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Exponential Decay Towards Equilibrium for the Inhomogeneous Aizenman-Bak Model

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Abstract: The Aizenman-Bak model for reacting polymers is considered for spatially inhomogeneous situations in which they diffuse in space with a non-degenerate size-dependent coefficient. Both the break-up and the coalescence of polymers are taken into account with fragmentation and coagulation constant kernels. We demonstrate that the entropy-entropy dissipation method applies directly in this inhomogeneous setting giving not only the necessary basic a priori estimates to start the smoothness and size decay analysis in one dimension, but also the exponential convergence towards global equilibria for constant diffusion coefficient in any spatial dimension or for non-degenerate diffusion in dimension one. We finally conclude by showing that solutions in the one dimensional case are immediately smooth in time and space while in size distribution solutions are decaying faster than any polynomial. Up to our knowledge, this is the first result of explicit equilibration rates for spatially inhomogeneous coagulation-fragmentation models.

Key words: Coagulation-fragmentation continuous in size models with spatial diffusion, exponential equilibration rate, entropy-entropy dissipation method, reaction-diffusion systems.

AMS subject classification: 35B40, 35B45, 82D60

1. Introduction

We analyze the spatial inhomogeneous version of a size-continuous model for reacting polymers or clusters of aggregates:

$$\partial_t f - a(y) \triangle_x f = Q(f,f) . \tag{1.1}$$

Here, $f = f(t,x,y)$ is the concentration of polymers/clusters with length/size $y \geq 0$ at time $t \geq 0$ and point $x \in \Omega \subset \mathbb{R}^d$, $d \geq 1$. These polymers/clusters diffuse in the environment Ω. This set is assumed to be a smooth bounded domain with normalized volume, i.e., $|\Omega| = 1$. In the one dimensional case, we will set $\Omega = (0,1)$. Equation (1.1) is to be considered with homogeneous Neumann boundary condition

$$\nabla_x f(t,x,y) \cdot \nu(x) = 0 \qquad \text{on } \partial\Omega \tag{1.2}$$

with ν the outward unit normal to Ω, so that there is no polymer flux through the physical boundary. We assume the diffusion coefficient $a(y)$ to be non-degenerate in the sense that there exist $a_*, a^* \in \mathbb{R}^+$ such that

$$0 < a_* \leq a(y) \leq a^*. \tag{1.3}$$

On the other hand, the reaction term $Q(f,f)$ of (1.1) models chemical degradation -break-up or fragmentation- and polymerization -coalescence or coagulation- of polymers/clusters. More precisely, the full collision operator reads as

$$\begin{aligned} Q(f,f) &= Q_c(f,f) + Q_b(f,f) = Q^+(f,f) - Q^-(f,f) \\ &= Q_c^+(f,f) - Q_c^-(f,f) + Q_b^+(f,f) - Q_b^-(f,f) \end{aligned} \tag{1.4}$$

with obvious definitions of the coagulation $Q_c(f,f)$, fragmentation or break-up $Q_b(f,f)$, loss $Q^-(f,f)$ and gain $Q^+(f,f)$ operators which are determined from the four basic terms in (1.4):

1. Coalescence of clusters of size $y' \leq y$ and $y - y'$ results into clusters of size y:

$$Q_c^+(f,f) := \int_0^y f(t,x,y-y') f(t,x,y')\, dy'. \tag{1.5}$$

2. Polymerization of clusters of size y with other clusters of size y' produces a loss in its concentration:

$$Q_c^-(f,f) := 2 f(t,x,y) \int_0^\infty f(t,x,y')\, dy'. \tag{1.6}$$

3. Break-up of clusters of size y' larger than y contributes to create clusters of size y:

$$Q_b^+(f,f) := 2 \int_y^\infty f(t,x,y')\, dy'. \tag{1.7}$$

4. Break-up of polymers of size y reduces its concentration:

$$Q_b^-(f,f) := y\, f(t,x,y). \tag{1.8}$$

This kind of models finds its application not only in polymers and cluster aggregation in aerosols [S16, S17, AB, Al, Dr] but also in cell physiology [PS], population dynamics [Ok] and astrophysics [Sa]. Here, fragmentation and coagulation kernels are all set up to constants as in the original Aizenman-Bak model [AB]. This will be of paramount importance in the basic a-priori estimates. The conservation of the total number of monomers at time $t \geq 0$ quantified by

$$\int_\Omega N(t,x)\,dx, \quad \text{where} \quad N(t,x) := \int_0^\infty y\, f(t,x,y)\,dy$$

is the basic conservation law satisfied by equation (1.1) since the reaction term (1.4) satisfies

$$\int_\Omega \int_0^\infty y\, Q(f,f)\,dy\,dx = 0,$$

and thus, assuming initially a positive total number of monomers, we formally conclude

$$\int_\Omega \int_0^\infty y\, f(t,x,y)\,dy\,dx = \int_\Omega N(t,x)\,dx = \int_\Omega N_0(x)\,dx := N_\infty > 0. \qquad (1.9)$$

Another macroscopic quantity of interest is the number density of polymers,

$$M(t,x) := \int_0^\infty f(t,x,y)\,dy, \qquad (1.10)$$

that together with the total number of monomers $N(t,x)$ satisfies the reaction-diffusion system

$$\partial_t N - \triangle_x \left(\int_0^\infty y a(y)\, f(t,x,y)\,dy \right) = 0\,, \qquad (1.11)$$

$$\partial_t M - \triangle_x \left(\int_0^\infty a(y)\, f(t,x,y)\,dy \right) = N - M^2\,, \qquad (1.12)$$

becoming a closed decoupled system in the constant diffusion case $(a(y) := a)$:

$$\partial_t N - a\,\triangle_x N = 0\,, \qquad (1.13)$$

$$\partial_t M - a\,\triangle_x M = N - M^2\,. \qquad (1.14)$$

The definition of the full collision operator has to be understood in the weak sense as

$$< Q(f,f), \varphi >= \int_0^\infty \int_0^\infty (f(y'') - f(y)f(y'))\,(\varphi(y) + \varphi(y') - \varphi(y''))\,dy\,dy' \quad (1.15)$$

for any smooth function $\varphi(y)$, where $y'' = y + y'$ and the dependence on (t,x) of the density function has been dropped for notational convenience. An alternative weak formulation that can be useful in several arguments below is obtained integrating by parts in the Q_b^+ part giving

$$< Q(f,f), \varphi >= -2 \int_0^\infty \varphi(y) f(y)\,dy \int_0^\infty f(y')\,dy' + \int_0^\infty \int_0^\infty f(y) f(y') \varphi(y'')\,dy\,dy'$$
$$+ 2 \int_0^\infty f(y)\,\Phi(y)\,dy - \int_0^\infty y\, f(y)\,\varphi(y)\,dy \qquad (1.16)$$

for any smooth function φ, the function Φ being the primitive of φ ($\partial_y \Phi = \varphi$) such that $\Phi(0) = 0$.

Let us consider the (free-energy) entropy functional associated to any positive density f as

$$H(f)(t,x) = \int_0^\infty (f \ln f - f)\, dy\,, \tag{1.17}$$

and the relative entropy $H(f|g) = H(f) - H(g)$ of two states f and g not necessarily with the same L^1_y-norm. Then, the entropy formally dissipates as

$$\begin{aligned} \frac{d}{dt}\int_\Omega H(f)\,dx = & -\int_\Omega \int_0^\infty a(y)\frac{|\nabla_x f|^2}{f}\,dy\,dx \\ & -\int_\Omega \int_0^\infty\!\!\int_0^\infty (f'' - ff')\ln\left(\frac{f''}{ff'}\right) dy\,dy'dx \;:= -D_H(f) \end{aligned} \tag{1.18}$$

with obvious notations.

Global existence and uniqueness of classical solutions has been studied in [Am, AW] for some particular cases, namely, for constant diffusion coefficient or dimension one with additional restrictions for the coagulation and fragmentation kernel not including the AB model. The initial boundary-value problem to (1.1)-(1.2) was then analyzed in [LM02-1], for much more general coagulation and fragmentation kernels including the AB model (1.5) - (1.8), proving the global existence of weak solutions satisfying the entropy dissipation inequality

$$\int_\Omega H(f(t))\,dx + \int_0^t D_H(f(s))\,ds \le \int_\Omega H(f_0)\,dx$$

for all $t \ge 0$.

