in particular in the cases $\beta \geq 0$ (see Corollary 2.1), but they tell little about the transient and qualitative behaviour. The aim of this section is to collect more information.

The second moment formula (27) allows to derive the following estimate

$$M\phi(t) = 2\exp[-\Phi(t)] \int_0^\infty \frac{X^2\rho(X,0)dX}{X^2 L(t)^{-2}\exp[-\Phi(t)] + 1} \leq M\exp[-\Phi(t)]\phi(0), \tag{34}$$

that is,

$$\phi(t) \leq \exp[-\Phi(t)]\phi(0). \tag{35}$$

Since $\Phi(t) \geqslant 4M\beta t$, we may continue to estimate

$$\phi(t) \leqslant \exp(-4M\beta t)\phi(0), \tag{36}$$

which is a better bound for short times than the bound (26).

Using (35) and the equation $\Phi'(t) = 4M\beta + 24M\delta\phi(t)$ gives a differential inequality for $\Phi(t)$,

$$\Phi'(t) \leq 4M\beta + 24M\delta\exp[-\Phi(t)]\phi(0),$$

which can be exploited by the following simple Gronwall type argument. If we multiply by $\exp[\Phi(t)]$ we can rewrite the differential inequality as

$$\frac{d}{dt}\exp[\Phi(t)] - 4M\beta\exp[\Phi(t)] \leqslant 24M\delta\phi(0).$$

Multiplying by $\exp(-4M\beta t)$, we find that

$$\frac{d}{dt}\exp[\Phi(t) - 4M\beta t] \leqslant 24M\delta\phi(0)\exp(-4M\beta t). \tag{37}$$

Provided that $\beta \neq 0$ we have, on integrating and using $\Phi(0) = 0$,

$$\exp[\Phi(t) - 4M\beta t] - 1 \leqslant \frac{6\delta\phi(0)}{\beta}\big[1 - \exp(-4M\beta t)\big].$$

Hence,

$$\exp[\Phi(t)] \leqslant \exp(4M\beta t)\Big[1 + \frac{6\delta\phi(0)}{\beta}\Big] - \frac{6\delta\phi(0)}{\beta}. \tag{38}$$

Inequality (37) and the basic estimate (38) yield the following lemma.

Lemma 3.1. *Let $\Phi(t)$ with $\Phi(0)=0$ satisfy the inequality*

$$\Phi'(t) \leq 4M\beta + 24M\delta \exp[-\Phi(t)]\phi(0).$$

Then, for all $t>0$, we have

$$\Phi(t) \leq \begin{cases} 4M\beta t + \log\Big[1+\dfrac{6\delta\phi(0)}{\beta}\Big] + \log\Big\{1-\dfrac{6\delta\phi(0)/\beta\ \exp(-4M\beta t)}{1+6\delta\phi(0)/\beta}\Big\} & \\ \qquad < 4M\beta t + \log\Big[1+\dfrac{6\delta\phi(0)}{\beta}\Big] & \beta>0, \\ \log\big[1+24M\delta\phi(0)t\big] & \beta=0, \\ \log\Big\{\dfrac{6\delta\phi(0)}{|\beta|} + \Big[1-\dfrac{6\delta\phi(0)}{|\beta|}\Big]\exp(-4M|\beta|t)\Big\} < \log\Big\{\dfrac{6\delta\phi(0)}{|\beta|}+1\Big\} & \beta<0. \end{cases} \tag{39}$$

Moreover, if $\beta < -6\delta\phi(0)$, that is, $\beta<0$ and $6\delta\phi(0)/|\beta|<1$, then the sharper bound $\Phi(t)<0$ holds for all $t>0$.

Remark 3.1. *If $-6\delta\phi(0) < \beta < 0$, Lemma 3.1 does not allow us to exclude the possibility that there is a time interval $(0,t_0)$ throughout which $\Phi(t)>0$.*

Proof. In the case $\beta>0$, inequality (38) can be written as

$$\exp[\Phi(t)] \leqslant \exp(4M\beta t)\Big[1+\frac{6\delta\phi(0)}{\beta}\Big]\Big\{1-\frac{6\delta\phi(0)/\beta}{1+6\delta\phi(0)/\beta}\exp(-4M\beta t)\Big\},$$

which becomes sharp in the limit $t\to 0^+$ and directly yields the first estimate (39) for $\beta>0$. The second upper bound for $\beta>0$ follows by discarding the lower-order, negative last term from the expression inside the braces.

In the case $\beta=0$, inequality (37) reduces to $\exp[\Phi(t)]-1 \leqslant 24\delta M\phi(0)t$, leading to

$$\Phi(t) = 24\delta M\int_0^t \phi(\tau)d\tau \leqslant \log\big[1+24\delta M\phi(0)t\big]$$

as stated in (39).

Finally, the case $\beta<0$ requires a more careful analysis. The inequality (38) becomes

$$\exp[\Phi(t)] \leqslant \exp(-4M|\beta|t)\Big[1-\frac{6\delta\phi(0)}{|\beta|}\Big] + \frac{6\delta\phi(0)}{|\beta|}, \tag{40}$$

which directly yields (39) in the case $\beta<0$.

By rewriting the right-hand side of (40) as $1+[6\delta\phi(0)/|\beta|-1]\,[1-\exp(-4M|\beta|t)]$, we find that if $6\delta\phi(0)/|\beta|<1$, then for all $t>0$ we have $\exp[\Phi(t)]<1$, that is, $\Phi(t)<0$. □

4. Some estimates for ϕ' and qualitative behaviour of ϕ

We begin by differentiating the expression (27) for $\phi(t)$ that we obtained in Lemma 2.2 and using (31), that is $L'(t) = -\frac{1}{2}\Phi'(t)L(t) - 4\delta ML(t)^3$:

$$M\phi'(t) = -2\int_0^\infty \frac{\rho(X,0)dX}{(L(t)^{-2}+X^{-2}\exp[\Phi(t)])^2}\left[\Phi'(t)\left(X^{-2}\exp[\Phi(t)]+L(t)^{-2}\right)+8\delta M\right] \tag{41}$$

$$= -\Phi'(t)M\phi(t) - 16\delta M\int_0^\infty \frac{\rho(X,0)dX}{(X^{-2}\exp[\Phi(t)]+L(t)^{-2})^2}. \tag{42}$$

The last term on the right hand side of Eq. (42) can in fact be rewritten in terms of the fourth-order moment $\mu_4(t)$:

$$M\mu_4(t) := \int_{-\infty}^{\infty} x^4 \rho(x,t)\,dx = \int_{-L}^{L} \frac{\rho(X,0)\frac{\partial X}{\partial x}\Big|_{t \text{ fixed}}\,dx}{(X^{-2}\exp[\Phi(t)] + L^{-2})^2} = 2\int_0^{\infty} \frac{\rho(X,0)\,dX}{(X^{-2}\exp[\Phi(t)] + L^{-2}(t))^2}, \tag{43}$$

where we have used (19) and a computation analogous to that in Lemma 2.2. We obtain

$$M\phi'(t) = -\underbrace{4M\left[\beta + 6\delta\phi(t)\right]}_{=\Phi'(t)} M\phi(t) - 8\delta M^2 \mu_4(t). \tag{44}$$

Note that the above differentiation is rigorously justified for all $t > 0$, since the fourth moment $\mu_4(t)$ is bounded in terms of the second-order moment $\phi(t)$ and the support $L(t)$, which is finite for all $t > 0$. However, at $t = 0$, the fourth-order moment is not necessarily bounded, that is, $\mu_4(0) = \infty$ is possible and, in these cases, $\phi'(t)$ diverges as $t \to 0^+$.

Nevertheless, eq. (44) allows us—independently of the boundedness of higher-order moments—to derive upper bounds for the evolution of $\Phi(t)$. From Jensen's inequality, we have

$$\mu_4(t) = 2\int_0^{\infty} \frac{M^{-1}\rho(X,0)dX}{(L(t)^{-2} + X^{-2}\exp[\Phi(t)])^2} \geqslant \left\{2\int_0^{\infty} \frac{M^{-1}\rho(X,0)dX}{L(t)^{-2} + X^{-2}\exp[\Phi(t)]}\right\}^2 = \phi(t)^2. \tag{45}$$

The inequality is strict unless $\rho(X,0)$ is a single Dirac delta function located within $[0,\infty)$ (and mirrored onto $(-\infty,0]$ by symmetry).

We now see that

$$\phi'(t) \leq -4M\phi(t)\left[\beta + 8\delta\phi(t)\right]. \tag{46}$$

Remark 4.1. *We have already shown in Corollary 2.1 that if $\beta > 0$, then $\rho(x,t)$ must collapse to a single delta function in the limit $t \to \infty$, that is, $\rho(x,t) \to M\boldsymbol{\delta}(x)$. The inequality* (46) *shows that for $\beta > 0$, so long as the system has not already collapsed to a delta function at the origin, $\phi(t)$ is strictly monotonic decreasing. Turning to the case $\beta < 0$, we find from the inequality* (46) *that we have monotonic decay of $\phi(t)$ so long as $\phi(t) > |\beta|/(8\delta)$, but we find the inequality uninformative for the interval $0 \leqslant \phi(t) \leqslant |\beta|/(8\delta)$. We note from the inequalities* (11) *that this interval is precisely the interval of allowed values of an equilibrium solution $\phi(\infty)$ for the second moment.*

Lemma 4.1 (Monotonicity and asymptotic behaviour of $\phi(t)$)**.**
The range of possible equilibrium values $\phi(\infty)$, that is,

$$0 \leqslant \phi(\infty) \leqslant \frac{|\beta|}{8\delta},$$

is globally and monotonic attractive for all initial moments $\phi(0) > |\beta|/(8\delta)$ and invariant for all initial moments $\phi(0) \leqslant |\beta|/(8\delta)$. More precisely, we have in case $\beta < 0$

$$\phi(t) - \frac{|\beta|}{8\delta} \leqslant \begin{cases} \dfrac{\phi(0) - |\beta|/(8\delta)}{1 + [\exp(4M|\beta|t) - 1]\,8\delta\phi(0)/|\beta|} = O(\exp(-4M|\beta|t)), & \phi(0) > \dfrac{|\beta|}{8\delta}, \\ 0, & \phi(0) \leqslant \dfrac{|\beta|}{8\delta}, \end{cases} \quad \textit{if} \quad \beta < 0; \tag{47}$$

and in the cases $\beta \geq 0$

$$\phi(t) \leqslant \begin{cases} \dfrac{\phi(0)}{1 + 32M\delta\phi(0)t} = O(t^{-1}) & \textit{if} \quad \beta = 0, \\ \dfrac{\phi(0)\exp(-4M\beta t)}{1 + [1 - \exp(-4M\beta t)]8\delta\phi(0)/\beta} = O(\exp(-4M\beta t)) & \textit{if} \quad \beta > 0. \end{cases} \tag{48}$$

The convergence of $\phi(t)$ to the equilibrium value 0 is strict monotonic decay for $\beta \geqslant 0$. For $\beta < 0$, we have strict monotonic decay of $\phi(t)$ so long as $\phi(t) > |\beta|/(8\delta)$, while if $\phi(\tau) \leqslant |\beta|/(8\delta)$, then $\phi(t) \leqslant |\beta|/(8\delta)$ for all $t > \tau$, but we lack information concerning monotonicity.

Remark 4.2. *The bounds* (47) *and* (48) *are sharp in the sense that if $\phi(0)$ lies in the interval for equilibrium second moments $\phi(\infty) \in [0, |\beta|/8\delta]$, then $\phi(t)$ remains in this interval for all times. Moreover, they improve upon earlier estimates of the rate of decay of $\phi(t) \to 0$ when $\beta \geqslant 0$.*

Proof. If we multiply the inequality (46) by $-\phi(t)^{-2}$ we find that

$$\frac{d}{dt}\frac{1}{\phi(t)} \geqslant \frac{4M\beta}{\phi(t)} + 32M\delta, \qquad \text{that is,} \qquad \frac{d}{dt}\Big[\frac{\exp(-4M\beta t)}{\phi(t)}\Big] \geqslant 32M\delta\exp(-4M\beta t)$$

and on integrating,

$$\frac{\exp(-4M\beta t)}{\phi(t)} - \frac{1}{\phi(0)} \geqslant \frac{8\delta}{\beta}[1 - \exp(-4M\beta t)].$$

We can rewrite this inequality as the upper bound

$$\frac{\phi(t)}{\phi(0)} \leqslant \frac{\exp(-4M\beta t)}{1 + [1 - \exp(-4M\beta t)]8\delta\phi(0)/\beta}. \tag{49}$$

Eq. (49) refines our earlier result (36) that $\phi(t) \leqslant \exp(-4M\beta t)\phi(0)$. On taking the limit $\beta \to 0^+$ we find that

$$\phi(t) \leqslant \frac{\phi(0)}{1 + 32M\delta\phi(0)t} \qquad \text{in the case } \beta = 0. \tag{50}$$

This is consistent with, but a stronger result than, the inequality on $\Phi(t)$ in (39) in the case $\beta = 0$.

