Exponential Decay toward Equilibrium via Entropy Methods for Reaction-Diffusion Equations

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Abstract

In this work, we show how the entropy method enables to get in an elementary way (and without linearization) estimates of exponential decay towards equilibrium for solutions of reaction-diffusion equations corresponding to a reversible reaction. Explicit rates of convergence combining the dissipative effects of diffusion and reaction are given.

Key words: Reaction-Diffusion, Entropy method, Exponential Decay

AMS subject classification: 35B40, 35K57

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1 Introduction

The entropy method for the study of the long-time asymptotics of a dissipative PDE consists in looking for a nonnegative Lyapounov functional $H \equiv H(f)$ and its nonnegative dissipation $D \equiv D(f)$ (i.e. functionals which satisfy

$$\frac{d}{dt} H(f(t)) = -D(f(t))$$

along the flow of the PDE), which are well-behaved in the following sense: first,

$$H(f) = 0 \iff f = f_\infty$$

for some equilibrium f_∞ (usually, such a result is true only when all the conserved quantities have been taken into account), and secondly,

$$D(f) \geq \Phi(H(f))$$

for some nonnegative function Φ such that $\Phi(x) = 0 \iff x = 0$.

If $\Phi'(0) \neq 0$, one usually gets exponential convergence toward f_∞ with a rate which can be explicitly estimated. This method, which is an alternative to the linearization around the equilibrium, has the advantage of being quite robust. This is due to the fact that it mainly relies on functional inequalities which have no direct link with the original PDE.

The entropy method has lately been used in many situations: nonlinear diffusion equations (such as fast diffusions [8, 7], equations of fourth order [3], Landau equation [9], etc.), integral equations (such as the spatially homogeneous Boltzmann equation [24, 25, 26]), or kinetic equations ([4], [10, 11], [13]).

We propose here to use the entropy method in the context of systems of reaction-diffusion equations. Several previous results on the long-time behavior of reaction-diffusion systems have been obtain by different (for instance, by linearization) methods (e.g. [5, 18, 1]).

In [5], exponential convergence to equilibrium for systems of reaction-diffusion equations (for which the solution trajectories remain in invariant domains) was shown provided that the diffusion term dominates over the reaction- (as well as convection-) terms. More precisely, the first non-zero eigenvalue of the diffusion term (with boundary conditions) multiplied by the minimal diffusion constant has to be bigger than the linearized effects of reaction (and convection) estimated within the invariant domain. The obtained convergence rate is then simply the difference of the two according values.

Entropy functionals were in previous works mostly used as a monotone Lyapounov functionals to prove, for instance, that the ω-limit set consists only of the steady state (see e.g. [21]).

Related to our work are [16, 17], where a lower bound of the entropy dissipation in terms of the entropy has been established, but in a non-constructive way, i.e. via a contradiction argument with no control on the constants.

Our aim is to provide quantitative exponential convergence to equilibrium with explicit rates and constants for reversible reaction processes in a bounded box $\Omega \subset \mathbb{R}^N$ ($N \geq 1$). More precisely, we consider a system of PDE's whose unknowns are $a_i \equiv a_i(t,x) \geq 0$, $i \in \{1, 2, \ldots, q\}$, where $t \geq 0$ and $x \in \Omega$. This system writes

$$\partial_t a_i - d_i \, \Delta_x a_i = (-1)^{s_i} (l \, a_1..a_p - k \, a_{p+1}..a_q) \tag{1}$$

with the homogeneous Neumann boundary condition $\nabla_x a_i \cdot n = 0$ (on $\partial\Omega$, with n the outward normal to Ω). Here, d_i are constant diffusion rates and $k > 0$, $l > 0$ are the assumed to be strictly positive reaction rates corresponding to a reversible reaction. Finally, $s_i = 1$ if $i = 1, .., p$ and $s_i = 0$ if $i = p + 1, .., q$.

Applications of systems like (1) have been stated to model reactions of chemical substances (see e.g. [21, 14] for the system (14)–(18) below and [15, 12, 27, 22] more generally).

In particular, we shall consider two typical situations. The first one corresponds to a system of two equations :

$$\partial_t a - d_a \triangle_x a = -\, a^2 + \, b, \tag{2}$$

$$\partial_t b - d_b \triangle_x b = \, a^2 - \, b. \tag{3}$$

They satisfy the homogeneous Neumann conditions

$$n(x) \cdot \nabla_x \, a = 0, \qquad n(x) \cdot \nabla_x \, b = 0 \qquad x \in \partial\Omega, \tag{4}$$

and the nonnegative initial condition

$$a(0,x) = a_0(x) \geq 0, \qquad b(0,x) = b_0(x) \geq 0. \tag{5}$$

We remark that compared to (1) and thanks to the rescaling $t \to \frac{1}{k} t$, $x \to |\Omega|^{\frac{1}{N}} x$, $(a,b) \to \frac{k}{l}(a,b)$, it is - without loss of generality - convenient to assume that

$$l = k = 1 \,, \qquad |\Omega| = 1 \,. \tag{6}$$

The flow of equations (2) – (5) conserves the total L^1 norm

$$M \equiv \int_\Omega (a(t,x)+b(t,x))\,dx = \int_\Omega (a_0(x)+b_0(x))\,dx\,, \tag{7}$$

which determines (at least formally) the unique equilibrium states (a_∞, b_∞) as the nonnegative constants satisfying $a_\infty + b_\infty = M$ and $a_\infty^2 = b_\infty$, i.e.

$$a_\infty = -\frac{1}{2} + \frac{1}{2}\sqrt{1+4M}\,, \qquad b_\infty = M - a_\infty\,. \tag{8}$$

Finally, we introduce the entropy functional associated to (2) – (5)

$$E(a,b) \equiv \int_\Omega \big(a\,(\ln(a^2)-2) + b\,(\ln b - 1)\big)\; dx \tag{9}$$

to state our main result for this system:

Theorem 1.1 *Let Ω be a bounded, connected, and regular open set of $\mathbb{R}^N$ ($N \geq 1$), and d_a, d_b be two strictly positive diffusivity constants. Let the initial data a_0, b_0 be two nonnegative functions of $L^\infty(\Omega)$ and denote $L_1 \equiv \|a_0\|_\infty + \|b_0\|_\infty$. Then, the unique nonnegative global solution $t \in \mathbb{R}_+ \mapsto (a(t), b(t))$ in $L^\infty(\Omega)$ to equations (2) – (6) obeys the following exponential decay toward equilibrium:*

$$\begin{aligned} &\|a(t,\cdot)-a_\infty\|^2_{L^1(\mathbb{R}^N)} + \frac{1}{2}\,\|b(t,\cdot)-b_\infty\|^2_{L^1(\mathbb{R}^N)} \\ &\leq \frac{(6+2\sqrt{2})M}{3+2\sqrt{2}}\,(E(a_0,b_0)-E(a_\infty,b_\infty))\,e^{-\frac{t}{K_1}\min\left\{4,\frac{8d_a}{P(\Omega)K_2}\right\}}, \end{aligned} \tag{10}$$

where $P(\Omega)$ is the Poincaré constant of Ω, and $K_1(L_1,M), K_2(M, d_a/d_b)$ are constants defined as follows: we introduce the function Φ (from $\mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{R}$) defined by

$$\Phi(x,y) = \frac{x\,(\ln(x)-\ln(y)) - (x-y)}{(\sqrt{x}-\sqrt{y})^2}. \tag{11}$$

