Long-time Asymptotics via Entropy Methods for Diffusion Dominated Equations

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Abstract. Uniform convergence rates of diffusion dominated equations towards their asymptotic profiles are quantified via entropy methods for bounded integrable non-negative initial data with finite entropy. Convergence rates are sharp since they coincide with the purely diffusive ones. The approach is applied to both convection- and absorption-diffusion equations. Finally, Wasserstein metrics are used to control the expansion of the support for the convection-diffusion case.

Key words: convection-diffusion, absorption-diffusion, entropy methods, large time behaviour, convergence rate.

AMS subject classification: 35B40, 35K65.

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1 Introduction and main result

This article contributes to the analysis of the long-time behaviour of non-negative solutions of diffusion dominated equations, particularly: the convection-diffusion equation

$$u_t = (u^m)_{xx} - (u^q)_x \,, \tag{1}$$

$$u(t=0,x) = u_0(x) \geq 0 \,, \tag{2}$$

for (t, x) in $[0, \infty) \times \mathbb{R}$ and for the range of exponents

$$m \geq 1 \,, \quad q > m+1 \,, \tag{3}$$

and the absorption-diffusion equation

$$u_t = (u^m)_{xx} - u^r \,, \tag{4}$$

$$u(t=0,x) = u_0(x) \geq 0 \,, \tag{5}$$

for (t, x) in $[0, \infty) \times \mathbb{R}$ and for the range of exponents

$$m \geq 1 \,, \quad r > m+2 \,. \tag{6}$$

Applications of (1) and (4) occur, for instance, in the modelling of transport processes through a porous medium (see e.g. [Kal]).

Regarding the convection-diffusion equation (1), the well-posedness of the Cauchy problem (1)-(2) as well as the long-time behaviour has been investigated by several authors (see e.g. [EZ, EVZ, DZ99, Zua] for the parabolic case $m = 1$ and [BT, Gil89, LS97, LS98, RV] for the degenerate parabolic case $m > 1$). It is well-known that solutions of (1) with appropriate data (2) (see section 2 for the precise integrability assumptions) preserve the initial mass

$$\int_{\mathbb{R}} u \, dx = \int_{\mathbb{R}} u_0 \, dx =: M$$

and, for large times, tend towards an asymptotic self-similar profile U_M, which is uniquely determined by the mass M. More precisely, for $p \in [1, \infty]$ asymptotic estimates of the form

$$\lim_{t \to \infty} t^{r_p(q,m)} \|u(t) - U_M(t)\|_p \leq C(M, p, q, m) \,, \tag{7}$$

are established, where $r_p(q, m) \geq 0$ denotes the L^p-convergence rate and $C(M, p, q, m)$ is a non-negative constant. With the assumed exponents $q > m+1$ we consider the diffusion dominated case of (1), since the profiles U_M are the source-type solutions of the purely diffusive heat ($m = 1$) respectively porous medium ($m > 1$) equation; i.e. the family of unique self-similar solutions of

$$(U_M)_t - (U_M^m)_{xx} = 0\,,$$
$$\lim_{t\to 0} \int_{\mathbb{R}} U_M(x,t)\,\theta(x)\,dx = M\,\theta(0)\,,$$

for all continuous and bounded functions $\theta(x)$ parametrised by the mass M. The self-similarity of the source-type solutions implies the invariance of U_M under the rescalings

$$u_\lambda(t,x) = \lambda^{\frac{1}{m+1}} u(\lambda t, \lambda^{\frac{1}{m+1}} x)\,, \tag{8}$$

for all $\lambda > 0$. Moreover, if u solves (1) then u_λ solves

$$(u_\lambda)_t - (u_\lambda^m)_{xx} = -\lambda^{\frac{m+1-q}{m+1}} (u_\lambda^q)_x\,, \tag{9}$$

with a negative exponent of λ on the r.h.s. of (9) and it is easy to see that (7) is formally equivalent to

$$u_\lambda(\cdot,t) \to U(\cdot,t) \qquad \lambda \to \infty\,.$$

This idea of proving (7) was made rigorous by establishing precise λ-invariant a-priori estimates on the decay of the solutions (see e.g. [EZ] resp. [RV]). Another approach [LS97, LS98] used directly the decay estimates of the solution to measure the L^p-distance to the asymptotic profile.

Similarly to equation (1), the large time behavior of the absorption-diffusion equation (4) is dominated by the diffusion term. The precise statement of the convergence, the relevant references and their discussion are postponed to the fourth section.

In this paper, we discuss an *entropy-entropy dissipation* approach to analyse the long-time behaviour of the Cauchy problems (1)-(2) and (4)-(5). Entropy methods have already been used by several authors in the purely diffusion case [CT00, CJMTU, DE, DP, FM, LM, Ott]. In a preliminary step, a time-dependent rescaling is used, which transforms (1) into a nonlinear Fokker-Planck type equation with quadratic confining potential. Moreover, the time-dependent asymptotic profiles U_M (shifted in time) are rescaled into stationary profiles.

The actual entropy approach quantifies the dissipation process, which morphs the solutions towards the asymptotic profiles. The main tools are the distinguished Liapunov functional of the purely diffusive case, called entropy functional, and its corresponding entropy dissipation. A quantitative estimate, which bounds the entropy dissipation below by the relative entropy with respect to the asymptotic stationary profiles, was already obtained in previous works [CJMTU, DP, Ott]. Exploiting this entropy-entropy dissipation estimate, one establishes firstly convergence in entropy, which secondly implies convergence in norms for the pure diffusive case.

In our context, after applying the above mentioned rescaling, we shall think about equations (1) and (4) as asymptotic perturbations of the purely diffusive nonlinear Fokker-Planck equations. We show that the behaviour of the relative entropy for large times is controlled by the purely diffusive case reflecting the perturbation character of these models. This control ultimately gives an exponential decay in the L^1-norm towards the stationary profiles. Backward rescaling leads finally to algebraic L^1-convergence rates in (7) and the L^p-rates follow by interpolation with a L^∞-bound.

It is interesting to study how the convergence rates in (7) depend especially on the order of bounded initial moments. An underlying result has been shown in [DZ94], where the solution of the heat equation was expanded in a sum of derivatives of the heat kernel with the i-th order initial moment as $i-$th coefficient (see also [PZ] for an expansion of a nonlinear equation). For the IVP (1)-(2) with L^1-integrable data it was shown in [EZ] ($m = 1$) respectively [LS98] ($m > 1$) that there is L^1-convergence but without actual rate in (7). On the other hand, assuming additionally a bounded first order moment leads in the parabolic case $m = 1$ [EZ, Zua] to the rates

$$r_1(q,1) = \begin{cases} \frac{1}{2} & \text{for } q > 3\,, \\ \frac{1}{2} - \varepsilon & \text{for } q = 3 \text{ and for all } \varepsilon > 0\,, \\ \frac{q}{2} - 1 & \text{for } 2 < q < 3\,. \end{cases} \tag{10}$$

For $q > 3$, the rate $\frac{1}{2}$ is optimal in the sense that it is indeed the convergence rate of solutions of the heat equation towards the heat kernel (see [Zua] for second order terms in the asymptotic expansion (7)). For $2 < q < 3$, the rates $\frac{q}{2} - 1$ are sharp in the class of integrable initial data with bounded first order moment as they can't be improved by the second order term.

The initial data in the proposed entropy approach will have to guarantee the natural conditions of propagation of finite initial entropy. Our main result will therefore assume

Theorem 1.1 *Let $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be non-negative with bounded second order moment $\int_{\mathbb{R}} u_0 |x|^2\, dx < \infty$. Then,*

$$r_1(q,m) = \begin{cases} \frac{1}{m+1} & \text{for} \quad 1 \le m < 2\,, \\ \frac{2}{m(m+1)} & \text{for} \quad 2 \le m\,. \end{cases} \tag{11}$$

As main improvement via the entropy method, we obtain the optimal rates (i.e. the convergence rates of the purely diffusive equation) uniformly in the diffusion dominated range $q > m + 1$. This is indeed a consequence of the bounded second order initial moment as will become clear in section 3, where we will prove (11) in the range $m + 1 < q < m + 3$ via an iteration process, which relies crucially on the bounded second order moment.

The remainder of the paper is organised as follows: In section 2, we restate the used existence theory of (1)-(2) and the a-priori estimates on the solutions. In section 3, we firstly demonstrate formally the entropy method before we prove theorem 3.1 rigorously as well as we discuss more general initial data and extensions to higher dimensions. Section 4 generalizes the procedure to obtain decay rates to the absorption-diffusion equation (4). Finally, in the one dimensional case of the convection diffusion equation, we illustrate the convergence in Wasserstein metrics in section 5. As a consequence, we deduce precise asymptotic estimates of the support growth.

Notations: for all $p \in [1, \infty]$, we use $\|\cdot\|_p$ for the usual $L^p(\mathbb{R})$-norm. $(\cdot)_x^2$ shall be the short-form of $((\cdot)_x)^2$. Beside commonly denoted function spaces, $C_b(\mathbb{R} \times (0, \infty))$ specifies the space of continuous and bound functions on $\mathbb{R} \times (0, \infty)$. Finally, $C(\cdot, \cdots, \cdot)$ or $C_i(\cdot, \ldots, \cdot)$ $i = 1, 2, 3\ldots$ will denote various constants depending on the stated arguments as well as on q and m, which we shall suppress.