The equilibrium states for which the entropy dissipation vanishes are better understood after applying the following remarkable inequality proven in [AB, Propositions 4.2 and 4.3] (reviewed in section 2):

$$\int_0^\infty\!\!\int_0^\infty (f'' - ff')\ln\left(\frac{f''}{ff'}\right) dy\,dy' \ge M\,H(f|f_{M,N}) + (M^2 - N)\ln\frac{M^2}{N}\,. \tag{1.19}$$

Herein, $f_{M,N}$ denotes a distinguished, exponential-in-size distribution with the very moments M and N :

$$f_{M,N}(t,x,y) = \frac{M^2}{N} e^{-\frac{M}{N}y}\,.$$

These distributions $f_{M,N}$ appear as analog to the so-called intermediate or local equilibria in the study of inhomogeneous kinetic equation (e.g. [DV01, CCG, FNS, DV05, FMS, NS]), but in contrast they persist dissipating entropy until $M^2 = N$. Finally, the conservation of mass (1.9) identifies (at least formally) the global equilibrium f_∞ with constant moments $M_\infty^2 = N = N_\infty$:

$$f_\infty = e^{-\frac{y}{\sqrt{N_\infty}}}\,. \tag{1.20}$$

The analogy to intermediate equilibria carries over to the following additivity of relative entropies

$$H(f|f_\infty) = H(f|f_{M,N}) + H(f_{M,N}|f_\infty)\,. \tag{1.21}$$

It is worthy to point out that even if $f_{M,N}$ and f_∞ do not have the same L^1_y-norm, its global relative entropy

$$\int_\Omega H(f_{M,N}|f_\infty)\,dx$$

is given by

$$\int_\Omega \int_0^\infty \left[f_{M,N}\ln f_{M,N} - f_\infty \ln f_\infty - (1+\ln f_\infty)(f_{M,N}-f_\infty)\right]\,dy\,dx$$

by the conservation law (1.9), and thus, it is a nonnegative quantity, as easily checked via Taylor expansion. In [LM02-1], it is proved that f_∞ attracts all global weak solutions in $L^1(\Omega\times(0,\infty))$ of (1.1)-(1.2) but no time decay rate is obtained. This result is the analog to convergence results along subsequences for the classical Boltzmann equation in [De].

Other existence and uniqueness results for inhomogeneous coagulation-fragmentation models were given in [CD] and the references therein. Finally, let us mention that the conservation law (1.9) is known not to hold for certain coagulation-fragmentation kernels, phenomena known as gelation [ELMP], and the convergence or not towards typical self-similar profiles for the pure coagulation models is a related issue; we refer to [Le, LM05, MP]. We refer finally to [LM02-1, LM03, LM04] for an extensive list of related literature.

Let us now discuss some works on the study of the long time asymptotics for related models. Qualitative results concerning a discrete version of a coagulation-fragmentation system, the Becker-Döring system, have been obtained in [CP, LW, LM02-2] and the references therein. We emphasize that global explicit decay estimates towards equilibrium were obtained for the Becker-Döring system without diffusion in [JN] by entropy-entropy dissipation methods. Other techniques have recently been developed for inhomogeneous kinetic equations. We refer to [MN] for a spectral approach and to [V06] for a general description of the concept of hypocoercivity. Note that the presence of diffusion instead of advection makes possible in our present context, not to use the concept of hypocoercivity.

In this work we prove exponential decay towards equilibrium with explicit rates and constants in the next two situations.

In the special case of size-independent diffusion coefficients $a(y) = a$, we prove in section 2 exponential decay towards equilibrium in all space dimensions $d \geq 1$ by exploiting the closed system (1.13)–(1.14) for N and M. Subsequently, improving the entropy estimates by using the arguments in Section 4, we get :

Theorem 1. *Let Ω be a smooth bounded connected open set of $\mathbb{R}^d$, $d \geq 1$ and let us assume that the nonnegative initial data $f_0 \neq 0$ is such that $(1+y+\ln f_0)f_0 \in L^1(\Omega\times(0,\infty))$ where $M_0(x)$ and $N_0(x)$ are $L^\infty(\Omega)$-functions.*

Then, the global weak solution to the initial boundary value problem (1.1)-(1.8) *with constant diffusion coefficient $a(y) = a > 0$ converges exponentially fast in time with explicitly computable constants C_1, C_2 and rate α towards the global equilibrium* (1.20)*, both in global relative entropy :*

$$\int_\Omega H(f(t)|f_\infty)\,dx \leq C_1 e^{-\alpha\, t}\,, \tag{1.22}$$

and in $L^1_{x,y}$ sense :

$$\|f(t,\cdot,\cdot) - f_\infty\|_{L^1_{x,y}} \le C_2\, e^{-\frac{\alpha}{2}\,t} \tag{1.23}$$

for all $t \ge 0$, where f_∞ is defined by (1.20) *and $N_\infty > 0$ is determined by the conservation of mass* (1.9).

In the case of general diffusion coefficients $a(y)$ satisfying (1.3) we prove in section 3 a-priori estimates in the one-dimensional case $d = 1$. Further, in section 4, we establish an entropy-entropy dissipation estimate for this case. This allows us to establish the following Theorem :

Theorem 2. *Let Ω be the interval* $(0,1)$ *and let us consider a diffusion coefficient satisfying* (1.3). *Let us also assume that $f_0 \ne 0$ is a nonnegative initial datum such that $(1 + y + \ln f_0)f_0 \in L^1((0,1) \times (0,\infty))$.*

Then, the global weak solutions $f(t,x,y)$ of (1.1)–(1.2) *decay exponentially to the global equilibrium state* (1.20) *with explicitly computable constants C_1, C_2 and rate α, both in global relative entropy :*

$$\int_0^1 H(f(t)|f_\infty)\,dx \le C_1 \int_0^1 H(f_0|f_\infty)\,dx\; e^{-\alpha\,t}\,, \tag{1.24}$$

and in $L^1_{x,y}$ sense :

$$\|f(t,\cdot,\cdot) - f_\infty\|_{L^1_{x,y}} \le C_2 \int_\Omega H(f_0|f_\infty)\,dx\, e^{-\frac{\alpha}{2}\,t} \tag{1.25}$$

for all $t \ge 0$, where f_∞ is defined by (1.20) and $N_\infty > 0$ is determined by the conservation of mass (1.9).

It is further possible to interpolate the exponential decay in a "weak" norm like L^1 with polynomially growing bounds in "strong" norms like (weighted) $L^1_y(H^1_x)$ in order to get an exponential decay in a "medium" norm like $L^1_y(L^\infty_x)$. Thus, the decay toward equilibrium can be extended to these stronger norms. The following proposition is proved at the end of section 4:

Proposition 1. *Under the assumptions of Theorem* 2*, for all $t_* > 0$ and $q \ge 0$, there are explicitly computable constants $C_3, \alpha > 0$ such that whenever $t \ge t_*$,*

$$\int_0^\infty (1+y)^q\, \|f(t,\cdot,y) - f_\infty(y)\|_{L^\infty_x}\,dy \le C_3\, e^{-\alpha\,t}. \tag{1.26}$$

A bootstrap argument in the spirit of the proof of Proposition 1 allows to replace the L^∞_x norm by any Sobolev norms in (1.26).

2. A Relevant Particular Case: Constant Diffusion Coefficient.

The aim of this section is the proof of theorem 1. We start by reminding the reader the following functional inequality:

Lemma 1. [AB, Proposition 4.3] *Let $g := g(y)$ be a function of $L^1_+((0,\infty))$ with finite entropy $g \ln g \in L^1((0,\infty))$, then*

$$\int_0^\infty\!\!\int_0^\infty g(y)g(y')\ln g(y+y')\,dy\,dy' \le \left(\int_0^\infty g(y)\,dy\right)\left(\int_0^\infty g(y')\ln g(y')\,dy'\right) - \left(\int_0^\infty g(y)\,dy\right)^2. \tag{2.1}$$

This inequality allows us to show the dissipation inequality (1.19), which is also contained in [AB] but we decided to include it for reader's sake. Expanding the expression of the second part of the entropy dissipation $D_H(f)$ in (1.18), we deduce that

$$\begin{aligned}\int_0^\infty\!\!\int_0^\infty (f''-ff')\ln\left(\frac{f''}{ff'}\right)dy\,dy' \;&=\; -\int_0^\infty\!\!\int_0^\infty ff'\ln f''\,dy\,dy' \\ &\quad + 2\int_0^\infty\!\!\int_0^\infty ff'\ln f\,dy\,dy' \\ &\quad + \int_0^\infty\!\!\int_0^\infty ff'\frac{f''}{ff'}\ln\left(\frac{f''}{ff'}\right)dy\,dy'.\end{aligned}$$