In the case $\beta < 0$, we have

$$\phi(t) \leqslant \frac{\phi(0)\exp(4M|\beta|t)}{1 + [\exp(4M|\beta|t) - 1]8\delta\phi(0)/|\beta|} \xrightarrow{t\to\infty} \frac{|\beta|}{8\delta}.$$

Hence, we calculate

$$\phi(t) - \frac{|\beta|}{8\delta} \leqslant \frac{\phi(0) - |\beta|/(8\delta)}{1 + [\exp(4M|\beta|t) - 1]8\delta\phi(0)/|\beta|} \xrightarrow{t\to\infty} 0,$$

which yields (47). The remaining results stated in the lemma are immediate consequences of Remark 4.1 and noting that if $\phi(\tau) = |\beta|/(8\delta)$ we can use a time shift to set up a new initial value problem with $\phi(0) = |\beta|/(8\delta)$. □

Corollary 4.1 (Refinements for $\Phi(t)$ in the cases $\beta \geqslant 0$)**.**
For $\beta > 0$, the estimates (48) *imply*

$$4M\beta t \leqslant \Phi(t) \leqslant 4M\beta t + \frac{3}{4}\log\Big\{1 + \frac{8\delta\phi(0)}{\beta}\Big[1 - \exp(-4M\beta t)\Big]\Big\} \leqslant 4M\beta t + \frac{3}{4}\log\Big\{1 + \frac{8\delta\phi(0)}{\beta}\Big\}, \tag{51}$$

and for $\beta = 0$

$$0 \leqslant \Phi(t) \leqslant \frac{3}{4}\log[1 + 32M\delta\phi(0)t], \tag{52}$$

and both upper bounds are improvements on the corresponding bounds in (39)*.*

Proof. For $\beta > 0$, we have

$$\begin{aligned}24M\delta\int_0^t \phi(\tau)d\tau &\leqslant 24M\delta\int_0^t \frac{\phi(0)\exp(-4M\beta t)d\tau}{1 + [1 - \exp(-4M\beta t)]8\delta\phi(0)/\beta} \\ &= \frac{3}{4}\log\Big\{1 + \frac{8\delta\phi(0)}{\beta}\Big[1 - \exp(-4M\beta t)\Big]\Big\},\end{aligned}$$

which yields (51). For $\beta = 0$, we obtain (52) by calculating

$$24M\delta \int_0^t \phi(\tau)d\tau \leqslant 24M\delta\phi(0)\int_0^\tau \frac{d\tau}{1+32M\delta\phi(0)t} = \frac{3}{4}\log[1+32M\delta\phi(0)t].$$

□

Corollary 4.2 (Refinements for $\Phi(t)$ in the case $\beta < 0$)**.**
For $\beta < 0$, inequality (47) allow two possible scenarios when $\phi(0) > |\beta|/(8\delta)$: either

(a) *$\phi(t) > |\beta|/(8\delta)$ for all t and $\phi(t) \to |\beta|/(8\delta)$ (monotonically) as $t \to \infty$;*
or
(b) *$\phi(t)$ decays monotonically until $\phi(t) = |\beta|/(8\delta)$ at some positive time t_*, and then from the bound (47), we have $\phi(t) \leqslant |\beta|/(8\delta)$ for all $t \geqslant t_*$.*

Moreover, we have

$$\lim_{t\to\infty} \Phi(t) \leq \lim_{t\to\infty} -M|\beta|t = -\infty, \tag{53}$$

which improves upon the upper bound (39).

Proof. We only need to show (53). Since (47) implies for all $\varepsilon > 0$ that $\phi(t) < (|\beta| + \varepsilon)/(8\delta)$ for sufficiently large $t > 0$, we have $\Phi'(t) < -M[|\beta| + 3\varepsilon]$, which yields (53). □

Remark 4.3. *In the introduction, we mentioned the condition $\beta + 6\delta\phi(t) = 0$, which in [20] separated the unimodal ($\beta + 6\delta\phi(\infty) \geqslant 0$) and bimodal ($\beta + 6\delta\phi(\infty) < 0$) stationary states of Eq. (2) with diffusion.*

Here, for Eq. (2) without diffusion, we see now that the (non-autonomous) pitchfork bifurcation at $\beta + 6\delta\phi(t) = 0$ in Eq. (12) governs a mechanism of the formation of a single versus two aggregates in the sense that $\beta \geqslant 0$ implies $\phi(\infty) = 0$ (so that $\beta + 6\delta\phi(\infty) \geqslant 0$) and thus a unimodal single Dirac at $x = 0$, while $\beta < 0$ implies that $0 \leqslant \phi(\infty) \leqslant |\beta|/(8\delta)$ and, therefore, $\beta + 6\delta\phi(\infty) \leqslant -\frac{1}{4}|\beta| < 0$, which yields two additional Diracs at locations $\xi^2 = |\beta|/[2\delta(1+6\lambda)]$ for $0 \leqslant \lambda \leqslant 1/2$. In fact, the stability criteria of [16] show that the third Dirac at $x = 0$ is instable and solutions will only converge towards the bimodal two Dirac steady state

$$\rho(x,\infty) = \frac{M}{2}\left[\boldsymbol{\delta}(x-\xi) + \boldsymbol{\delta}(x+\xi)\right], \quad \xi = \left[\frac{|\beta|}{8\delta}\right]^{1/2}, \qquad \text{with} \quad \phi(\infty) = \frac{|\beta|}{8\delta}.$$

However, this is not (possible to be) part of our analysis, since we have no way of excluding the unstable three Dirac steady states for $0 < \lambda < 1/2$ on the basis of only considering second- and third-order moments.

On the other hand, the above bimodal two Dirac steady state corresponds also to the zero diffusivity limit obtained in [20], which found two equal delta functions at $x = \pm[|\beta/(8\delta)]^{1/2}$ if $\beta + 6\delta\phi(\infty) < 0$, and a single delta function at the origin if $\beta + 6\delta\phi(\infty) \geqslant 0$. In [20], we also gave a formula for the possible locations of delta functions with constraints related to the partitioning of mass. Having $\phi(t) \to |\beta|/(8\delta)$ places the delta functions at the endpoints of the feasible interval and forces all mass to lie at these endpoints, with no mass at the origin.

5. Monotonicity and concavity of $\Phi(t)$

The aim of this section is to study the qualitative behaviour of the quantity $\Phi(t)$. We have already established that in case $\beta \geqslant 0$

- $\Phi(t)$ is monotonic increasing (and indeed strictly increasing unless $\beta = 0$ and the collapsed state in which $\phi(t) = 0$ has already been attained);
- $\Phi(t) \sim 4M\beta t$ as $t \to \infty$ for $\beta > 0$ (from the inequalities (51)) and there is at most logarithmic asymptotic growth of $\Phi(t)$ when $\beta = 0$ (from the inequality (52));
- $\phi(t)$ is strictly monotonic decreasing when $\phi(t) > 0$, $\lim_{t\to\infty} \phi(t) = 0$, and $\lim_{t\to\infty} \Phi'(t) = 4M\beta$.

Hence our further discussion is confined to the case $\beta < 0$.

We begin by recalling that

$$\Phi'(t) = 4M[\beta + 6\delta\phi(t)], \quad \text{so that} \quad \Phi''(t) = 24M\delta\phi'(t),$$

and also by recalling the bound (46), which we rewrite as

$$\phi'(t) \leq -\phi(t)\Phi'(t) - 8M\delta\phi^2(t), \tag{54}$$

Hence

$$\Phi''(t) \leqslant -24M\delta[\phi(t)\Phi'(t) + 8M\delta\phi^2(t)].$$

If $\bar{t}$ is to be a stationary point for $\Phi(t)$, so that $\Phi'(\bar{t}) = 0$, we find that $\Phi''(\bar{t}) < 0$ (unless collapse to a single delta function at the origin has already occurred) and so *the only possible stationary points for* $\Phi(t)$ *are local maxima. This observation allows to characterise the monotonicity behaviour of* $\Phi(t)$ *by distinguishing two cases:* (i) $\Phi'(0) \leq 0$ *and* (ii) $\Phi'(0) > 0$.
$\beta < 0$ Case (i): First, we assume that

$$\Phi'(0) = 4M[\beta + 6\delta\phi(0)] < 0, \qquad \text{which requires} \qquad \beta < 0 \quad \text{and} \quad \phi(0) < \frac{|\beta|}{6\delta}.$$

Then, by continuity of $\Phi'(t)$ and of $\phi(t)$, there exists an interval $[0, t_0]$ in which $\Phi'(t) < 0$ and $\phi(t) < |\beta|/(6\delta)$. Since $\Phi(0) = 0$ we also have $\Phi(t) < 0$ in this interval. If $\Phi'(t)$ is to vanish for the first time at some positive time $\bar{t}$, at that time Φ has to have either a local minimum or a point of horizontal inflection, since $\Phi(t)$ is a decreasing function on $[0, \bar{t}]$. But we have shown that the only possible stationary points for $\Phi(t)$ are local maxima. Hence $\Phi(t)$ *is strictly decreasing for all times.* The exclusion of $\Phi'(t) = 0$ at any positive time rules out $\beta + 6\delta\phi(t) = 0$ at any positive time, and so $\phi(t) < |\beta|/(6\delta)$ for all time. Actually the inequality (47) already contains a sharper result: if $|\beta|/(8\delta) \leqslant \phi(0) \leqslant |\beta|/(6\delta)$, then for all time $\phi(t) \leqslant \phi(0) \leqslant |\beta|/(6\delta)$, while if $\phi(0) \leqslant |\beta|/(8\delta)$, then $\phi(0) \leqslant |\beta|/(8\delta)$ for all time.

To complete the analysis of Case (i) we assume that

$$\Phi'(0) = 4M[\beta + 6\delta\phi(0)] = 0, \qquad \text{which for} \quad \beta < 0 \quad \text{requires} \quad \phi(0) = \frac{|\beta|}{6\delta}.$$

Since $\phi'(t) \leq -4M\phi(t)\,[\beta + 8\delta\phi(t)]$ from the bound (46), we have $\phi'(0) \leq -2M|\beta|^2/(9\delta) < 0$, enforcing the strict decrease of $\phi(t)$ in the neighbourhood of the origin, and so there is an interval $(0, t_0)$ in which we have

$$\Phi'(t) = 4M[\beta + 6\delta\phi(t)] < 4M[\beta + 6\delta\phi(0)] = 0.$$

The initial decrease of $\Phi(t)$ precludes subsequent the attainment of a local maximum, which is the only possible type of stationary point for $\Phi(t)$. Hence $\Phi'(t) < 0$ for all $t > 0$.
$\beta < 0$ Case (ii): Here we have

$$\Phi'(0) = 4M[\beta + 6\delta\phi(0)] > 0, \qquad \text{which for} \quad \beta < 0 \quad \text{requires} \qquad \phi(0) > \frac{|\beta|}{6\delta}.$$

Similar continuity considerations to those used for Case (i) establish an initial interval in which $\Phi(t) > 0$ and $\phi(t) > |\beta|/(6\delta)$. But we showed above that if $\phi(0) > |\beta|/(8\delta)$, then $\phi(t)$ decays monotonically, and either approaches $|\beta|/(8\delta)$ asymptotically from above, or attains the value $|\beta|/(8\delta)$ in finite time. From the intermediate value theorem, we conclude that $\phi(t) = |\beta|/(6\delta)$ occurs at finite time, and at this time $\Phi(t)$ attains a maximum, which is its only extremum.

Altogether, we have proven the following lemma.