2 Preliminaries and rescaling

2.1 Existence, apriori estimates, source-type solutions

Proposition 2.1 ***(Existence for the linear diffusion case)*** *[EZ, DZ99] Let u_0 be non-negative in $L^1(\mathbb{R}^N)$ for $N \geq 1$. Then, the IVP (1)-(2) with $m = 1$, $q > 2$ has a unique classical solution $u \in C([0, \infty), L^1(\mathbb{R}^N)) \cap C((0, \infty), W^{2,p}(\mathbb{R}^N) \cap C^1((0, \infty), L^p(\mathbb{R}^N))$ for every $p \in (1, \infty)$ satisfying the conservation of mass*

$$\int_{\mathbb{R}} u(t, x)\, dx = \int_{\mathbb{R}} u_0(x)\, dx = M\,,$$

and

$$\|u(t)\|_p \;\leq\; C(M, p)\, t^{-\frac{N}{2}(1-\frac{1}{p})} \qquad t > 0\,, \tag{12}$$

with a constant C depending only on M and p. If moreover $u_0 \in L^p(\mathbb{R}^N)$ for $p \in (\frac{q}{3}, \infty]$, then [DZ99]

$$\|u(t)\|_p \;\leq\; \|u_0\|_p \qquad t \geq 0\,. \tag{13}$$

Proposition 2.2 ***(Existence for the degenerate diffusion case)***
(i) *Let u_0 be non-negative in $L^1(\mathbb{R})$. Then, the IVP (1)-(2) with $m > 1, q > m+1$ has a unique mild solution $u \in C([0,\infty), L^1(\mathbb{R}))$ in the sense of nonlinear semigroup theory [BT]. Moreover, for $u_{0,1}, u_{0,2} \in L^1(\mathbb{R})$ and $(\cdot)_+$ denoting* $\max\{0,\cdot\}$*, we have the L^1-contraction property*

$$\|(u_1(t) - u_2(t))_+\|_1 \leq \|(u_{0,1} - u_{0,2})_+\|_1 \,, \tag{14}$$

which ensures also that $u(t)$ remains non-negative for $t > 0$.
(ii) *If $u_0 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ for $p \in (1,\infty)$, then [RV]*

$$\|u(t)\|_p \leq \|u_0\|_p \qquad t \geq 0\,. \tag{15}$$

(iii) *Let u_0 be additionally in $L^\infty(\mathbb{R})$ [LS98]. Then, $u \in L^\infty((0,\infty) \times \mathbb{R})$ satisfies the conservation of mass*

$$\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} u_0(x)\,dx = M\,,$$

and

$$\|u(t)\|_\infty \leq \|u_0\|_\infty \qquad t \geq 0\,. \tag{16}$$

(iv) *Let u_0 be non-negative in $C_b(\mathbb{R})$. Then, there exists a unique generalised solution of the IVP (1), (2) in the sense of [Gil89].*
(v) *Let u_0 be non-negative in $\chi_{m,q}$ with*

$$\chi_{m,q} = \{v \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\,, v^m \in W^{1,\infty}_{loc}(\mathbb{R}), (v^m)_x - v^q \in W^{1,1}(\mathbb{R})\}\,.$$

Then, $u(t)$ is in $C_b([0,\infty), L^1(\mathbb{R}))$ [BT], the mild solution is the generalised solution and there exists $p_0 \in (1,2)$ such that for any $\tau > 0$ and any compact set $K \subset \mathbb{R} \times (0,\infty)$,

$$\|(u^q)_x\|_{L^\infty(K)} + \|(u^m)_{xx}\|_{L^{p_0}(K)} + \|(u_t)_x\|_{L^{p_0}(K)} \leq C(K, \|u(\tau)\|_{L^\infty(\mathbb{R})})\,, \tag{17}$$

where C is a constant just depending on $K, \|u(\tau)\|_{L^\infty(\mathbb{R})}$ and τ [LS98].
(vi) *Solutions with compactly supported initial data remain compactly supported for all $t \geq 0$ [Gil88].*

Remark 2.1 *The space $\chi_{m,q}$ is dense in $L^1(\mathbb{R})$ [BT].*

Proposition 2.3 ***(Temporal decay estimates)***
Let $m = 1$ and $u_0 \in L^1(\mathbb{R}^N)$ for $N \geq 1$, then for $t_0 > 0$ and $p \in [1,\infty)$ [EZ]

$$\|u(t)\|_\infty \leq C(M,t_0)\,t^{-\frac{N}{2}} \qquad t \geq t_0 > 0\,, \tag{18}$$

$$\|u_x(t)\|_\infty \leq C(M,t_0)\,t^{-\frac{N}{2}-\frac{1}{2}} \qquad t \geq t_0 > 0\,, \tag{19}$$

while if moreover $u_0 \in L^r(\mathbb{R}^N)$ *with* $r > \frac{q}{3}$, *then [DZ99]*

$$\|u(t)\|_\infty \;\leq\; C(M)\, t^{-\frac{N}{2}} \qquad t > 0\,. \tag{20}$$

Let $m > 1$ *and* $u_0 \in L^1(\mathbb{R})$ *and* $p \in [1,\infty)$, *then there holds [LS98]*

$$\|u(t)\|_p \;\leq\; C(M,p)\, t^{-\frac{1}{m+1}(1-\frac{1}{p})} \qquad t > 0\,, \tag{21}$$

$$\|u(t)\|_\infty \;\leq\; C(M)\, t^{-\frac{1}{m+1}} \qquad t > 0\,. \tag{22}$$

Moreover, it follows from [Rey, Theorem 3.1], that for $t_0 > 0$

$$\|(u^{m-1})_x(t)\|_\infty \;\leq\; C(M,t_0)\, t^{-\frac{m}{m+1}} \qquad t \geq t_0 > 0\,. \tag{23}$$

Proposition 2.4 ***(Source-type solutions as asymptotic profiles)***
The source-type solutions in the case $m = 1$ *are mass-multiples of the heat kernel, while in the case* $m > 1$, *they are the Barenblatt-Pattle solutions of the porous medium equation:*

$$U_M(t,x) = \begin{cases} M\,(4\pi t)^{-\frac{1}{2}}\, e^{-\frac{|x|^2}{4t}}, & m = 1\,, \\ t^{-\frac{1}{m+1}} \left(\beta_M^2 - c_m^2 |x|^2 t^{-\frac{2}{m+1}}\right)_+^{\frac{1}{m-1}} & m > 1\,, \end{cases} \tag{24}$$

where $c_m^2 = \frac{m-1}{2m(m+1)}$ *and* β_M^2 *is a constant such that* $\|U_M\|_1 = M$.

2.2 Rescaling

As mentioned in the introduction, it is convenient to rescale the variables and the solution as

$$x = yR(t)\,, \qquad s = \frac{1}{\lambda}\ln(1+\lambda t)\,, \qquad \rho(s,y) = R(t)u\,, \tag{25}$$

where $R(t)$ is defined as

$$R(t) := (\lambda t + 1)^{\frac{1}{\lambda}}\,, \qquad R(s) = e^s\,. \tag{26}$$

We emphasise that (25) leaves the initial data $\rho_0 = u_0$ unchanged and preserves the L^1-norm $\|\rho(t)\|_1 = \|u(t)\|_1$ for all $t \geq 0$.

The rescaling (25) transforms the convection-diffusion equation (1) into a nonlinear Fokker-Planck type equation

$$\rho_s = (\rho y)_y + (\rho^m)_{yy} - R^{-\delta}(\rho^q)_y\,, \tag{27}$$

$$\rho(s=0,y) = u_0(y)\,, \tag{28}$$

where we have set

$$\lambda = m+1\,, \qquad \delta = q - (m+1) > 0\,.$$

Proposition 2.5 ***(Rescaled existence results and decay estimates)*** *Let $m=1$ and u_0 be non-negative in $L^1(\mathbb{R}^N)$. Then, $\rho \in C([0,\infty), L^1(\mathbb{R}^N)) \cap C((0,\infty), W^{2,p}(\mathbb{R}^N)) \cap C^1((0,\infty), L^p(\mathbb{R}^N))$ for every $p \in (1,\infty)$, preserves the mass $\int_{\mathbb{R}} \rho(s,y)\,dy = M$ and satisfies, for $s_0>0$ and $p\in[1,\infty)$, the bounds*

$$\begin{aligned} \|\rho(s)\|_p &\leq C(M,p)\, s^{-\frac{N}{2}(1-\frac{1}{p})} \qquad s>0\,, & (29)\\ \|\rho(s)\|_\infty &\leq C(M,s_0) \qquad s\geq s_0>0\,, & (30)\\ \|\rho_x(s)\|_\infty &\leq C(M,s_0) \qquad s\geq s_0>0\,. & (31) \end{aligned}$$

If moreover $u_0 \in L^p(\mathbb{R})$ with $p\in(\frac{q}{3},\infty]$, then

$$\begin{aligned} \|\rho(s)\|_\infty &\leq C(M,s_0)\, s^{-\frac{N}{2}} \qquad 0<s\leq s_0\,, & (32)\\ \|\rho(s)\|_p &\leq \|u_0\|_p\, e^{\frac{p-1}{p}\lambda s} \qquad s\geq 0\,. & (33) \end{aligned}$$