Using Lemma 1 for the first term and collecting terms, we get

$$\begin{aligned}\int_0^\infty\!\!\int_0^\infty (f''-ff')\ln\left(\frac{f''}{ff'}\right)dy\,dy' \;&\ge M^2 + M\int_0^\infty f\ln f\,dy \\ &\quad + \int_0^\infty\!\!\int_0^\infty ff'\frac{f''}{ff'}\ln\left(\frac{f''}{ff'}\right)dy\,dy'.\end{aligned}$$

Using Jensen's inequality for the last term, i.e., normalizing the densities by M^2 and using the convexity of $x\ln x$, and pointing out that

$$\int_0^\infty\!\!\int_0^\infty f''\,dy\,dy' = N,$$

we deduce

$$\begin{aligned}\int_0^\infty\!\!\int_0^\infty (f''-ff')\ln\left(\frac{f''}{ff'}\right)dy\,dy' \;&\ge M^2 + M\int_0^\infty f\ln f\,dy \\ &\quad + N\ln\left(\frac{N}{M^2}\right)dy\,dy'.\end{aligned}$$

Taking into account the definition of the intermediate equilibria $f_{M,N}$ and the relative entropy, we finally conclude the estimate

$$\int_0^\infty\!\!\int_0^\infty (f''-ff')\ln\left(\frac{f''}{ff'}\right)dy\,dy' \ge M\,H(f|f_{M,N}) + (M^2-N)\ln\frac{M^2}{N}\,. \tag{2.2}$$

For the subsequent large-time analysis, we will rather study the relative entropy with respect to the global equilibrium, which dissipates according to (1.18) and (2.2) as

$$\frac{d}{dt}\int_\Omega H(f|f_\infty)\,dx \le -\int_\Omega\int_0^\infty a(y)\frac{|\nabla_x f|^2}{f}\,dy\,dx - \int_\Omega \left[M\,H(f|f_{M,N}) + (M^2-N)\ln\frac{M^2}{N}\right] dx \;:= -D(f). \tag{2.3}$$

In the constant diffusion case, the equations for the first two moments $M(t,x)$ and $N(t,x)$ become the closed system (1.13)-(1.14). This system can be studied by a direct application of the techniques in [DF05, DF06], and the following result is obtained:

Proposition 2. *Let Ω be a smooth bounded connected open set of $\mathbb{R}^d$, $d \ge 1$ and let us assume that the initial data $M_0(x)$ and $N_0(x) \neq 0$ are nonnegative $L^\infty(\Omega)$-functions. Then, the unique global bounded solution of the system* (1.13)-(1.14) *satisfies:*

1. *There exist increasing functions $t \mapsto M_*(t)$, $N_*(t)$ and decreasing functions $t \mapsto M^*(t)$, $N^*(t)$ such that*
$$0 < M_*(t) \le M(t,x) \le M^*(t) < \infty\,, \tag{2.4}$$
$$0 < N_*(t) \le N(t,x) \le N^*(t) < \infty\,, \tag{2.5}$$
for all $t > 0$.
2. *$M(t,x) \to M_\infty = \sqrt{N_\infty}$ and $N(t,x) \to N_\infty$ as $t \to \infty$ in $L^1(\Omega)$ exponentially fast as a consequence of $H(f_{M,N}|f_\infty) \to 0$ exponentially fast as $t \to \infty$ with explicit constants.*

Proof.- The existence and uniqueness of global, classical solutions with global L^∞ bounds from below and above are standard thanks to the maximum principle applied to the equations for N and further for M. We refer, for instance, to [Ro, Ki] for details.

Now, the relative entropy between the intermediate equilibrium and the global one is given by

$$\int_\Omega H(f_{M,N}|f_\infty)dx = \int_\Omega \left[2\sqrt{N}\,(\xi\ln\xi - \xi + 1) + \frac{\left(\sqrt{N}-\sqrt{N_\infty}\right)^2}{\sqrt{N_\infty}} \right] dx, \tag{2.6}$$

where $\xi = \frac{M}{\sqrt{N}}$. It is obvious that the global entropy

$$\int_\Omega H(f_{M,N}|f_\infty)\,dx$$

is positive, using that the minimum of $\xi\ln\xi - \xi + 1$ is zero, and it can be written as

$$\int_\Omega H(f_{M,N}|f_\infty)\,dx = \int_\Omega \left[M\ln\frac{M^2}{N} - 2(M-\sqrt{N}) + 2\left(\sqrt{N_\infty}-\sqrt{N}\right)\right] dx\,,$$

by using the conservation of mass (1.9). One can check, by straightforward manipulations and integration-by-parts, that it dissipates as

$$\frac{d}{dt}\int_\Omega H(f_{M,N}|f_\infty)\,dx = -\int_\Omega (M^2-N)\ln\frac{M^2}{N}\,dx \\ -\frac{a}{2}\int_\Omega M\left|\nabla_x \ln\frac{M^2}{N}\right|^2 dx - \frac{a}{2}\int_\Omega M\frac{|\nabla_x N|^2}{N^2}\,dx\,. \quad (2.7)$$

Now, we estimate the dissipation term on the right hand side of (2.7) as

$$\int_\Omega (M^2-N)\ln\frac{M^2}{N}\,dx \ge 4\int_\Omega \left(M-\sqrt{N}\right)^2 dx\,, \\ \frac{a}{2}\int_\Omega M\frac{|\nabla_x N|^2}{N^2}\,dx \ge \frac{a}{2}\frac{M_*}{N^*}\int_\Omega \left(\sqrt{N}-\overline{\sqrt{N}}\right)^2 dx\,,$$

by the elementary inequality $(x-y)(\ln x - \ln y) \ge 4(\sqrt{x}-\sqrt{y})^2$, by Poincaré's inequality for $|\nabla_x \sqrt{N}|^2$. Here, the notation

$$\overline{\sqrt{N}} = \int_\Omega \sqrt{N}\,dx$$

has been used where we remind that the normalization $|\Omega| = 1$ was fixed. Next, we estimate the relative entropy (2.6) from above using that the function $\Phi(x,y) = (x(\ln x - \ln y) - (x-y))/(\sqrt{x}-\sqrt{y})^2$ is continuous on $(0,\infty)^2 \to \mathbb{R}$, satisfies $\Phi(x,x) = 2$ and $\Phi(x,y) = \Phi(x/y,1)$, and is increasing in the first argument (see [DF05, Lemma 2.1])

$$\int_\Omega \left[2M\ln\frac{M}{\sqrt{N}} - 2\left(M-\sqrt{N}\right)\right] dx \le 2\Phi(M^*/N_*,1)\int_\Omega \left(\sqrt{M}-\sqrt[4]{N}\right)^2 dx\,.$$

Moreover, by the conservation law (1.9)

$$\int_\Omega 2\left(\sqrt{N_\infty}-\sqrt{N}\right) dx = 2\left(\sqrt{\overline{N}}-\overline{\sqrt{N}}\right)\,.$$

In order to join the above estimates, we use

$$\int_\Omega \left(M-\sqrt{N}\right)^2 dx \ge \left(\sqrt{M_*}+\sqrt[4]{N_*}\right)^2 \int_\Omega \left(\sqrt{M}-\sqrt[4]{N}\right)^2 dx\,, \\ \int_\Omega \left(\sqrt{N}-\overline{\sqrt{N}}\right)^2 dx = \overline{N}-\overline{\sqrt{N}}^2 \ \ge\ 2\sqrt{N_*}\left(\sqrt{\overline{N}}-\overline{\sqrt{N}}\right)\,,$$

and obtain therefore

$$-\frac{d}{dt}\int_\Omega H(f_{M,N}|f_\infty)\,dx \ge k(t)\int_\Omega H(f_{M,N}|f_\infty)\,dx\,, \\ k(t) = \min\left\{\frac{2\left(\sqrt{M_*(t)}+\sqrt[4]{N_*(t)}\right)^2}{\Phi(M^*(t)/N_*(t),1)}, \frac{a}{2}\frac{M_*(t)\sqrt{N_*(t)}}{N^*(t)}\right\},$$

where $k(t)$ is positive and increasing with respect to time. In fact, for a given $t_* > 0$, from (2.4)-(2.5), we have $0 < \mathcal{M}_* \le M(t,x) \le \mathcal{M}^* < \infty$ and $0 < \mathcal{N}_* \le N(t,x) \le \mathcal{N}^* < \infty$ for all $t \ge t_*$, and thus

$$k(t) \ge \min\left\{ \frac{2\left(\sqrt{\mathcal{M}_*} + \sqrt[4]{\mathcal{N}_*}\right)^2}{\Phi(\mathcal{M}^*/\mathcal{N}_*, 1)}, \frac{a}{2}\frac{\mathcal{M}_*\sqrt{\mathcal{N}_*}}{\mathcal{N}^*} \right\} = k_*,$$

for all $t \ge t_*$. Denoting the primitive $K(t) = \int_{t_*}^t k(s)\,ds$, we conclude

$$\begin{aligned} \int_\Omega H(f_{M,N}(t)|f_\infty)\,dx &\le \int_\Omega H(f_{M,N}(t_*)|f_\infty)\,dx\, e^{-K(t)} \\ &\le \int_\Omega H(f_{M,N}(t_*)|f_\infty)\,dx\, e^{-k_*\,(t-t_*)}\,. \end{aligned} \tag{2.8}$$