Lemma 5.1 (Monotonicity and concavity of $\Phi(t)$)**.**
The function $\Phi(t)$ *as defined in* (14)*, or equivalently by* $\Phi(0) = 0$ *and* $\Phi'(t) = 4M[\beta + 6\delta\phi(t)]$*, is a strictly monotonic function except under precisely determined conditions when it attains a single local maximum.*

More precisely, for $t \in (0,\infty)$*,*

$$\beta > 0 \implies \begin{cases} \Phi(t) \text{ is strictly positive, concave and strictly monotonic increasing;} \\ \Phi(t) \sim 4M\beta t \text{ as } t \to \infty \text{ [that is, } \Phi'(\infty) = 4M\beta] \text{ and } \phi(\infty) = 0. \end{cases}$$

$$\beta = 0 \implies \begin{cases} \Phi(t) \text{ is non-negative, concave and strictly monotonic increasing;} \\ \Phi(t) = O(\log(t)) \text{ as } t \to \infty, \text{ [that is, } \Phi'(\infty) = 0] \text{ and } \phi(\infty) = 0. \end{cases}$$

$$\left.\begin{matrix} \beta < 0 \\ \Phi'(0) \le 0 \end{matrix}\right\} \implies \begin{cases} \Phi(t) \text{ is strictly negative and strictly monotonic decreasing;} \\ \Phi(t) \to -\infty \text{ as } t \to \infty \text{ with } \Phi'(\infty) = 4M[\beta + 6\delta\phi(\infty)] \in [-4M|\beta|, -M|\beta|]. \end{cases}$$

$$\left.\begin{matrix} \beta < 0 \\ \Phi'(0) > 0 \end{matrix}\right\} \implies \begin{cases} \exists t_1 > 0 \text{ such that } \Phi(t) > 0 \text{ and } \Phi(t) \text{ is strictly monotonic increasing on } (0,t_1); \\ \Phi(t_1) \text{ is a unique positive maximum where } \phi(t_1) = |\beta|/(6\delta); \\ \Phi(t) \text{ is strictly monotonic decreasing on } (t_1,\infty); \\ \Phi(t) \to -\infty \text{ as } t \to \infty \text{ with } \Phi'(\infty) = 4M[\beta + 6\delta\phi(\infty)] \in [-4M|\beta|, -M|\beta|]. \end{cases}$$

Note that $\Phi(t)$ *is necessarily concave whenever* $\phi(t) > \beta/(6\delta)$ *and that the asymptotic slope* $\Phi'(\infty)$ *characterises the equilibrium values second-order moment, that is,*

$$\phi(\infty) = \frac{\Phi'(\infty) - 4M\beta}{24M\delta} \begin{cases} = 0 & \beta \ge 0, \\ \in [0, |\beta|/(8\delta)] & \beta < 0. \end{cases}$$

Remark 5.1. *Lemma 5.1 shows that knowledge of the function* $\Phi(t)$ *is equivalent to essentially knowing those parts of dynamics of the evolution problem* (2)*, which are independent of the initial distribution* $\rho(x,0)$ *and the history dependence of* (2)*.*

Accordingly, we have, for instance, no way of reasoning to which equilibrium $\phi(\infty)$ *from* (9) *the second-order moment* $\phi(t)$ *converges to as* $t \to \infty$ *in the case* $\beta < 0$*.*

The following calculation derives a formula for $\Phi(t)$ and shows that it can be interpreted as a kind of moment potential function, as claimed in Theorem 1.1.

We recall the formula (27) for $\phi(t)$ and calculate with $\frac{d}{dt}\left(L^{-2}\exp[-\Phi(t)]\right) = 8\delta M \exp[-\Phi(t)]$:

$$\begin{aligned} M\phi(t) &= \int_{-\infty}^{\infty} \frac{X^2 \exp[-\Phi(t)]\rho(X,0)dX}{1 + X^2 L(t)^{-2}\exp[-\Phi(t)]} = \frac{1}{8\delta M}\int_{-\infty}^{\infty} \frac{\frac{d}{dt}\left(X^2 L^{-2}\exp[-\Phi(t)]\right)\rho(X,0)dX}{1 + X^2 L(t)^{-2}\exp[-\Phi(t)]} \\ &= \frac{d}{dt}\frac{1}{8\delta M}\int_{-\infty}^{\infty} \log\left[1 + X^2 L(t)^{-2}\exp[-\Phi(t)]\right]\rho(X,0)dX \end{aligned}$$

Hence, with $\Phi'(t) = 4M\beta + 24\delta M\phi(t)$ and recalling that $L^{-2}(0) = 0$, we integrate and deduce that

$$\Phi(t) = 4M\beta t + \frac{3}{M}\int_{-\infty}^{\infty} \log\left[1 + X^2 L(t)^{-2}\exp[-\Phi(t)]\right]\rho(X,0)dX. \tag{55}$$

On the other hand, given the formula (55), we can calculate $\phi(t)$ as

$$M\phi(t) = \frac{\Phi'(t)}{24\delta} - \frac{M\beta}{6\delta},$$

which shows that $\Phi(t)$ can be interpreted as a potential function to the second-order moment $\phi(t)$.

Note that despite (55) being a nice formula, it seems difficult to extract upper bounds for $\Phi(t)$ which are as good as (39). Indeed, a natural way to exploit (55) would be to use Jensen's inequality, leading to

$$\Phi(t) \leqslant 4M\beta t + 3\log\left\{1 + L(t)^{-2}\exp[-\Phi(t)]\phi(0)\right\}.$$

Let us see what this produces in the case $\beta < 0$, when a further improvement of (39) would certainly be of interest. But when we use the lower bound $\Phi(t) \geq -4M|\beta|t$ to estimate

$$L(t)^{-2}\exp[-\Phi(t)] \leq 8\delta M \int_0^t \exp[4M|\beta|\tau]\,d\tau = \frac{2\delta}{|\beta|}\left[\exp[4M|\beta|t] - 1\right],$$

we obtain

$$\Phi(t) \leq 4M\beta t + \underbrace{3\log\left[1 + \frac{2\delta}{|\beta|}\left[\exp[4M|\beta|t] - 1\right]\phi(0)\right]}_{= O(12M|\beta|t) \text{ as } t\to\infty} = O(8M|\beta|t) \text{ as } t \to \infty$$

which is a linear growing and hence weaker upper bound than (39).

6. Lower bound and convexity estimate for $\phi''(t)$

Differentiating our expression (42) for $\phi'(t)$ and using Eq. (31) to eliminate $L'(t)$ we find that

$$\begin{aligned}\phi''(t) &= -\Phi''(t)\phi(t) - \Phi'(t)\phi'(t)\\ &\quad + 16\delta M \int_{-\infty}^{\infty} \frac{M^{-1}\rho(X,0)}{(L(t)^{-2} + X^{-2}\exp[\Phi(t)])^3}\Big\{\Phi'(t)(L(t)^{-2} + X^{-2}\exp[\Phi(t)]) + 8\delta M\Big\}dX, \end{aligned} \tag{56}$$

By noting that Eq. (44) can be rewritten as

$$8\delta M \int_{-\infty}^{\infty} \frac{M^{-1}\rho(X,0)dX}{(L(t)^{-2} + X^{-2}\exp[\Phi(t)])^2} = -\phi'(t) - \Phi'(t)\phi(t), \tag{57}$$

and that the last integral on the right hand side of (56) can be written analogously to Lemma 2.2, in terms of the sixth-order moment

$$\mu_6(t) := \int_{-\infty}^{\infty} \frac{x^6\rho(x,t)\,dx}{M} = 2\int_0^L \frac{M^{-1}\rho(X,0)\frac{\partial X}{\partial x}\big|_{t\text{ fixed}}\,dx}{(X^{-2}\exp[\Phi(t)] + L^{-2})^3} = 2\int_0^{\infty} \frac{M^{-1}\rho(X,0)\,dX}{(X^{-2}\exp[\Phi(t)] + L^{-2}(t))^3}, \tag{58}$$

we have

$$\begin{aligned}\phi''(t) &= -\Phi''(t)\phi(t) - \Phi'(t)\phi'(t) + 2\Phi'(t)[-\phi'(t) - \Phi'(t)\phi(t)] + 128(\delta M)^2\mu_6(t)\\ &= -\Phi''(t)\phi(t) - 3\Phi'(t)\phi'(t) - 2\Phi'(t)^2\phi(t) + 128(\delta M)^2\mu_6(t).\end{aligned} \tag{59}$$

We remark that Eq. (59) shows that differentiation of (44) is justified for all $t > 0$ since the sixth-order moment (58) is bounded in terms of the finite support $L(t) < \infty$ for all $t > 0$ (see Lemma 2.1).

Using Jensen's inequality on the sixth-order moment analogue to (45), that is, $\mu_6(t) \geqslant \phi^3(t)$, gives

$$\phi''(t) \geqslant -\Phi''(t)\phi(t) - 3\Phi'(t)\phi'(t) - 2\Phi'(t)^2\phi(t) + 128(\delta M)^2\phi(t)^3.$$

If we now recall that $\Phi'(t) = 4M[\beta + 6\delta\phi(t)]$ and $\Phi''(t) = 24M\delta\phi'(t)$ we arrive at

$$\phi''(t) \geqslant -12M[\beta + 8\delta\phi(t)]\phi'(t) - 32M^2\phi(t)[\beta + 8\delta\phi(t)][\beta + 4\delta\phi(t)]. \tag{60}$$

Useful conclusions can be drawn if we impose the restriction that $\beta + 8\delta\phi(t) > 0$, that is, in either of the cases $\beta > 0$, or $\beta \leqslant 0$ with $\phi(t) > |\beta|/(8\delta)$. The inequality (46) tells us that $-\phi'(t) \geqslant 4M\phi(t)\,[\beta + 8\delta\phi(t)]$ and so we can eliminate $\phi'(t)$ from the inequality (60) and we find after a little algebra that

$$\phi''(t) \geqslant 16M^2\phi(t)[\beta + 8\delta\phi(t)][\beta + 16\delta\phi(t)] > 0, \qquad \text{whenever} \quad \phi(t) > -\frac{\beta}{8\delta}. \tag{61}$$

We now see that if $\beta \geqslant 0$, the second-order moment $\phi(t)$ decays to zero as a convex function for every nonzero $\phi(0)$. The case $\beta < 0$ is more subtle: if we are given that $\phi(0) > |\beta|/(8\delta)$, the second-order moment $\phi(t)$ decays as a convex function so long as $\phi(t) > |\beta|/(8\delta)$. Since for all possible equilibria, $\phi(t) \leqslant |\beta|/(8\delta)$ we conclude that if $\phi(t) > |\beta|/(8\delta)$ then either $\phi(t)$ decays monotonically until we achieve $\phi(t) = |\beta|/(8\delta)$ in finite time and thereafter, $\phi(t) \leqslant |\beta|/(8\delta)$; or else $\phi(t)$ decays monotonically for all time, with $\lim_{t\to\infty} \phi(t) = |\beta|/(8\delta)$.

We conclude this section with a brief illustration of possible transient growth of $\Phi(t)$ and $\phi(t)$ when $\beta < 0$. We observe first that Eq. (44), which we can write as

$$\phi'(t) = -4M\left[\beta + 6\delta\phi(t)\right]\phi(t) - 8\delta M\mu_4(t) \tag{62}$$

shows us that if the initial fourth-order moment

$$\mu_4(0) = 2\int_0^\infty X^4 M^{-1}\rho(X,0)dX \tag{63}$$

is infinite, then $\phi'(t) \to -\infty$ as $t \to 0^+$. Hence, transient growth of $\Phi(t)$ and $\phi(t)$ is best seen when assuming compactly supported initial data.

Lemma 6.1. *Suppose that $\phi(0) > 0$, which excludes the trivial initial condition $\rho(x,0) = \boldsymbol{\delta}(x)$.*

(i) *A sufficient condition for $\phi'(0) < 0$ is that $\beta + 8\delta\phi(0) > 0$.*

(ii) *Sufficient conditions for $\phi'(0) > 0$ are that* supp $\rho(X,0) \in [-K_0, K_0]$ *for some $K_0 > 0$ and $\beta < -6\delta\phi(0) - 2\delta K_0^2$.*

Proof. Since $\mu_4(0) \geqslant \phi(0)^2$ from Jensens' inequality, we find from Eq. (62) that

$$\phi'(0)/\phi(0) \leqslant -4M[\beta + 8\delta\phi(0)]$$

and the result (i) follows. In the case of compact support with supp $\rho(X,0) \in [-K_0, K_0]$, we can obtain an upper bound for $\mu_4(0)$ by replacing X^4 in Eq. (63) by $K_0^2X^2$, giving $\mu_4(0) \leqslant K_0^2\phi(0)$. We find from Eq. (62) that

$$\phi'(0)/\phi(0) \geqslant -4M[\beta + 6\delta\phi(0) + 2\delta K_0^2]$$

and the result (ii) follows. □

Of course, Lemma 6.1 only gives us new information in the case $\beta < 0$. We compare the sufficient conditions for initial growth or initial decay of $\phi(t)$ with exact results for a specific compact support example in Section 8.2.