*Let $m>1$. Then, the statements **(i)**, **(iv)-(vi)** of proposition 2.2 remain unchanged for ρ replacing u. Only the items **(ii)**, **(iii)** changes to*

$$\|\rho(s)\|_p \leq \|u_0\|_p\, e^{\frac{p-1}{p}\lambda s} \qquad s\geq 0\,. \qquad (34)$$

for $u_0 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ with $p\in(1,\infty]$. The rescaled bounds of proposition 2.3 guarantee still $\rho\in L^\infty((0,\infty)\times\mathbb{R})$. For $p\in[1,\infty)$, we have

$$\begin{aligned} \|\rho(s)\|_p &\leq C(M,p)\, s^{-\frac{1}{m+1}(1-\frac{1}{p})} \qquad s>0\,, & (35)\\ \|\rho(s)\|_\infty &\leq C(M,s_0)\, s^{-\frac{1}{m+1}} \qquad 0<s\leq s_0\,, & (36)\\ \|\rho(s)\|_\infty &\leq C(M,s_0) \qquad s\geq s_0>0\,, & (37)\\ \|(\rho^{m-1})_x(s)\|_\infty &\leq C(M,s_0) \qquad s\geq s_0>0\,. & (38) \end{aligned}$$

Remark 2.2 *It is worthwhile to mention that the proof of the L^1-L^∞ smoothing effect (22) leading to (37) (based on a technique of [BB] and adapted in [LS97]) works only for the u variables, but not in the rescaled setting.*

Proposition 2.6 ***(Rescaled asymptotic profiles)***
The soure-type solutions (24) rescales to the profiles

$$\rho_M(s,y) = \begin{cases} M(2\pi)^{-\frac{1}{2}}\left(1-e^{-2s}\right)^{-\frac{1}{2}} e^{-\frac{|y|^2}{2}\left(1-e^{-2s}\right)^{-1}} & m=1\,,\\ \lambda^{\frac{1}{\lambda}}\left(1-e^{-s\lambda}\right)^{-\frac{1}{\lambda}}\left(\beta_M^2 - c_m^2\lambda^{\frac{2}{\lambda}}\left(1-e^{-s\lambda}\right)^{-\frac{2}{\lambda}}|y|^2\right)_+^{\frac{1}{m-1}} & m>1\,, \end{cases} \qquad (39)$$

which tend, as $s\to\infty$, towards stationary profiles, which are directly obtained by rescaling the time shifted profiles

$$U_M(x,t+\lambda^{-1}) \;\to\; \rho_{M,\infty}(y) = \begin{cases} M(2\pi)^{-\frac{1}{2}} e^{-\frac{y^2}{2}} & m=1\,,\\ \lambda^{\frac{1}{\lambda}}\left(\beta_M^2 - c_m^2\lambda^{\frac{2}{\lambda}}\,y^2\right)_+^{\frac{1}{m-1}} & m>1\,. \end{cases} \qquad (40)$$

Direct expansion of the time shifted profiles yields the convergence rates

$$\|u_M(t)-u_M(t+\lambda^{-1})\|_1 = O\left(\lambda t+1\right)^{-1}, \qquad \|\rho_M-\rho_{M,\infty}\|_1 = O\left(e^{-\lambda s}\right). \quad (41)$$

3 Convergence Rates

Our approach uses the entropy functionals

$$H(\rho) = \begin{cases} \int_{\mathbb{R}} (\rho \frac{|y|^2}{2} + \rho \ln \rho)\, dy & m = 1\,, \\ \int_{\mathbb{R}} (\rho \frac{|y|^2}{2} + \frac{1}{m-1}\rho^m)\, dy & m > 1\,, \end{cases} \quad (42)$$

for which the following results are proved in [CJMTU, DP, Ott].

Proposition 3.1 ***(Entropy-entropy dissipation estimates),***
(i) *The stationary profile $\rho_{M,\infty}$ is the minimiser of the entropy functional H in the class of all admissible comparison functions χ_M*

$$\chi_M := \left\{ \rho \in L^1(\mathbb{R}^N) : \rho \geq 0, \int_{\mathbb{R}} \rho(y)\, dy = M \right\}$$

(ii) *For $\rho, \rho_{M,\infty} \in \chi_M$ and with the* relative entropy

$$H\left(\rho|\rho_{M,\infty}\right) = H(\rho) - H(\rho_{M,\infty}) \geq 0\,, \quad (43)$$

the following Csiszár-Kullback type inequality *is valid with a constant C.*

$$\|\rho - \rho_{M,\infty}\|_1 \leq C\, H\left(\rho|\rho_{M,\infty}\right)^{\alpha} \qquad \text{with} \quad \alpha = \begin{cases} \frac{1}{2} & 1 \leq m \leq 2\,, \\ \frac{1}{m} & 2 < m\,. \end{cases} \quad (44)$$

(iii) *The* generalised logarithmic Sobolev inequality

$$H\left(\rho|\rho_{M,\infty}\right) \leq \frac{1}{2} \int_{\mathbb{R}} \rho\, \xi(y,\rho)_y^2\, dy =: D(\rho)\,, \quad (45)$$

where

$$\xi(y,\rho) := \begin{cases} \frac{|y|^2}{2} + \ln(\rho) & m = 1\,, \\ \frac{|y|^2}{2} + \frac{m}{m-1}\rho^{m-1} & m > 1\,, \end{cases} \quad (46)$$

holds as long as the right hand side of (45), the so called entropy dissipation *term $D(\rho)$, is finite.*

Idea of the proof

To demonstrate formally our approach, it will be useful to rewrite (27) as

$$\rho_s = \left(\rho\,\xi(y,\rho)_y - R^{-\delta}\rho^q\right)_y \,. \tag{47}$$

Multiplication of (47) with $\xi(y,\rho)$ and integration by parts yields

$$\frac{d}{ds}H(\rho) = -\int_{\mathbb{R}} \rho\,\xi(y,\rho)_y^2\,dy + R^{-\delta}\int_{\mathbb{R}} \rho^q y\,dy\,. \tag{48}$$

Let $u_0(1+|x|^2) \in L^1(\mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R})$ and consider firstly $0 < s_0 < s_1$. Then, by (33) resp. (34) and by Young's inequality, it follows that

$$\|\rho\|_\infty^{q-1}\int_{\mathbb{R}} \rho|y|\,dy \;\le\; C(s_1, \|u_0\|_\infty)\left(M + \int_{\mathbb{R}} \rho y^2\,dy\right) \qquad 0 \le s \le s_1\,.$$

Since the second order moment is bound in terms of the entropy

$$\int_{\mathbb{R}} \rho y^2\,dy \le CH(\rho)\,, \tag{49}$$

with a constant C (which is well-known also for $m=1$, see e.g. [CJMTU]) a Gronwall lemma applied on (48) yields the propagation of the initial entropy:

$$H(\rho(s)) \le C(s_1, M, \|u_0\|_\infty)H(u_0) \qquad 0 \le s \le s_1\,. \tag{50}$$

Next, for $s_0 \le s \le s_1$, the global L^∞-bounds (30), resp. (37) and the global $W^{1,\infty}$-bounds (31) resp. (38) justify the logarithmic Sobolev inequality (45) for all $s_1 > s_0$ such that

$$\begin{aligned} \frac{d}{ds}H(\rho) &\le -2H(\rho|\rho_{M,\infty}) + R^{-\delta}\int_{\mathbb{R}} \rho^q y\,dy \qquad s \ge s_0\,, \\ &\le -2H(\rho) + 2H(\rho_{M,\infty}) + R^{-\delta}\|\rho\|_\infty^{q-1}\frac{1}{2}\left(M + CH(\rho)\right), \end{aligned} \tag{51}$$

by (43), (49) and by Young's inequality. Moreover, using the entropy bound (50) as well as the global L^∞-bounds of (30) resp. (37), we estimate that

$$\frac{d}{ds}H(\rho) \le \left(-2 + R^{-\delta}C_1(s_0, M)\right)H(\rho) + C_2(s_0, M) \qquad s_0 \le s \le s_1\,.$$

By choosing s_0 such that $R^{-\delta}(s_0)\,C_1(s_0, M) < 1$, a Gronwall lemma yields

$$H(\rho(s)) \le C(s_0, M, H(\rho(s_0))) \qquad s_0 \le s \le s_1\,,$$

for all $s_1 > s_0$. Hence, in combination with (50), the entropy and therefore the second order moment is globally bound:

$$\int_{\mathbb{R}} \rho y^2\, dy \le CH(\rho(s)) \le C(s_0, M, \|u_0\|_\infty, H(u_0)) \qquad s \ge 0\,. \tag{52}$$

Considering the relative entropy, we may now estimate the second term on the r.h.s. (51) using the global entropy bound (52) such that we obtain

$$\frac{d}{ds} H\left(\rho|\rho_{M,\infty}\right) \le -2H\left(\rho|\rho_{M,\infty}\right) + R^{-\delta} C(s_0, M, \|u_0\|_\infty, H(u_0)) \qquad s \ge s_0\,,$$

and therefore, by a Gronwall lemma,

$$\begin{aligned} H\left(\rho|\rho_{M,\infty}\right) &\le C(s_0, M, \|u_0\|_\infty, H(u_0))\, e^{-\min\{\delta,2\}\, s} \qquad s \ge s_0\,, \\ \|\rho - \rho_{M,\infty}\|_1 &\le C_1(s_0, M, \|u_0\|_\infty, H(u_0))\, e^{-\min\{\delta,2\}\alpha\, s} \qquad s \ge s_0\,, \end{aligned}$$

where we have used the Csiszár-Kullback type inequality (44).

