# DISSERTATION

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“Normal crossings in local analytic geometry”

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# Introduction

## Motivation and Problem

Consider a divisor (=hypersurface) $D$ in a complex manifold $S$ of dimension $n$. Then $D$ is said to have *normal crossings* at a point $p$ if locally at $p$ there exist complex coordinates $(x_1, \ldots, x_n)$ such that $D$ is defined by the equation $x_1 \cdots x_m = 0$ for some $0 \leq m \leq n$. In general there is no algorithm to find these coordinates. Hence the question arises: is there an *effective algebraic* characterization of a divisor with normal crossings?

Normal crossing divisors appear in many contexts in algebraic and analytic geometry, for example in the embedded resolution of singularities [53], in compactification problems [26, 38] or in cohomology computations [29]. One of the most striking results is the Theorem of Hironaka [53], namely, that any algebraic variety over a field of characteristic zero can be transformed by a sequence of blowups into a divisor with normal crossings. However, given an (algebraic or analytic) variety, it is not clear how to determine effectively if this variety has normal crossings: only in case the decomposition into irreducible components is known, the normal crossings property can be checked rather easily (see e.g. [9]).

The main objective of this thesis is to give an effective algebraic characterization of normal crossing divisors in complex manifolds. By "effective" is meant that one should be able to decide from data derived from a local defining equation of the divisor whether it has normal crossings at a point. The guiding idea for our investigations is that the singular locus should carry all information about a divisor having normal crossings. This point of view is inspired by the Theorem of Mather–Yau [61], which says that an isolated hypersurface singularity in $(\mathbb{C}^n, 0)$ is uniquely determined by its Tjurina algebra, i.e., the quotient of the power series ring in $n$ variables by the ideal generated by a (reduced) local defining equation and its partial derivatives. On the other hand the tangent behaviour along the divisor, via so-called logarithmic vector fields, will give us means to control the normal crossings property. Here the rich theory of logarithmic vector fields (differential forms) and free divisors, initiated by K. Saito in the 1980's [81], will be the other main ingredient for an algebraic criterion characterizing normal crossing divisors.

# Overview

The main result of this thesis is that a divisor $D$ in a complex manifold $S$ of dimension $n$ has normal crossings at a point $p$ if (and only if) it is free at $p$, the Jacobian ideal of $D$ at $p$ is a radical ideal and its normalization $\widetilde{D}$ is Gorenstein. Another way to phrase this is that $D$ is either smooth at $p$ or its Tjurina algebra is reduced and Cohen–Macaulay of Krull-dimension $n-2$ and the normalization $\widetilde{\mathcal{O}}_D$ is a Gorenstein ring or, another equivalent formulation, either $D$ is smooth at $p$ or the Jacobian ideal of $D$ at $p$ is radical , perfect and of depth 2 in the local ring $\mathcal{O}_{S,p} \cong \mathbb{C}\{x_1, \ldots, x_n\}$ and the normalization $\widetilde{\mathcal{O}}_D$ is a Gorenstein ring. The additional condition on the normalization is technical and we do not know if it is necessary.

Our approach to prove the above statement originates from K. Saito's theory of free divisors, a class of divisors, which includes normal crossing divisors. Since a normal crossing divisor is free, one is led to impose additional conditions on free divisors in order to single out the ones with normal crossings. It turns out that the radicality of the Jacobian ideal is the right property. Since there is an interpretation of free divisors by their Jacobian ideals (due to Terao, Aleksandrov and Simis), one so obtains a purely algebraic characterization of normal crossing divisors. Moreover, two other characterizations of normal crossing divisors in terms of logarithmic differential forms (resp. vector fields) and the logarithmic residue are shown. The second one makes use of the dual logarithmic residue, introduced by Granger and Schulze [43]. As an application of the second one, a question about the logarithmic residue

posed by Saito in [81,92] is answered (which was first answered in [43]). Along the way two generalizations of the concept of normal crossings are considered: splayed divisors and mikado divisors. We introduce the former as unions of "transversally" intersecting (possibly singular) hypersurfaces. Here two hypersurfaces intersect "transversally" if their defining equations can be chosen in separated variables. Mikado divisors on the other hand, are constituted by smooth hypersurfaces all whose intersections have to be smooth. This notion was introduced by H. Hauser in [49] and appears in connection with resolution of singularities. We prove that one can read off the Jacobian ideal of a plane curve whether the curve is mikado. We also prove that the Jacobian ideal of a divisor determines whether it is splayed. Moreover, it is shown that the Hilbert–Samuel polynomials of splayed divisors satisfy a certain additivity relation: the Hilbert–Samuel polynomial of a splayed divisor is the sum of the Hilbert–Samuel polynomial of its splayed components minus the Hilbert–Samuel polynomial of their intersection.
The contents of the thesis are:

In Chapter 1 we recall the notions of logarithmic differential forms and vector fields, free divisors and the logarithmic residue. Here the important class of Euler-homogeneous free divisors is considered. In particular, we show a characterization of an Euler–homogeneous divisor in terms of a basis of the module of logarithmic differential forms $\Omega^1_S(\log D)$ (Prop. 1.29). Moreover, we exhibit problems when working with divisors defined by non-reduced holomorphic equations. Our results in this chapter are: A characterization of a normal crossing divisor in terms of logarithmic differential forms (resp. vector fields), more precisely, a divisor $D$ has normal crossings at a point $p$ if and only if it is free at $p$ and the module of logarithmic one-forms $\Omega^1_{S,p}(\log D)$ has a basis of closed forms (resp. the module of logarithmic vector fields $\mathrm{Der}_{S,p}(\log D)$ has a basis of commuting vector fields), see Thm. 1.52 and Prop. 1.54. Then, following Granger and Schulze, we describe a normal crossing divisor in terms of the logarithmic residue in Thm. 1.63, namely, $D$ has normal crossings at a point $p$ if and only if $D$ is free at $p$, the residue of logarithmic one-forms is equal to the ring of weakly holomorphic functions on $D$ and the normalization $\widetilde{D}$ is Gorenstein. Finally, as an application of Thm. 1.63, the question of K. Saito is considered whether the equality of the logarithmic residue and the ring of weakly holomorphic functions of a divisor implies that the divisor has normal crossings in codimension one. This question was first affirmatively answered by

Granger and Schulze [43]. It also has a topological counterpart, which was proven by Saito and Lê–Saito in [81] and [92]. We review the background and history of the problem, and also give a positive answer to Saito's question in Thm. 1.82.