The case (ii) in Lemma 6.1 shows that for compactly supported initial data, there is initial growth of $\phi(0)$ for negative β of sufficiently large magnitude. Suppose that the sufficient conditions (ii) for initial growth of $\phi(t)$ from compactly supported initial data are fulfilled. Then

$$\Phi'(0) = 4M[\beta + 6\delta\phi(0)] \leqslant -8\delta M K_0^2 < 0.$$

Thus having $\phi'(0) > 0$ need not imply that $\Phi'(0) > 0$.

7. The curvature of $\rho(x,t)$ at the origin

We know from Eq. (22) that if the initial profile is twice-differentiable in an open interval containing the origin, then at any given positive time t there will also be an open interval containing the origin in which the initial profile is twice-differentiable, and we have

$$\rho(0,t) = \exp[\Phi(t)/2]\rho(0,0), \qquad \rho_x(0,t) = 0, \tag{64}$$

and

$$\rho_{xx}(0,t) = \exp[\Phi(t)/2]\Big\{\exp[\Phi(t)]\rho_{xx}(0,0) + \frac{3\rho(0,0)}{L(t)^2}\Big\}. \tag{65}$$

Solutions for the functions $\Phi(t)$ and $L(t)$ and the numerical values of $\rho(0,0)$ and $\rho_{xx}(0,0)$ completely determine the values of $\rho(0,t)$ and the sign of the curvature at the origin. We would expect to find $\rho_{xx}(0,t) \geqslant 0$ at large times when $\beta < 0$ if the profile $\rho(x,t)$ is to converge to two Dirac delta functions symmetrically place around the origin. Though perhaps not intuitively obvious, in the case $\beta \geqslant 0$, starting with $\rho_{xx}(0,0) < 0$ under some parameter conditions we can still evolve to $\rho_{xx}(0,0) > 0$ at sufficiently long times, as we demonstrate in the following theorem.

Theorem 7.1. *Suppose that $\rho(x,0)$ is twice-differentiable at in an open interval containing $x = 0$ and that $\rho(0,0) > 0$.*

(i) *If $\beta < 0$, then $\liminf_{t\to\infty} \rho_{xx}(0,t) \geqslant 0$.*
(ii) *If $\beta = 0$, then $\rho_{xx}(0,t) \to \infty$ as $t \to \infty$.*
(iii) *If $\beta > 0$ and $\rho_{xx}(0,0) \geqslant 0$, then $\rho_{xx}(0,t) \to \infty$ as $t \to \infty$.*
(iv) *If $\beta > 0$ and $\rho_{xx}(0,0) < 0$, then there exists $\beta^\dagger > 0$ such that $\rho_{xx}(0,t) \to \infty$ as $t \to \infty$ for $0 < \beta < \beta^\dagger$.*

Note that no restrictions are placed on the sign of $\rho_{xx}(0,0)$ in the results (i) and (ii). The cases (ii)–(iv) contain some quite surprising predictions concerning the convergence to a single Dirac at the origin. In particular (iv) implies that even for unimodal initial data the solution will aggregate into an (at least) bimodal time-evolving profile as it collapses asymptotically to a single Dirac at the origin. This will be illustrated below by numerical examples.

Proof. (i) For $\beta < 0$ Corollary 4.2 shows that $\Phi(t) \to -\infty$ as $t \to \infty$, so $\rho_{xx}(0,0)\exp[\Phi(t)] \to 0$ and we find that $\rho_{xx}(0,t) \sim 3\rho(0,0)\exp[\Phi(t)/2]/L(t)^2$, so $\rho_{xx}(0,t)$ is non-negative for sufficiently large t, giving $\liminf_{t\to\infty} \rho_{xx}(0,t) \geqslant 0$. Noting that $\phi(t) \leqslant L(t)^2$, if we had a rigorous proof either that $L(t)$ has a positive lower bound, or that $\phi(t)$ has a positive lower bound, we could deduce the stronger result that $\rho_{xx}(0,t) \to 0$ as $t \to \infty$.

(ii) For $\beta = 0$, using the bound on $L(t)$ from Lemma 2.1 we have

$$\rho_{xx}(0,t) \geqslant \exp[\Phi(t)/2]\Big\{\exp[\Phi(t)]\rho_{xx}(0,0) + 24M\delta t\rho(0,0)\Big\}.$$

From Corollary 4.1, we have $1 \leqslant \exp[\Phi(t)] \leqslant [1 + 32M\delta\phi(0)t]^{3/4}$. When $\rho_{xx}(0,0) \geqslant 0$, we now know that $\rho_{xx}(0,t) \geqslant 24M\delta t\rho(0,0)$ and the divergence of $\rho_{xx}(0,t)$ to infinity is clear. For $\rho_{xx}(0,0) < 0$, we have

$$\begin{aligned}\rho_{xx}(0,t) &\geqslant \exp[\Phi(t)/2]\Big\{24M\delta t\rho(0,0) - \exp[\Phi(t)]\,|\rho_{xx}(0,0)|\Big\}\\ &\geqslant \exp[\Phi(t)/2]\Big\{24M\delta t\rho(0,0) - [1 + 32M\delta\phi(0)t]^{3/4}\,|\rho_{xx}(0,0)|\Big\}.\end{aligned}$$

Because $[1+32M\delta\phi(0)t]^{3/4} = O(t^{3/4})$, the expression in braces is dominated at large t by the positive term, and $\exp[\Phi(t)/2] \geqslant 1$, so we conclude that $\rho_{xx}(0,t) \to \infty$.

(iii) For $\beta > 0$, from Corollary 4.1 we have $\exp(4M\beta t) \leqslant \exp[\Phi(t)] \leqslant [1 + 8\delta\phi(0)/\beta]^{3/4}\exp(4M\beta t)$, while Lemma 2.1 gives $1/L(t)^2 \geqslant (2\delta/\beta)[\exp(4M\beta t) - 1]$. Thus for $\rho_{xx} \geqslant 0$ we have

$$\rho_{xx}(0,t) \geqslant \frac{6\delta}{\beta}\rho(0,0)\exp(2M\beta t)[\exp(4M\beta t) - 1],$$

establishing the claimed divergence of $\rho_{xx}(0,t)$.

(iv) For our final case, in which $\beta > 0$ and $\rho_{xx}(0,0) < 0$, using the bounds on $\exp[\Phi(t)]$ and $1/L(t)^2$ from part (iii) we have we have

$$\rho_{xx}(0,t) \geqslant \exp[\Phi(t)/2]\Big\{-\Big[1 + \frac{8\delta\phi(0)}{\beta}\Big]^{3/4}\exp[4M\beta t]|\,\rho_{xx}(0,0)| + \frac{6\delta}{\beta}[\exp(4M\beta t) - 1]\rho(0,0)\Big\}.$$

Let

$$\kappa = \frac{6\delta}{\beta}\rho(0,0) - \Big[1 + \frac{8\delta\phi(0)}{\beta}\Big]^{3/4}|\rho_{xx}(0,0)|. \tag{66}$$

If $\kappa > 0$, then noting that $\exp[\Phi(t)/2] \geqslant \exp(2M\beta t)$ we find that $\rho_{xx}(0,t) \geqslant \kappa\exp(6M\beta t)[1 + o(1)]$ as $t \to \infty$, establishing the divergence of $\rho_{xx}(0,t)$. We conclude by noting that in the limit $\beta \to 0^+$ the first term dominates the second in the right-hand side of Eq. (66), making κ positive and establishing the existence of the parameter interval $0 < \beta < \beta^\dagger$ in which $\rho_{xx} \to \infty$ as $t \to \infty$. $\square$

8. Numerical solutions for smooth initial data

We consider two specific examples with symmetric initial data $\rho(x,0)$ to illustrate the quality of the bounds obtained and show the qualitative features of the collapse of the initial profile to one or two Dirac masses. In the first example (Section 8.1), we consider smooth initial data $\rho(x,0)$ with unbounded support and $\rho_{xx}(0,0)<0$. The prospect of a change of sign of the curvature at the origin, that is, a change in the sign of $\rho_{xx}(0,t)$ therefore arises (as discussed in Section 7). In the second example (Section 8.2), we consider a uniform distribution on a finite interval, so that $\rho_{xx}(0,0)=0$ ensuring that $\rho_{xx}(x,t)>0$ for all $t>0$.

For convenience in constructing numerical solutions, we introduce a dimensionless time

$$T = 24\delta M\phi(0)t \tag{67}$$

and a single control parameter

$$k = \frac{\beta}{6\delta\phi(0)}. \tag{68}$$

which is positive for an attractive quadratic coefficient β in the potential and negative for a repulsive quadratic coefficient β.

When expressed in terms of k and T, our bounds (25) on $L(t)$ obtained in Lemma 2.1 and our bounds (47) and (48) on the second moment $\phi(t)$ obtained in Lemma 4.1 become

$$\frac{L(t)}{\sqrt{\phi(0)}} \leqslant \begin{cases} (3|k|)^{1/2}[1-\exp(-|k|T)]^{-1/2} & \text{if } k<0, \\ (3/T)^{1/2} & \text{if } k=0, \\ (3k)^{1/2}[\exp(kT)-1]^{-1/2} & \text{if } k>0, \end{cases} \tag{69}$$

and

$$\frac{\phi(t)}{\phi(0)} \leqslant \begin{cases} \dfrac{3|k|}{4} & \text{if } k\leqslant -\dfrac{4}{3} \\ \dfrac{3|k|}{4-(4-3|k|)\exp(-|k|T)} & \text{if } -\dfrac{4}{3}<k<0, \\ \dfrac{3}{3+4T} & \text{if } k=0, \\ \dfrac{3k\exp(-kT)}{3k+4[1-\exp(-kT)]} & \text{if } k>0. \end{cases} \tag{70}$$

Remark 8.1. *As discussed in the Introduction, for $\beta<0$ (corresponding here to $k<0$) the only possible symmetric equilibrium states $\rho(x,\infty)$ that can arise are linear combinations of a Dirac component $\delta(x)$ at the origin and a pair of Diracs placed symmetrically about the origin, $\delta(x+\xi)+\delta(x-\xi)$, with $[|\beta|/(8\delta)]^{1/2}\leqslant \xi\leqslant [|\beta|/(2\delta)]^{1/2}$, with the lower bound for ξ being attained if and only if there is no mass at the origin. In dimensionless form, the bounds on the distance of a possible Dirac component from the origin are*

$$\frac{(3|k|)^{1/2}}{2} \leqslant \frac{\xi}{\sqrt{\phi(0)}} \leqslant (3|k|)^{1/2}, \tag{71}$$

and here again the lower bound is attained if and only if there is no mass at the origin. Because the support of $\rho(x,t)$ is always confined to $[-L(t),L(t)]$, we know that $\xi\leqslant \lim_{t\to\infty}L(t)$ and so numerical evidence that $L(t)\to(3|k|)^{1/2}/2$ would support the belief that there is no mass at the origin in the equilibrium state to which the system converges. This matches the results of [16] on the instability of three Dirac stationary states.

We obtain numerical solutions of our differential equations for $\Phi(t)$ and $U(t)$, expressed in terms of the dimensionless time coordinate T, for various values of the dimensionless parameter k using the function `NDSolve` in Mathematica (Wolfram Research, Version 11).

8.1. **Example with an unbounded initial support**

Consider the initial mass distribution

$$\rho(x,0)=\frac{2M\ell^3}{\pi(x^2+\ell^2)^2}, \tag{72}$$

($M>0$ and $\ell>0$), for which it is easily verified that

$$\int_{-\infty}^{\infty}\rho(x,0)dx=M \quad \text{and} \quad \phi(0)=\frac{1}{M}\int_{-\infty}^{\infty}x^2\rho(x,0)dx=\ell^2. \tag{73}$$

If we let $X=\ell z$ and define $\lambda(t)=\ell^{-1}L(t)\exp[\Phi(t)/2]$, then by residue calculus, completing the contour of integration in the upper half-plane in the standard way, we find that

$$\int_0^{\infty}\frac{L(t)^2X^2\rho(X,0)dX}{X^2+L(t)^2\exp[\Phi(t)]}=\frac{ML(t)^2}{\pi}\int_{-\infty}^{\infty}\frac{z^2dz}{(z^2+1)^2[z^2+\lambda(t)^2]}=\frac{ML(t)^2}{2[\lambda(t)+1]^2}. \tag{74}$$

We see from Eq. (29) that

$$\Phi'(t)=4M\beta+\frac{24\delta M\ell^2}{[\exp[\Phi(t)/2]+\ell/L(t)]^2}.$$

If we scale the function $S(t)=1/L(t)^2$ introduced earlier, defining

$$U(t)=\ell^2S(t)=\ell^2/L(t)^2, \tag{75}$$

we have the relatively innocuous initial value problem

$$\Phi'(t)=4M\beta+\frac{24\delta M\ell^2}{[\exp[\Phi(t)/2]+U(t)^{1/2}]^2}, \qquad \Phi(0)=0, \tag{76}$$

$$U'(t)=\Phi'(t)U(t)+8\ell^2\delta M, \qquad U(0)=0. \tag{77}$$

This autonomous system of differential equations has no equilibrium solutions, because $\Phi'(t)=0$ enforces $U'(t)=8\ell^2\delta M>0$. For $\beta>0$ it is clear that $\Phi(t)\sim 4M\beta t$, consistent with our general discussion. For $\beta<0$, if we are to have, with a slight abuse of notation, $U(t)\to U(\infty)$ and $\Phi'(t)\to\Phi'(\infty)<0$ [so that $\Phi(t)\to-\infty$], we require

$$U(\infty)\Big\{4M\beta+\frac{24\delta M\ell^2}{U(\infty)}\Big\}+8\ell^2\delta M=0.$$

This implies that $U(\infty)=8\ell^2\delta/|\beta|$, so we predict that

$$L(t)\to[|\beta|/(8\delta)]^{1/2} \quad \text{and} \quad \Phi'(t)\to-M|\beta| \quad \text{as } t\to\infty. \tag{78}$$

The prediction concerning $L(t)$ is consistent with Lemma 2.1, which gave $\liminf L(t)\leqslant[|\beta|/(2\delta)]^{1/2}$, now seen to be an overgenerous estimate by a factor of 2. The prediction for the limiting behaviour of $\Phi'(t)$ is consistent with Lemma 5.1, which established that $\Phi'(\infty)\in[-4M|\beta|,-M|\beta|]$.