Then

$$K_1(L_1,M) = \max\left\{\tfrac{2\Phi(L_1,a_\infty)}{a_\infty}, \Phi(L_1,b_\infty)\right\} = O(\ln(L_1)) \quad \textit{for large } L_1\,, \tag{12}$$

$$K_2(M, d_a/d_b) = \tfrac{d_a}{d_b}\sqrt{1+4M} + \sqrt{\tfrac{d_a^2}{d_b^2}(1+4M) + \tfrac{d_a}{d_b}(\sqrt{1+4M}-1)}\,. \tag{13}$$

The second situation we wish to investigate corresponds to a system of three equations:

$$\partial_t a - d_a \triangle_x a = -ab + c, \tag{14}$$

$$\partial_t b - d_b \triangle_x b = -ab + c, \tag{15}$$

$$\partial_t c - d_c \triangle_x c = ab - c, \tag{16}$$

with a, b, c satisfying homogeneous Neumann conditions

$$n(x)\cdot\nabla_x a = 0, \qquad n(x)\cdot\nabla_x b = 0, \qquad n(x)\cdot\nabla_x c = 0 \qquad x \in \partial\Omega, \tag{17}$$

and the nonnegative initial condition

$$a(0,x) = a_0(x) \geq 0, \qquad b(0,x) = b_0(x) \geq 0, \qquad c(0,x) = c_0(x) \geq 0\,. \tag{18}$$

As above, due to the rescaling $t \to \frac{1}{k}t$, $x \to |\Omega|^{\frac{1}{N}}x$, $(a,b,c) \to \frac{k}{l}(a,b,c)$, it means no restriction for (14) – (16) to assume that

$$l = k = 1\,, \qquad |\Omega| = 1\,. \tag{19}$$

The following conservation laws hold for solutions of (14) – (18):

$$M_1 \equiv \int_\Omega (a(t,x) + c(t,x))\,dx = \int_\Omega (a_0(x) + c_0(x))\,dx\,, \tag{20}$$

$$M_2 \equiv \int_\Omega (b(t,x) + c(t,x))\,dx = \int_\Omega (b_0(x) + c_0(x))\,dx\,, \tag{21}$$

characterizing the unique equilibrium $(a_\infty, b_\infty, c_\infty)$ as the unique nonnegative constants satisfying $a_\infty + c_\infty = M_1$, $b_\infty + c_\infty = M_2$, and $a_\infty\, b_\infty = c_\infty$, i.e.

$$\begin{gathered} c_\infty = \tfrac{1}{2}(1 + M_1 + M_2) - \tfrac{1}{2}\sqrt{(1 + M_1 + M_2)^2 - 4M_1M_2}\,, \\ a_\infty = M_1 - c_\infty\,, \qquad b_\infty = M_2 - c_\infty\,. \end{gathered} \tag{22}$$

Introducing the entropy functional associated to (14) – (18)

$$E(a,b,c) \equiv \int_\Omega (a(\ln(a) - 1) + b(\ln(b) - 1) + c(\ln(c) - 1))\,dx\,, \tag{23}$$

our main theorem in this case writes:

Theorem 1.2 *Let Ω be a bounded, connected, and regular (C^3 if $N > 5$) open set of $\mathbb{R}^N$ ($N \geq 1$), and d_a, d_b, d_c be three strictly positive diffusivity constants. Let the initial data a_0, b_0, c_0 be three nonnegative functions of $L^\infty(\Omega)$ (if $N > 5$, we suppose moreover that a_0, b_0, c_0 are $C^3(\overline{\Omega})$). Then, the unique nonnegative global solution $t \in \mathbb{R}_+ \mapsto (a(t), b(t), c(t))$ in $L^\infty(\Omega)$ to equations (14) – (19) satisfies the following estimate of exponential decay toward equilibrium:*

$$\begin{gathered} \tfrac{1}{2M_1}\|a(t,\cdot) - a_\infty\|_{L^1(\mathbb{R}^N)} + \tfrac{1}{2M_2}\|b(t,\cdot) - b_\infty\|_{L^1(\mathbb{R}^N)} \\ + \tfrac{1}{M_1+M_2}\|c(t,\cdot) - c_\infty\|_{L^1(\mathbb{R}^N)} \leq \tfrac{9+2\sqrt{2}}{3+2\sqrt{2}}(E(a_0,b_0,c_0) - E(a_\infty,b_\infty,c_\infty))\,e^{-K_1 t}\,, \end{gathered} \tag{24}$$

with

$$K_1 = \frac{1}{K_2}\min\left\{4, \frac{4d_a}{P(\Omega)(\frac{b_\infty}{4}+K_3)}, \frac{4d_b}{P(\Omega)(\frac{a_\infty}{4}+K_4)}, \frac{4d_c}{P(\Omega)(2+K_5)}\right\}, \quad (25)$$

where $P(\Omega)$ is the Poincaré constant of Ω, and K_1,..,K_5 are constants (depending only on d_a, d_b, d_c, and M_1, M_2, and the global L^∞ bound L_2 (see (41) below)), whose complicated expressions are given in (45) and (47) – (49).

Notations: In the formulas for K_1,..,K_5 as well as in all the following, we introduce capital letters as a short notation for square roots of lower case concentrations

$$A \equiv \sqrt{a}, \quad A_\infty \equiv \sqrt{a_\infty}, \quad B \equiv \sqrt{b}, \quad B_\infty \equiv \sqrt{b_\infty}, \quad C \equiv \sqrt{c}, \quad C_\infty \equiv \sqrt{c_\infty},$$

and overlines for spatial averaging (remember that $|\Omega| = 1$): $\overline{A} = \int_\Omega A\,dx, \ldots$ Despite we prefer different letters for different unknowns, there are some points where an index notation is more convenient: $a_1 \equiv a$, $a_2 \equiv b$, $a_3 \equiv c$. There will be no confusion with K_i with i integer denoting various constants. Moreover, we denote $\|f\|_2^2 = \int_\Omega f^2\,dx$ for a given function $f : \Omega \to \mathbb{R}$.

Outline: In section 2, we prove theorem 1.1 and make some remarks. Next, in section 3, we state the proof of theorem 1.2.

2 The case of two equations

We begin with an elementary lemma that will be useful in sections 2 and 3:

Lemma 2.1 *We consider the function Φ defined by (11). Then, Φ is continuous on $]0,+\infty[^2$. For all $y > 0$, $\Phi(\cdot, y)$ is strictly increasing on $]0,+\infty[$, and satisfies $\lim_{x\to 0}\Phi(x,y) = 1$, $\Phi(y,y) = 2$, and $\Phi(x,y) \sim \ln x$ for $x \to \infty$. Finally, for all $x > 0$, $\Phi(x,\cdot)$ is strictly decreasing.*

Proof of the lemma 2.1: We notice that $\partial_x\Phi(x,y) > 0$ if and only if

$$1 > \ln\left(\frac{x}{y}\right)\left(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}}\right)^{-1}. \quad (26)$$

Then, remembering that $\ln a < \sqrt{a} - \frac{1}{\sqrt{a}}$ for $a > 1$, we see that $\partial_x\Phi(x,y) > 0$ for all $x \in]0,+\infty[-\{y\}$. Similarly, we notice that $\partial_y\Phi(x,y) < 0$ if and only if (26) holds and therefore $\partial_y\Phi(x,y) < 0$ for all $y \in]0,+\infty[-\{x\}$. $\square$

Before we start to prove the theorem, we note that the system (2) – (5) has a unique solution such that

$$0 \le a(t), b(t) \le L_1 \equiv \|a_0\|_\infty + \|b_0\|_\infty \qquad \text{for} \quad t \ge 0\,, \tag{27}$$

as can be shown by a direct application of the maximum principle or by comparison with the diffusionless system (see e.g. [19, 2]).