In the second chapter, our attention is drawn to singularities and Jacobian ideals of divisors. Here commutative algebra is used to characterize normal crossing divisors. The chapter is devoted to prove our main theorem: a divisor has normal crossings at point if and only if it is free at the point and its Jacobian ideal is radical and its normalization $\widetilde{D}$ is Gorenstein, see Thm. 2.1.
We recall Aleksandrov's characterization of free divisors in terms of their Jacobian ideals (Theorem 2.6): a divisor $D$ in a complex manifold $S$ of dimension $n$ is free at a point $p$ if and only if either $D$ is smooth at $p$ or $\mathcal{O}_{\operatorname{Sing} D,p}$ (the Tjurina-Algebra of $D$) is Cohen–Macaulay of Krull-dimension $n-2$. Together with Thm. 2.1 this yields a purely algebraic characterization of normal crossing divisors. As an illustration of the main theorem we first deal with the problem for some special cases: for curves in smooth surfaces it can be directly shown that the Jacobian ideal is radical already implies the normal crossings property (Prop. 2.15). This result is used to establish that for a divisor $D$ in a manifold $S$ of dimension $n \geq 2$ the Tjurina-Algebra $\mathcal{O}_{\operatorname{Sing} D,p}$ is reduced of Krull-dimension $n-2$ and Gorenstein if and only if $(\operatorname{Sing} D, p)$ is smooth; hence $D$ is locally the union of two transversally intersecting hyperplanes (Prop. 2.18). Moreover, the assertion of Thm. 2.1 is shown for hyperplane arrangements and generalizations thereof (Prop. 2.32). Note that for the proofs of these special cases the Gorenstein assumption on the normalization of the divisors is not needed.
To prove Thm. 2.1 in general, the problem is reduced to the analytically irreducible case: if a divisor $(D,p) = \cup_{i=1}^{m}(D_i,p)$ has a radical Jacobian ideal, then all its components $D_i$ also have a radical Jacobian ideal, see Prop. 2.48. In order to show this we introduce splayed divisors, which are a generalization of normal crossing divisors allowing singular components. They can be characterized by the "Leibniz property" of their Jacobian ideals, namely, for a splayed divisor $D = D_1 \cup D_2$ defined locally by the equation $gh = 0$ one has the decomposition of the Jacobian ideal $J_{gh} = gJ_h + hJ_g$, see Thm. 2.43. Finally, the irreducible case of Theorem 2.1 is shown with a theorem by R. Piene about ideals in a desingularization [76][1], similar to the results on the logarithmic residue

[1] Here thanks to David Mond, who pointed out the use of this theorem.

in chapter 1.

In the last chapter further-reaching questions are considered: first it is asked which type of radical ideals occur as Jacobian ideals of divisors. For manifolds of dimensions 2 and 3 divisors with radical ideal can be nearly completely described (Prop. 3.2). For higher dimensional ambient spaces it is shown that the Jacobian ideal of a divisor $D$ is radical and defines a complete intersection if and only if $D$ is isomorphic to the cylinder over a lower-dimensional $A_1$-singularity (Prop. 3.10). In the sequel, we study splayed divisors in more detail, in particular, yet another characterization of splayed divisors in terms of their Jacobian ideals is established (Prop. 3.37) and it is shown that their Hilbert–Samuel polynomials satisfy the following additivity property: if $D = D_1 \cup D_2$ is splayed then $\chi_{D,p}(t) + \chi_{D_1 \cap D_2,p}(t) = \chi_{D_1,p}(t) + \chi_{D_2,p}(t)$, where $\chi_D$ denotes the Hilbert–Samuel polynomial of the divisor $D$ (see Prop. 3.33). Ultimately, another possible generalization of normal crossing divisors, so-called mikado divisors, is considered. Using Teissier's generalized Milnor numbers [93] we are able to give a criterion for plane curves being mikado in terms of their Jacobian ideals (Thm. 3.49).

We have included an appendix (Appendix A) in which the most important notions and theorems quoted in the text can be found. This appendix is divided into a commutative algebra and a local analytic geometry section.
Eventually, this thesis is about geometry: there is a second appendix (Appendix B), where pictures of some recurring examples of divisors in two and three-dimensional manifolds are displayed.

About the notation (cf. Appendix A): unless otherwise stated, $(S, D)$ denotes a complex manifold $S$ of dimension $n$ together with a divisor $D$ in $S$. The complex coordinates at a point $p \in S$ are denoted by $(x_1, \ldots, x_n)$ and the ring of holomorphic functions at $p$ is denoted by $\mathcal{O}_{S,p} \cong \mathbb{C}\{x_1, \ldots, x_n\}$. Mostly $D$ is considered to be locally at $p$ defined by a reduced holomorphic function $h \in \mathcal{O}_{S,p}$. The Jacobian ideal of $h$ is the ideal generated by the partial derivatives of $h$ and denoted by $J_h = (\partial_{x_1} h, \ldots, \partial_{x_n} h)$. We always consider the singular locus $(\mathrm{Sing}\, D, p)$ as given by the (possibly non-reduced) Jacobian ideal, with ring $\mathcal{O}_{\mathrm{Sing}\, D,p} = \mathcal{O}_{S,p}/((h) + J_h)$ (the Tjurina algebra).

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# Contents

# Chapter 1

# Normal crossing and free divisors

In this chapter normal crossing divisors are studied from the "geometric" point of view, namely with logarithmic differential forms and vector fields along the divisors. We call this approach geometric because logarithmic vector fields correspond precisely to tangent vectors at smooth points of the divisor. After introducing the basic notions of K. Saito's theory of logarithmic differential forms and logarithmic derivations, we derive two criteria for a divisor to have normal crossings: the first one is in terms of a basis of the module of logarithmic differential forms resp. vector fields and the second one uses the logarithmic residue. As an application of the characterization by the logarithmic residue, we answer in section 1.4 a question posed by K. Saito in his 1980 paper [81].

## 1.1 Theory of logarithmic differential forms and logarithmic vector fields

Logarithmic differential forms along normal crossing divisors were first considered by P. Deligne [28]. He computed the cohomology of the complement of a normal crossing divisor with the help of logarithmic differential forms. In 1980, K. Saito [81] generalized the notions of Deligne to arbitrary (reduced) divisors in a complex manifold. He called divisors whose sheaf of logarithmic differential forms is locally free, *free* divisors. He was the first able to prove that the discriminant of a

versal deformation of an isolated singularity is always a free divisor. Since then free divisors have been an active area of research, e.g. in the theory of hyperplane arrangements, in connection with the logarithmic comparison theorem and also in deformation theory [13, 22, 94]. There are still many questions open, for example, one lacks a classification of free divisors.
For our study we will mainly need the basic notions of the theory; the exposition follows loosely the seminal paper by K. Saito [81].