Using the scalings (67) and (68) we have

$$\frac{d\Phi}{dT}=k+\frac{1}{[\exp[\Phi/2]+U^{1/2}]^2}, \qquad \Phi|_{T=0}=0; \tag{79}$$

$$\frac{dU}{dT}=kU+\frac{U}{[\exp[\Phi/2]+U^{1/2}]^2}+\frac{1}{3}, \qquad U|_{T=0}=0. \tag{80}$$

We expect to see

$$\lim_{T\to\infty}U=\frac{4}{3|k|} \quad \text{and} \quad \lim_{T\to\infty}\frac{d\Phi}{dT}=-\frac{|k|}{4} \quad \text{for } k<0, \tag{81}$$

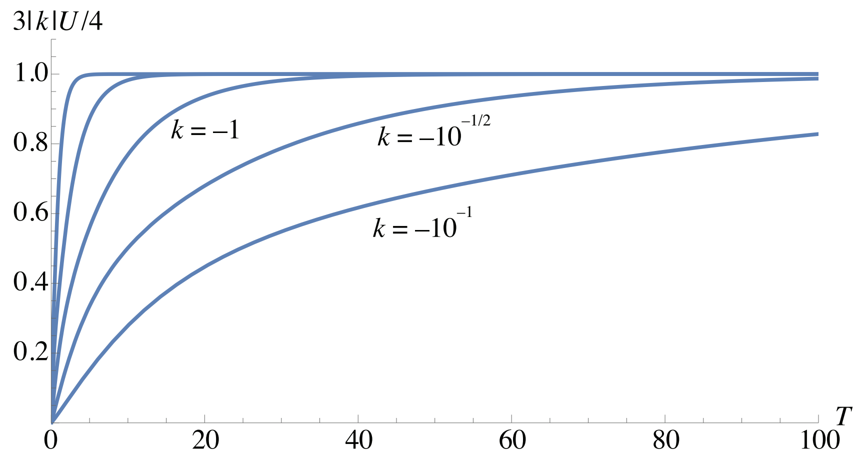

FIGURE 2. For the function $U(t)$ associated with our numerical study for the initial profile (72) of Section 8.1, we predict that $3|k|U(t)/4 \to 1$ as $t \to \infty$ when $k < 0$. Here we confirm this numerically for the cases $k \in \{-10, -10^{1/2}, -1, -10^{-1/2}, 10^{-1}\}$. The top curve is for $k = -10$ and the lowest for $k = -1/10$. The natural dimensionless timescale in the problem is $1/|k|$ and although not shown here, using $|k|T$ as the horizontal axis brings the five curves into near coincidence for $|k|T \geqslant 30$.

corresponding to the limits (78). The asymptotic limit for U when $k < 0$ is numerically confirmed and illustrated in Figure 2.

We recall that $\Phi'(t) = 4M\beta + 24M\delta\phi(t)$, which rescales into $d\Phi/dT = k + \phi(t)/\phi(0)$, and so

$$\frac{\phi(t)}{\phi(0)} = \frac{1}{\{\exp[\Phi(t)/2] + U(t)^{1/2}\}^2}. \tag{82}$$

It is therefore straightforward to determine $\phi(t)$ numerically. It is convenient to compare the numerical results for $L(t)$ (inferred from the computed $U(t)$) and $\phi(t)$ with our bounds (69) for $L(t)$ and (70) for $\phi(t)$, which are the results of Lemma 2.1 Lemma 4.1, respectively, expressed in scaled variables. To accomodate the comparisons in a single plot for a given value of k, we show the scaled root-mean square spread $[\phi(t)/\phi(0)]^{1/2}$ and the scaled maximum support radius $L(t)/[\phi(0)]^{1/2}$.

In Figure 3 we show results for the cases in which there is either no quadratic component to the potential ($k = 0$) or the quadratic component is attractive ($k > 0$). The bounds are always qualitatively similar to the computed curves, and the bound for $L(t)$ improves markedly as k increases. The strict monotonic decay to 0 of $\phi(t)$, and the convexity of $\phi(t)$, established for $\beta \geqslant 0$ in Lemma 4.1 and (61), respectively, are clearly seen in the computed curves for $[\phi(t)/\phi(0)]^{1/2}$.

In Figure 4 we show results for four cases in which the quadratic component is repulsive ($k < 0$). In addition to plotting the same four curves as in Figure 3, we show horizontal lines that mark the lower and upper bounds (71) for the possible distance ξ of equilibrium Diracs from the origin. In each case, we find that the upper bound for $L(t)$ appears to converge to the upper bound for ξ, while the computed $L(t)$ appears to converge to the lower bound for ξ, so the upper bound on $L(t)$ is much less informative than it was seen to be when $k \geqslant 0$. However, if we recall Remark 8.1, the numerical evidence that $L(t)/\phi(0)^{1/2} \to (3|k|)^{1/2}/2$ tells us that at equilibrium there is no Dirac mass at the origin.

The bound (70) is much more informative. It tells us that $[\phi(t)/\phi(0)]^{1/2}$ converges to the lower bound $(3|k|)^{1/2}/2$ on the distance of a Dirac component from the origin, as does the computed root-mean-square spread $\sqrt{\phi(t)}$. Note that for $k \leqslant -4/3$ the upper bound (70) for $[\phi(t)/\phi((0)]^{1/2}$ and the minimum Dirac distance from the origin coincide for all t, and the upper bound for $L(t)$ and for the equilibrium Dirac position fall outside the range displayed, so that the bottom two panels in Figure 4 each exhibit fewer curves than the upper two panels. The strict decay and convexity of $\phi(t)$ for $\beta < 0$ whenever $\phi(t) > |\beta|/(8\delta)$, and the trapping forever of $\phi(t)$ below $|\beta|/(8\delta)$ once $\phi(t)$ ever falls below $|\beta|/(8\delta)$ established in Lemma 4.1 and (61) are also manifest in the results in Figure 4.

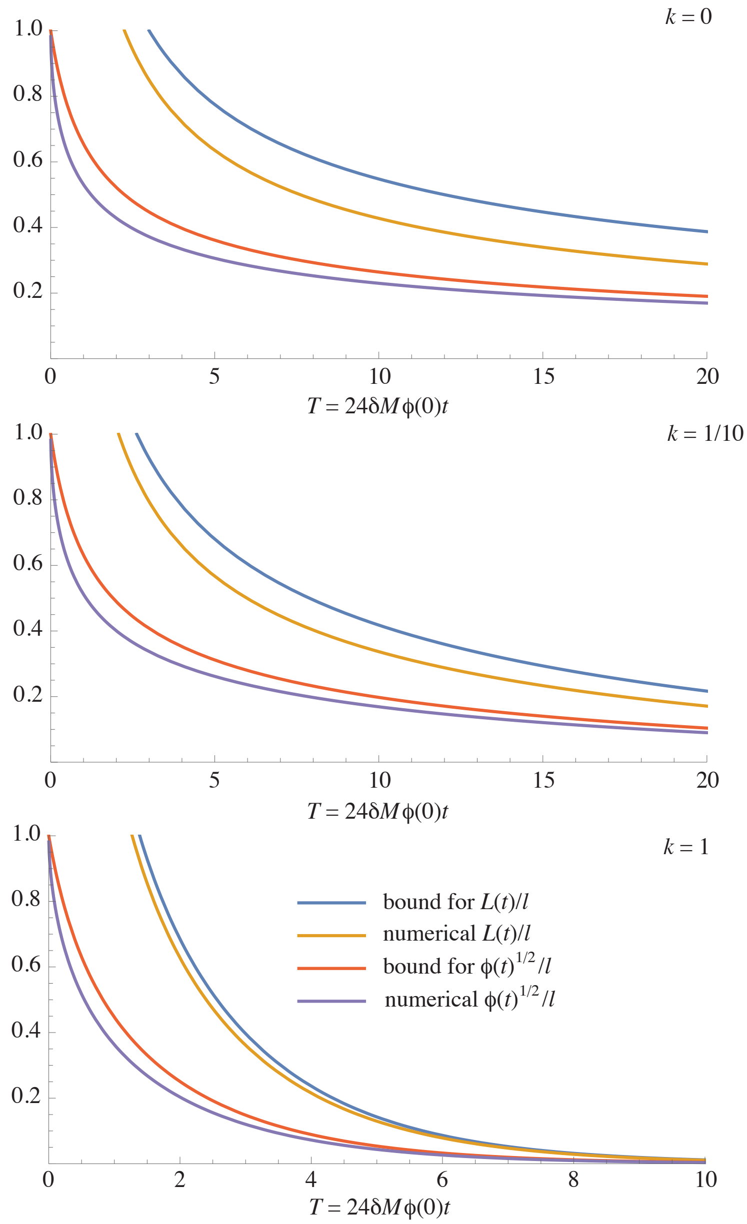

Figure 3. For the unbounded support initial distribution (84) when the quadratic term in the potential is non-negative (i.e., purely attractive potential), we show four curves, listed here ordered from top to bottom: the upper bound (69) for the maximum support radius $L(t)$, $L(t)$ itself, the upper bound (70) for the root-mean-square spread $\sqrt{\phi(t)}$, and $\sqrt{\phi(t)}$ itself. For $k > 1$ the top two curves become increasing difficult to distinguish as k increases, as do the lower two curves.

The time-dependent mass profile, given by Eq. (22), becomes for the present example

$$\frac{\rho(x,t)}{M/\ell} = \frac{2}{\pi} \frac{[1 - U(t)(x/\ell)^2]^{1/2}}{\left[1 + \{\exp[\Phi(t)] - U(t)\}(x/\ell)^2\right]^2} \exp\left[\frac{\Phi(t)}{2}\right], \tag{83}$$

where $\ell = \sqrt{\phi(0)}$. We show how the scaled mass profile (83) evolves with time for $k \in \{-1, 0, 1\}$ in Figure 5. For $k = -1$ we see strong evidence of the profile converging to a pair of delta functions placed symmetrically