Proof of theorem 1.1: We recall the entropy for equation (2) – (5)

$$E(a,b) \equiv \int_\Omega \big(a\,(\ln(a^2) - 2) + b\,(\ln(b) - 1)\big)\,dx\,,$$

and introduce the entropy dissipation

$$D(a,b) = 2d_a \int_\Omega \frac{|\nabla_x a|^2}{a}\,dx + d_b \int_\Omega \frac{|\nabla_x b|^2}{b}\,dx + \int_\Omega (a^2 - b)\ln\frac{a^2}{b}\,dx. \tag{28}$$

It is clear that (for nonnegative functions a, b such that identity $\int_\Omega (a+b) = M$ holds) $D(a,b) = 0$ if and only if $(a,b) = (a_\infty, b_\infty)$. In the following, we prove a quantitative lower bound of the entropy dissipation in terms of the relative entropy with respect to the equilibrium - called sometimes the entropy/entropy-dissipation estimate. Note that this estimate is valid for functions which may have nothing to do with the solutions of eq. (2) – (6).

Lemma 2.2 *For all (measurable) functions $a, b : \Omega \to \mathbb{R}$, which satisfy that $0 \le a \le L_1$, $0 \le b \le L_1$, and $\int_\Omega (a+b) = M$,*

$$D(a,b) \ge \frac{1}{K_1}\min\left\{4, \frac{8d_a}{P(\Omega)K_2}\right\}(E(a,b) - E(a_\infty, b_\infty)), \tag{29}$$

where $P(\Omega)$ is the Poincaré constant of Ω, a_∞, b_∞ are given by (8), and the explicit constants $K_1(L_1, M)$, $K_2(M, d_a/d_b)$ are defined by the formulas (12) and (13).

Proof of lemma 2.2: Recalling the notation $A = \sqrt{a}$, we start with the identity $|\nabla_x a|^2/a = 4\,|\nabla_x A|^2$, and apply Poincaré's inequality. Using then the inequality $(a-b)(\ln(a) - \ln(b)) \ge 4(A-B)^2$, we get

$$D(a,b) \ge 4\left\|A^2 - B\right\|_2^2 + \frac{8d_a}{P(\Omega)}\left\|A - \overline{A}\right\|_2^2 + \frac{4d_b}{P(\Omega)}\left\|B - \overline{B}\right\|_2^2. \tag{30}$$

We shall prove in the sequel that the r.h.s. of (30) is bounded below by (some constant times) the relative entropy $E(a,b) - E(a_\infty, b_\infty)$.

Firstly, we use the conservation law (7) to rewrite the relative entropy as

$$E(a,b) - E(a_\infty, b_\infty) = \int_\Omega \Bigg(a \ln\left(\frac{a^2}{a_\infty^2}\right) - 2\,(a - a_\infty) + b \ln\left(\frac{b}{b_\infty}\right) - (b - b_\infty) \Bigg)\, dx\,,$$

and use lemma 2.1 as well as the global bound (27) to obtain

$$E(a,b) - E(a_\infty, b_\infty) \le K_1(L_1, M) \left(A_\infty^2 \, \|A - A_\infty\|_2^2 + \|B - B_\infty\|_2^2 \right), \quad (31)$$

with $K_1(L_1, M)$ given in (12).

Defining now (for some $\gamma > 0$)

$$K_2(\gamma) = A_\infty \gamma, \qquad K_3(\gamma) = 2B_\infty + 1 + \frac{A_\infty}{\gamma}\,, \quad (32)$$

we prove that the quantity Γ defined below is nonnegative:

$$0 \le \Gamma \equiv \|A(A - A_\infty)\|_2^2 + 2A_\infty \int_\Omega A(A - A_\infty)^2\, dx + K_2\, \|A - \overline{A}\|_2^2 + K_3\, \|B - \overline{B}\|_2^2 + 2 \underbrace{\int_\Omega (A^2 - A_\infty^2)(B_\infty - B)\, dx}_{*}\,. \quad (33)$$

Note that in (33) only $*$ may be nonpositive. We distinguish three cases:

1. We suppose that $B_\infty - \overline{B} > 0$ and $A_\infty - \overline{A} > 0$. Then, the conservation law (7), i.e. $\int_\Omega (A^2 - A_\infty^2)\, dx = \int_\Omega (B_\infty^2 - B^2)\, dx$ yields

$$* = (B_\infty - \overline{B})^2 \int_\Omega (B_\infty + B)\, dx - (B_\infty - \overline{B}) \|B - \overline{B}\|_2^2 \quad (34)$$
$$+ \int_\Omega \left(A(A - A_\infty)(\overline{B} - B) + A_\infty (A - \overline{A})(\overline{B} - B) \right)\, dx$$
$$\ge -B_\infty \|B - \overline{B}\|_2^2 - \frac{1}{2} \|A(A - A_\infty)\|_2^2 - \frac{1}{2} \|B - \overline{B}\|_2^2 \quad (35)$$
$$- \frac{A_\infty \gamma}{2} \|A - \overline{A}\|_2^2 - \frac{A_\infty}{2\gamma} \|B - \overline{B}\|_2^2\,,$$

thanks to Young's inequality (and for all $\gamma > 0$). By comparing (35) with (33), we obtain the constants (32).

2. We now suppose that $B_\infty - \overline{B} > 0$ and $A_\infty - \overline{A} < 0$. We observe that

$$\begin{aligned}(B_\infty - \overline{B})^2 \textstyle\int_\Omega (B_\infty + B)\, dx - (B_\infty - \overline{B})\|B - \overline{B}\|_2^2 \\ = (B_\infty - \overline{B}) \textstyle\int_\Omega (A^2 - A_\infty^2)\, dx \\ = (B_\infty - \overline{B}) \left(\|A - \overline{A}\|_2^2 + (\overline{A} - A_\infty)(\overline{A} + A_\infty)\right) \geq 0\,.\end{aligned}$$

As a consequence, according to (34),

$$* \;\geq\; \int_\Omega \left(A(A - A_\infty)(\overline{B} - B) + A_\infty(A - \overline{A})(\overline{B} - B)\right) dx \quad (36)$$

and (35) still holds.

3. Finally, if $B_\infty - \overline{B} < 0$, then $A_\infty - \overline{A} > 0$ because of (7) and the line (34) is obviously nonnegative (as in the second case), so that (35) holds again.