Finally, the convergence towards equilibrium for M and N follows directly from (2.6). In fact, a simple Taylor expansion shows that

$$\|\sqrt{N(t)} - M(t)\|_{L^2_x}^2 + \|\sqrt{N(t)} - \sqrt{N_\infty}\|_{L^2_x}^2 \le L \int_\Omega H(f_{M,N}(t)|f_\infty)\,dx\,,$$

with

$$L = \max\left(1, \frac{\mathcal{M}^*}{\sqrt{\mathcal{N}_*}}, \sqrt{\mathcal{N}^*}\right),$$

that implies by trivial arguments the exponential convergence in $L^1(\Omega)$ towards equilibrium for M and N. □

We now turn back to the

Proof of Theorem 1.- Let us fix $t_* > 0$. From (2.4)-(2.5), we have $0 < \mathcal{M}_* \le M(t,x) \le \mathcal{M}^* < \infty$ and $0 < \mathcal{N}_* \le N(t,x) \le \mathcal{N}^* < \infty$ for all $t \ge t_*$, and thus

$$\mathcal{M}_* \int_\Omega H(f|f_{M,N})\,dx \le \int_\Omega M\, H(f|f_{M,N})\,dx \le D(f)$$

for all $t \ge t_*$. As a consequence, going back to (2.3), we infer that

$$\begin{aligned} \frac{d}{dt}\int_\Omega H(f|f_\infty)\,dx &\le -\,\mathcal{M}_* \int_\Omega H(f|f_{M,N})\,dx \\ &= \mathcal{M}_* \int_\Omega H(f_{M,N}|f_\infty)\,dx - \mathcal{M}_* \int_\Omega H(f|f_\infty)\,dx \\ &\le \mathcal{M}_* \int_\Omega H(f_{M,N}(t_*)|f_\infty)\,dx\, e^{-k_*(t-t_*)} - \mathcal{M}_* \int_\Omega H(f|f_\infty)\,dx \end{aligned}$$

using (1.21) and (2.8), for all $t \ge t_*$. A Gronwall lemma implies that

$$\begin{aligned} \int_\Omega H(f|f_\infty)\,dx \le& \int_\Omega H(f|f_\infty)(t_*)\,dx\, e^{-\mathcal{M}_*(t-t_*)} \\ &+ \mathcal{M}_* \int_\Omega H(f_{M,N}(t_*)|f_\infty)\,dx\, \frac{e^{-k_*(t-t_*)} - e^{-\mathcal{M}_*(t-t_*)}}{\mathcal{M}_* - k_*} \end{aligned} \tag{2.9}$$

(for $\mathcal{M}_* \neq k_*$, which can be assumed without loss of generality and) for all $t \geq t_*$. Noticing that the global entropy

$$\int_\Omega H(f(t)|f_\infty)\,dx$$

is non-increasing in time, we deduce estimate (1.22).

Next, convergence in L^1 as stated in Theorem 1 follows from the functional inequality of Csiszar-Kullback type [Cs, Ku] :

$$\|f(t,\cdot,\cdot) - f_\infty\|^2_{L^1_{x,y}} \leq 2\left\{\int_\Omega M(t,x)\,dx + \sqrt{N_\infty}\right\}\int_\Omega H(f(t)|f_\infty)\,dx\,. \quad (2.10)$$

The proof is standard (see [CCD] for related inequalities) and it is shown via a Taylor expansion of the function $\varphi(f) = f\ln(f) - f$ up to second order around f_∞. Indeed, for a function $\zeta(x,y) \in (\inf\{f(x,y), f_\infty(y)\}, \sup\{f(x,y), f_\infty(y)\})$, we get

$$\int_\Omega H(f|f_\infty)\,dx = \int_\Omega\int_0^\infty \left[-\frac{y}{\sqrt{N_\infty}}(f-f_\infty) + \frac{1}{2\zeta}(f-f_\infty)^2\right]dy\,dx$$

and the first term vanishes due to the conservation law (1.9). For the second term, we apply Hölder's inequality

$$\|f - f_\infty\|^2_{L^1_{x,y}} \leq \|\zeta\|_{L^1_{x,y}}\int_0^1\int_0^\infty \frac{1}{\zeta}(f-f_\infty)^2\,dy\,dx \text{ with } \|\zeta\|_{L^1_{x,y}} \leq \int_\Omega M\,dx + \sqrt{N_\infty}.$$

Noticing that $t \in [0,t_*] \mapsto f(t,\cdot,\cdot) \in L^1_{x,y}$ is bounded from the first item of Proposition 2, we finally get (1.23), which concludes the proof of Theorem 1. □

3. A-priori Estimates

In the sequel, we shall discuss the general diffusion coefficient, i.e., size dependent verifying (1.3) but we restrict to the one dimensional case, $d = 1$ (we shall not recall this fact in the various Lemmas). We begin the proof of Theorem 2.

Lemma 2. *Assume that $f_0 \neq 0$ is a non-negative initial datum such that $(1+y)f_0 \in L^1((0,1)\times(0,\infty))$. Then, there exists $\mathcal{M}_0^* > 0$ such that solutions of (1.1)–(1.8) satisfy*

$$\sup_{t\geq 0}\int_\Omega\int_0^\infty f(t,x,y)\,dy\,dx \;\leq\; \int_\Omega M(t,x)\,dx \;\leq \mathcal{M}_0^* \quad (3.1)$$

Proof.- We estimate the $L^1(\Omega)$-norm of $M(t,x)$ by integrating equality (1.14), obtaining

$$\begin{aligned}\frac{d}{dt}\int_\Omega M(t,x)\,dx &= \int_\Omega N(t,x)\,dx - \int_\Omega M(t,x)^2\,dx \\ &\leq \int_\Omega N_0(x)\,dx - \left(\int_\Omega M(t,x)\,dx\right)^2\end{aligned}$$

by Hölder's inequality and the conservation of mass (1.9). Therefore, for all $t \geq 0$

$$\int_\Omega M(t,x)\,dx \leq \max\left\{\int_\Omega M_0(x)\,dx, \left(\int_\Omega N_0(x)\,dx\right)^{1/2}\right\} := \mathcal{M}_0^*$$

showing (3.1). □

We now turn to a control of the $L^1_y(L^\infty_x)$-norm of f:

Lemma 3. *Assuming that the nonnegative initial datum* $f_0 \neq 0$ *satisfies* $(1 + y + \ln f_0)f_0 \in L^1((0,1)\times(0,\infty))$. *Then, the number density of polymers* $M \in L^1 + L^\infty(0,\infty; L^\infty(0,1))$. *More precisely, there exist* $m_\infty > 0$ *and an* $L^1_+(0,\infty)$-*function* $m_1(t)$ *such that the solution of* (1.1)–(1.8) *satisfies*

$$\int_0^\infty \sup_{0<x<1} f(t,x,y)\,dy \;\leq\; m_\infty + m_1(t) \tag{3.2}$$

and as a consequence,

$$\|M(t,\cdot)\|_{L^\infty_x} \leq m_\infty + m_1(t)$$

a.e. $t \geq 0$.