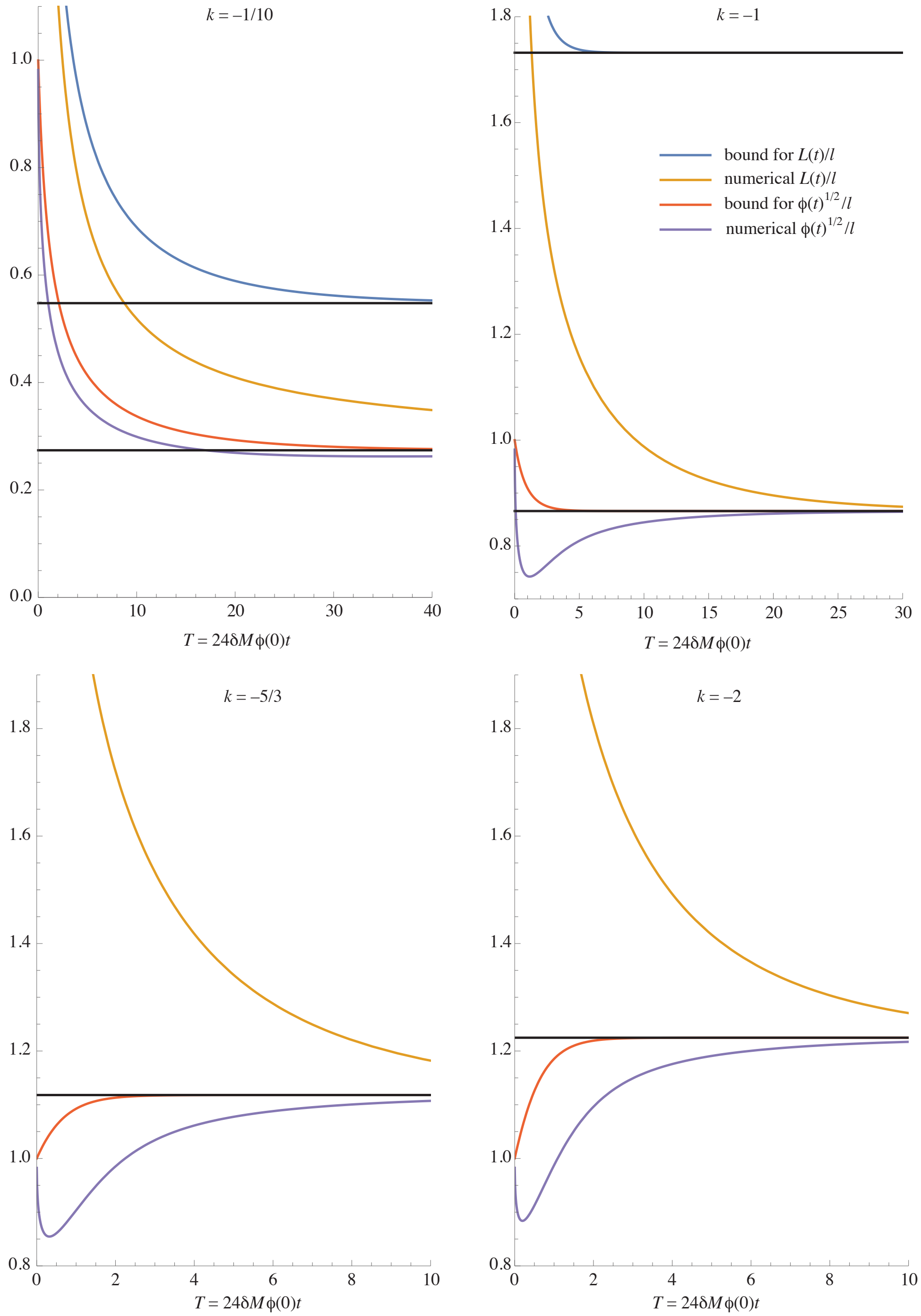

Figure 4. For the unbounded support initial distribution (84) with a repulsive quadratic potential component, we show in colour (ordered top to bottom) the upper bound (69) for the maximum support radius $L(t)$ (off scale for $k < -1$), $L(t)$ itself, the upper bound (70) for the root-mean-square spread $\sqrt{\phi(t)}$, and $\sqrt{\phi(t)}$ itself. We also show (black lines) bounds (71) on the allowed equilibrium Dirac distance ξ from the origin. For $k \leqslant -4/3$, the upper bound (70) and the lower bound for ξ coincide.

about the origin, and the curvature at the origin is consistent with the relevant prediction of Theorem 7.1 that $\liminf_{t\to\infty} \rho_{xx}(0,t) \geqslant 0$. We find from our numerical solutions that the time at which the curvature changes sign is $T \approx 2.38186$.

For $k = 0$ the change of curvature at the origin predicted by Theorem 7.1 is also evident, occurring at some scaled time $T \in (5, 15)$. From inequalities used in the proof of Theorem 7.1 and a little straightforward algebra, we can deduce that for the initial mass profile under discussion, the curvature sign change for $k = 0$ must occur for some $T \leqslant T^\dagger$, where $T^\dagger$ is the positive solution of the equation $T - 4[1 + 4T/3]^{3/4} = 0$. Numerical solution of the equation in Mathematica gives $T^\dagger \approx 609.059$, but the numerically determined time at which the curvature changes sign is much earlier ($T \approx 9.82507$), so the inequality arguments used in Section 7 are not particularly sharp. This probably reflects our overgenerous upper bounds for $L(t)$.

For $k = 1$ we see no change in the curvature at the origin for the times displayed and the collapse to a delta function is clear. Theorem 7.1 does imply that there exists $k^\dagger > 0$ such that for $0 < k < k^\dagger$ there will be a sign change of $\rho_{xx}(0,t)$ at large time. In the present example, by calculating the constant κ defined by Eq. (66), we are able to show that $k^\dagger \geqslant 0.00164$, but this turns out to be a very pessimistic estimate. In Figure 5(d) we show the numerically determined time T_0 at which curvature changes sign for $-1 \leqslant k \leqslant 0.1012$. The increase of T_0 with k is very rapid for $k > 0.1$ (for $k = 0.101781$, $T_0 \approx 1859.36$) and it would be natural to conjecture that $k^\dagger < \infty$ and that $T_0 \to \infty$ as $k \to k^\dagger$ from below, but we have not attempted to estimate the value of $k^\dagger$.

8.2. **Example with a compact initial support**

Consider the compactly supported uniform initial distribution

$$\rho(x,0) = \begin{cases} M/(2K_0) & \text{for } |x| < K_0 \\ 0 & \text{otherwise} \end{cases} \tag{84}$$

for which

$$\phi(0) = \frac{1}{MK_0}\int_{-\infty}^{\infty} x^2 \rho(x,0)dx = \frac{K_0^2}{3}. \tag{85}$$

In using the semi-analytical solution (22), we note that the finiteness of the initial support shows that $\rho(x,t) > 0$ if and only if

$$\frac{|x|\exp[\Phi(t)/2]}{\sqrt{1 - x^2/L(t)^2}} < K_0.$$

This condition can be rewritten as

$$|x| < \frac{1}{\sqrt{\exp[\Phi(t)]K_0^{-2} + L(t)^{-2}}} = K(t), \tag{86}$$

so that $K(0) = K_0$. The actual support $[-K(t), K(t)]$ of $\rho(x,t)$ at time $t > 0$ is always well inside the maximal possible support $[-L(t), L(t)]$. We find that

$$\rho(x,t) = \begin{cases} \dfrac{M}{2K_0}\exp\Big[\dfrac{\Phi(t)}{2}\Big]\Big[1 - \dfrac{x^2}{L(t)^2}\Big]^{-3/2} & \text{for } |x| < K(t), \\ 0 & \text{otherwise.} \end{cases} \tag{87}$$

Direct calculation of $\phi(t) = 2M^{-1}\displaystyle\int_0^\infty x^2\rho(x,t)dx$ using the substitution $x = K(t)\sin(\theta)$ yields

$$\phi(t) = L(t)^2\Big\{1 - \frac{\exp[\Phi(t)/2]L(t)}{K_0}\arctan\Big(\frac{K_0}{\exp[\Phi(t)/2]L(t)}\Big)\Big\} \tag{88}$$

$$= \frac{K_0^2\exp[-\Phi(t)]}{3} - \frac{K_0^4\exp[-2\Phi(t)]}{5L(t)^2} + O\Big(\frac{K_0^6\exp[-3\Phi(t)]}{L(t)^4}\Big) \text{ as } t \to 0^+. \tag{89}$$

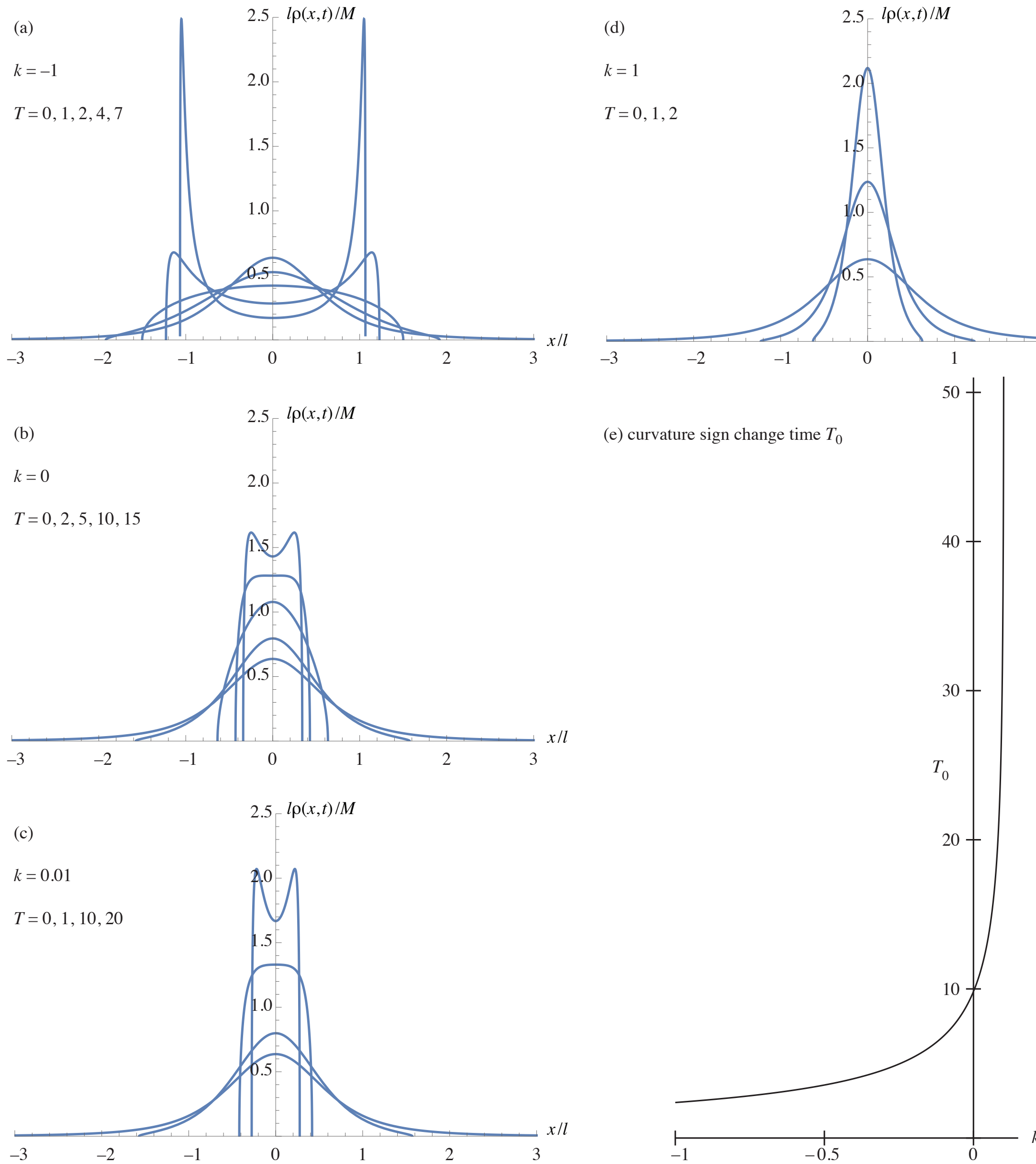

FIGURE 5. (a)–(d) The evolution of the scaled mass profile $\ell\rho(x,t)/M$ with time for $k \in \{-1, 0, , 0.01, 1\}$. For $k = 1$ numerical solutions do not show a change in change in the sign of the curvature at the origin for $T \leqslant 1000$, and if a sign change were to occur, the corresponding T would be enormous. (e) The time at which the curvature changes for $-10 \leqslant k \leqslant 0.1012$. The curve becomes increasing steep, and for $k = 0.101781$, the sign change occurs for $T \approx 1859.36$.

From the small-t asymptotic behaviour (17) of $L(t)$, noting that $\Phi'(0) = 4M[\beta + 6\delta\phi(0)]$, we find that

$$\phi(t) = \frac{K_0^2}{3}\Big\{1 - 4M\Big[\beta + \frac{16\delta K_0^2}{5}\Big]t + O(t^2)\Big\} \quad \text{as } t \to 0^+. \tag{90}$$

This means that there is initial growth of $\phi(t)$ if and only if $\beta < -(16/5)\delta K_0^2 = -(48/5)\delta\phi(0)$, corresponding to $k < -8/5$. In scaled terms, Lemma 6.1 tells us that $\phi'(0) < 0$ when $k > -4/3$ and that $\phi'(0) > 0$ when

$k < -2$, so that neither our general sufficient condition for initial decay of $\phi(t)$ nor our general initial conditions for initial growth of $\phi(t)$ from compact initial conditions in Lemma 6.1 is sharp.