### 1.1.1 Definitions and most important theorems

In this section the notions logarithmic differential forms, logarithmic vector fields and free divisors are introduced. The most important results are proven (duality between logarithmic differential forms and vector fields, Saito's criterion) and also non-reduced divisors and Euler-homogeneous free divisors are considered.

The notion of logarithmic differential forms was first used by K. Saito in order to study the Gauß–Manin connection of the singularity $A_3$ (see [80]). He introduced the analytic sheaves $\Omega^q_S(\log D)$ and $\mathrm{Der}_S(\log D)$ of a reduced divisor $D$ in a smooth complex manifold $S$. The hypersurfaces $D$ for which the sheaves of $\mathcal{O}_S$-modules $\Omega^q_S(\log D)$ and $\mathrm{Der}_S(\log D)$ are locally free are called free divisors. Saito gave a criterion, see Thm. 1.19, to decide whether given logarithmic differential forms (resp. vector fields) form a basis of $\Omega^1_S(\log D)$ (resp. $\mathrm{Der}_S(\log D)$). Saito's theory works in full generality only for reduced divisors, so we shall exhibit problems when dealing with non-reduced divisors, in particular, we show that the duality between logarithmic vector fields and differential forms breaks down in this case. We will also look at the class of Euler-homogeneous divisors. These divisors are of special interest in various applications and many results can be formulated more easily for them. We conclude this section with some historical remarks and applications of free divisors.

**Definition 1.1.** Let $U$ be a domain in $\mathbb{C}^n$ and let $D \subseteq U$ be a divisor of $\mathbb{C}^n$, defined by a reduced equation $h = 0$, where $h$ is a holomorphic function on $U$. A meromorphic $q$-form $\omega$ on $U$ is called a *logarithmic $q$-form (along $D$)* if $h\omega$ and $hd\omega$ are holomorphic on $U$. Since most of the time the divisor $D$ is fixed, we will simply speak of logarithmic $q$-forms.
Now let $S$ be a complex manifold of dimension $n$ and $x = (x_1, \ldots, x_n)$

local complex coordinates around a point $p \in S$. Let $h_p = h_p(x) = 0$ be a local (reduced) equation for $D$ (in the sequel we will often only write $h$, if the meaning is clear). A meromorphic $q$-form $\omega$ is logarithmic (along $D$) at a point $p$ if $\omega h_p$ and $h_p d\omega$ are holomorphic in an open neighbourhood around $p$. We denote

$$\Omega^q_{S,p}(\log D) = \{\omega : \omega \text{ germ of a logarithmic } q\text{-form at } p\},$$

We set

$$\Omega^q_S(\log D) = \bigcup_{p \in S} \Omega^q_{S,p}(\log D),$$

that is, $\Omega^q_S(\log D)$ is the sheaf whose stalks are precisely $\Omega_{S,p}(\log D)$ (for this definition of a sheaf see Appendix A). Note that $\Omega^q_S(\log D)$ is an *analytic* sheaf, i.e., its stalk $\Omega^q_{S,p}(\log D)$ at a point $p$ is a $\mathcal{O}_{S,p}$-module.

**Lemma 1.2.** *We denote by $(S, D)$ the pair of a complex manifold $S$ of complex dimension $n$ and a fixed divisor $D$ in $S$, by $p$ a point in $S$ with complex coordinates $(x_1, \ldots, x_n)$ and by $h_p = h$ the reduced equation of $D$ at $p$. Let $\omega$ be a meromorphic differential form at $p$. The following conditions are equivalent:*
*(i) $\omega$ is a logarithmic $q$-form.*
*(ii) $h\omega$ and $dh \wedge \omega$ are holomorphic at $p$.*
*(iii) There exists a holomorphic function germ $g \in \mathcal{O}_{S,p}$, a holomorphic $(q-1)$-form $\xi$ in $\Omega^{q-1}_{S,p}$ and a holomorphic $q$-form $\eta \in \Omega^q_{S,p}$ such that*

$$\dim \mathcal{O}_{D,p}/(g)\mathcal{O}_{D,p} \leq n - 2, \quad \textit{and}$$

$$g\omega = \frac{dh}{h} \wedge \xi + \eta.$$

*(iv) There exists an $(n-2)$-dimensional analytic set $A \subseteq D$ such that for $p \in D \backslash A$ the germ $\omega_p$ is an element of $\Omega^{q-1}_{U,p} \wedge \frac{dh}{h} + \Omega^q_{U,p}$.*

*Proof.* Here are only shown the implications which are needed later (for a complete proof see [81]). In particular, we will mostly be concerned with logarithmic 1-forms and therefore show $(ii) \Rightarrow (iii)$ only for 1-forms.
The equivalence of $(i)$ and $(ii)$ follows from the formula $d(h\omega) = dh \wedge \omega + hd\omega$.
$(ii) \Rightarrow (iii)$: Since $\omega h$ is contained in $\Omega^1_{S,p}$, one can present $\omega$ as $\frac{1}{h}\sum_{i=1}^n a_i(x)dx_i$, where all $a_i$ are holomorphic function germs. Since

$\omega \wedge dh$ is holomorphic by (ii), the 2-form $\omega h \wedge dh$ is divisible by $h$. We compute this expression: $\omega h \wedge dh = \sum_{i=1}^n a_i dx_i \wedge \sum_{j=1}^n (\partial_{x_j} h) dx_j = \sum_{i<j} (a_i(\partial_{x_j} h) - a_j(\partial_{x_i} h)) dx_i \wedge dx_j =: \sum_{i<j} b_{ij} h dx_i \wedge dx_j$ (we may set $b_{ij} := -b_{ji}$ for $i \geq j$). The $b_{ij}$ are in $\mathcal{O}_{S,p}$. Therefore computing $(\partial_{x_j} h)\omega$ for some $j \in \{1, \ldots, n\}$ yields

$$(\partial_{x_j} h)\omega = \frac{\sum_{i=1}^n a_i (\partial_{x_j} h) dx_i}{h} = a_j \frac{dh}{h} + \sum_{i=1}^n b_{ji} dx_i.$$

Since possibly after a coordinate change one can always find a $j$ such that $\dim(\{h = \partial_{x_j} h = 0\}) \leq n-2$ (see Lemma A.19), one can take $g := \partial_{x_j} h$ for such a $j$, $\xi := a_j$ and $\eta := \sum_{i=1}^n b_{ji} dx_i$. □

*Remark* 1.3. The proof above shows that the holomorphic function $g$ from *(iii)* of Lemma 1.2 can always be chosen as a suitable partial derivative of $h$, but possibly only after a change of coordinates. An example therefore is the normal crossing divisor $h = x_1 x_2$ in $\mathbb{C}\{x_1, x_2\}$. The divisors defined by the partial derivatives $\partial_{x_1} h = x_2$ and $\partial_{x_2} h = x_1$ have both a common component with $D$. However, after a coordinate change $x_1 = y_1 - y_2, x_2 = y_1 + y_2$, one sees that e.g. $\partial_{y_1} h = \partial_{y_1}(y_1^2 - y_2^2) = 2y_1$ has the desired property.