In the cases $\delta < 2$, the convergence rate can be improved iteratively by the following argument: Since $\int_{\mathbb{R}} \rho_{M,\infty}^q y\, dy = 0$, the Taylor expansion $\rho^q = \rho_{M,\infty}^q + q\zeta^{q-1}(\rho - \rho_{M,\infty})$ with $|\zeta| \le \max\{\|\rho(t)\|_\infty, \|\rho_{M,\infty}\|_\infty\}$ gives

$$\int_{\mathbb{R}} \rho^q y\, dy \le \|q\zeta\|_\infty^{q-1} \|(\rho - \rho_\infty) y^2\|_1^{\frac{1}{2}} \|\rho - \rho_\infty\|_1^{\frac{1}{2}} \le C(s_0, M, \|u_0\|_\infty, H(u_0))\, e^{-\frac{\delta\alpha}{2} s}\,,$$

where we have used the L^∞-bounds (30) resp. (37) together with the entropy bound (52). Hence the last term on the r.h.s. of (51) decays actually with the rate $\delta_1 = \delta(1 + \frac{\alpha}{2})$. Iterating this argument, the improvement of the L^1-convergence rate stops after finitely many iterations at the optimal rate 2α as soon as $\delta_k = \delta_{k-1}(1 + \frac{\alpha}{2}) \ge 2$.

Finally, rescaling backwards together with (41) gives the L^1-convergence rates stated in theorem 3.1. The convergence rates r_p in other L^p-norms follow by interpolation with the L^∞-bounds (30) resp.(37)

$$\|u - u_M\|_p \le R^{\frac{1-p}{p}}(t) \|\rho - \rho_M\|_\infty^{\frac{1-p}{p}} \|u - u_M\|_1^{\frac{1}{p}} \le C(1+t)^{-\frac{p-1}{p(m+1)} - \frac{r_1}{p}}. \tag{53}$$

3.1 Proof of the main result

Theorem 3.1 *Let $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be non-negative with bounded second order moment $\int_{\mathbb{R}} u_0 |x|^2\, dx < \infty$. Then, the solutions u of (1)-(2) satisfies*

$$\|u - U_M\|_1 \le C(M, \|u_0\|_\infty, H(u_0))(1+t)^{-r_1}\,,$$

with a constant C *depending on* M, $\|u_0\|_\infty$ *and* $H(u_0)$ *and the* L^1*-convergence rates*

$$r_1(q,m) = \begin{cases} \frac{1}{m+1} & 1 \le m < 2\,, \\ \frac{2}{m(m+1)} & 2 \le m\,. \end{cases} \tag{54}$$

Moreover, for $p \in (1, \infty)$,

$$\|u - U_M\|_p \le C(p, M, \|u_0\|_\infty, H(u_0))(1+t)^{-r_p}\,,$$

with

$$r_p = \frac{p-1}{p(m+1)} + \frac{r_1}{p}\,. \tag{55}$$

Remark 3.1 *In the degenerate parabolic case* $m > 1$, *the* L^1*-rates (54) improve by interpolation the known* L^p*-rates of [LS98].*

Proof of theorem 3.1: In the linear diffusion case $m = 1$, the regularity of the classical solution (proposition 2.5) justifies the formal integration by parts and therefore the global bound of the second order moment (52). Together with the L^∞-bound (30) and the $W^{1,\infty}$-bound (31) this ensures that the entropy dissipation $D(\rho)$ is globally bound. Hence, the logarithmic Sobolev inequality (45) is justified, which completes the proof in this case.

In the degenerate diffusion case $m > 1$, we restrict firstly to compactly supported initial data in $\chi_{m,q}$. Then, for $s_0 > 0$, the mild solution $u(s_0)$ is compactly supported in $C_b(\mathbb{R})$. Taking $u(s_0)$ as initial data, we consider an approximating sequence $u(s_0)_k$ introduced in [Gil89, pp. 182-183] (cf. also [LS98]) with

$$\frac{1}{k} \le u(s_0)_k \le 2\|u(s_0)\|_\infty\,, \qquad u(s_0)_{k+1} \le u(s_0)_k\,,$$

and $u(s_0)_k$ converges to $u(s_0)$ in $C(\mathbb{R})$. By denoting u_k the classical solution of (1) with initial data $u(s_0)_k$, it follows that

$$\frac{1}{k} \le u_k \le 2\|u(s_0)\|_\infty \qquad t \ge 0\,, \tag{56}$$

and that the limit

$$\lim_{k\to\infty} u_k(x,t) = \overline{u}(x,t) \tag{57}$$

exists for all $(x,t) \in \mathbb{R} \times [0,\infty)$ such that $\overline{u}(x,t) = u(x, t+s_0)$ [Gil89]. We denote in the following with ρ_k the rescaled approximative solutions. Since the approximative solutions are no longer integrable, we use a sequence of cutoff-functions $\theta_n = \theta(\frac{x}{n})$ for $n \in \mathbb{N}$, where $\theta \in \mathcal{D}(\mathbb{R})$ with $0 \le \theta \le 1$ and

$\theta(y) = 1$ for $|y| \leq 1$ and $\theta(y) = 0$ for $|y| \leq 2$. Since the approximation alters only the initial data, we have of

$$(\rho_k)_s = \left(\rho_k \left(\frac{y^2}{2} + \frac{m}{m-1}\rho_k^{m-1}\right)_y + R^{-\delta}\rho_k^q\right)_y, \tag{58}$$

for which multiplication with $(\frac{y^2}{2} + \frac{m}{m-1}\rho_k^{m-1})\theta_n$ and integration by parts gives

$$\begin{aligned}\frac{d}{ds}\int_{\mathbb{R}} \left(\rho_k \frac{y^2}{2} + \frac{1}{m-1}\rho_k^m\right)\theta_n\,dy = &-\int_{\mathbb{R}} \rho_k \left(\frac{y^2}{2} + \frac{m}{m-1}\rho_k^{m-1}\right)_y^2 \theta_n\,dy \\ &-\int_{\mathbb{R}} \rho_k\,\left(y + m\rho_k^{m-2}(\rho_k)_y\right)\left(\frac{y^2}{2} + \frac{m}{m-1}\rho_k^{m-1}\right)(\theta_n)_y\,dy \\ &+R^{-\delta}\int_{\mathbb{R}} \rho_k^q y\,\theta_n\,dy - R^{-\delta}\frac{m}{q+m-1}\int_{\mathbb{R}} \rho_k^{q+m-1}\,(\theta_n)_y\,dy \\ &+R^{-\delta}\int_{\mathbb{R}} \left(\rho_k^q \frac{y^2}{2} + \frac{m}{m-1}\rho_k^{q+m-1}\right)(\theta_n)_y\,dy\,. \end{aligned} \tag{59}$$

Expanding the first term on the right hand side of (59) leads to

$$-\int_{\mathbb{R}} \rho_k y^2 \theta_n\,dy - 2\int_{\mathbb{R}} (\rho_k^m)_y y\,\theta_n\,dy - \frac{m^2}{(m-1)^2}\int_{\mathbb{R}} \rho_k (\rho_k^{m-1})_y^2\,\theta_n\,dy\,. \tag{60}$$

For the first term, the pointwise convergence (57) of $\rho_k \to \rho$ and the uniform L^∞-bound (56) on ρ_k justify to pass to the limit $k \to \infty$ for all $n \in \mathbb{N}$. For the second term of (60), (17) implies since $q > m+1$ that ρ_k^m is uniformly bound in $W^{2,p_0}(K) \cap W^{1,\infty}(K)$ [LS98]. By a compactness argument [LS98] and by the pointwise convergence (57), there exists a subsequence ρ_k such that

$$\lim_{k\to\infty} \rho_k = \rho \quad \text{in} \quad C(\mathbb{R} \times (0,\infty))\,, \tag{61}$$

$$\lim_{k\to\infty} (\rho_k^m)_y(s) = (\rho^m)_y(s) \quad \text{in} \quad L^{p_0}_{loc}(\mathbb{R}) \quad \text{for almost every } s > 0\,, \tag{62}$$

and therefore

$$\lim_{k\to\infty} \int_{\mathbb{R}} (\rho_k^m)_y\,y\,\theta_n\,dy = \int_{\mathbb{R}} (\rho^m)_y\,y\,\theta_n\,dy \qquad \forall n \in \mathbb{N},$$

for almost every $s > 0$. For the third term of (60), the uniform L^∞-bound (38) ensures weak convergence of $(\rho_k^{m-1})_y$ in L^2_{loc} to some limit Ψ and hence

$$\lim_{k\to\infty} \int_{\mathbb{R}} \rho_k (\rho_k^{m-1})_y^2\,\theta_n\,dy = \int_{\mathbb{R}} \rho\Psi\,\theta_n\,dy \qquad \forall n \in \mathbb{N}, \tag{63}$$

such that it remains to identify Ψ as $(\rho^{m-1})_y^2$ in integral on the r.h.s. of (63). By using (38), then

$$\lim_{k\to\infty}\int_{\{\rho=0\}}\rho_k(\rho_k^{m-1})_y^2\,\theta_n\,dy=0\qquad\forall n\in\mathbb{N},$$

and thus, contributions to this integral come only from the open set $\{\rho>0\}$ (since ρ is continuous), on which (61) implies $\rho_k(s,y)\searrow\rho(s,y)$ for all (s,y) and (62) yields, for another subsequence,

$$(\rho_k^m)_y(s,y)=\frac{m}{m-1}\rho_k(\rho_k^{m-1})_y\to\frac{m}{m-1}\rho(\rho^{m-1})_y=(\rho^m)_y(s,y)\quad\text{a.e.}\,.$$