Next, using (33) and (31), we observe that

$$\begin{aligned}&\frac{1}{K_1}\left(E(a,b) - E(a_\infty, b_\infty)\right) \leq A_\infty^2\|A - A_\infty\|_2^2 + \|B - B_\infty\|_2^2 + \Gamma \\ &= \|A^2 - A_\infty^2 + B_\infty - B\|_2^2 + K_2\|A - \overline{A}\|_2^2 + K_3\|B - \overline{B}\|_2^2\,, \qquad (37)\end{aligned}$$

and recall that $A_\infty^2 = B_\infty$. To conclude the proof of the lemma, it remains to compare (37) with (30), which gives (29) after choosing γ in order to set the fraction $K_2/K_3 = 2d_a/d_b$ according to (30), i.e. by taking

$$\gamma = \frac{2d_a}{d_b}\left(A_\infty + \frac{1}{2A_\infty}\right) + \sqrt{\frac{4d_a^2}{d_b^2}\left(A_\infty + \frac{1}{2A_\infty}\right)^2 + \frac{2d_a}{d_b}}\,, \qquad (38)$$

so that (13) follows (32) and $a_\infty = -\frac{1}{2} + \frac{1}{2}\sqrt{1+4M}$ in (8). □

We now turn to another lemma, which plays here the same role as the Cziszar-Kullback-Pinsker inequality ([6] and [20]) in information theory. That is, we show that the relative entropy $E(a,b) - E(a_\infty, b_\infty)$ controls (from above) the squares of the L^1-distances to the equilibrium.

Lemma 2.3 *For all (measurable) functions $a, b : \Omega \to \mathbb{R}$ such that $0 \leq a$, $0 \leq b$ and $\int_\Omega (a+b) = M$,*

$$E(a,b) - E(a_\infty, b_\infty) \geq \frac{3 + 2\sqrt{2}}{(6 + 2\sqrt{2})M}\left(\|a - a_\infty\|_1^2 + \frac{1}{2}\,\|b - b_\infty\|_1^2\right), \qquad (39)$$

where a_∞ and b_∞ are defined by (8).

Proof of lemma 2.3: Recalling $a_1 \equiv a$ and $a_2 \equiv b$ as well as $\overline{a_i} = \int_\Omega a_i\,dx$, we define $q(a_i) \equiv a_i \ln a_i - a_i$ to write

$$\begin{aligned} E(a,b) - E(a_\infty, b_\infty) &= 2\int_\Omega a\,\ln(\frac{a}{\overline{a}})\,dx + \int_\Omega b\,\ln(\frac{b}{\overline{b}})\,dx \\ &\quad + 2\,(q(\overline{a}) - q(a_\infty)) + (q(\overline{b}) - q(b_\infty))\,. \end{aligned}$$

We first note that thanks to the Cziszar-Kullback-Pinsker inequality,

$$\int_\Omega a\,\ln(\frac{a}{\overline{a}})\,dx \geq \frac{1}{2\overline{a}}\,\|a-\overline{a}\|_1^2, \qquad \int_\Omega b\,\ln(\frac{b}{\overline{b}})\,dx \geq \frac{1}{2\overline{b}}\,\|b-\overline{b}\|_1^2\,,$$

and moreover $\overline{a_i} \leq M$ by the conservation of mass (7). Then, we consider $Q(\overline{a_i}) \equiv 2\,q(\overline{a_i}) + q(M - \overline{a_i})$ for both $a_i \in [0, M]$. Since

$$Q''(\overline{a_i}) = \frac{2}{\overline{a_i}} + \frac{1}{M - \overline{a_i}} \geq \frac{3+2\sqrt{2}}{M},$$

and

$$\frac{3+2\sqrt{2}}{M} \leq \frac{1}{\overline{a_i}} + \frac{2}{M - \overline{a_i}} = Q''(M - \overline{a_i})\,,$$

we estimate

$$2\,(q(\overline{a}) - q(a_\infty)) + q(\overline{b}) - q(b_\infty) \geq \frac{3+2\sqrt{2}}{3M}\,|\overline{a} - a_\infty|^2 + \frac{3+2\sqrt{2}}{6M}\,|\overline{b} - b_\infty|^2\,.$$

Finally, we conclude the proof of the lemma by observing that

$$\|a - a_\infty\|_1^2 \leq \frac{6+2\sqrt{2}}{3+2\sqrt{2}}\left(\|a-\overline{a}\|_1^2 + \frac{3+2\sqrt{2}}{3}\,|\overline{a} - a_\infty|^2\right),$$

and

$$\|b - b_\infty\|_1^2 \leq \frac{6+2\sqrt{2}}{3+2\sqrt{2}}\left(\|b-\overline{b}\|_1^2 + \frac{3+2\sqrt{2}}{3}\,|\overline{b} - b_\infty|^2\right).$$

□

End of the proof of theorem 1.1: We observe that

$$\frac{d}{dt}(E(a(t),b(t)) - E(a_\infty,b_\infty)) = -\,D(a(t),b(t))\,.$$

Using lemma 2.2 and Gronwall's lemma, we see that

$$E(a(t),b(t)) - E(a_\infty,b_\infty) \leq (E(a_0,b_0) - E(a_\infty,b_\infty))\,e^{-\frac{t}{K_1}\min(4,\frac{8d_a}{P(\Omega)K_2})}, \quad (40)$$

and we obtain theorem 1.1 by combining lemma 2.3 and estimate (40). ■

Remark 2.1 ***(Decay rate)***
The result of theorem 1.1 express, up to our knowledge, the first explicit convergence to equilibrium rates for reaction-diffusion systems. The rate $1/K_1 \min\{4, 8d_a/P(\Omega)K_2\}$ *obtained in lemma 2.2 via the entropy method reflects the combined dissipative effects of reaction (i.e.* 4 *due to the rescaling (6)) and the diffusion (i.e.* $8d_a/P(\Omega)K_2$*).*

This is an improvement compared to classical linearization results like [5], where the diffusion term had to dominate over the reaction, which was estimated like a perturbation within a invariant region.

Nevertheless, the obtained rate is not sharp (which is obvious, for instance, in the estimate of case 1 in lemma 2.2).

Remark 2.2 ***(Example)***
We give a numerical example of the rate of exponential decay in theorem 1.1, in order to show that the rates obtained by our method are of order 1 when the data also are of order 1. For $L = 3M$, $M = 2$, $a_\infty = 1 = b_\infty$, $d_a = d_b$, *we get*

$$\approx \min\left\{0.731; 0.231\frac{d_a}{P(\Omega)}\right\}.$$

3 The case of three equations

Proof of theorem 1.2: Under the assumptions of theorem 1.2, the system (14), (15), (16) with boundary condition (17) and initial data (18) has a unique nonnegative globally bound solution (see [21] for dimension $d \leq 5$ and [14] in all dimensions under the additional assumptions of $C^{2+\alpha}$-boundaries ($0 < \alpha < 1$) and correspondingly smooth initial data (18)). We denote by L_2 the global bound for this system :

$$L_2 \equiv \sup_{t\geq 0} \{\|a(t,\cdot)\|_\infty, \|b(t,\cdot)\|_\infty, \|c(t,\cdot)\|_\infty\} < \infty\,. \tag{41}$$

We recall the entropy functional $E(a, b, c)$ associated to (14) – (19)

$$E(a, b, c) \equiv \int_\Omega (a\,(\ln a - 1) + b\,(\ln b - 1) + c\,(\ln c - 1))\,dx\,,$$

and introduce the corresponding entropy dissipation

$$\begin{aligned} D(a, b, c) \;&=\; d_a \int_\Omega \frac{|\nabla_x a|^2}{a}\,dx + d_b \int_\Omega \frac{|\nabla_x b|^2}{b}\,dx + d_c \int_\Omega \frac{|\nabla_x c|^2}{c}\,dx \\ &\quad + \int_\Omega (ab - c)(\ln(ab) - \ln c)\,dx\,. \end{aligned}$$

Note that $D(a,b,c) = 0$ if and only if $(a,b,c) = (a_\infty, b_\infty, c_\infty)$ (provided that the conservation laws (20) and (21) hold).