Proof.- In order to estimate the L^∞_x-norm of $M(t,x)$, we first use the entropy dissipation (1.18) of $H(f)$ to deduce that

$$\begin{aligned} 2\int_0^\infty\!\!\int_0^1\!\!\int_0^\infty \left(\partial_x\sqrt{f}\right)^2 dy\,dx\,dt &\leq \int_0^\infty\!\!\int_0^1\!\!\int_0^\infty \frac{a(y)}{2\,a_*}\frac{(\partial_x f)^2}{f}\,dy\,dx\,dt \\ &\leq \frac{H(f_0)}{2\,a_*} := \mu_1. \end{aligned} \tag{3.3}$$

Now, we integrate

$$\sqrt{f}(t,x,y) - \sqrt{f}(t,\tilde{x},y) = \int_{\tilde{x}}^x \partial_x\sqrt{f}(t,\xi,y)\,d\xi$$

with respect to $\tilde{x} \in (0,1)$ and estimate

$$\sup_{0<x<1}\left(\sqrt{f}(t,x,y) - \int_0^1 \sqrt{f}(t,\tilde{x},y)\,d\tilde{x}\right)^2 \leq \int_0^1 \left(\partial_x\sqrt{f}(t,\xi,y)\right)^2 d\xi.$$

Hence, after further integration with respect to $y \in (0,\infty)$, we apply Young's and Hölder's inequalities to show

$$\int_0^\infty \sup_{0<x<1} f(t,x,y)\,dy \leq \underbrace{2\int_0^\infty\!\!\int_0^1 \left(\partial_x\sqrt{f}(t,\xi,y)\right)^2 d\xi\,dy}_{:=\,m_1(t)} + \underbrace{2\int_0^\infty\!\!\int_0^1 f(t,\tilde{x},y)\,d\tilde{x}\,dy}_{\leq\, m_\infty}.$$

In particular, we have, due to (3.1) and (3.3), that

$$\int_0^\infty m_1(t)\,dt \leq \mu_1 = \frac{H(f_0)}{2\,a_*}, \qquad m_\infty = 2\mathcal{M}_0^*.$$

Finally,

$$\|M(t,\cdot)\|_{L^\infty_x} \le \int_0^\infty \sup_{0<x<1} f(t,x,y)\,dy\,,$$

which completes the proof of Lemma 3. □

Note that the estimates (3.1) and (3.2) are somehow in duality, a fact that will become essential below. We now prove a Lemma showing that the total number of clusters $\int_0^1 M(t,x)\,dx$ is bounded below by a strictly positive constant :

Lemma 4. *Assume that* $f_0 \neq 0$ *is a nonnegative initial datum such that* $(1 + y + \ln f_0) f_0 \in L^1((0,1)\times(0,\infty))$. *Then, there exists a constant* $M_{0*} > 0$ *such that for all times* $t \ge 0$, *one has*

$$\int_\Omega M(t,x)\,dx \ge \mathcal{M}_{0*}, \tag{3.4}$$

where f *is a solution of* (1.1)–(1.8).

Proof.- We remind that

$$\frac{d}{dt}\int_0^1 M(t,x)\,dx = \int_0^1 N_0(x)\,dx - \int_0^1 M(t,x)^2\,dx,$$

so that

$$\begin{aligned}\frac{d}{dt}\int_0^1 M(t,x)\,dx &\ge \int_0^1 N_0(x)\,dx - \|M(t,\cdot)\|_{L^\infty_x} \int_0^1 M(t,x)\,dx \\ &\ge \int_0^1 N_0(x)\,dx - (m_\infty + m_1(t)) \int_0^1 M(t,x)\,dx.\end{aligned}$$

Then,

$$\frac{d}{dt}\left[e^{\int_0^t (m_\infty + m_1(s))\,ds} \int_0^1 M(t,x)\,dx\right] \ge \int_0^1 N_0(x)\,dx\, e^{\int_0^t (m_\infty + m_1(s))\,ds},$$

and, recalling $\mu_1 \ge \int_0^{+\infty} m_1(s)\,ds$), we deduce

$$\begin{aligned}\int_0^1 M(t,x)\,dx &\ge \int_0^1 M_0(x)\,dx\, e^{-\int_0^t (m_\infty + m_1(\sigma))\,d\sigma} \\ &\quad + \int_0^1 N_0(x)\,dx \int_0^t e^{-\int_s^t (m_\infty + m_1(\sigma))\,d\sigma}\,ds \\ &\ge e^{-\mu_1 - m_\infty t} \int_0^1 M_0(x)\,dx + \int_0^t e^{-(t-s)\,m_\infty - \mu_1}\,ds \int_0^1 N_0(x)\,dx \\ &\ge e^{-\mu_1}\left[e^{-m_\infty t} \int_0^1 M_0(x)\,dx + \frac{1 - e^{-m_\infty t}}{m_\infty} \int_0^1 N_0(x)\,dx\right].\end{aligned}$$

Distinguishing here between $t < 1$ and $t \ge 1$, for instance, we obtain

$$\mathcal{M}_{0*} := e^{-\mu_1} \inf\left\{e^{-m_\infty} \int_0^1 M_0(x)\,dx, \frac{1 - e^{-m_\infty}}{m_\infty} \int_0^1 N_0(x)\,dx\right\},$$

which concludes the proof of lemma 4. □

Next, we show the uniform control in time of all moments with respect to size y of the solutions. Let us define the moment of order $p > 1$ by

$$M_p(f)(t) := \int_0^1 \int_0^\infty y^p \, f(t,x,y) \, dy \, dx$$

for all $t \geq 0$.

Lemma 5. *We assume that $f_0 \neq 0$ is a nonnegative initial datum such that $(1 + y + \ln f_0) f_0 \in L^1((0,1) \times (0,\infty))$. Then, the solution f of (1.1)–(1.8) has moments $M_p(f)(t)$ uniformly bounded in time $t > t_* > 0$ and for any $p > 1$, i.e., there exist explicit constants $\mathcal{M}_p^*(f_0, m_\infty, m_1, p)$ such that*

$$M_p(f)(t) \leq \mathcal{M}_p^* \,, \qquad \text{for a.e. } t > t_* > 0. \tag{3.5}$$

Proof.- We proceed in two steps:

Step 1.- We first assume that $M_p(f)(t_*) < \infty$ for certain $p > 1$ and $t_* > 0$. Using the weak formulation (1.16), it is easy to check that

$$< Q(f,f), y^p > = -\, 2 \int_0^\infty y^p f(y) \, dy \, M(t,x) + \int_0^\infty \int_0^\infty f(y) f(z) (y+z)^p \, dy \, dz \\ - \frac{p-1}{p+1} \int_0^\infty f(y) \, y^{p+1} \, dy.$$

Taking into account Lemma 3 and $(y+z)^p \leq C_p' \, (y^p + z^p)$, we deduce

$$< Q(f,f), y^p > \leq \; 2(C_p' - 1) \int_0^\infty y^p f(y) \, dy \; [m_\infty + m_1(t)] - \frac{p-1}{p+1} \int_0^\infty f(y) \, y^{p+1} \, dy$$

for all $p > 1$. Integrating in space, we find that the evolution of the moment of order $p > 1$ is given by

$$\frac{d}{dt} M_p(f)(t) \leq 2(C_p' - 1) \, M_p(f)(t) \; [m_\infty + m_1(t)] - \frac{p-1}{p+1} M_{p+1}(f)(t). \tag{3.6}$$

Trivial interpolation of the $p+1$-order moment with the moment of order one implies

$$M_p(f)(t) \leq \frac{1}{\epsilon^{p-1}} \int_\Omega N_0(t,x) \, dx + \epsilon \, M_{p+1}(f)(t)$$

for all $\epsilon > 0$, and thus

$$\frac{d}{dt} M_p(f)(t) \leq 2(C_p' - 1) \, M_p(f)(t) \; [m_\infty + m_1(t)] - \frac{p-1}{p+1} \frac{1}{\epsilon} M_p(f)(t) + D_\epsilon$$

for certain constant D_ϵ. Choosing $\epsilon > 0$ such that

$$2(C_p' - 1) m_\infty - \frac{p-1}{p+1} \frac{1}{\epsilon} \leq - \frac{1}{2\epsilon}$$

we obtain

$$\frac{d}{dt}M_p(f)(t) \le -\frac{1}{2\epsilon}M_p(f)(t) + 2(C_p' - 1)\, m_1(t)\, M_p(f)(t) + D_\epsilon$$

for a.e. $t > t_*$. According to Duhamel's formula,

$$M_p(f)(t) \le M_p(f)(t_*) \,\exp\left(2(C_p'-1)\int_{t_*}^t m_1(s)\,ds - \frac{t-t_*}{2\epsilon}\right) + D_\epsilon \int_{t_*}^t \exp\left(2(C_p'-1)\int_s^t m_1(\tau)\,d\tau - \frac{t-s}{2\epsilon}\right) ds \qquad (3.7)$$

which shows that the moment $M_p(f)(t)$ is bounded by a constant $\mathcal{M}_p^*$ for a.e. $t > t_*$ since $m_1(t) \in L^1((0,\infty))$ by Lemma 3.