Hence, we conclude almost everywhere convergence of $(\rho_k^{m-1})_y^2\to(\rho^{m-1})_y^2$ on the set $\{\rho>0\}$ and the Lebesgue dominated convergence theorem yields

$$\lim_{k\to\infty}\int_{\mathbb{R}}\rho_k(\rho_k^{m-1})_y^2\,\theta_n\,dy=\int_{\mathbb{R}}\rho(\rho^{m-1})_y^2\,\theta_n\,dy\qquad\forall n\in\mathbb{N}.$$

Returning to (59), the second term on the right hand side expands with further integration by parts to

$$-\int_{\mathbb{R}}\left(\rho_k\frac{y^3}{2}+\frac{1}{m-1}\rho_k^m y\right)(\theta_n)_y\,dy+\int_{\mathbb{R}}\left(\rho_k^m\frac{y^2}{2}+d_m\rho_k^{2m-1}\right)(\theta_n)_{yy}\,dy\,,\quad(64)$$

with $d_m=m^2/(2m^2-3m+1)$ and (56) allows to pass to the limit in k for all $n\in\mathbb{N}$. Similarly, it is justified to drop the subscript k in the remainder terms of (59).

For the limit $n\to\infty$ of

$$\begin{aligned}\frac{d}{ds}\int_{\mathbb{R}}\left(\rho\frac{y^2}{2}+\frac{1}{m-1}\rho^m\right)\theta_n\,dy=&-\int_{\mathbb{R}}\rho\left(\frac{y^2}{2}+\frac{m}{m-1}\rho^{m-1}\right)_y^2\theta_n\,dy\\&+R^{-\delta}\int_{\mathbb{R}}\rho^q y\,\theta_n\,dy+\int_{\mathbb{R}}O(\rho y^3,\rho^m y,\rho^{q+m-1},\rho^q y^2)(\theta_n)_y\,dy\\&+\int_{\mathbb{R}}O(\rho^m y^2,\rho^{2m-1})(\theta_n)_{yy}\,dy\,,\end{aligned}$$

the bounds (37) and (38) imply again, since ρ is compactly supported, that each integrant is bounded uniformly in n by integrable functions and hence, by the dominated convergence theorem, that

$$\frac{d}{ds}H(\rho)=-\int_{\mathbb{R}}\rho\left(\frac{y^2}{2}+\frac{m}{m-1}\rho^{m-1}\right)_y^2dy+R^{-\delta}\int_{\mathbb{R}}\rho^q y\,dy\qquad s\geq s_0\,,$$

with a global bound on the entropy dissipation $D(\rho)$ justifying the generalised logarithmic Sobolev inequality (45). Finally, Gronwall's lemma yields with (52)

$$H(\rho) \leq C(M, \|u_0\|_\infty, H(u_0))\, e^{-\min\{2,\delta\}\, s} + H(\rho_\infty), \qquad s \geq s_0\,, \tag{65}$$

for ρ compactly supported in $\chi_{m,q}$ and therefore in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ by a density argument (cf. remark 2.1).

In order to obtain the result for all $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with finite entropy, we consider smooth truncations u_{0k} with compact support and reduced mass $M_k = \int_{\mathbb{R}} u_{0k}\, dx$ such that

$$\begin{gathered} u_{0k} \to u_0 \quad \text{in} \quad L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\ , \\ H(u_{0k}) \to H(u_0) \quad \text{in} \quad \mathbb{R}, \qquad M_k \nearrow M \quad \text{in} \quad \mathbb{R}, \end{gathered}$$

and therefore $H(\rho_{M_k,\infty}) \nearrow H(\rho_{M,\infty})$. Then, the L^1-contraction (14) implies that, for a subsequence, ρ_k converges to ρ almost everywhere and hence

$$\lim_{k\to\infty} \left(\rho_k \frac{y^2}{2} + \frac{1}{m-1}\rho_k^m\right) = \left(\rho\frac{y^2}{2} + \frac{1}{m-1}\rho^m\right) \quad \text{almost everywhere}\,.$$

Since the r.h.s. of (65) (for ρ_k instead of ρ) depends continuously on the approximation, we have the uniform bound

$$H(\rho_k) \leq 2C(M, \|u_0\|_\infty, H(u_0))\, e^{-\min\{2,\delta\}\, s} + H(\rho_{M,\infty}),$$

for k large enough and Fatou's lemma implies

$$H(\rho|\rho_{M,\infty}) \leq \liminf_{k\in\mathbb{N}} H(\rho_k|\rho_{M,\infty}) \leq 2C(M, \|u_0\|_\infty, H(u_0))\, e^{-\min\{2,\delta\}\, s},$$

and hence the decay of $H(\rho|\rho_{M,\infty})$ as stated in the theorem. The improvement of the decay rates follows from the formal argument. ■

3.2 Generalisations

Initial data

In section 3, the assumption $u_0 \in L^\infty(\mathbb{R})$ was only used to prove the propagation of the initial entropy until some positive time $s_0 > 0$, since for positive times, the propagation of entropy follows from the propositions 2.3 and 2.5.

Lemma 3.1 *Let $u_0 \in L^1(\mathbb{R}) \cap L^{2q-1}(\mathbb{R})$ be non-negative with bounded second order moment $\int_{\mathbb{R}} u_0 |x|^2\, dx < \infty$. Then, for $s_0 > 0$, holds*

$$H(\rho(s)) \leq C(M, \|u_0\|_{2q-1}) H(u_0) \qquad 0 < s \leq s_0\,.$$

Proof: It is sufficient to show the propagation of the second order moment:

$$\frac{d}{ds}\int_{\mathbb{R}} \rho \frac{y^2}{2}\, dy = -2\int_{\mathbb{R}} \rho \frac{y^2}{2}\, dy + \int_{\mathbb{R}} \rho^m\, dy + R^{-\delta}\int_{\mathbb{R}} \rho^q y\, dy\,,$$

Using Young's inequality for the last term on the r.h.s., we estimate

$$\frac{d}{ds}\int_{\mathbb{R}} \rho \frac{y^2}{2}\, dy \leq -2(1-R^{-\delta})\int_{\mathbb{R}} \rho \frac{y^2}{2}\, dy + \int_{\mathbb{R}} \rho^m\, dy + R^{-\delta}\int_{\mathbb{R}} \rho^{2q-1}\, dy\,,$$

and the lemma follows by the propagation of the L^{2q-1}-norm (33) resp. (34) and a Gronwall lemma. □

Corollary 3.1 *Let $u_0 \in L^1(\mathbb{R}) \cap L^{2q-1}(\mathbb{R})$ be non-negative with bounded second order moment $\int_{\mathbb{R}} u_0 |x|^2\, dx < \infty$. Then, theorem 3.1 holds.*

Theorem 3.2 *Let $q < m+2$ and $u_0 \in L^1(\mathbb{R}) \cap L^m(\mathbb{R})$ be non-negative with bounded second order moment $\int_{\mathbb{R}} u_0|x|^2\, dx < \infty$. Then theorem 3.1 holds.*

Proof: For $s_0 > 0$, we estimate for (48) that

$$\frac{d}{ds}H(\rho) \leq \left|\int_{\mathbb{R}} \rho^q y\, dy\right| \leq \|\rho\|_\infty^{q-1}\left(\int_{\mathbb{R}} \rho\, dy + \int_{\mathbb{R}} \rho y^2\, dy\right) \qquad 0 < s \leq s_0\,,$$

and therefore, using the sharp bounds (32) resp. (36) as well as (49),

$$\frac{d}{ds}H(\rho) = \frac{d}{ds}(M + H(\rho)) \leq C(M)\, s^{-\frac{q-1}{m+1}}\,(M + H(\rho)) \qquad 0 < s \leq s_0\,.$$

Since $q < m+2$, $s^{-\frac{q-1}{m+1}}$ is integrable on the interval $(0, s_0)$ and a general Gronwall lemma implies the propagation of entropy and the theorem. ■