We now state the entropy/entropy-dissipation lemma for our model. Note once again that this lemma applies for functions which are not necessarily solutions of system (14) – (19).

Lemma 3.1 *Let a, b, c be (measurable) functions from Ω to $\mathbb{R}$ such that $0 \le a \le L_2$, $0 \le b \le L_2$, $0 \le c \le L_2$ and $\int_\Omega a + c = M_1$, $\int_\Omega b + c = M_2$. Then,*

$$D(a,b,c) \ge K_1(E(a,b,c) - E(a_\infty, b_\infty, c_\infty)) \tag{42}$$

with K_1 defined by (25) (and (45), (47) – (49)), and a_∞, b_∞, c_∞ defined by (22).

Proof of lemma 3.1: Let still square roots be denoted by capital letters $A = \sqrt{a}$, $B = \sqrt{b}$, $C = \sqrt{c}$. Using Poincaré's inequality and $(ab-c)(\ln(ab) - \ln c) \ge 4(AB - C)^2$, we obtain the estimate

$$D(a,b,c) \ge 4\left\|AB - C\right\|_2^2 + \frac{4d_a}{P}\left\|A - \overline{A}\right\|_2^2 + \frac{4d_b}{P}\left\|B - \overline{B}\right\|_2^2 + \frac{4d_c}{P}\left\|C - \overline{C}\right\|_2^2 . \tag{43}$$

Analog to the proof of lemma 2.2, we show in the sequel that the r.h.s. of (43) is bounded below by the relative entropy $E(a,b,c) - E(a_\infty, b_\infty, c_\infty)$.

First, we use the conservation laws (20), (21) to rewrite the relative entropy as

$$E(a,b,c) - E(a_\infty, b_\infty, c_\infty) = \int_\Omega \left(a \ln\left(\frac{a}{a_\infty}\right) - (a - a_\infty) \right.$$
$$\left. + b \ln\left(\frac{b}{b_\infty}\right) - (b - b_\infty) + c \ln\left(\frac{c}{c_\infty}\right) - (c - c_\infty) \right) dx,$$

and we use lemma 2.1 as well as the global bound (41) to estimate

$$\begin{aligned} E(a,b,c) - E(a_\infty, b_\infty, c_\infty) \le\ & K_2 \left(\frac{B_\infty^2}{4}\|A - A_\infty\|_2^2 + \frac{A_\infty^2}{4}\|B - B_\infty\|_2^2 \right. \\ & \left. + \|C - C_\infty\|_2^2 \right), \end{aligned} \tag{44}$$

with

$$K_2(L_2, M_1, M_2) = \max\left\{ \frac{4}{b_\infty}\Phi(\|a\|_\infty, a_\infty), \frac{4}{a_\infty}\Phi(\|b\|_\infty, b_\infty), \Phi(\|c\|_\infty, c_\infty) \right\} . \tag{45}$$

The statement of lemma 3.1 with the constant K_1 given by (25) follows from the following lemma, which provides an upper bound for the r.h.s. of (44) in terms of the r.h.s. of (43).

Lemma 3.2 *Let A, B, and C be (measurable) functions from Ω to $\mathbb{R}_+$ such that (20) $\overline{A^2}+\overline{C^2}=M_1$ and (21) $\overline{B^2}+\overline{C^2}=M_2$. Then, the estimate*

$$\begin{aligned}\tfrac{B_\infty^2}{4}\|A-A_\infty\|_2^2+\tfrac{A_\infty^2}{4}\|B-B_\infty\|_2^2+\|C-C_\infty\|_2^2 &\le \|AB-C\|_2^2\\ +(\tfrac{B_\infty^2}{4}+K_3)\|A-\overline{A}\|_2^2+(\tfrac{A_\infty^2}{4}+K_4)\|B-\overline{B}\|_2^2&+(2+K_5)\|C-\overline{C}\|_2^2 \end{aligned}\tag{46}$$

holds, with the constants

$$K_3=\max\left\{\begin{array}{l} C_\infty+\frac{A_\infty B_\infty}{2}+\frac{2C_\infty^2}{M_1},\\ \sqrt{\min\{M_1,M_2\}}+\frac{A_\infty B_\infty}{2},\\ C_\infty+\frac{A_\infty\sqrt{M_2}}{2}+\frac{A_\infty(\sqrt{M_2}+B_\infty)C_\infty}{M_1}+\frac{A_\infty^2}{2}+\frac{B_\infty(\sqrt{M_2}-B_\infty)}{2},\\ C_\infty+\frac{\sqrt{M_1}B_\infty}{2}-\frac{B_\infty^2}{2}-\frac{A_\infty(\sqrt{M_1}-A_\infty)}{2},\\ C_\infty+\frac{A_\infty(\sqrt{M_2}-B_\infty)}{4}+\frac{(\sqrt{M_1}-A_\infty)B_\infty}{4} \end{array}\right\},\tag{47}$$

$$K_4=\max\left\{\begin{array}{l} C_\infty+\frac{A_\infty B_\infty}{2}+\frac{2C_\infty^2}{M_2},\\ \sqrt{\min\{M_1,M_2\}}+\frac{A_\infty B_\infty}{2},\\ C_\infty+\frac{A_\infty\sqrt{M_2}}{2}-\frac{A_\infty^2}{2}-\frac{B_\infty(\sqrt{M_2}-B_\infty)}{2},\\ C_\infty+\frac{\sqrt{M_1}B_\infty}{2}+\frac{(\sqrt{M_1}+A_\infty)B_\infty C_\infty}{M_2}+\frac{B_\infty^2}{2}+\frac{A_\infty(\sqrt{M_1}-A_\infty)}{2},\\ C_\infty+\frac{A_\infty(\sqrt{M_2}-B_\infty)}{4}+\frac{(\sqrt{M_1}-A_\infty)B_\infty}{4} \end{array}\right\},\tag{48}$$

and

$$K_5=\max\left\{\ 2C_\infty^2\left(\frac{1}{M_1}+\frac{1}{M_2}\right),\frac{A_\infty(\sqrt{M_2}+B_\infty)C_\infty}{M_1},\frac{(\sqrt{M_1}+A_\infty)B_\infty C_\infty}{M_2}\ \right\}.\tag{49}$$

Proof of lemma 3.2: In a first step, we reformulate (46) in order to isolate the "bad" terms (i.e. the analogs to $*$ in equation (33) in section 2). In a second step, we shall control these terms using the conservations laws $\overline{A^2}+\overline{C^2}=M_1$ and $\overline{B^2}+\overline{C^2}=M_2$.