**Lemma 1.4.** *(i)* $\Omega_S^q(\log D)$ *is a coherent sheaf of* $\mathcal{O}_S$*-modules for* $q = 0, \ldots, n$.
*(ii)* $\bigoplus_{i=0}^n \Omega_S^q(\log D)$ *is an exterior algebra over* $\mathcal{O}_S$ *and closed under exterior differentiation.*
*(iii)* $\Omega_{S,p}^0(\log D) = \Omega_{S,p}^0 = \mathcal{O}_{S,p}$ *and* $\Omega_{S,p}^n(\log D) = \frac{\Omega_{S,p}^n}{h}$. *If* $\omega_1, \ldots, \omega_n$ *are in* $\Omega_{S,p}^1(\log D)$, *then*

$$\omega_1 \wedge \cdots \wedge \omega_n = f \frac{\bigwedge_{i=1}^n dx_i}{h},$$

*for some* $f \in \mathcal{O}_{S,p}$.

*Proof.* (i) Since $\Omega_S^q(\log D)$ is a finitely generated subsheaf of the coherent free sheaf $\frac{1}{h}\Omega_S^q$ for any $q = 0, \ldots, n$, it follows by the Meta-Theorem for coherent sheaves, Thm. A.15, that $\Omega_S^q(\log D)$ is also coherent.
(ii) This can be easily checked by using the description (iv) of Lemma 1.2 for logarithmic differential forms.
(iii) The two equalities follow from the definitions of $\Omega_{S,p}^0$ resp. $\Omega_{S,p}^n(\log D)$. Since by (ii) $\bigoplus_{i=0}^n \Omega_S^q(\log D)$ is an exterior $\mathcal{O}_{S,p}$-algebra, $\omega_1 \wedge \cdots \wedge \omega_n \in \Omega_{S,p}^n(\log D) = \frac{1}{h}\Omega_{S,p}^n$, which implies the claim. □

We can also study logarithmic vector fields (= logarithmic derivations) along a divisor in a complex manifold. These vector fields $\delta$ appear naturally as tangent vectors $\delta(p), p \in D$ to the divisor $D$ in its smooth points. Later it is shown that the module of logarithmic vector fields at a point is dual to the module of logarithmic differential 1-forms (Lemma 1.9).
Logarithmic vector fields can also be studied as Lie algebras, since the module of logarithmic vector fields is equipped with the usual Lie bracket of vector fields and is trivially stable under this Lie bracket. From considering the Lie algebra of logarithmic vector fields one can derive many properties of the divisor corresponding to this Lie algebra, as studied by Hauser and Müller, see [50]: in this article logarithmic vector fields are considered as "tangent vector fields" along varieties of any codimension. The Lie algebra structure of logarithmic vector fields is also considered by Granger and Schulze [42]. Furthermore, there is an interest in the generalization of the modules of logarithmic vector fields to so-called *tangential idealizers* in an algebraic context, see [65, 89]. Here we will study logarithmic derivations in Saito's spirit:

**Definition 1.5.** Let $(S, D)$ be as in Lemma 1.2. A *logarithmic vector field* (or *logarithmic derivation*) is a holomorphic vector field on $S$, that is, an element of $\mathrm{Der}_S$ satisfying one of the two equivalent conditions:
(i) For any smooth point $p$ of $D$, the vector $\delta(p)$ of $p$ is tangent to $D$,
(ii) For any point $p$, where $(D, p)$ is given by $h = 0$, the germ $\delta(h)$ is contained in the ideal $(h)$ of $\mathcal{O}_{S,p}$. The module of germs of logarithmic derivations (of D) $D$ at $p$ is denoted by

$$\mathrm{Der}_{S,p}(\log D) = \{\delta : \delta \in \mathrm{Der}_{S,p} \text{ such that } \delta h \in (h)\},$$

and the sheaf of $\mathcal{O}_S$-modules, whose stalk at a point $p$, is $\mathrm{Der}_{S,p}(\log D)$ is

$$\mathrm{Der}_S(\log D) = \bigcup_{p \in S} \mathrm{Der}_{S,p}(\log D).$$

**Definition 1.6.** Let $S$ be an $n$-dimensional complex manifold and let $x = (x_1, \ldots, x_n)$ be the complex coordinates around a point $p \in S$. Let $\xi = \sum_{i=1}^n \xi^i \partial_{x_i}, \eta = \sum_{i=1}^n \eta^i \partial_{x_i}$ be in $\mathrm{Der}_{S,p}$. Then the *Lie bracket* $[\xi, \eta]$ is defined as $[\xi, \eta] = \xi \circ \eta - \eta \circ \xi$. In the local coordinates this looks as follows:

$$[\xi, \eta]^k = \sum_{i=1}^n (\xi^i \partial_{x_i}(\eta^k) - \eta^i \partial_{x_i}(\xi^k)), \quad \text{for } k = 1, \ldots, n.$$

**Lemma 1.7.** *Let $(S, D)$, $p$ and $h$ be defined as in Lemma 1.2. Some useful properties of* $\mathrm{Der}_S(\log D)$*:*
*(i)* $\mathrm{Der}_S(\log D)$ *is a coherent $\mathcal{O}_S$-submodule of* $\mathrm{Der}_S$*, the sheaf of holomorphic vector fields on $S$.*
*(ii)* $\mathrm{Der}_S(\log D)$ *is closed under the bracket* $[\cdot, \cdot]$.
*(iii) For any vector fields $\delta_1, \dots, \delta_n \in \mathrm{Der}_{S,p}(\log D)$ with $\delta_i = \sum_{j=1}^n a_{ij}\partial_{x_j}$ the determinant of their coefficients $\det((a_{ij})_{i,j=1,\dots,n})$ is contained in the ideal $(h) \subseteq \mathcal{O}_{S,p}$.*