Higher dimensions

The entropy method of section 3 is essential independent of the space dimension and applies formally straight-forward to the multidimensional IVP's

$$u_t = \triangle u^m - d\cdot \nabla u^q\,, \qquad (66)$$

$$u(t=0,x) = u_0(x) \geq 0\,, \qquad (67)$$

for $(t,x) \in [0,\infty) \times \mathbb{R}^N$ and $d \in \mathbb{R}^N$ with $N > 1$, i.e. to the rescaled IVP's

$$\rho_s = \nabla\cdot(\rho y + \nabla(\rho^m)) - R^{-\delta} d\cdot\nabla\rho^q\,,$$

$$\rho(s=0,y) = u_0(y) \geq 0\,.$$

Theorem 3.3 *Let $m = 1$ and Let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for $N > 1$ be non-negative with bounded second order moment $\int_{\mathbb{R}} u_0|x|^2\,dx < \infty$. Then, the solutions u of (66)-(67) satisfies*

$$\|u - U_M\|_1 \leq C(M, \|u_0\|_\infty, H(u_0))(1+t)^{-r_1}\,, \tag{68}$$

with a constant C depending on M, $\|u_0\|_\infty$ and $H(u_0)$ and the L^1-convergence rates

$$r_1(q,m) = \begin{cases} \frac{1}{N(m-1)+2} & 1 \leq m < 2\,, \\ \frac{2}{m}\frac{1}{N(m-1)+2} & 2 \leq m\,. \end{cases} \tag{69}$$

Moreover, for $p \in (1,\infty)$,

$$\|u - U_M\|_p \leq C(p, M, \|u_0\|_r, H(u_0))(1+t)^{-r_p}\,, \tag{70}$$

with

$$r_p = \frac{N}{N(m-1)+2}\frac{p-1}{p} + \frac{r_1}{p}\,. \tag{71}$$

Proof: The proof of the theorem 3.3 is analog to theorem 3.1, since in the case $m = 1$, the existence theory (proposition 2.1) and the temporal decay estimates (proposition 2.3) have been equally well established for all $N \geq 1$ [EZ, DZ99]. ■

Remark 3.2 ***(Multidimensional degenerate diffusion)***
For $m > 1$ and $N > 1$, existence theory and sufficient L^∞-decay estimates have been shown in [RV]. Nevertheless, to prove theorem 3.3, we lack a multi-dimensional analog of (23) in order to control the entropy dissipation.

4 Absorption-diffusion equation

The entropy approach, which was presented in section 3, applies in a modified way to the absorption-diffusion equation

$$u_t = (u^m)_{xx} - u^r\,,$$
$$u(t=0,x) = u_0(x) \geq 0\,,$$

for (t,x) in $[0,\infty) \times \mathbb{R}$ and for the range of exponents

$$m \geq 1\,, \quad r > m+2\,,$$

for which the large-time behaviour is also dominated by the diffusion term (see e.g. [GV84, KP, Vaz]). For non-negative $u_0 \in L^1(\mathbb{R})$, it was shown that

$$\lim_{t\to\infty} t^{\frac{1}{m+1}}|u(t) - U_{M(\infty)}(t)| \to 0\,,$$

uniformly on the sets $\{|x| \le a t^{\frac{1}{m+1}}\}$ for $a \ge 0$ and the positive limiting mass

$$M(\infty) := \|u_0\|_1 - \int_0^\infty \int_{\mathbb{R}} u^r \, dx \, dt > 0 \,.$$

Assuming non-negative integrable initial data, existence and uniqueness of non-negative bounded and continuous weak solutions of (4) can be found, for instance, in [Kal, GV91] and the references therein. We shall consider non-negative $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, by comparison with the heat resp. porous media equation, the solutions of (4) obey the L^∞-decay (20) resp. (22) in proposition 2.3. The $W^{1,\infty}$-decay (19) resp. (23) of proposition 2.3 has been shown in [HV].

The rescaling (25) transforms (4) into

$$\begin{aligned} \rho_s &= (\rho y)_y + (\rho^m)_{yy} - R^{-\delta} \rho^r \,, & (72)\\ &= \left(\rho\, \xi(y,\rho)_y\right)_y - R^{-\delta} \rho^r \,, & (73) \end{aligned}$$

where $\delta = r - (m+2) > 0$ and $\xi(y,\rho)$ as in (46) and the solution $\rho(s)$ satisfies in particular

$$\|\rho(s)\|_\infty \le C(\|u_0\|_1, \|u_0\|_\infty) \qquad s > 0 \,. \tag{74}$$

Moreover, the absorption term leads obviously to a decay of the mass $M(s) := \int_{\mathbb{R}} \rho(s,y)\, dy$:

$$\frac{d}{ds} M(s) = -R^{-\delta} \int_{\mathbb{R}} \rho^r \, dy \,. \tag{75}$$

By (74) and (75), the L^p-norms are globally bound for all $p \in [1,\infty]$. Hence, by multiplying (72) with y^{2n} for $n \in \mathbb{N}$, integration by parts yields

$$\frac{d}{ds} \left\|\rho y^{2n}\right\|_1 \le \left(-2n + R^{-\delta} \|\rho\|_\infty^{r-1}\right) \left\|\rho y^{2n}\right\|_1 + 2n(2n-1) \|\rho\|_\infty^{m-1} \left\|\rho y^{2(n-1)}\right\|_1$$

and therefore, by iteration $2(n-1) \to 2n$, the propagation of the moments

$$\left\|\rho y^{2n}\right\|_1 (s) \le C(\|u_0\|_1, \|u_0\|_\infty) \left\|u_0(y) y^{2n}\right\|_1 \qquad s > 0 \,, n \in \mathbb{N} \,. \tag{76}$$

The entropy approach is based on the generalised logarithmic Sobolev inequality (45) and the Csiszár-Kullback type inequality (44), which are valid on the set of admissible comparison functions χ_M with the same mass M (see proposition 3.1). We shall therefore consider the relative entropy with respect to the asymptotic profile $\rho_{M(s),\infty}$ depending the actual mass $M(s)$

$$H(\rho|\rho_{M(s),\infty}) := H(\rho) - H(\rho_{M(s),\infty}) \ge 0 \qquad s \ge 0 \,. \tag{77}$$

For the time-derivative of the entropy $H(\rho)$, we multiply (73) with $\xi(y,\rho)$ and integrate by parts. Using (75) in the case $m=1$, we obtain

$$\frac{d}{ds}H(\rho) = -D(\rho) - R^{-\delta}\int_{\mathbb{R}} \rho^r \times \begin{cases} \left(\frac{y^2}{2} + \ln\rho + 1\right) dy & m=1\,, \\ \left(\frac{y^2}{2} + \frac{m}{m-1}\rho^{m-1}\right) dy & m>1\,, \end{cases} \tag{78}$$

and, as a direct consequence, the propagation of the entropy

$$H(\rho(s)) \le C(\|u_0\|_1, \|u_0\|_\infty) H(u_0) \qquad s>0\,. \tag{79}$$

In order to calculating the time-derivative of the asymptotic entropy $H(\rho_{M(s),\infty})$, we observe that $D(\rho_{M(s),\infty}(y)) = 0$ implies $\rho_{M(s),\infty}\,\xi(\rho_{M(s),\infty})_y = 0$ and therefore, on the support of $\rho_{M(s),\infty}$,

$$\xi_{M(s)} := \xi(\rho_{M(s),\infty}) = \left\{ \begin{array}{ll} \ln(\frac{M(s)}{\sqrt{2\pi}}) & m=1 \\ \frac{m}{m-1}\lambda^{\frac{m-1}{\lambda}}\beta^2_{M(s)} & m>1 \end{array} \right\} \text{ where } \ \rho_{M(s),\infty}(y) > 0\,. \tag{80}$$

In the case $m=1$, we have $H(\rho_{M(s),\infty}) = M\xi_M$, while in the case $m>1$, we use $H(\rho_{M(s),\infty}) = M\xi_M - \int (\rho_{M(s),\infty})^m\,dy$ and obtain by straightforward calculations

$$\frac{d}{ds}H(\rho_{M(s),\infty}) = \frac{dM}{ds} \times \begin{cases} (\ln(\frac{M(s)}{\sqrt{2\pi}}) + 1) & m=1\,, \\ \xi_{M(s)} & m>1\,. \end{cases} \tag{81}$$

By combining (78) with (81) (and recalling (40) for $m=1$), we obtain for the time-derivative of the relative entropy

$$\frac{d}{ds}H(\rho|\rho_{M(s),\infty}) = -D(\rho) - R^{-\delta}\int_{\mathbb{R}} \rho^r \times \begin{cases} \left(\ln\rho - \ln\rho_{M(s),\infty}\right) dy & m=1\,, \\ \left(\frac{y^2}{2} + \frac{m}{m-1}\rho^{m-1} - \xi_{M(s)}\right) dy & m>1\,. \end{cases} \tag{82}$$