We start with

$$\begin{aligned}\|AB-C\|_2^2 &= \|AB-A_\infty B_\infty\|_2^2+\|C-C_\infty\|_2^2 &(50)\\ &\quad +2\int_\Omega (AB-A_\infty B_\infty)(C_\infty-C)\,dx\,, &(51)\end{aligned}$$

where we split (51) using $C_\infty-C=(C_\infty-\overline{C})+(\overline{C}-C)$ and calculate for the first part

$$\begin{aligned}2\int_\Omega(AB-A_\infty B_\infty)(C_\infty-\overline{C}) = (C_\infty-\overline{C})\Big(2\int_\Omega(A-\overline{A})(B-\overline{B})\,dx &\quad (52)\\ +(\overline{A}-A_\infty)(\overline{B}+B_\infty)+(\overline{B}-B_\infty)(\overline{A}+A_\infty)\Big), &\quad (53)\end{aligned}$$

while we estimate by Young's inequality for the second part

$$2\int_\Omega (AB - A_\infty B_\infty)(\overline{C} - C)\,dx \geq -\frac{1}{2}\|AB - A_\infty B_\infty\|_2^2 - 2\|C - \overline{C}\|_2^2 . \quad (54)$$

After inserting (54) into (51), there remains $\frac{1}{2}\|AB-A_\infty B_\infty\|_2^2$, which we split again as the sum of two halfs. Expanding the first half using $-A_\infty B + A_\infty B$ and the second half using $-AB_\infty + AB_\infty$ yields

$$\frac{1}{2}\|AB - A_\infty B_\infty\|_2^2 =$$

$$\frac{1}{4}\|(A - A_\infty)B\|_2^2 + \frac{1}{2}\int_\Omega (A - A_\infty)BA_\infty(B - B_\infty)\,dx + \frac{A_\infty^2}{4}\|B - B_\infty\|_2^2 \quad (55)$$

$$+\frac{1}{4}\|A(B - B_\infty)\|_2^2 + \frac{1}{2}\int_\Omega A(B - B_\infty)(A - A_\infty)B_\infty\,dx + \frac{B_\infty^2}{4}\|A - A_\infty\|_2^2 \quad (56)$$

Next, the integrals in (55) and (56) are expanded using $B - B_\infty = (B-\overline{B}) + (\overline{B} - B_\infty)$ (respectively $A - A_\infty = (A-\overline{A}) + (\overline{A} - A_\infty)$) and the first of these further parts are estimated thanks to

$$\frac{1}{2}\int_\Omega (A - A_\infty)BA_\infty(B - \overline{B})\,dx \geq -\frac{1}{4}\|(A - A_\infty)B\|_2^2 - \frac{A_\infty^2}{4}\|B - \overline{B}\|_2^2 , \quad (57)$$

$$\frac{1}{2}\int_\Omega A(B - B_\infty)(A - \overline{A})B_\infty\,dx \geq -\frac{1}{4}\|A(B - B_\infty)\|_2^2 - \frac{B_\infty^2}{4}\|A - \overline{A}\|_2^2 . \quad (58)$$

Altogether, we obtain from (55)–(58) that

$$\frac{1}{2}\|AB - A_\infty B_\infty\|_2^2 \geq \quad (59)$$

$$\frac{B_\infty^2}{4}\|A - A_\infty\|_2^2 + \frac{A_\infty^2}{4}\|B - B_\infty\|_2^2 - \frac{A_\infty^2}{4}\|B - \overline{B}\|_2^2 - \frac{B_\infty^2}{4}\|A - \overline{A}\|_2^2$$

$$+\frac{1}{2}(A_\infty(\overline{B} - B_\infty) + (\overline{A} - A_\infty)B_\infty)\int_\Omega (A - \overline{A})(B - \overline{B})\,dx$$

$$+\frac{1}{2}(\overline{A} - A_\infty)(\overline{B} - B_\infty)(A_\infty\overline{B} + B_\infty\overline{A}) . \quad (60)$$

After inserting (50)–(60) into (46), it remains (as second step) to show that

$$K_3\|A - \overline{A}\|_2^2 + K_4\|B - \overline{B}\|_2^2 + K_5\|C - \overline{C}\|_2^2 +$$

$$\left(2(C_\infty - \overline{C}) + \frac{1}{2}(A_\infty(\overline{B} - B_\infty) + (\overline{A} - A_\infty)B_\infty)\right)\int_\Omega (A - \overline{A})(B - \overline{B})\,dx$$

$$+(C_\infty - \overline{C})\left((\overline{A} - A_\infty)(\overline{B} + B_\infty) + (\overline{B} - B_\infty)(\overline{A} + A_\infty)\right)$$

$$+\frac{1}{2}(\overline{A} - A_\infty)(\overline{B} - B_\infty)(A_\infty\overline{B} + B_\infty\overline{A}) \geq 0 , \quad (61)$$

for which we are going to distinguish five cases:

1. For the case $\overline{A} < A_\infty$, $\overline{B} < B_\infty$, and $\overline{C} < C_\infty$, it is sufficient to show

$$K_3\|A - \overline{A}\|_2^2 + K_4\|B - \overline{B}\|_2^2 + K_5\|C - \overline{C}\|_2^2$$

$$-(2C_\infty + A_\infty B_\infty)\left(\frac{1}{2}\|A - \overline{A}\|_2^2 + \frac{1}{2}\|B - \overline{B}\|_2^2\right) \quad (62)$$

$$-(C_\infty - \overline{C})(A_\infty - \overline{A})(\overline{B} + B_\infty) \quad (63)$$

$$-(C_\infty - \overline{C})(B_\infty - \overline{B})(\overline{A} + A_\infty) \geq 0 . \quad (64)$$

Since (63) and (64) are symmetric in A and B, we choose (64) to show how the conservation (21) - rewritten in the form

$$\|B-\overline{B}\|_2^2+\|C-\overline{C}\|_2^2 = (B_\infty-\overline{B})(\overline{B}+B_\infty)+(C_\infty-\overline{C})(\overline{C}+C_\infty), \quad (65)$$

allows to prove that (remember that $\overline{A} < A_\infty$, $\overline{B} < B_\infty$, $\overline{C} < C_\infty$)

$$\frac{2C_\infty^2}{M_2}(\|B-\overline{B}\|_2^2 + \|C-\overline{C}\|_2^2) - (C_\infty - \overline{C})(B_\infty - \overline{B})(\overline{A} + A_\infty) \geq$$
$$\frac{2C_\infty^2}{M_2}((B_\infty - \overline{B})B_\infty + (C_\infty - \overline{C})C_\infty) - 2A_\infty(C_\infty - \overline{C})(B_\infty - \overline{B})\,.$$

This last expression is a linear function of $\overline{B}$ (which will be denoted by $\Psi(\overline{B})$) which is nonnegative on $0 \leq \overline{B} \leq B_\infty$ because

$$\Psi(B_\infty) \geq \frac{2C_\infty^2}{M_2}(C_\infty - \overline{C})C_\infty \geq 0\,, \quad (66)$$

$$\Psi(0) \geq \frac{2A_\infty B_\infty}{\frac{B_\infty^2}{C_\infty - \overline{C}} + C_\infty}(B_\infty^2 + (C_\infty - \overline{C})C_\infty) \quad (67)$$

$$-2A_\infty B_\infty(C_\infty - \overline{C}) = 0\,, \quad (68)$$

where we have used $A_\infty B_\infty = C_\infty$ and $C_\infty \geq C_\infty - \overline{C}$.

For (63), there is a symmetric estimate based on the conservation (20). Adding these two estimates together with the coefficient of (62) gives the first lines for the constants K_3 (47) and K_4 (48) and the first expression for K_5 (49).

2. For the second case $\overline{A} < A_\infty$, $\overline{B} < B_\infty$, and $\overline{C} > C_\infty$, we proceed in a similar way to (62)–(64), but instead of line (62), we find here

$$-(2\overline{C} + A_\infty B_\infty)\left(\frac{1}{2}\|A-\overline{A}\|_2^2 + \frac{1}{2}\|B-\overline{B}\|_2^2\right)\,,$$

while the lines (63) and (64) are nonnegative in this case and thus neglected. Using the estimate $\overline{C}^2 \leq \min\{M_1, M_2\}$ (due to (20) and (21)), we get the second line of (47) and (48).