*Proof.* (i): Elements $\delta$ of $\mathrm{Der}_{S,p}(\log D)$ are in one-to-one correspondence to syzygies $(a_1, \dots, a_n, b)$ of the coherent ideal $(\partial_{x_1}h, \dots, \partial_{x_n}h, h)$ via

$$\delta \longrightarrow (\delta(x_1), \dots, \delta(x_n), -\frac{\delta(h)}{h})$$
$$\sum_{i=1}^n a_i \partial_{x_i} \longleftarrow (a_1, \dots, a_n, b).$$

Thus the sheaf $\mathrm{Der}_S(\log D)$ is locally isomorphic to the module of syzygies of the coherent ideal $(\partial_{x_1}h, \dots, \partial_{x_n}h, h)$ and hence itself coherent.
(ii): By definition we have for any $\delta, \eta \in \mathrm{Der}_{S,p}(\log D)$ that $\delta(h) = fh$ and $\eta(h) = gh$ for some $f, g \in \mathcal{O}_{S,p}$. Hence

$$\begin{aligned}[\delta, \eta](h) &= \delta(\eta(h)) - \eta(\delta(h)) = \delta(gh) - \eta(fh) \\ &= (\delta(g) + g^2 - \eta(f) - f^2)h \in (h).\end{aligned}$$

(iii): The determinant of the $\delta_i$ is equal to $\delta_1 \wedge \cdots \wedge \delta_n = f\partial_{x_1} \wedge \cdots \wedge \partial_{x_n}$. Since at any smooth point $p \in D$ the vectors $\delta_i(p)$ are tangent to $D$, they have to be linearly dependent. This means nothing else but $\det((a_{ij}(p))) = 0$ for all $p \in D$. Hence $\det((a_{ij}(x)_{i,j=1,\dots,n}) \in (h)\mathcal{O}_{S,p}$ and thus $f \in (h)$. □

**Definition 1.8.** Let $p = (x_1, \dots, x_n)$ be a point in an $n$-dimensional complex manifold $S$. The *pairing* of vector fields and differential $q$-forms is denoted by

$$\mathrm{Der}_{S,p} \times \Omega^q_{S,p} \to \Omega^{q-1}_{S,p}, (\delta, \omega) \mapsto \delta \cdot \omega.$$

In coordinates: for $\delta = \sum_{i=1}^n \delta^i \partial_{x_i}$ and $\omega = \sum_J \omega_J dx_{j_1} \wedge \cdots \wedge dx_{j_q}$, where $J = \{j_1, \dots, j_q\}$ with $1 \leq j_1 < \cdots < j_q$, we have $\delta \cdot \omega = \sum_{i=1}^n (\sum_{J: i \in J} \delta^i \omega_J dx_{j_1} \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{j_q})$. Especially for the product of derivations with 1-forms this means $\delta \cdot \omega = \sum_{i=1}^n (\sum_{j=1}^n \delta^j \partial_{x_j}(\omega_i dx_i))$ $= \sum_{i,j=1}^n \delta^j \omega_i \delta_{ij} = \sum_{i=1}^n \delta^i \omega_i \in \mathcal{O}_{S,p}$.

The pairing can be defined for logarithmic vector fields and differential forms. A priori, the product of some $\delta \in \operatorname{Der}_{S,p}(\log D)$ and $\omega \in \Omega^1_{S,p}(\log D)$ may only be meromorphic. The following lemma of Saito [81, Lemma 1.6] shows that $\delta \cdot \omega$ is actually holomorphic.

**Lemma 1.9.** *Let $D$ be a reduced divisor and $p$ a point in $S$. The pairing*

$$\operatorname{Der}_{S,p}(\log D) \times \Omega^q_{S,p}(\log D) \to \Omega^{q-1}_{S,p}(\log D); (\delta, \omega) \mapsto \delta \cdot \omega$$

*is well defined. In particular, by the pairing, $\operatorname{Der}_{S,p}(\log D)$ and $\Omega^1_{S,p}(\log D)$ are dual $\mathcal{O}_{S,p}$-modules.*

*Proof.* Suppose that $h = 0$ is the local equation of $D$ at $p$. First we show that for $\delta \in \operatorname{Der}_{S,p}(\log D)$ and $\omega \in \Omega^q_{S,p}(\log D)$ the product $\delta \cdot \omega$ is contained in $\Omega^{q-1}_{S,p}(\log D)$: by Lemma 1.2 we may represent $\omega$ as $\frac{\xi}{g} \wedge \frac{dh}{h} + \frac{\eta}{g}$ for some holomorphic $(q-1)$-form $\xi$ and $q$-form $\eta$ and a holomorphic $g$, which does not vanish on any irreducible component of $D$. By definition we have $g(\delta \cdot \omega) = \delta \cdot (g\omega)$ and hence

$$\delta \cdot (g\omega) = \delta \cdot \xi \wedge \frac{dh}{h} + (-1)^{q-1} \xi \wedge \delta(\frac{dh}{h}) + \delta \cdot \eta.$$

Here $\delta(\frac{dh}{h}) = \frac{\delta h}{h}$, $\delta \cdot \xi$ and $\delta \cdot \eta$ are holomorphic. Thus $\delta \cdot \omega$ has the required representation

$$g(\delta \cdot \omega) = \xi' \wedge \frac{dh}{h} + \eta',$$

with $\xi' = \delta \cdot \xi$ and $\eta' = \delta \cdot \eta + (-1)^{q-1} \frac{\delta h}{h} \xi$. The non-singularity of the pairing $\operatorname{Der}_{S,p}(\log D) \times \Omega^1_{S,p}(\log D)$ is shown as follows: for the differential forms one has

$$\Omega^1_{S,p} \subseteq \Omega^1_{S,p}(\log D) \subseteq \frac{1}{h} \Omega^1_{S,p},$$

by definition of $\Omega^1_{S,p}(\log D)$. Taking duals $*$ results in

$$\operatorname{Der}_{S,p} \supseteq (\Omega^1_{S,p}(\log D))^* \supseteq h \operatorname{Der}_{S,p}.$$

Thus an element $\delta \in (\Omega^1_{S,p}(\log D))^*$ can be seen as an element in $\operatorname{Der}_{S,p}$. The differential form $\frac{dh}{h}$ is logarithmic. By duality $\delta \cdot \frac{dh}{h}$ must be contained in $\mathcal{O}_{S,p}$. However, $\delta \cdot \frac{dh}{h} = \frac{\delta h}{h}$ and thus $\delta h \in (h)\mathcal{O}_{S,p}$. This proves that $\delta$ is also contained in $\mathrm{Der}_{S,p}(\log D)$. For the other inclusion, consider

$$h\,\mathrm{Der}_{S,p} \subseteq \mathrm{Der}_{S,p}(\log D) \subseteq \mathrm{Der}_{S,p}\,.$$