Moreover, it follows from (3.6) that the boundedness of $M_p(t_*)$ immediately implies that

$$\int_{t_*}^T M_{p+1}(f)(t)\,dt < \infty$$

for all $T > 0$, and thus the finiteness of $M_{p+1}(f)(t)$ for a.e. $t > t_*$ and a simple induction argument enables then to conclude the bounds on all higher moments.

Step 2.- It remains to show that for given nontrivial initial data $y\, f_0 \in L^1_{x,y}$ and for a $p > 1$ and a time $t_* > 0$ we have that $M_p(t_*) < \infty$. We start with the following observation [MW, Appendix A]: For a nonnegative integrable function $g(y) \neq 0$ on $(0,\infty)$, there exists a concave function $\Phi(y)$, depending on g, smoothly increasing from $\Phi(0) > 0$ to $\Phi(\infty) = \infty$ such that

$$\int_0^\infty \Phi(y)\, g(y)\, dy < \infty.$$

Moreover, the function Φ can be constructed to satisfy

$$\Phi(y) - \Phi(y') \ge C\frac{y - y'}{y\ln^2(e+y)} \qquad (3.8)$$

for $0 < y' < y$ with C not depending on g. We refer to [MW, Appendix A] for all the details of this "by-now standard" construction.

To show now that $M_p(t_*) < \infty$ for a $p > 1$ and a time $t_* > 0$, we take functions $\Phi(x,y)$ constructed for nontrivial $y\, f_0(x,y) \in L^1_y(0,\infty)$ a.e. $x \in (0,1)$ and calculate - similar to Step 1 - the moment

$$M_{1,\Phi}(f)(t) = \int_\Omega \int_0^\infty y\,\Phi(x,y) f(x,y)\,dy\,dx.$$

For the fragmentation part, we use (3.8) for $0 < y' < y$ and estimate

$$\begin{aligned} y\,\Phi(y)\,Q_f(f) &= 2\int_0^y y'(\Phi(y') - \Phi(y))\,dy'\,f(y) \\ &\le -C\,\ln^{-2}(e+y)y^{-1}\int_0^y y'(y-y')\,dy'\,f(y) \\ &= -C\,\ln^{-2}(e+y)\frac{y^2}{6}\,f(y) \le -C_\delta\, y^{2-\delta}\,f(y)\,, \end{aligned}$$

for all $\delta > 0$ and a positive constant C_δ, where the (t,x)-dependence has been dropped for notational convenience. Hence, by estimating the coagulation part similar to Step 1 making use of the concavity of Φ, we obtain that

$$\frac{d}{dt} M_{1,\Phi}(f)(t) \le 3(m_\infty + m_1(t)) M_{1,\Phi}(f)(t) - C_\delta M_{2-\delta}(f)(t)\,,$$

and boundedness of the moment $M_{1,\Phi}$ follows by interpolation as well as the finiteness of $M_{2-\delta}(f)(t_*)$ analogously to Step 1. □

Next, we show that M and N are bounded below uniformly (with respect to t and x) for all $t \ge t_* > 0$.

Proposition 3. *Under the assumptions of theorem 2, let $t_* > 0$ be given. Then, there are strictly positive constants $\mathcal{M}_*$ and $\mathcal{N}_*$ such that for all $t \ge t_* > 0$,*

$$M(t,x) \ge \mathcal{M}_* \qquad \textit{and} \qquad N(t,x) \ge \mathcal{N}_*.$$

Proof.- We write the equation satisfied by f in this way :

$$\partial_t f - a(y)\,\partial_{xx} f = g_1 - y\, f - \|M(t,\cdot)\|_{L^\infty_x}\, f,$$

where g_1 is nonnegative. Then

$$(\partial_t + a(y)\,\partial_{xx})\Big(f\, e^{ty + \int_0^t \|M(s,\cdot)\|_{L^\infty_x}\, ds} \Big) = g_2,$$

where g_2 is nonnegative.

Now, we recall that the solution $h := h(t,x)$ of the heat equation

$$\partial_t h - a\,\partial_{xx} h = G,$$

with homogeneous Neumann boundary condition on the interval $(0,1)$, where $a > 0$ is a constant and $G := G(t,x) \in L^1$, is given by the formula

$$\begin{aligned} h(t,x) = {} & \frac{1}{2\sqrt{\pi}} \int_{-1}^{1} \tilde{h}(0,z) \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{a\,t}}\, e^{-\frac{(2k+x-z)^2}{4a\,t}}\, dz \\ & + \frac{1}{2\sqrt{\pi}} \int_0^t \int_{-1}^{1} \tilde{G}(s,z) \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{a\,(t-s)}}\, e^{-\frac{(2k+x-z)^2}{4a\,(t-s)}}\, dz ds, \end{aligned}$$

with $\tilde{h}$ and $\tilde{G}$ denoting the “evenly mirrored around 0 in the x variable” functions h and G.

Therefore, for all $t_1, t \ge 0$, and $x \in (0,1)$, $y \in \mathbb{R}_+$,

$$f(t_1+t,x,y)\, e^{(t_1+t)\,y + \int_0^{t_1+t} \|M(s,\cdot)\|_{L^\infty_x}\, ds}$$

$$\ge \frac{1}{2\sqrt{\pi}} \int_{-1}^{1} \tilde{f}(t_1,z,y)\, \frac{1}{\sqrt{a(y)\,t}}\, e^{-\frac{(x-z)^2}{4a(y)\,t}}\, e^{t_1\,y + \int_0^{t_1} \|M(s,\cdot)\|_{L^\infty_x}\, ds}\, dz,$$

so that when $t \in [t_*, 2t_*]$ (and since $|x-z|<2$) :

$$f(t_1+t,x,y) \geq \frac{1}{\sqrt{2\pi\, a^*\, t_*}} \int_0^1 f(t_1,z,y)\, e^{-\frac{1}{a_*\, t_*}}\, e^{-2t_*\, y - 2t_*\, m_\infty - \mu_1}\, dz$$
$$\geq C \int_0^1 f(t_1,z,y)\, e^{-2t_*\, y}\, dz,$$

where $C>0$ depends on the constants a_*, a^*, m_∞, μ_1 and $t_*>0$.

We recall that for all $t \geq t_*$,

$$\int_0^1\!\!\int_0^\infty y^2\, f(t,x,y)\, dy\, dx \leq \mathcal{M}_2^*,$$

and thus, for any $A>0$, we deduce

$$\begin{aligned} N(t_1+t,x) &\geq C\, e^{-2t_*\, A} \int_0^1\!\!\int_0^A f(t_1,z,y)\, y\, dydz \\ &\geq C\, e^{-2t_*\, A} \left(\int_0^1 N(t_1,x)\, dx - \mathcal{M}_2^*/A \right) \\ &= C\, e^{-2t_*\, A} \left(\int_0^1 N_0(x)\, dx - \mathcal{M}_2^*/A \right), \end{aligned}$$

due to the conservation law (1.9) and

$$\int_0^1\!\!\int_A^\infty y\, f(t,x,y)\, dy\, dx \leq \mathcal{M}_2^*/A.$$

Choosing now A, we get that $N(t_1+t,x) \geq \mathcal{N}_*$ for some $\mathcal{N}_*>0$ which does not depend on t_1. Using Lemma 4,

$$\begin{aligned} M(t_1+t,x) &\geq C \int_0^1\!\!\int_0^\infty f(t_1,z,y)\, e^{-2y}\, dydz \\ &\geq C\, e^{-2t_*\, A} \int_0^1\!\!\int_0^A f(t_1,z,y)\, dydz \\ &\geq C\, e^{-2t_*\, A} \left(\mathcal{M}_{0*} - \mathcal{M}_2^*/A^2 \right). \end{aligned}$$

Once again choosing A, we get that $M(t_1+t,x) \geq \mathcal{M}_*$. Since $\mathcal{M}_*$ does not depend on t_1, we get Proposition 3. □

4. Entropy entropy dissipation estimate

Please note that in this section we will systematically use the shortcuts :

$$\overline{M} = \int_0^1\!\!\int_0^\infty f(x,y)\, dydx\,, \qquad \overline{N} = \int_0^1\!\!\int_0^\infty yf(x,y)\, dydx\,.$$

We introduce a Lemma enabling to estimate the entropy of f by means of its entropy dissipation. This is a functional estimate, that is, the function f in this Lemma does not depend on t and has not necessarily something to do with the solution of our equation.