By the generalised logarithmic Sobolev inequality (45) and since the solution is bounded in L^p for $p \in [1,\infty]$, we estimate

$$\frac{d}{ds}H(\rho|\rho_{M(s),\infty}) \le -2H(\rho|\rho_{M(s),\infty}) + R^{-\delta}\,C(\|u_0\|_1, \|u_0\|_\infty)\,,$$

and the Csiszár-Kullback type inequality (44) implies that $\|\rho - \rho_{M(s),\infty}\|_1$ decays exponentially with rate $\min\{2,\delta\}\alpha$. Analog to section 3, we use a bootstrapping argument to improve this rate: Considering firstly the case $m=1$, where

$$\begin{aligned} \rho^2\ln\rho - \rho^2\ln(\rho_{M,\infty}) &= \rho^2\ln\rho - \rho^2_{M,\infty}\ln(\rho_{M,\infty}) + \ln(\rho_{M,\infty})(\rho^2_{M,\infty} - \rho^2) \\ &= \left(\zeta(y)(1 + 2\ln\zeta(y)) + \left(-\ln(\frac{M}{\sqrt{2\pi}}) + \frac{y^2}{2}\right)(\rho + \rho_{M,\infty})\right)(\rho - \rho_{M,\infty})\,, \end{aligned}$$

with $\zeta(y)$ is in the interval $(\rho(y), \rho_{M,\infty}(y))$, we estimate with Hölder's inequality that

$$\int_{\mathbb{R}} \rho^{r-2} \left(\zeta(1+2\ln\zeta) + (\rho+\rho_{M,\infty}) \left(-\ln(\frac{M}{\sqrt{2\pi}}) + \frac{y^2}{2} \right) \right) (\rho - \rho_{M,\infty})\, dy$$
$$\leq \left(\|\rho\|_\infty^{r-2} \|\zeta(1+2\ln\zeta)\|_\infty + \|\rho\|_\infty^{r-2} \|\rho+\rho_{M,\infty}\|_\infty \ln(\tfrac{M}{\sqrt{2\pi}}) \right) \|\rho-\rho_{M,\infty}\|_1$$
$$+ \|\rho\|_\infty^{r-2} \|\rho+\rho_{M,\infty}\|_\infty \tfrac{1}{4} \, \|(\rho-\rho_{M,\infty}) y^4\|_1^{\frac{1}{2}} \, \|\rho-\rho_{M,\infty}\|_1^{\frac{1}{2}} \,.$$

In the case $m > 1$, we use $\rho^r_{M(s),\infty}\xi_{M(s)} = \rho^r_{M(s),\infty}\left(\frac{y^2}{2} + \frac{m}{m-1}\rho^{m-1}_{M(s),\infty}\right)$ to expand the r.h.s. of (82) as

$$\int_{\mathbb{R}} \left((\rho^r - \rho^r_{M,\infty}) \frac{y^2}{2} + \frac{m}{m-1} (\rho^{r+m-1} - \rho^{r+m-1}_{M,\infty}) - \xi_M (\rho^r - \rho^r_{M,\infty}) \right) dy$$
$$\leq C\left(p, \|\zeta\|_\infty, \|(\rho-\rho_{M,\infty})y^4\|_1^{\frac{1}{2}}\right) \|\rho-\rho_{M,\infty}\|_1^{\frac{1}{2}} + C(p, \|\zeta\|_\infty) \|\rho-\rho_{M,\infty}\|_1$$

We conclude that the integral on the r.h.s of (82) decays as $\|\rho - \rho_{M(s),\infty}\|_1^{\frac{1}{2}}$ provided that the initial fourth order moment $\|u_0 x^4\|_1$ is bounded. Analog to section 3, this improves for $\delta < 2$ the exponential decay rate of $\|\rho - \rho_{M(s),\infty}\|_1$ to $\min\{2, \delta_1\}\alpha$ with $\delta_1 = \delta(1+\frac{\alpha}{2})$ and, after finitely many steps, we obtain the optimal rate 2α as stated in

Theorem 4.1 ***(Decay to the time-dependent asymptotic profile)*** *Let $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be non-negative with bounded fourth order moment $\int_{\mathbb{R}} u_0 |x|^4\, dx < \infty$. Let $U_{M(t)}$ denote the rescaled time-dependent asymptotic profile $\rho_{M(s),\infty}$. Then, the solutions u of (4)-(5) satisfies*

$$\left\|u - U_{M(t)}\right\|_1 \leq C(\|u_0\|_1, \|u_0\|_\infty, \|u_0 x^4\|_1)(1+t)^{-r_1}\,,$$

with a constant C depending on $\|u_0\|_1$, $\|u_0\|_\infty$ and $\|u_0 x^4\|_1$ and with the L^1-convergence rates

$$r_1(q,m) = \begin{cases} \frac{1}{m+1} & 1 \leq m < 2\,, \\ \frac{2}{m(m+1)} & 2 \leq m\,. \end{cases} \tag{83}$$

Moreover, for $p \in (1,\infty)$,

$$\left\|u - U_{M(t)}\right\|_p \leq C(p, \|u_0\|_1, \|u_0\|_\infty, H(u_0))(1+t)^{-r_p}\,,$$

with

$$r_p = \frac{1}{m+1}\frac{p-1}{p} + \frac{r_1}{p}\,. \tag{84}$$

Remark 4.1 *The assumption* $\int_{\mathbb{R}} u_0 |x|^4 \, dx < \infty$ *in relation to the propagation of the even moments (76) has been made for the sake of simplicity of the above arguments and can easily be relaxed to* $\int_{\mathbb{R}} u_0 |x|^{2+\varepsilon} \, dx < \infty$ *for* $\varepsilon > 0$.

Proof: The proof of theorem 4.1 is overall similar to theorem 3.1 but it is also alternative for $m > 1$ by applying the logarithmic Sobolev inequalities directly to the regularised problems, i.e. (4) on a bounded interval $(-n, n)$ with homogeneous Dirichlet data (see also [KPV]) and a sequence of data $\rho_{0,n} \searrow \rho_0$ pointwise and monotonely from above such that

$$\|\rho_{0,n}\|_1 = \left(1 + \frac{1}{n}\right) \|\rho_0\|_1 .$$

Then, the solutions ρ_n are positive and smooth on $(-n, n)$ and satisfy the mass decay

$$\frac{d}{ds} M_n(s) \le R^{-\delta} \int_{-n}^{n} \rho_n^r dy,$$

since $\int_{-n}^{n} (\rho_n^m)_{yy} \, dy \le 0$. Proceeding analog to the above formal calculations, it is straightforward to show that the approximating relative entropy $H_n(\rho_n | \rho_{n,M_n(s),\infty}) = \int_{-n}^{n} \rho_n (\frac{y^2}{2} + \frac{1}{m-1} \rho_n^{m-1}) \, dy$ decays equally or less as stated in (82), such that a generalized logarithmic Sobolev inequality on $(-n, n)$ [CJMTU] and a bootstrapping argument analog to above imply convergence of ρ_n to $\rho_{n,M_n(s),\infty}$ as stated in the theorem, which follows by the limit $n \to \infty$. ■

Remark 4.2 ***(Multidimensional absorption diffusion)***
Theorem 4.1 holds also for the N*-dimensional absorption-diffusion equation*

$$u_t = \triangle(u^m) - u^r ,$$

with $(t, x) \in [0, \infty) \times \mathbb{R}^N$ *and* $N > 1$*, provided the changed* L^p*-rates*

$$r_p = \frac{N}{N(m-1)+2} \frac{p-1}{p} + \frac{r_1}{p} . \tag{85}$$

Corollary 4.1 ***(Decay to the stationary asymptotic profile)***
Let $N \ge 1$*. Let* $U_{M(\infty)}$ *denote the stationary asymptotic profile with mass* $M(\infty)$*. Then, since* $\int_{\mathbb{R}} \rho^r_{M(\infty),\infty} \, dy > 0$*, it follows from (75) that*

$$\left\| u - U_{M(\infty)} \right\|_1 \le C(\|u_0\|_1, \|u_0\|_\infty, \|u_0 x^4\|_1)(1+t)^{-r_1} ,$$

with

$$r_1(q, m) = \begin{cases} \min\{1, \delta\} \frac{1}{N(m-1)+2} & 1 \le m < 2 , \\ \min\{\frac{2}{m}, \delta\} \frac{1}{N(m-1)+2} & 2 \le m . \end{cases} \tag{86}$$

5 Convergence in Wasserstein metric

The results of this section are proven formally in such a way that it should remain only a matter of adapting the existence theory by providing appropriate smoothings of the solutions such as in [RV, CGT, CT03, LT] to make them rigorous.

We are going to compare two solutions ρ_i with $i = 1, 2$ of

$$\tfrac{\partial \rho_i}{\partial s} = (\rho_i y)_y + (\rho_i^m)_{yy} - \theta_i R^{-\delta} (\rho_i^q)_y \,, \qquad \theta_i \in \{0,1\}\,, \tag{87}$$

$$\rho_i(s=0,y) = u_{0,i}(y) \geq 0\,, \qquad i = 1,2\,, \tag{88}$$

subject to non-negative initial data $u_{0,i} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with normalised masses $\|u_{0,i}\|_1 = 1$ and with all moments bound, i.e. $\int_\mathbb{R} u_{0,i}|x|^k\,dx < \infty$ for all $k \in \mathbb{N}$. By setting $\theta_i = 0$, we drop the convection term of (27) (i.e. $\theta_i = 1$) and consider mass-preserving solutions ρ_i of the rescaled heat $(m = 1)$ respectively porous medium equation $(m > 1)$.