Dualizing yields

$$\frac{1}{h}\Omega^1_{S,p} \supseteq (\mathrm{Der}_{S,p}(\log D))^* \supseteq \Omega^1_{S,p}.$$

Hence an element $\omega \in (\mathrm{Der}_{S,p}(\log D))^*$ can be written as $\frac{1}{h}\sum_{i=1}^n a_i dx_i$. In order to show that $dh \wedge \omega$ is holomorphic we first observe that the holomorphic vector fields $\delta_{ij} = (\partial_{x_i} h)\partial_{x_j} - (\partial_{x_j} h)\partial_{x_i}$ are elements of $\mathrm{Der}_{S,p}(\log D)$ for all $1 \leq i, j \leq n$. Therefore, by duality, $\delta_{ij} \cdot \omega = \frac{1}{h}(a_j(\partial_i h) - a_i(\partial_j h))$ must be holomorphic. A computation of

$$dh \wedge \omega = \frac{1}{h}\left(\sum_{i<j}(a_j(\partial_{x_i} h) - a_i(\partial_{x_j} h))dx_i \wedge dx_j\right)$$

shows that $dh \wedge \omega$ is holomorphic and thus $\omega \in \Omega^1_{S,p}(\log D)$. □

**Corollary.** *$\Omega^1_{S,p}(\log D)$ and $\mathrm{Der}_{S,p}(\log D)$ are reflexive $\mathcal{O}_{S,p}$-modules. If the dimension $\dim S$ equals 2, then $\mathrm{Der}_{S,p}(\log D)$ and $\Omega^1_{S,p}(\log D)$ are locally free modules.*

*Proof.* Since $\mathcal{O}_{S,p}$ is a regular local ring of Krull-dimension 2, we can apply Theorem A.3 and are finished. □

Let us see some examples of divisors and modules of logarithmic differential forms resp. derivations. In particular, $\Omega^1_{S,p}(\log D)$ is in general not a *free* $\mathcal{O}_{S,p}$-module if $\dim S \geq 3$.

*Example* 1.10. (The normal crossing divisor) This is the most basic example, which was originally considered by Deligne [28] in order to study logarithmic differential forms. Let $D$ be a normal crossing divisor in a manifold $S$ of dimension $n$ and suppose that locally at a point $p = (x_1, \ldots, x_n)$ the divisor is given as $D = \{x_1 \cdots x_k = 0\}$ where $0 \leq k \leq n$. Then clearly $\Omega^1_{S,p}(\log D)$ is free and has the basis $\frac{dx_1}{x_1}, \ldots, \frac{dx_k}{x_k}, dx_{k+1}, \ldots, dx_n$. The dual basis of $\mathrm{Der}_{S,p}(\log D)$ is given by $x_1\partial_{x_1}, \ldots, x_k\partial_{x_k}, \partial_{x_{k+1}}, \ldots, \partial_{x_n}$. In particular, if $D = \{x_1 = 0\}$ is smooth, then $\Omega^1_{S,p}(\log D)$ is free and any $\omega \in \Omega^1_{S,p}(\log D)$ has a representation $\omega = a_1\frac{dx_1}{x_1} + \sum_{i=2}^n a_i dx_i$, with $a_i \in \mathcal{O}_{S,p}$, of Lemma 1.2.

*Example* 1.11. (The cusp) The corollary above shows that any divisor in $S$ is free if $\dim S = 2$. The *cusp singularity* $(D, p)$ in $(S, p)$ is given at $p = (x, y)$ by $h = x^3 - y^2$. A basis of $\mathrm{Der}_{S,p}(\log D)$ is formed by $\delta_1 = 2x\partial_x + 3y\partial_y$ and $\delta_2 = 2y\partial_x + 3x^2\partial_y$. A basis of $\Omega^1_{S,p}(\log D)$ is obtained by forming the dual basis to $(\delta_1, \delta_2)$.

*Example* 1.12. (Isolated surface singularity) Consider the normal surface singularity $E_8$ in $(\mathbb{C}^3, 0)$, given by $h = x^2 + y^3 + z^5$. Here it is not so easy to determine a system of generators of $\mathrm{Der}_{S,p}(\log D)$. However, in Thm. 1.42 it will be shown that $\Omega^1_{S,p}(\log D)$ is generated by $\frac{dh}{h} = \frac{1}{h}(2xdx + 3y^2dy + 5z^4dz)$, $dx, dy$, and $dz$.

*Example* 1.13. (The Whitney Umbrella) The Whitney Umbrella $D$ is the surface in $\mathbb{C}^3$ given by the equation $x^2 - y^2z = 0$. It has the $z$-axis as singular locus. Later it will be proven that for all points $p$ of $\mathbb{C}^3 \backslash \{0\}$ the module of logarithmic derivations at $p$ is free, see Example 2.9. However, a computation (of the syzygies of the Jacobian ideal of $x^2 - y^2z$) shows that $\mathrm{Der}_{\mathbb{C}^3,0}(\log D)$ is minimally generated by the four vector fields $\delta_1 = 2x\partial_x + y\partial_y + 2z\partial_z$, $\delta_2 = -y\partial_y + 2z\partial_z$, $\delta_3 = -y^2\partial_x + 2x\partial_z$, $\delta_4 = x\partial_y - yz\partial_x$ and hence not free.

**Definition 1.14.** A hypersurface $D$ in $S$ is called a *free divisor* if $\Omega^1_S(\log D)$ is a locally free $\mathcal{O}_S$-module. We say that $D$ is *free at* $p$ (or $(D, p)$ is free) if $\Omega^1_{S,p}(\log D)$ or the dual module $\mathrm{Der}_{S,p}(\log D)$) is a free $\mathcal{O}_{S,p}$-module.

One can construct free divisors in any dimension as cylinders over plane curves. However, it is not so easy to find more interesting examples of free divisors in dimension greater than 2. Free divisors in dimension $\geq 3$ have not yet been classified and only some classes of examples are known but there is no general theory, see e.g. [40, 67, 86].

*Example* 1.15. (Hyperplane arrangements) Consider the divisor $H$ that is given globally by $xyz(x + y)$ in $\mathbb{C}^3$, i.e., $H$ is a union of four hyperplanes. With the help of Saito's criterion (Thm. 1.19) it can be shown that $H$ is a free divisor, a so-called free hyperplane arrangement. In Chapter 2 we will see more of hyperplane arrangements. Freeness of hyperplane arrangements has been studied by Terao [94].