3. In the third case $\overline{A} < A_\infty$, $\overline{B} > B_\infty$, the latter hypothesis implies $\overline{C} < C_\infty$ by the conservation law (21). As above, we estimate

$$K_3\|A-\overline{A}\|_2^2 + K_4\|B-\overline{B}\|_2^2 + K_5\|C-\overline{C}\|_2^2 \quad (69)$$

$$-(2C_\infty + A_\infty\sqrt{M_2})\left(\tfrac{1}{2}\|A-\overline{A}\|_2^2 + \tfrac{1}{2}\|B-\overline{B}\|_2^2\right) \quad (70)$$

$$-(C_\infty - \overline{C})(A_\infty - \overline{A})(\overline{B} + B_\infty) \quad (71)$$

$$+\tfrac{1}{2}(\overline{A} - A_\infty)(\overline{B} - B_\infty)(A_\infty\overline{B} + B_\infty\overline{A}) \geq 0 \quad (72)$$

where we have used the conservation law (21) to estimate

$$\overline{B} \le \sqrt{\overline{B^2}} \le \sqrt{M_2}\,,$$

in the coefficient of (70). An analog argument to (65)–(68) shows for the term (71) that

$$\begin{aligned} \frac{A_\infty(\sqrt{M_2}+B_\infty)C_\infty}{M_1}\left(\|A-\overline{A}\|_2^2+\|C-\overline{C}\|_2^2\right) & \\ -(C_\infty-\overline{C})(A_\infty-\overline{A})(\overline{B}+B_\infty) \quad \ge \quad 0\,. & \qquad (73) \end{aligned}$$

For (72), we use

$$\|A-\overline{A}\|_2^2 - \|B-\overline{B}\|_2^2 = (A_\infty-\overline{A})(\overline{A}+A_\infty) - (B_\infty-\overline{B})(\overline{B}+B_\infty)$$

in order to calculate

$$\begin{aligned} &(\overline{A}-A_\infty)(\overline{B}-B_\infty)(A_\infty\overline{B}+A_\infty B_\infty-2A_\infty B_\infty+A_\infty B_\infty+B_\infty\overline{A}) \\ &\quad = A_\infty(\overline{A}-A_\infty)^2(\overline{A}+A_\infty)+B_\infty(\overline{B}-B_\infty)^2(\overline{B}+B_\infty) \\ &\qquad\qquad -2A_\infty B_\infty(\overline{A}-A_\infty)(\overline{B}-B_\infty) \qquad (74) \\ &\quad +(A_\infty(\overline{A}-A_\infty)-B_\infty(\overline{B}-B_\infty))\left(\|A-\overline{A}\|_2^2-\|B-\overline{B}\|_2^2\right)\,, \qquad (75) \end{aligned}$$

where the line (74) is nonnegative in the considered case. For (75), we observe that in the present case, $\|A-\overline{A}\|_2^2 - \|B-\overline{B}\|_2^2$ is nonnegative and hence that (75) is bounded below by

$$-(A_\infty^2 + B_\infty(\sqrt{M_2}-B_\infty))\left(\|A-\overline{A}\|_2^2-\|B-\overline{B}\|_2^2\right)\,. \qquad (76)$$

Hence, combining (72) and (73)–(76) yields the third contributions to K_3 (47), K_4 (48) and the second to K_5 (49).

4. Concerning the fourth case $\overline{A} > A_\infty$ implying $\overline{C} < C_\infty$ and $\overline{B} < B_\infty$ (by (20)), we proceed in a symmetric way compared to case three, which leads to the fourth contributions to K_3 (47), K_4 (48) and the third to K_5 (49).

5. In the final case, we consider $\overline{A} > A_\infty$, $\overline{B} > B_\infty$ implying $\overline{C} < C_\infty$. Therefore, (61) is bounded below by

$$-(C_\infty+\frac{A_\infty(\sqrt{M_2}-B_\infty)}{4}+\frac{(\sqrt{M_1}-A_\infty)B_\infty}{4})\left(\|A-\overline{A}\|_2^2+\|B-\overline{B}\|_2^2\right),$$

which completes the formulas for K_3 (47) and K_4 (48).

This ends the proof of lemma 3.2. □

According to estimate (44) and lemma 3.2, we obtain lemma 3.1. □

We now write down the lemma which plays the role of Cziszar-Kullback-Pinsker inequality in information theory.

Lemma 3.3 *For all (measurable) functions* $a, b, c : \Omega \to \mathbb{R}$ *such that* $0 \le a$, $0 \le b$, $0 \le c$ *and* $\int_\Omega (a+c) = M_1$, $\int_\Omega (b+c) = M_2$,

$$\begin{aligned} E(a,b,c) - E(a_\infty, b_\infty, c_\infty) &\geq \tfrac{3+2\sqrt{2}}{2M_1(9+2\sqrt{2})} \|a - a_\infty\|_1^2 \\ + \tfrac{3+2\sqrt{2}}{2M_2(9+2\sqrt{2})} \|b - b_\infty\|_1^2 &+ \tfrac{3+2\sqrt{2}}{(M_1+M_2)(9+2\sqrt{2})} \|c - c_\infty\|_1^2, \end{aligned} \tag{77}$$

where a_∞, b_∞ *and* c_∞ *are defined by (22).*

Proof of lemma 3.3: As in lemma 2.3, we define $q(a_i) = a_i \ln a_i - a_i$ for $a_1 \equiv a$, $a_2 \equiv b$, and $a_3 \equiv c$, and rewrite

$$E(a,b,c) - E(a_\infty, b_\infty, c_\infty) = \int_\Omega a \ln \frac{a}{\overline{a}}\, dx + \int_\Omega b \ln \frac{b}{\overline{b}}\, dx + \int_\Omega c \ln \frac{c}{\overline{c}}\, dx \tag{78}$$

$$+q(\overline{a}) - q(a_\infty) + q(\overline{b}) - q(b_\infty) + q(\overline{c}) - q(c_\infty). \tag{79}$$

Using the conservations (20) and (21), and defining

$$Q(M_i, \overline{a_i}) = q(\overline{a_i}) + \frac{1}{2} q(M_i - \overline{a_i}) \qquad \text{for } \overline{a_i} \in [0, M_i]\,, \quad i = 1,2\,,$$

$$R(M_i, \overline{a_i}) = \frac{1}{2} q(\overline{a_i}) + q(M_i - \overline{a_i}) \qquad \text{for } \overline{a_i} \in [0, M_i]\,, \quad i = 1,2\,,$$

we can write the line (79) in two ways as

$$\begin{aligned} &Q(M_1, \overline{a}) - Q(M_1, a_\infty) + Q(M_2, \overline{b}) - Q(M_2, b_\infty) \\ &= R(M_1, \overline{c}) - R(M_1, c_\infty) + R(M_2, \overline{c}) - R(M_2, c_\infty)\,. \end{aligned} \tag{80}$$