Lemma 6. *Assume $\Omega = (0,1)$ and (1.3). Let $f := f(x,y) \geq 0$ be a measurable function with moments satisfying $0 < \mathcal{M}_* \leq M(x) \leq \|M\|_{L^\infty_x}$ and $0 < \mathcal{N}_* \leq N(x)$. Let $f_{M,N}$ denote the function $M^2/N \exp(-yM/N)$. Then, the following entropy entropy dissipation estimate holds (with notations (1.17) and (2.3))*

$$D(f) \geq \frac{C(\mathcal{M}_*, \mathcal{N}_*, a_*)}{\ln\left(\|M\|_{L^\infty_x}\right) \|M\|_{L^\infty_x}} \int_0^1 H(f|f_\infty)\, dx\,, \tag{4.1}$$

with a constant $C(\mathcal{M}_*, \mathcal{N}_*, a_*)$ depending only on $\mathcal{M}_*$, $\mathcal{N}_*$, and a_*.

With respect to (2.3), we have in particular that

$$\begin{aligned} \int_0^1\!\!\int_0^\infty a(y) \frac{|\nabla_x f|^2}{f}\, dydx + \int_0^1 (M^2 - N) \ln \frac{M^2}{N}\, dx \\ \geq \frac{C(\mathcal{N}_*, a_*)}{\ln\left(\|M\|_{L^\infty_x}\right) \|M\|_{L^\infty_x}} \int_0^1 H(f_{M,N}|f_\infty)\, dx\,, \end{aligned} \tag{4.2}$$

for the right-hand side

$$\int_0^1 H(f_{M,N}|f_\infty)\, dx = \int_0^1 \left[M \ln \frac{M^2}{N} - 2(M - \sqrt{N}) \right] dx + 2\left(\overline{\sqrt{N}} - \sqrt{\overline{N}}\right). \tag{4.3}$$

Proof.- We prove first the inequality (4.2) divided into four steps corresponding to the four terms of (4.2) and (4.3). Then, estimate (4.1) follows in step 5.

Step 1 - The first, "Fisher"-type term of (4.2) controls the variance of M : Denoting with $P(\Omega)$ the constant of Poincaré's inequality, we estimate using Cauchy-Schwartz

$$\begin{aligned} \int_0^1\!\!\int_0^\infty a(y) \frac{|\nabla_x f|^2}{f}\, dydx &\geq \frac{a_*}{\|M\|_{L^\infty_x}} \int_0^1 \left(\int_0^\infty \frac{|\nabla_x f|^2}{f}\, dy \right) \left(\int_0^\infty f\, dy \right) dx \\ &\geq \frac{a_*}{\|M\|_{L^\infty_x}} \int_0^1 \left| \int_0^\infty \nabla_x f\, dy \right|^2 dx = \frac{a_*}{\|M\|_{L^\infty_x}} \int_0^1 |\nabla_x M|^2\, dx \\ &\geq \frac{a_*}{P(\Omega) \|M(t,\cdot)\|_{L^\infty_x}} \|M - \overline{M}\|^2_{L^2_x}\,. \end{aligned} \tag{4.4}$$

We remark that the seemingly more natural estimate

$$\int_0^1\!\!\int_0^\infty a(y) \frac{|\nabla_x f|^2}{f}\, dydx \geq \frac{4}{P(\Omega)} \|\sqrt{M} - \overline{\sqrt{M}}\|^2_{L^2_x}\,,$$

provides a bound which does not seem sufficient to conclude like in step 3.

Step 2 - The "reactive" term of (4.2) controls the L^2-distance of M to $\sqrt{N}$: We use the elementary inequality $(x - y) \ln(x/y) \geq 4(\sqrt{x} - \sqrt{y})^2$ to estimate

$$\int_0^1 (M^2 - N) \ln \frac{M^2}{N}\, dx \geq 4 \|M - \sqrt{N}\|^2_{L^2_x}\,. \tag{4.5}$$

Step 3 - The second term of (4.3) is dominated in terms of (4.4) and (4.5) :

$$\sqrt{\overline{N}} - \overline{\sqrt{N}} \le \frac{2}{\sqrt{\overline{N}}} \left(\|M - \sqrt{N}\|^2_{L^2_x} + \|M - \overline{M}\|^2_{L^2_x} \right) . \tag{4.6}$$

Indeed, since $\sqrt{N} - \overline{\sqrt{N}}$ is orthogonal to $\overline{\sqrt{N}} - \overline{M}$ in L^2_x, we have

$$\sqrt{\overline{N}} - \overline{\sqrt{N}} \le \frac{\overline{N} - \overline{\sqrt{N}}^2}{\sqrt{\overline{N}}} = \frac{1}{\sqrt{\overline{N}}} \|\sqrt{N} - \overline{\sqrt{N}}\|^2_{L^2_x} \le \frac{1}{\sqrt{\overline{N}}} \|\sqrt{N} - \overline{M}\|^2_{L^2_x} ,$$

and further, we obtain (4.6) by expanding $\|\sqrt{N} - M\|^2_{L^2_x}$ and Young's inequality

$$\frac{1}{2}\|\sqrt{N} - \overline{M}\|^2_{L^2_x} - \|M - \overline{M}\|^2_{L^2_x} \le \|\sqrt{N} - \overline{M} - M + \overline{M}\|^2_{L^2_x} .$$

Step 4 - The first term of (4.3) is dominated in terms of (4.4) and (4.5) : We begin estimating the first term of (4.3) as

$$\int_0^1 \left[M \ln \frac{M^2}{N} - 2(M - \sqrt{N}) \right] dx \le 2\Phi(\|M(t,\cdot)\|_{L^\infty_x}/\mathcal{N}_*, 1) \|\sqrt{M} - \sqrt[4]{N}\|^2_{L^2_x} , \tag{4.7}$$

where the function $\Phi(x,y) = (x \ln(x/y) - (x-y))/(\sqrt{x} - \sqrt{y})^2$ - as already introduced in section 2 - is continuous and monotone increasing in the first argument and satisfies $\Phi(x,y) = \Phi(x/y,1) = O(\ln(x/y))$.

Further, one possiblity to dominate (4.7) in terms of (4.5) uses the estimate $(\sqrt{\mathcal{M}_*} + \sqrt[4]{\mathcal{N}_*})\|\sqrt{M} - \sqrt[4]{N}\|^2_{L^2_x} \le \|M - \sqrt{N}\|^2_{L^2_x}$. Instead, the following estimate (proven below) avoids the lower bounds and controls (4.7) in terms of (4.4) and (4.5) :

$$\|\sqrt{M} - \sqrt[4]{N}\|^2_{L^2_x} \le \frac{4}{\sqrt{\overline{N}}} \|M - \sqrt{N}\|^2_{L^2_x} + \frac{6}{\sqrt{\overline{N}}} \|\sqrt{N} - \sqrt{\overline{N}}\|^2_{L^2_x} , \tag{4.8}$$

and further $\|\sqrt{N} - \sqrt{\overline{N}}\|^2_{L^2_x} = 2\sqrt{\overline{N}}(\sqrt{\overline{N}} - \overline{\sqrt{N}})$ together with the estimate (4.6).

In order to prove the inequality (4.8), we exploit the ansatz $M = \sqrt{\overline{N}}(1 + \mu(x))$. Since obviously $-1 \le \mu(x)$, we expand the left-hand side of (4.8) as

$$\begin{aligned} \|\sqrt{M} - \sqrt[4]{\overline{N}} + \sqrt[4]{\overline{N}} - \sqrt[4]{N}\|^2_{L^2_x} &\le 2\|\sqrt{M} - \sqrt[4]{\overline{N}}\|^2_{L^2_x} + 2\|\sqrt[4]{\overline{N}} - \sqrt[4]{N}\|^2_{L^2_x} \\ &\le 2\sqrt{\overline{N}}\, \|\mu\|^2_{L^2_x} + \frac{2}{\sqrt{\overline{N}}} \|\sqrt{\overline{N}} - \sqrt{N}\|^2_{L^2_x} , \end{aligned}$$

and the first term on the right-hand side of (4.8) using Young's inequality

$$\|M - \sqrt{N}\|^2_{L^2_x} = \|\sqrt{\overline{N}} - \sqrt{N} + \sqrt{\overline{N}}\,\mu\|^2_{L^2_x} \ge -\|\sqrt{\overline{N}} - \sqrt{N}\|^2_{L^2_x} + \frac{\overline{N}}{2}\,\|\mu\|^2_{L^2_x} ,$$

which proves (4.8). Alltogether, the inequality (4.2) follows from (4.4), (4.5), (4.6), (4.7) with the property that $\Phi(x/y,1) = O(\ln(x/y))$, and (4.8) with (4.6).