*Example* 1.16. (The 4-lines) Consider the divisor $D$ in $(\mathbb{C}^3, 0)$ given by $h = xy(x + y)(x + yz)$. This example is originally from Whitney [100] and serves as the prototypic example of an analytic variety that is not analytically trivial along a smooth subvariety (here: along the $z$-axis). It is also a source of examples and counterexamples in the theory of free divisors and logarithmic differential operators, see e.g. [18,19,71]. "The 4-lines" divisor is a mild generalization of a hyperplane arrangement since it consists generically at points of the singular locus of four smooth surfaces, which intersect pairwise transversally. A computation of the syzygies of the Jacobian ideal shows that the three logarithmic vector fields $\delta_1 = xy\partial_x + y^2\partial_y - 4(x+yz)\partial_z$, $\delta_2 = x(x+3y)\partial_x - y(3x+y)\partial_y + 4x(z-1)\partial_z$ and $\delta_3 = x\partial_x + y\partial_y$ form a basis of $\mathrm{Der}_{\mathbb{C}^3,0}(\log D)$. So $D$ is a free divisor.

*Example* 1.17. (Discriminants) Another source of nontrivial examples of free divisors are discriminants of deformations. A remarkable result in singularity theory is that the discriminant of a versal deformation of any isolated hypersurface singularity is a free divisor. This was first proven by Saito [82], for different proofs see [4, 60]. This result has been generalized in many directions, see e.g. [13, 25, 60, 68] and more references therein. The first interesting example is the discriminant of the versal deformation of an $A_3$-singularity defined by $w^4 = 0$. A versal deformation of this singularity is given by $X = \{w^4+xw^2+yw+z = 0\}$, cf. [27, 60], which has discriminant

$$h = 256z^3 - 128x^2z^2 + 16x^4z + 144xy^2z - 4x^3y^2 - 27y^4.$$

The polynomial $h$ defines a free divisor in $\mathbb{C}^3$. A basis of $\mathrm{Der}_{\mathbb{C}^3,0}(\log D)$ is given by $\delta_1 = 6y\partial_x + (8z - 2x^2)\partial_y - xy\partial_z$, $\delta_2 = (4x^2 - 48z)\partial_x + 12xy\partial_y + (9y^2 - 16xz)\partial_z$ and $\delta_3 = 2x\partial_x + 3y\partial_y + 4z\partial_z$. A versal deformation of the $A_4$-singularity $w^5 = 0$ is given by $X = \{w^5 + xw^3 + yw^2 + zw + u = 0\}$. Its discriminant is defined by a quasi-homogeneous polynomial in four variables and gives rise to a free divisor in $\mathbb{C}^4$.

*Remark* 1.18. Although a divisor may not be free at all of its points, the set of free points is open and dense in $S$ and also in $D$. This follows from the general fact that for a coherent sheaf $\mathcal{F}$ on a complex space $X$ the set $\{x \in X : \mathcal{F}_x \text{ is not a free } \mathcal{O}_{X,x}\text{-module}\}$ is a proper analytic subset in $X$ (cf. [27, Thm. 6.2.11.]). Note that the set of non-free points of $D$ is contained in $\mathrm{Sing}\, D$, the singular locus of $D$.

**Theorem 1.19** (Saito's criterion)**.** *Let $(S, D)$, $p$ and $h$ be defined as in Lemma 1.2. The $\mathcal{O}_{S,p}$-module $\mathrm{Der}_{S,p}(\log D)$ is free if and only if there exist $n$ vector fields $\delta_i = \sum_{j=1}^n a_{ij}(x)\partial_{x_j}$ in $\mathrm{Der}_{S,p}(\log D)$, $i = 1, \dots, n$, such that $\det(a_{ij}(x))$ is equal to $h$ up to an invertible factor. Moreover, then the vector fields $\delta_1, \dots, \delta_n$ form a basis for $\mathrm{Der}_{S,p}(\log D)$.*
*$\Omega^1_{S,p}(\log D)$ is a free $\mathcal{O}_{S,p}$-module if and only if one has $\bigwedge^n \Omega^1_{S,p}(\log D) =$*

$\Omega^n_{S,p}(\log D)$. *This means that there exist* $n$ *elements* $\omega_i \in \Omega^1_{S,p}(\log D)$ *such that*

$$\omega_1 \wedge \ldots \wedge \omega_n = u\frac{dx_1 \wedge \ldots \wedge dx_n}{h},$$

*where* $u$ *is a unit in* $\mathcal{O}_{S,p}$, *i.e.,* $u \in \mathcal{O}^*_{S,p}$. *Then the* $\omega_1, \ldots, \omega_n$ *form an* $\mathcal{O}_{S,p}$*-basis for* $\Omega^1_{S,p}(\log D)$ *and one can write*

$$\Omega^q_{S,p}(\log D) = \sum_{i_1 < \cdots < i_q} \mathcal{O}_{S,p}\ \omega_{i_1} \wedge \cdots \wedge \omega_{i_q},$$

*for all* $q = 1, \ldots, n$.

*Proof.* Differential forms: First suppose that $\Omega^1_{S,p}(\log D)$ is a free $\mathcal{O}_{S,p}$-module. By Lemma 1.4 (i), the sheaf $\Omega^1_S(\log D)$ is coherent, thus there exists a neighbourhood $U$ of $p \in S$ such that $\Omega^1_S(\log D)|_U$ is $\mathcal{O}_S$-free. For any point $q \in U$, $q \notin D$, it is clear that $\Omega^1_{S,q}(\log D) = \Omega^1_{S,q}$. From this it follows that $\Omega^1_S(\log D)|_U$ has a basis $\omega_1, \ldots, \omega_n$ consisting of $n$ elements. Again by Lemma 1.4 (iii), one knows that

$$\omega_1 \wedge \cdots \wedge \omega_n = f\frac{\bigwedge_{i=1}^n dx_i}{h},$$

where $f$ is a holomorphic function. For any point $q \in U\backslash D$, $\omega_1 \wedge \cdots \wedge \omega_n$ has to be a unit multiple of the $n$-form $\bigwedge_{i=1}^n dx_i$. For a smooth point $q \in D \cap U$ with coordinates $(y_1, \ldots, y_n)$ we may suppose that $D$ looks locally like $\{y_1 = 0\}$. Then by the equivalent characterizations of logarithmic differential forms (Lemma 1.2) any $\omega \in \Omega^1_{S,q}(\log D)$ is of the form $\omega = \xi\frac{dy_1}{y_1} + \eta$, where $\xi$ is a holomorphic function and $\eta$ is a holomorphic differential 1-form. From this it follows that $\frac{dy_1}{y_1}, dy_2, \ldots, dy_n$ form a free basis of $\Omega^1_{S,q}(\log D)$. By the implicit function theorem $\omega_1 \wedge \cdots \wedge \omega_n = \frac{1}{y_1}\bigwedge_{i=1}^n dy_i = u\frac{1}{h}\bigwedge_{i=1}^n dx_i$ holds for some unit $u \in \mathcal{O}^*_{S,p}$. Thus $\omega_1 \wedge \cdots \wedge \omega_n$ is a unit multiple of $\frac{1}{h}\bigwedge_{i=1}^n dx_i$. Hence the holomorphic function $f$ can only vanish on a set of codimension greater than of equal to 2 in $U$, which implies that $f(x)$ does not vanish at all on $U$.