The propagation of the k-th order moment follows formally by multiplying (87) with $|y|^k$, provided that the $(k-1)$-th and the $(k-2)$-th order moment are bound:

$$\begin{aligned} \frac{d}{ds}\int_\mathbb{R} \rho_i |y|^k\,dy &= -k\int_\mathbb{R} \rho_i |y|^k\,dy + k(k-1)\int_\mathbb{R} \rho_i^m |y|^{k-2}\,dy \\ &\quad -k\int_\mathbb{R} \rho_i^m |y|^{k-1}\delta(y)\,dy + \theta_i k R^{-\delta}\int_\mathbb{R} \rho_i^q |y|^{k-1}\mathrm{sign}(y)\,dy\,, \end{aligned}$$

where the third term on the r.h.s. – involving the δ-distribution – is bounded for $k = 1$ by the L^∞-bound on the solution. Hence, by iteration and via a smoothing argument, all moments of ρ_i remain bounded:

$$\int_\mathbb{R} \rho_i(s,y)|y|^k\,dy < \infty \qquad s > 0\,, \quad k \in \mathbb{N}\,, \quad i = 1,2\,. \tag{89}$$

The Wasserstein p-distances of two one-dimensional probability measures ρ_1, ρ_2 with bounded moments (89) can be written as [Vil]

$$W_p(\rho_1(s), \rho_2(s))^p = \int_0^1 |F_1^{-1}(s,z) - F_1^{-1}(s,z)|^p\,dz \qquad p \in [1,\infty)\,,$$

in terms of the generalised inverses F_i^{-1} of the ρ_i-distribution functions F_i:

$$F_i^{-1}(s,z) = \inf\{y : F_i(y) > z\}\,, \qquad F_i(s,y) := \int_{-\infty}^y \rho_i\,dy\,, \qquad i = 1,2\,.$$

Theorem 5.1 ***(Decay of the Wasserstein p-distances)***
Let $0 \leq u_{0,i} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with normalised masses $\|u_{0,i}\|_1 = 1$ and bounded moments $\int_{\mathbb{R}} u_{0,i}|x|^k\,dx < \infty$ for all $k \in \mathbb{N}$ and $i = 1,2$. Then, for the solutions ρ_1, ρ_2 of (87) and independent of $\theta_i \in \{0,1\}$

$$W_p(\rho_1(s), \rho_2(s)) \leq C(\|u_{0,1}\|_\infty, \|u_{0,2}\|_\infty)\, W_p(u_{0,1}, u_{0,2})\, e^{-s}\,,$$

for all $p \in [1,\infty)$ and with a constant C depending on $\|u_{0,1}\|_\infty$, $\|u_{0,2}\|_\infty$. Backward rescaling yields

$$W_p(u_1(t), u_2(t)) \leq C(\|u_{0,1}\|_\infty, \|u_{0,2}\|_\infty)\, W_p(u_{0,1}, u_{0,2})\,.$$

Formal proof: Note firstly that taking derivatives of $F_i(s, F_i^{-1}(s,z)) = z$ with respect to s and z yields the relations

$$\frac{\partial F_i}{\partial s}(s,y) = -\rho_i(s,y)\frac{\partial F_i^{-1}}{\partial s}(s,z)\,, \qquad \rho_i(s,y) = \left(\frac{\partial F_i^{-1}}{\partial z}(s,z)\right)^{-1}, \tag{90}$$

where $y = F_i^{-1}(s,z)$. Next, integrating $\int_\infty^y$ (87) dy yields

$$\frac{\partial F_i}{\partial s} = y\rho_i(s,y) + (\rho_i^m)_y(s,y) - \theta_i R^{-\delta}\rho_i^q(s,y)$$

and using (90) as well as $\frac{\partial \rho_i}{\partial y} = -\frac{\partial^2 F_i^{-1}}{\partial z^2}(s,z)(\frac{\partial F_i^{-1}}{\partial z}(s,z))^{-3}$ we derive an evolution equation for $F_i^{-1}(s,z)$:

$$\frac{\partial F_i^{-1}}{\partial s} = -F_i^{-1} - \frac{\partial}{\partial z}\left(\frac{\partial F_i^{-1}}{\partial z}\right)^{-m} + \theta_i R^{-\delta}\left(\frac{\partial F_i^{-1}}{\partial z}\right)^{1-q}. \tag{91}$$

The evolution of the Wasserstein p-distances follows by subtracting (91) for $i = 1,2$, multiplying with $p\left|F_1^{-1} - F_2^{-1}\right|^{p-1}\operatorname{sign}(F_1^{-1} - F_2^{-1})$ and integrating:

$$\frac{d}{ds}\int_0^1 \left|F_1^{-1} - F_2^{-1}\right|^p\,dz = -p\int_0^1 \left(F_1^{-1} - F_2^{-1}\right)^p\,dz \tag{92}$$
$$+p(p-1)\int_0^1\left(\left(\tfrac{\partial F_1^{-1}}{\partial z}\right)^{-m} - \left(\tfrac{\partial F_2^{-1}}{\partial z}\right)^{-m}\right)\left|F_1^{-1} - F_2^{-1}\right|^{p-2}\left(\tfrac{\partial F_1^{-1}}{\partial z} - \tfrac{\partial F_2^{-1}}{\partial z}\right)dz$$
$$+pR^{-\delta}\int_0^1\left(\theta_1\left(\tfrac{\partial F_1^{-1}}{\partial z}\right)^{1-q} - \theta_2\left(\tfrac{\partial F_2^{-1}}{\partial z}\right)^{1-q}\right)\left|F_1^{-1} - F_2^{-1}\right|^{p-1}\operatorname{sgn}(F_1^{-1} - F_2^{-1})dz$$

The second term on the r.h.s. of (92) is non-positive since $(x^{-m} - y^{-m})(x - y) \leq 0$, while the third term can be estimated with the L^∞-bound (37) since $\rho_i^{-1} = \frac{\partial F_2^{-1}}{\partial z}$. With Hölder's inequality for the third term, we obtain

$$\frac{d}{ds}\int_0^1 \left(F_1^{-1} - F_2^{-1}\right)^p dz \leq -p\int_0^1 \left(F_1^{-1} - F_2^{-1}\right)^p dz$$
$$+pR^{-\delta}\left\|\theta_1\rho_1^{q-1} - \theta_2\rho_2^{q-1}\right\|_\infty \left(\int_0^1 \left(F_1^{-1} - F_2^{-1}\right)^p dz\right)^{\frac{p-1}{p}}.$$

Hence, by a Gronwall lemma, $W_p(\rho_1, \rho_2)^p$ is globally bounded for all $p \in [1, \infty)$, which implies further that $W_p(\rho_1, \rho_2)^p$ decays actually exponentially with rate $\min\{p, \delta\}$. Improving iteratively the rate, we conclude the optimal rate p for the decay of $W_p(\rho_1, \rho_2)^p$. The result stated in the theorem follows using

$$W_p(u_1(t,x), u_2(t,x)) = W_p(\rho_1(s,y), \rho_2(s,y))\, e^s\,,$$

which is easily shown via a change of variables. ■

Concerning our last result, we recall the finite speed of propagation in the case $m > 1$, which is that initially compactly supported solutions of (87) for either $\theta_i = 0, 1$ remain compactly supported in time [Vaz, Gil88]. Moreover, we observe that the Wasserstein p-distances $W_p(\rho_1, \rho_2)$ are increasing with respect to p such that the limit

$$W_\infty(\rho_1, \rho_2) := \lim_{p \uparrow \infty} W_p(\rho_1, \rho_2) = \sup_{z \in (0,1)} \mathrm{ess}\, |F_1^{-1}(s,z) - F_2^{-1}(s,z)|\,,$$

is well-defined and W_∞ satisfies the same decay estimates as W_p.

Corollary 5.1 ***(Estimate on the propagation of the support)*** *Let $m > 1$ and $0 \le u_{0,i} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $\|u_{0,i}\|_1 = 1$ be compactly supported. Let $\Omega_i(s)$ resp. $\Omega_i(t)$ denote to supports of $\rho_i(s,y)$ resp. $u_i(t,x)$. Moreover, define $I_i(\cdot) := \inf\{\Omega_i(\cdot)\}$ as well as $S_i(\cdot) := \sup\{\Omega_i(\cdot)\}$. Then,*

$$\max\{|I_1(s) - I_2(s)|, |S_1(s) - S_2(s)|\} \le C(\|u_{0,1}\|_\infty, \|u_{0,2}\|_\infty)\, W_\infty(u_{0,1}, u_{0,2})\, e^{-s},$$

respectively

$$\max\{|I_1(t) - I_2(t)|, |S_1(t) - S_2(t)|\} \le C(\|u_{0,1}\|_\infty, \|u_{0,2}\|_\infty)\, W_\infty(u_{0,1}, u_{0,2})\,.$$

Proof: Since the inverse $F_i^{-1}(s,z)$ of distributions $F_i = \int_{-\infty}^{y} \rho_i\, dy$ with integrable compactly supported densities $\rho_i(s,y)$ is continuous at $z = 0, 1$, we estimate

$$\sup_{z \in (0,1)} \mathrm{ess}\, |F_1^{-1}(s,z) - F_2^{-1}(s,z)| \ge \max\left\{|F_1^{-1}(s,0) - F_2^{-1}(s,0)|,\right.$$
$$\left.|F_1^{-1}(1) - F_2^{-1}(1)|\right\} \ge \max\left\{|I_1(s) - I_2(s)|, |S_1(s) - S_2(s)|\right\}\,.$$

■

Remark 5.1 *By taking $\theta_1 = 1$ such that ρ_1 is the solution of the convection-diffusion equation (27) and by choosing $\rho_2 = \rho_{1,\infty}$ the Barenblatt-Pattle solution for $\theta_2 = 0$, corollary 5.1 implies exponential decay of the boundaries of the support of ρ_1 towards the boundaries of the support of the Barenblatt-Pattle profile. In the original variables, this implies that the boundaries of the support remain bounded to each-other, while the solutions spread out in time at the rate given by the Barenblatt-Pattle profile.*

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