Although we shall use the same scaled time T [Eq. (67)] and scaled quadratic coefficient k Eq. (67)] as we did in Section 8.1, it will be convenient to scale lengths against the initial support radius K_0 rather than against $\phi(0)^{1/2} = K_0/\sqrt{3}$. For this reason we write

$$U(t) = K_0^2 S(t) = \frac{K_0^2}{L(t)^2}, \quad k = \frac{\beta}{2\delta K_0^2}, \tag{91}$$

and our evolution equations (32) and (33) reduce to

$$\frac{d\Phi}{dT} = k + \frac{3}{U}\Big\{1 - \frac{\exp[\Phi/2]}{\sqrt{U}} \arctan(\sqrt{U}\exp[-\Phi/2])\Big\}, \tag{92}$$

$$\frac{dU}{dT} = \frac{d\Phi}{dT}U + 1. \tag{93}$$

Our bounds (25) on $L(t)$ obtained in Lemma 2.1 and our bounds (47) and (48) on the second moment $\phi(t)$ become

$$\frac{L(t)}{K_0} \leqslant \begin{cases} |k|^{1/2}[1-\exp(|k|T)]^{-1/2} & \text{if } k < 0 \\ T^{-1/2} & \text{if } k = 0, \\ k^{1/2}[\exp(kT)-1]^{-1/2} & \text{if } k > 0, \end{cases} \tag{94}$$

and

$$\frac{\phi(t)}{K_0^2} \leqslant \begin{cases} \dfrac{3|k|}{4} & \text{if } k \leqslant -\dfrac{4}{3} \\ \dfrac{3|k|}{4} + \dfrac{1-9|k|/4}{3+(4/|k|)[\exp(|k|T)-1]} & \text{if } -\dfrac{4}{3} < k < 0 \\ \dfrac{1}{3+4T} & \text{if } k = 0, \\ \dfrac{\exp(-kT)}{3+(4/k)[1-\exp(-kT)]} & \text{if } k > 0. \end{cases} \tag{95}$$

In terms of the scaling (91), the bounds (11) on possible equilibrium locations of Dirac masses not at the origin when $\beta < 0$ become $|k|^{1/2}/2 \leqslant \xi/K_0 \leqslant |k|^{1/2}$. We observe that for $k \leqslant -4/3$, the upper bound for $\phi(t)^{1/2}$ (the root-mean-square spread) and the lower bound for ξ take the common value $|k|^{1/2}K_0/2$.

We show bounds and values of the support radius and root-mean-square spread in Figure 6 ($k \geqslant 0$) and Figure 7 ($k < 0$). The colours assigned to curves and the order in which five curves appear are the same in both figures, but for $k < 0$ we also show the bounds on Dirac locations as horizontal lines. The universal upper bound on the maximum support radius $L(t)$, which is the inequality (94) in the present example, is shown as the uppermost (blue) curve in each figure, while the computed maximum support radius $L(t)$ is the second-highest (brown) curve and the actual radius support radius $K(t)$ is the third-highest (green) curve.

For the purely attractive case $k \geqslant 0$ (Figure 6), the quality of the bound (94) increases as k increases, while the bound (95) for $\phi(t)$ performs very well for all $k \geqslant 0$. For $k < 0$ (Figure 7) the bound (94) for $L(t)$ converges to the largest possible position of an equilibrium Dirac, and this becomes increasing less informative about the actual $L(t)$ obtained by numerical solution of the differential equations as $|k|$ increases. The upper bound (95) for $\phi(t)$ is always informative, and the convergence of $\sqrt{\phi(t)}$ to the lower bound $|k|^{1/2}K_0/2$ on positive Dirac mass positions as $t \to \infty$ is readily apparent.

We noted above that for this specific initial condition the change in the sign of $\phi'(0)$ as k varies occurs at $k = -8/5$. The plots of $\phi(t)$ in Figures 6 and 7 reflect this.

In Figure 8(a) we show the actual support radius $K(t)$ for $k \in \{-16, -9, -4, -1, 0, 1\}$ as a function of time. The horizontal asymptotes for the negative k cases (the top five curves) appear to be the minimum equilibrium Dirac position $\sqrt{|k|}/2$ (though for $k = -1$ and $k = -1/4$ extension of the plots to somewhat larger values of t, not shown here, is needed for definitive confirmation).

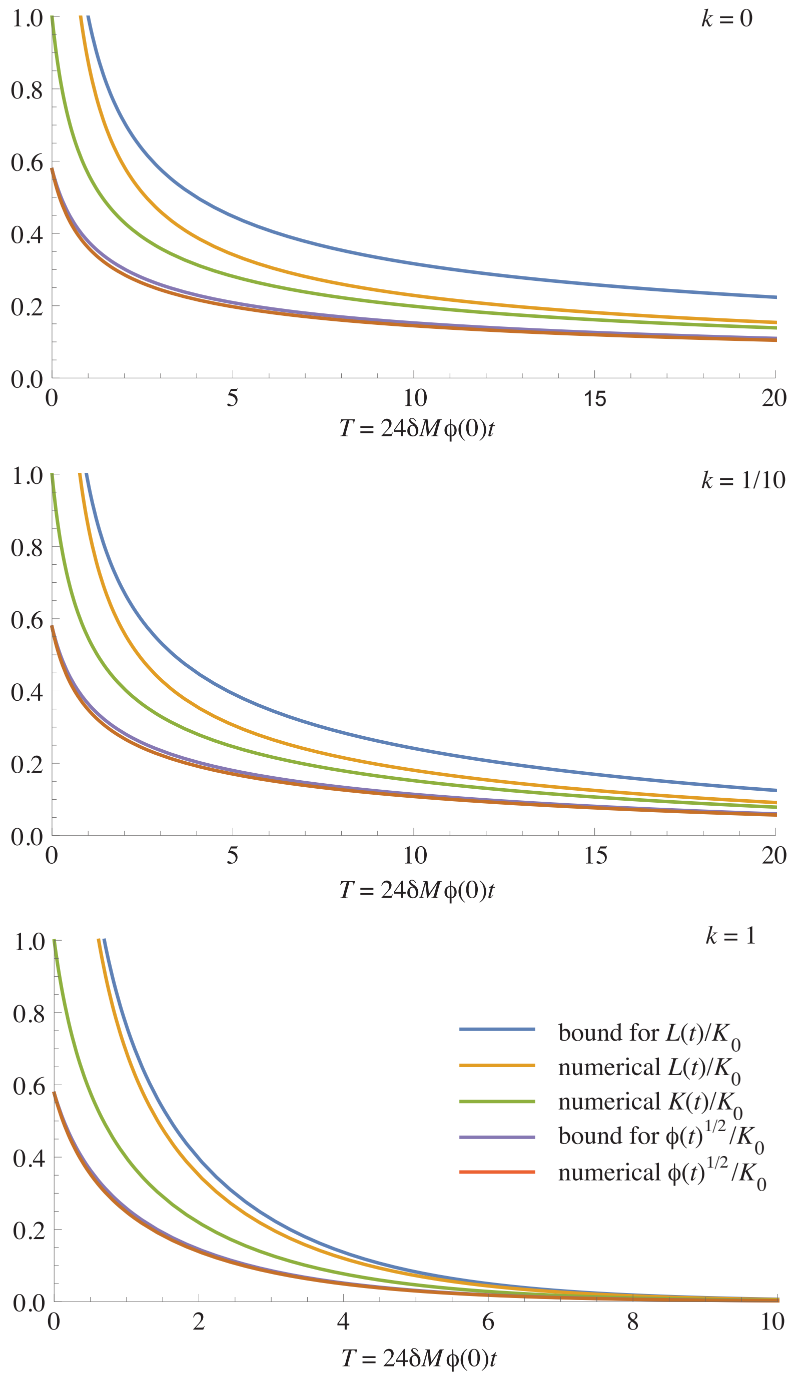

FIGURE 6. For the compact support initial distribution (84) when the quadratic term in the potential is non-negative (i.e., purely attractive potential), we show in colour, ordered from top to bottom: the upper bound (94) for the maximum support radius $L(t)$, $L(t)$ itself, the actual support radius $K(t)$, the upper bound (95) for the root-mean-square spread $\sqrt{\phi(t)}$, and $\sqrt{\phi(t)}$ itself. For $k > 1$ the top two curves become increasing difficult to distinguish as k increases, as do the lower two curves.

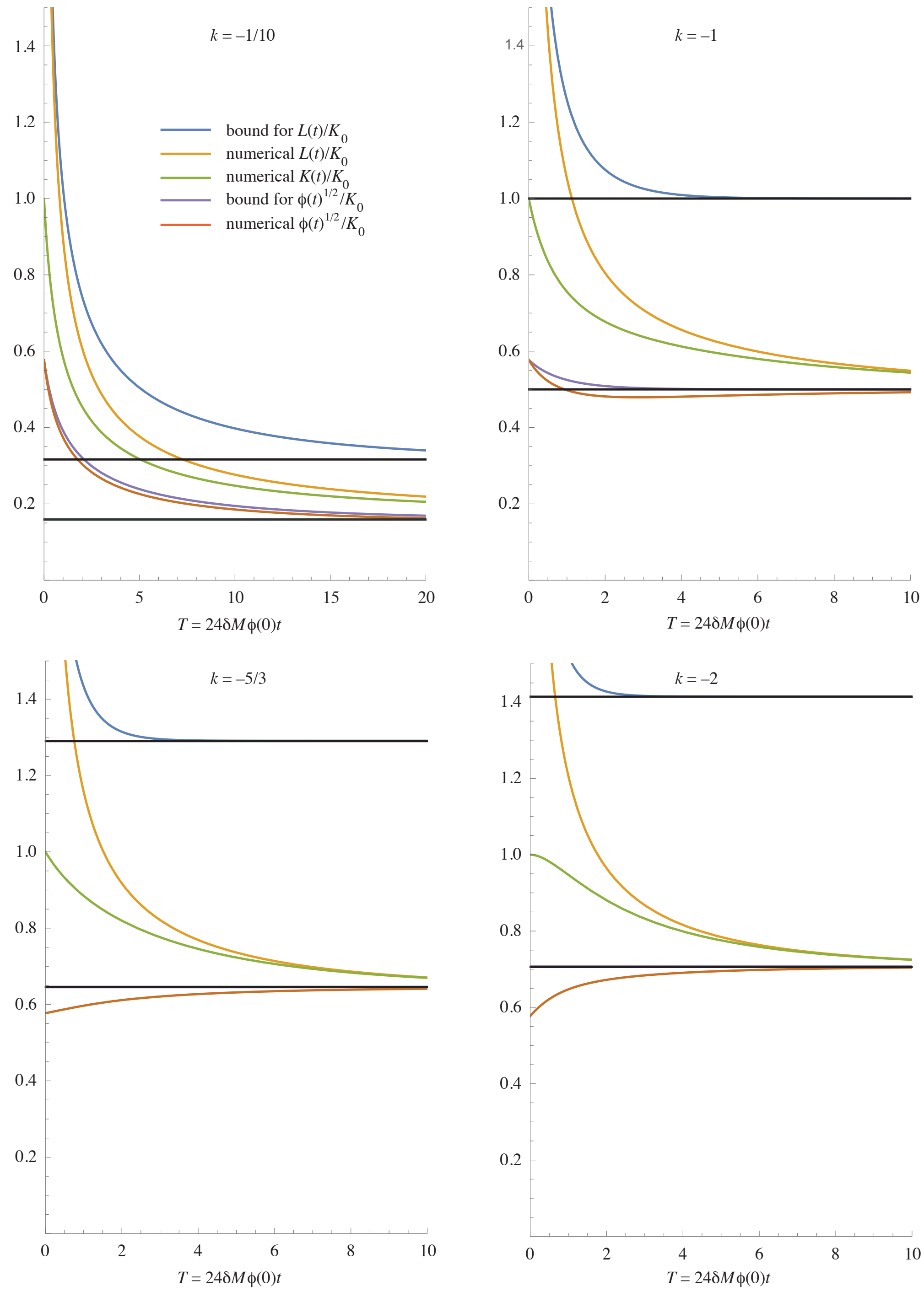

FIGURE 7. For the compact support initial distribution (84) in the case when the quadratic term in the potential is negative (a repulsive quadratic potential component), we show the following five curves, listed here ordered from top to bottom: the upper bound (94) for the maximum support radius $L(t)$, $L(t)$ itself, the actual support radius $K(t)$, the upper bound (95) for the root-mean-square spread $\sqrt{\phi(t)}$, and $\sqrt{\phi(t)}$ itself. We also show (black lines) bounds on the allowed equilibrium Dirac locations. For $k \leqslant -4/3$, the upper bound (70) and the lower bound for ξ coincide.