Step 5 - We estimate the entropy dissipation (2.3) using (4.2) and $M \geq \mathcal{M}_*$:

$$\begin{aligned} D(f) \geq & \int_0^1\!\!\int_0^\infty a(y)\frac{|\nabla_x f|^2}{f}\,dydx + \int_0^1 (M^2-N)\ln\frac{M^2}{N}\,dx \\ & +\mathcal{M}_*\int_0^1 H(f|f_{M,N})\,dx \\ & \geq \min\left\{\frac{C(\mathcal{N}_*,a_*)}{\ln\left(\|M\|_{L^\infty_x}\right)\|M\|_{L^\infty_x}},\mathcal{M}_*\right\}\int_0^1 \left(H(f|f_{M,N})+H(f_{M,N}|f_\infty)\right)\,dx \\ & = \frac{C(\mathcal{M}_*,\mathcal{N}_*,a_*)}{\ln\left(\|M\|_{L^\infty_x}\right)\|M\|_{L^\infty_x}}\int_0^1 H(f|f_\infty)\,dx\,, \end{aligned}$$

thanks to the additivity (1.21). This concludes the proof of lemma 6. □

We now turn back to the

Proof of Theorem 2.- According to Lemma 6, we have

$$\frac{d}{dt}\int_0^1 H(f|f_\infty)\,dx \leq -D(f) \leq -\frac{C}{\ln\left(\|M\|_{L^\infty_x}\right)\|M\|_{L^\infty_x}}\int_0^1 H(f|f_\infty)\,dx\,,$$

where $\|M\|_{L^\infty_x}(t) \leq m_\infty + m_1(t)$ is in $L^1_t + L^\infty_t$ by Lemma 3. Hence, for $t_* > 0$,

$$\int_0^1 H(f(t)|f_\infty)\,dx \leq \int_0^1 H(f(t_*)|f_\infty)\,dx\,\exp\left(\int_{t_*}^t -\frac{C}{\ln\left(\|M\|_{L^\infty_x}\right)\|M\|_{L^\infty_x}}\,ds\right).$$

Knowing that $m_1(t) \in L^1_t$ with $\int_0^\infty m_1(t)\,dt \leq \mu_1$, we consider the sets $A := \{s > 0 : m_1(s) \geq 1\}$ and $B_t := \{s \in [0,t] : m_1(s) < 1\}$. We readily find that

$$|A| = \int_A ds \leq \int_0^\infty m_1(t)\,dt \leq \mu_1 \qquad \text{and} \qquad |B_t| = t - \int_{A\cap[0,t]} ds \geq t - \mu_1.$$

Moreover,

$$\begin{aligned} \int_{t_*}^t -\frac{C}{\ln\left(\|M\|_{L^\infty_x}\right)\|M\|_{L^\infty_x}}\,ds & \leq \int_{B_t} -\frac{C}{\ln\left(\|M\|_{L^\infty_x}\right)\|M\|_{L^\infty_x}}\,ds \\ & \leq -\frac{C}{\ln(1+m_\infty)(1+m_\infty)}(t-\mu_1)\,. \end{aligned}$$

The rest of the proof of theorem 2 is skipped since it follows the same lines as in Theorem 1 by using the Csiszar-Kullback type inequality (2.10). □

Finally, we show Proposition 1. Let us denote by C_T any constant of the form $C(t)\,(1+T)^s$, where $s \in \mathbb{R}$ and $C(t)$ is bounded on any interval $[t_*, +\infty)$ with $t_* > 0$.

Proof of Proposition 1.- We observe using the bounds (3.5) and (3.1) that for all $q \geq 0$,

$$\int_0^T\int_0^1\int_0^\infty (1+y)^q\, Q^+(f,f)\, dy\, dx\, dt \leq \int_0^T\int_0^1\int_0^\infty \frac{(1+y)^{q+1}}{q+1}\, f(t,x,y)\, dy\, dx\, dt$$
$$+ \int_0^T\int_0^1\int_0^\infty\int_0^\infty (1+y+z)^q\, f(t,x,y)\, f(t,x,z)\, dz\, dy\, dx\, dt$$
$$\leq 2^{q+1}\,(\mathcal{M}_0^* + \mathcal{M}_{q+1}^*)\, T + 2^q \int_0^T \|M(t,\cdot)\|_{L^\infty_x} (\mathcal{M}_0^* + \mathcal{M}_q^*)\, dt \leq C_T.$$

According to the properties of the heat kernel (Cf. [DF06] for example), we know that for any $\varepsilon > 0$ and $t_* > 0$,

$$\|f(\cdot,\cdot,y)\|_{L^{3-\varepsilon}([t_*,T]\times\Omega)} \leq C_T \left(\|f(0,\cdot,y)\|_{L^1_x} + \|Q^+(f,f)(\cdot,\cdot,y)\|_{L^1([0,T]\times\Omega)} \right).$$

As a consequence,

$$\int_0^\infty (1+y)^q\, \|f(\cdot,\cdot,y)\|_{L^{3-\varepsilon}([t_*,T]\times\Omega)}\, dy \leq C_T.$$

Then, for all $r \in [2,3[$

$$\int_0^\infty (1+y)^q\, \|Q^+(f,f)(\cdot,\cdot,y)\|_{L^{r/2}([t_*,T]\times\Omega)}\, dy$$
$$\leq \int_0^\infty \frac{(1+y)^{q+1}}{q+1}\, \|f(\cdot,\cdot,y)\|_{L^r([t_*,T]\times\Omega)}\, dy$$
$$+ \int_0^\infty (1+y)^q\, \| \int_0^\infty f(\cdot,\cdot,y')\, f(\cdot,\cdot,y-y')\, dy' \|_{L^{r/2}([t_*,T]\times\Omega)}\, dy$$
$$\leq C_T + \int_0^\infty \int_0^\infty (1+y+z)^q \|f(\cdot,\cdot,y)\, f(\cdot,\cdot,z)\|_{L^{r/2}([t_*,T]\times\Omega)}\, dydz$$
$$\leq C_T + \left(2^{q-1} \int_0^\infty (1+y)^q\, \|f(\cdot,\cdot,y)\|_{L^r([t_*,T]\times\Omega)} dy \right)^2 \leq C_T.$$

Using again the properties of the heat kernel (still described in [DF06]), we see that for any $s \in [1,\infty)$ and $t_* > 0$,

$$\int_0^\infty (1+y)^q\, \|f(\cdot,\cdot,y)\|_{L^s([t_*,T]\times\Omega)} dy \leq C_T.$$

The above argument can now be used with $r = 4$ and shows that

$$\int_0^\infty (1+y)^q\, \|Q^+(f,f)(\cdot,\cdot,y)\|_{L^2([t_*,T]\times\Omega)}\, dy \leq C_T.$$

As a consequence, the standard energy estimate on the heat kernel implies that

$$\int_0^\infty (1+y)^q\, \|f(T,\cdot,y)\|_{H^1_x}\, dy \leq C_T.$$

Then, using a Gagliardo-Niremberg type interpolation and Theorem 2, we obtain

$$\int_0^\infty (1+y)^q \,\|f(T,\cdot,y) - f_\infty(y)\|_{L^\infty_x}\, dy \le \int_0^\infty \left[(1+y)^q \,\|f(T,\cdot,y) - f_\infty(y)\|_{H^1_x}^{3/4}\right] \times \left[\|f(T,\cdot,y) - f_\infty(y)\|_{L^1_x}^{1/4}\right] dy$$

$$\le \left[\int_0^\infty (1+y)^{4q/3} \|f(T,\cdot,y) - f_\infty(y)\|_{H^1_x} dy\right]^{3/4} \left[\int_0^\infty \|f(T,\cdot,y) - f_\infty(y)\|_{L^1_x} dy\right]^{1/4}$$

$$\le C_T^{3/4}\, \exp(-Cst\, T) \le Cst \exp(-Cst\, T),$$

which concludes the proof of proposition 1. □

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