Since

$$Q'(M_1, a_\infty) + Q'(M_2, b_\infty) = -R'(M_1, c_\infty) - R'(M_2, c_\infty) = \ln \frac{a_\infty b_\infty}{c_\infty} = 0$$

and (for $\overline{a_i} \in [0, M_i], i = 1, 2$)

$$Q''(M_i, \overline{a_i}),\ R''(M_i, \overline{a_i}) \geq \frac{K_6}{2M_i} \qquad \text{with} \qquad K_6 = 3 + 2\sqrt{2}\,,$$

we bound (80) from below by

$$\frac{K_6}{4M_1}\left(\theta|\overline{a}-a_\infty|^2+(1-\theta)|\overline{c}-c_\infty|^2\right)+\frac{K_6}{4M_2}\left(\theta|\overline{b}-b_\infty|^2+(1-\theta)|\overline{c}-c_\infty|^2\right)\,,$$

for all $0 \le \theta \le 1$. Using the Cziszar-Kullback-Pinsker inequality and choosing $\theta = \frac{1}{3}$, we obtain

$$E(a,b,c)-E(a_\infty,b_\infty,c_\infty)\ge \tfrac{1}{2M_1}\left(\|a-\overline{a}\|_1^2+\tfrac{K_6}{6}|\overline{a}-a_\infty|^2\right)$$
$$+\tfrac{1}{2M_2}\left(\|b-\overline{b}\|_1^2+\tfrac{K_6}{6}|\overline{b}-b_\infty|^2\right)+\tfrac{1}{M_1+M_2}\left(\|c-\overline{c}\|_1^2+\tfrac{K_6}{6}|\overline{c}-c_\infty|^2\right)\,.$$

Then,

$$\|a-a_\infty\|_1^2\le(1+\frac{6}{K_6})\left(\|a-\overline{a}\|_1^2+\frac{K_6}{6}|\overline{a}-a_\infty|^2\right)\,,$$

and the same holds for b and c, so that we get

$$E(a,b,c)-E(a_\infty,b_\infty,c_\infty)\ge \tfrac{1}{2M_1}\tfrac{K_6}{K_6+6}\|a-a_\infty\|_1^2+\tfrac{1}{2M_2}\tfrac{K_6}{K_6+6}\|b-b_\infty\|_1^2$$
$$+\tfrac{1}{M_1+M_2}\tfrac{K_6}{K_6+6}\|c-c_\infty\|_1^2\,.$$

This ends the proof of lemma 3.3. □

End of the proof of theorem 1.2: Noting that the solution to the system (14) – (19) satisfies the entropy equality

$$\frac{d}{dt}(E(a(t),b(t),c(t))-E(a_\infty,b_\infty,c_\infty))=-\,D(a(t),b(t),c(t))\,,$$

we see that theorem 1.2 is a direct consequence of lemma 3.1, lemma 3.3 and Gronwall's lemma. ■

References

[1] A. Barabanova, *On the global existence of solutions of a reaction-diffusion equation with exponential nonlinearity.* Proc. Amer. Math. Soc. **122**, no. 3 (1994), pp. 827–831.

[2] D. Bothe, D. Hilhorst, *A reaction-diffusion system with fast reversible reaction.* J. Math. Anal. Appl., **286**, no. 1 (2003), pp. 125–135.

[3] M. Caceres, J. Carrillo, G. Toscani, *Long-time behavior for a nonlinear fourth order parabolic equation* preprint HYKE2003-048.

[4] M. Caceres, J. Carrillo, T. Goudon, *Equilibration rate for the linear inhomogeneous relaxation-time Boltzmann equation for charged particles.* Comm. Partial Differential Equations **28**, no. 5-6 (2003), pp. 969–989.

[5] E. Conway, D. Hoff, J. Smoller, *Large time behaviour of solutions of systems of nonlinear reaction-diffusion equations.* SIAM J. Appl. Math. **35**, no. 1 (1978), pp. 1–16.

[6] I. Csiszár, *Eine informationstheoretische Ungleichung und ihre Anwendung auf den Beweis von Markoffschen Ketten.* Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), pp. 85–108.

[7] J. Carrillo, J.L. Vazquez, *Fine asymptotics for fast diffusion equations.* Comm. Partial Differential Equations **28**, no. 5-6 (2003), pp. 1023–1056.

[8] M. Del Pino, J. Dolbeault, *Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions.* J. Math. Pures Appl. **81**, no. 9 (2002), pp. 847–875.

[9] L. Desvillettes, C. Villani, *On the spatially homogeneous Landau equation for hard potentials. II. H-theorem and applications.* Comm. Partial Differential Equations **25**, no. 1-2 (2000), pp. 261–298.

[10] L. Desvillettes, C. Villani, *On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation.* Comm. Pure Appl. Math. **54**, no. 1 (2001), pp. 1–42.

[11] L. Desvillettes, C. Villani, *On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation.* Preprint HYKE2003-011.

[12] P. Érdi, J. Tóth, Mathematical models of chemical reactions. Nonlinear Science: theory and applications, Manchester University Press (1989).

[13] K. Fellner, L. Neumann, C. Schmeiser, *Convergence to global equilibrium for spatially inhomogeneous kinetic models of non-micro-reversible processes*, Online First Monatsh. d. Math.

[14] W. Feng, *Coupled system of reaction-diffusion equations and applications in carrier facilitated diffusion.* Nonlinear Analysis, Theory, Methods & Applications **17**, no. 3 (1991), pp. 285–311.

[15] P. C. Fife, Mathematical Aspects of Reacting and Diffusing Systems. Lecture Notes in Biomathematics, Vol. 28, Springer, Berlin (1979).

[16] K. Gröger, *Free energy estimates and asymptotic behaviour of reaction-diffusion processes.* Preprint 20, Institut für Angewandte Analysis und Stochastik, Berlin, 1992.

[17] A. Glitzky, K. Gröger, R. Hünlich, *Free energy and dissipation rate for reaction-diffusion processes of electrically charged species.* Appl. Anal. **60**, no. 3-4 (1996), pp. 201–217.

[18] K. Masuda, *On the global existence and asymptotic behavior of solution of reaction-diffusion equations.* Hokkaido Math. J. **12**, (1983), pp. 360–370.

[19] M. Kirane, *On stabilization of solutions of the system of parabolic differential equations describing the kinetics of an autocatalytic reversible chemical reaction*. Bull. Inst. Mat. Acad. Sin. **18**, no. 4 (1990), pp. 369–377.

[20] S. Kullback, Information Theory and Statistics, John Wiley, (1959).

[21] F. Rothe, Global Solutions of Reaction-Diffusion Systems. Lecture Notes in Mathematics, Springer, Berlin, (1984).

[22] J. Smoller, Shock Waves and Reaction-Diffusion Equations. Grundlehren der mathematischen Wissenschaften, Vol. 258, Springer, New-York (1983).

[23] M. E. Taylor, Partial Differential Equation III – Nonlinear Equations. Springer Series Applied Mathematical Sciences Vol. 117, New York, (1996).

[24] G. Toscani, C. Villani, *Sharp entropy dissipation bounds and explicit rate of trend to equilibrium for the spatially homogeneous Boltzmann equation.* Comm. Math. Phys. **203**, no. 3 (1999), pp. 667–706.

[25] G. Toscani, C. Villani, *On the trend to equilibrium for some dissipative systems with slowly increasing a priori bounds.* J. Statist. Phys. **98**, no. 5-6 (2000), pp. 1279–1309.

[26] C. Villani, *Cercignani's conjecture is sometimes true and always almost true.* Comm. Math. Phys. **234**, no. 3 (2003), pp. 455–490.

[27] A. Volpert, Vitaly A. Volpert, Vladimir A. Volpert, Traveling Wave Solutions of Parabolic Systems. Translations of Mathematical Monographs, Vol. 140, AMS, (1994).