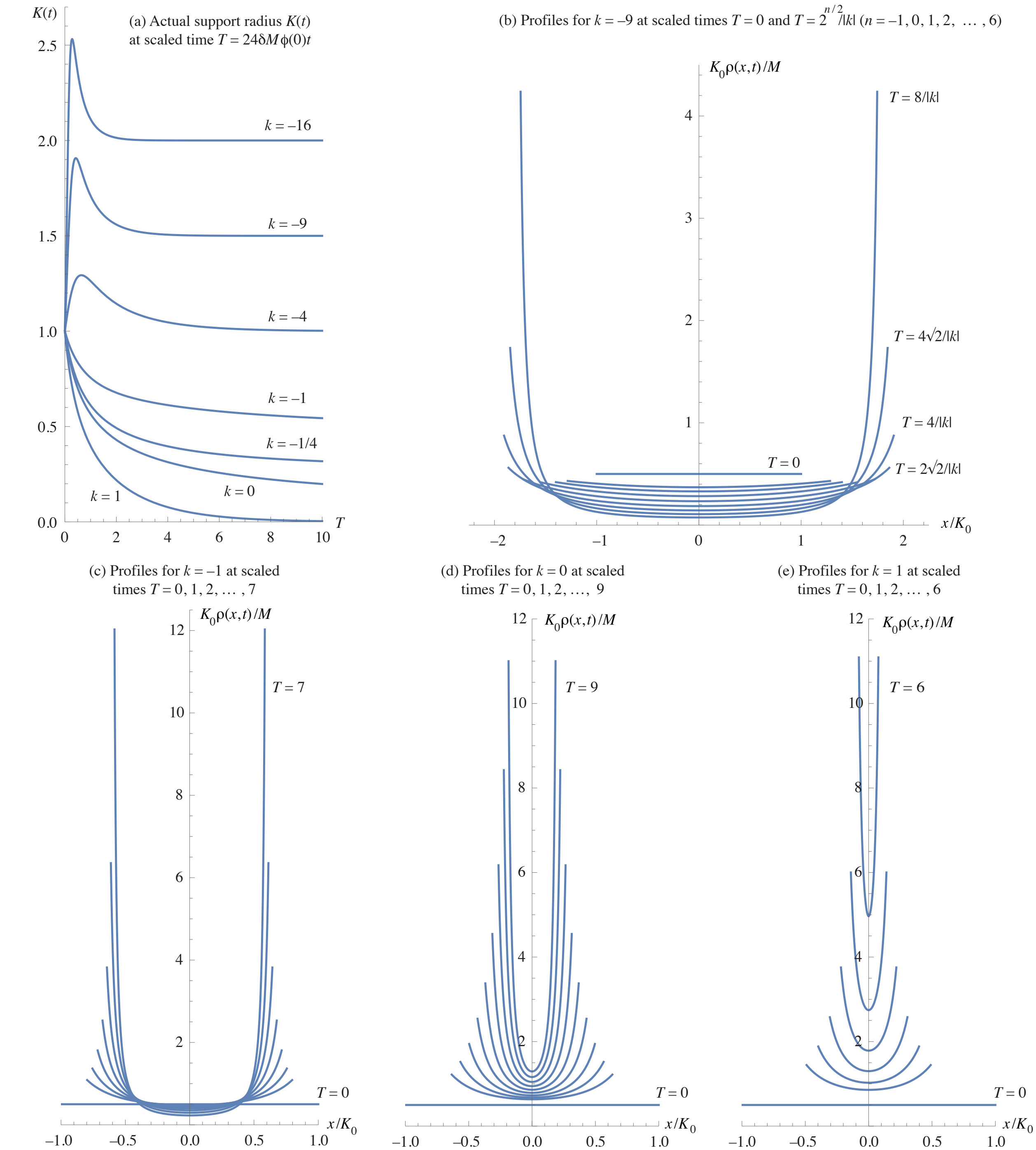

FIGURE 8. Evolution from the compact initial profile (84). (a) Actual support radius $K(t)$ for seven values of k: horizontal asymptotes for $k < 0$ cases (top five curves) are the minimum equilibrium Dirac position $\sqrt{|k|}/2$. (b)–(e) Scaled mass distribution $K_0\rho(x,t)/M$ as a function of scaled position x/K_0 at times shown.

9. Generalisation to $\psi \neq 0$

As mentioned in the Introduction, for asymmetric densities with $\psi \neq 0$, the positioning of equilibrium Diracs is given by the solutions of the cubic equation (8), that is,

$$x^3 + \Big[3\phi(\infty) + \frac{\beta}{2\delta}\Big]x - \psi(\infty) = 0. \tag{96}$$

Let x_i, $i = 1\ldots 3$ and ρ_i, $i = 1\ldots 3$ denote the positions and associated masses of an equilibrium of the form

$$\rho(x,\infty) = \sum_{i=1}^{3} \rho_i \boldsymbol{\delta}(x - x_i), \qquad \text{where} \quad M = \sum_{i=1}^{3} \rho_i. \tag{97}$$

Then, we have

$$M\phi(\infty) = \sum_{i=1}^{3} x_i^2 \rho_i, \qquad M\psi(\infty) = \sum_{i=1}^{3} x_i^3 \rho_i.$$

Evaluating (96) at $x = x_i$, multiplying by $x_i\rho_i$ and summing over $i = 1\ldots 3$ yields

$$\sum_{i=1}^{3} x_i^4 \rho_i + \Big[3\phi(\infty) + \frac{\beta}{2\delta}\Big]\phi(\infty) = \psi(\infty)\sum_{i=1}^{3} x_i\rho_i = 0, \tag{98}$$

since we have normalised the centre of mass to zero. Next, by using Jensen's inequality

$$\sum_{i=1}^{3} x_i^4 \frac{\rho_i}{M} \geq \left(\sum_{i=1}^{3} x_i^2 \frac{\rho_i}{M}\right)^2 = \phi(\infty)^2,$$

we estimate (98) as

$$\Big[4\phi(\infty) + \frac{\beta}{2\delta}\Big]\phi(\infty) \leq 0,$$

which implies $\phi(\infty) = 0$ for $\beta \geq 0$ and $0 \leq \phi(\infty) \leqslant |\beta|/(8\delta)$ for $\beta < 0$, which is the same interval of possible equilibrium values $\phi(\infty)$ as in (11) for symmetric densities ρ.

In the following, we show that not only the interval of equilibrium values $\phi(\infty)$, but all the statements of Theorem 1.1 hold equally for general densities $\rho(x,t)$ as stated in Corollary 1.1. For general $\rho(x,t)$, however, we do not have the explicit formulas via the method of characteristics like for $\phi(t)$ and $\phi'(t)$ in (27) and (41), (42) in the symmetric case. We will be able to show, however, that all estimates concerning $\phi(t)$ and $\phi'(t)$ remain correct since the corresponding formulas do not depend on the asymmetric part of $\rho(x,t)$.

We begin by deriving the evolution equations for the second-order moment $\phi(t)$, by multiplying equation (2) by x^2 and integrating over $\mathbb{R}$:

$$\begin{aligned} M\phi'(t) &= \int_{-\infty}^{\infty} x^2 \frac{\partial \rho(x,t)}{\partial t}\,dx = -2\int_{-\infty}^{\infty} x\rho(x,t)\Big[\int_{-\infty}^{\infty} W'(x-y)\rho(y,t)dy\Big]dx \\ &= -2\int_{-\infty}^{\infty} x\rho(x,t)\Big[2\beta M x + 4\delta M x^3 + 12\delta M\phi(t)x - 4\delta M\psi(t)\Big]dx \\ &= -\underbrace{4M^2\phi(t)[\beta + 6\delta\phi(t)]}_{=\Phi'(t)M\phi(t)} - 8\delta M^2\mu_4(t), \end{aligned} \tag{99}$$

which is identical to (44). Since (44) only involves $\phi(t)$ and $\Phi(t)$ all the entailed results of Sections 4 and 5 hold equally true for general densities $\rho(x,t)$. On the other hand, Eq. (99) also implies $\phi'(t) \leq \Phi'(t)\phi(t)$ and, therefore,

$$\phi(t) \leq \exp[-\Phi(t)]\phi(0),$$

which is inequality (35) (without having used the explicit formula as in (34)) and the starting point of the results of Section 3. Hence, all the statements of Sections 3, 4 and 5 remain true for general densities $\rho(x,t)$.

The results of Section 6, however, are changed by an additional term of the order $\psi^2(t)$, which appears in the second derivative of the second-order moment. Indeed, we calculate

$$\begin{aligned}M\mu_4'(t) &= \int_{-\infty}^{\infty} x^4 \frac{\partial\rho(x,t)}{\partial t}\,dx = -4\int_{-\infty}^{\infty} x^3\rho(x,t)\Big[\int_{-\infty}^{\infty} W'(x-y)\rho(y,t)dy\Big]dx \\ &= -4\int_{-\infty}^{\infty} x^3\rho(x,t)\Big[2\beta Mx + 4\delta Mx^3 + 12\delta M\phi(t)x - 4\delta M\psi(t)\Big]dx \\ &= -\underbrace{8M^2\mu_4(t)[\beta+6\delta\phi(t)]}_{=2\Phi'(t)M\mu_4(t)} - 16\delta M^2\mu_6(t) + 16\delta M^2\psi^2(t). \end{aligned} \tag{100}$$

Hence,

$$\phi''(t) = -\Phi''(t)\phi(t) - \Phi'(t)\phi'(t) + 16\delta M\Phi'(t)\mu_4(t) + 128\delta^2M^2\mu_6(t) - 128\delta^2M^2\psi^2(t), \tag{101}$$

and the last negative term proportional to $\psi^2(t)$ prevents an estimation like in Section 6 in order to show convexity of $\phi(t)$ when $\phi(t) > |\beta|/(8\delta)$.

Since we expect $\psi \to 0$ as $t \to \infty$, we might want to look into the evolution of $\psi(t)$:

$$\begin{aligned}M\psi'(t) &= \int_{-\infty}^{\infty} x^3 \frac{\partial\rho(x,t)}{\partial t}\,dx = -3\int_{-\infty}^{\infty} x^2\rho(x,t)\Big[2\beta Mx + 4\delta Mx^3 + 12\delta M\phi(t)x - 4\delta M\psi(t)\Big]dx \\ &= -\underbrace{6M^2\psi(t)[\beta+6\delta\phi(t)]}_{=\frac{3}{2}\Phi'(t)M\psi(t)} - 12\delta M^2\mu_5(t) + 12\delta M^2\psi(t)\phi(t). \end{aligned} \tag{102}$$

Any direct use of (102) seems unclear since the terms μ_5 and ψ have unknown sign in general and an estimate of the form $\mu_5(t) \geqslant \psi(t)\phi(t)$ is not even true on the set of non-symmetric two Dirac equilibria.

When investigating $\psi^2(t)$, we can multiply (102) with ψ and calculate

$$\frac{1}{2}\left(\psi^2\right)'(t) = -\frac{3}{2}\Phi'(t)\psi^2(t) - 12\delta M\left[\mu_5(t)\psi(t) + \psi^2(t)\phi(t)\right], \tag{103}$$

but again the behaviour of the right hand side term seems unclear. It already doesn't have a sign on the set of asymmetric two Dirac equilibria. Even if it would have a sign estimate (e.g. after some time) then (103) yields decay of $\psi^2(t)$ only when $\Phi'(t) > 0$, that is, for $\phi(t) > |\beta|/(6\delta)$, which is not enough to have decay down to the equilibrium values.

10. Discussion

We have shown how the practical solution of the problem of one-dimensional aggregation under potential $W(x) = \beta x^2 + \delta x^4$ with $\delta > 0$ with an arbitrary symmetric initial mass distribution $\rho(x,0)$ can be reduced to the numerical solution of a simple initial value problem for a pair of first-order ordinary differential equations. Moreover, these differential equations are amenable to rigorous analysis leading to a number of informative bounds on the evolution of the second moment of the mass distribution and the radius of its support and to sufficient conditions for a counterintuitive concavity change at the centre of mass.

Despite the apparent simplicity of the potential $W(x) = \beta x^2 + \delta x^4$ with $\delta > 0$, a number of questions for the associated aggregation problem remain unresolved. Although we have been able to determine sufficient conditions for the counter-intuitive convexity change of solution, our numerical solutions show that these conditions are unduly restrictive, and we do not have strong time-dependent results that exclude the formation of a Dirac component at the origin when $\beta < 0$. Although, as we have shown in Section 9, some of the results we have proved earlier in the paper under assumed symmetry of $\rho(x,0)$ survive relaxation of the requirement of symmetry, extending some of our other results to asymmetric cases would appear to be a challenging problem.

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