Conversely, suppose that there exist $n$ logarithmic differential one-forms $\omega_1, \ldots \omega_n$, such that

$$\omega_1 \wedge \cdots \wedge \omega_n = \frac{dx_1 \wedge \cdots \wedge dx_n}{h}.$$

Take any $\omega \in \Omega^q_{S,p}(\log D)$, $q = 1, \ldots, n$ and form

$$\omega \wedge \omega_{i_1} \wedge \cdots \wedge \omega_{i_{n-q}} = a_I\frac{\bigwedge_{i=1}^n dx_i}{h},$$

where the $a_I$ are in $\mathcal{O}_{S,p}$ and $I$ is the multi-index $(i_1, \ldots, i_{n-q})$ with $1 \leq i_1, \ldots, i_{n-q} \leq n$. We set

$$\omega' = \omega - \sum \operatorname{sgn}(I) a_I \omega_{j_1} \wedge \cdots \wedge \omega_{j_q},$$

where $\operatorname{sgn}(I)$ denotes the sign of the permutation $\binom{1\cdots n}{i_1 \cdots i_{n-q} j_1 \cdots j_q}$. It can easily be seen that $\omega' \wedge \omega_{i_1} \wedge \cdots \wedge \omega_{i_{n-q}} = 0$ for $1 \leq i_1 < \ldots < i_{n-q} \leq n$. Hence $\omega'$ is $\mathcal{O}_{S,p}$-linearly dependent on $\omega_{i_1}, \ldots, \omega_{i_{n-q}}$ for points in $S \backslash D$ and hence in the span of these $\omega_{i_j}$'s. But this means that $\omega = \omega' + \sum \operatorname{sgn}(I) a_I \omega_{j_1} \wedge \cdots \wedge \omega_{j_q}$ is an $\mathcal{O}_{S,p}$-combination of the $\omega_{j_1} \wedge \cdots \wedge \omega_{j_q}$. In particular, any $\omega \in \Omega^1_{S,p}(\log D)$ is a linear combination of the $\omega_i$'s.

For the logarithmic derivations we apply the duality proven above and a trick used in [74]: First suppose that $\operatorname{Der}_{S,p}(\log D)$ is free and has a basis $\delta_1, \ldots, \delta_n$ with $\delta_i = \sum_{j=1}^n a_{ij} \partial_{x_j}$. By the duality between logarithmic derivations and logarithmic differential 1-forms, $\Omega^1_{S,p}(\log D)$ is also free, which means that there exists a basis $\omega_1, \ldots, \omega_n \in \Omega^1_{S,p}(\log D)$ with $\langle \omega_i, \delta_j \rangle = \delta_{ij}$. If we write $\omega_i = \sum_{j=1}^n b_{ij} dx_i$ then by the first part of the theorem we have $\det(b_{ij}) = \frac{1}{h}$. Since the matrix $A = (a_{ij})$ is the adjoint matrix to $B = (b_{ij})$, it follows that $\det(A) = \det(B^{-1}) = h$, which is what had to be shown.

For the other implication suppose that we have $\delta_1, \ldots, \delta_n \in \operatorname{Der}_{S,p}(\log D)$ with $\delta_i = \sum_{j=1}^n a_{ij} \partial_{x_j}$, such that $\det(a_{ij}) = h$. We show that any $\delta = \sum_{i=1}^n c_i \partial_{x_i} \in \operatorname{Der}_{S,p}(\log D)$ can be written as an $\mathcal{O}_{S,p}$-linear combination of the $\delta_i$'s: First it follows by Cramer's rule that the derivations $h\partial_{x_j}$ are $\mathcal{O}_{S,p}$-linear combinations of the $\delta_i$'s for all $j = 1, \ldots, n$, hence so is $h\delta = \sum_{i=1}^n f_i \delta_i$. By Lemma 1.7, $\delta_1 \wedge \cdots \wedge \delta_{i-1} \wedge \delta \wedge \delta_{i+1} \wedge \cdots \wedge \delta_n = g_i h \partial_{x_1} \wedge \cdots \wedge \partial_{x_n}$ for some $g_i \in \mathcal{O}_{S,p}$ and thus

$$h(\delta_1 \wedge \cdots \wedge \delta_{i-1} \wedge \delta \wedge \delta_{i+1} \wedge \cdots \wedge \delta_n) = \delta_1 \wedge \cdots \wedge \delta_{i-1} \wedge h\delta \wedge \delta_{i+1} \wedge \cdots \wedge \delta_n,$$
$$h^2 g_i \partial_{x_1} \wedge \cdots \wedge \partial_{x_n} = f_i \delta_1 \wedge \cdots \wedge \delta_n,$$

which implies that $h^2 g_i = f_i h$. This means that all $f_i/h = g_i$ are holomorphic and hence $\delta = \sum_{i=1}^n g_i \delta_i$ is an $\mathcal{O}_{S,p}$-linear combination of the $\delta_i$, which is what had to be shown. □

### Problems with non-reduced divisors

In this section we briefly comment on why we always assume that the divisors $D$, we are dealing with are defined by *reduced* holomorphic equations. In short, the reason is that if $D$ is given by a non-reduced equation, then the duality between the modules $\Omega^1_{S,p}(\log D)$

and $\mathrm{Der}_{S,p}(\log D)$ does not hold.
If $D$ is given by a non-reduced equation $h^\alpha = h_1^{\alpha_1} \cdots h_m^{\alpha_m}$ with $\alpha_i \geq 1$ and $\sum \alpha_i > m$ at a point $p$, then the module $\mathrm{Der}_{S,p}(\log D)$ is $\mathrm{Der}_{S,p}(\log D_{red})$, where $D_{red}$ denotes the divisor with local equation $h = h_1 \cdots h_m$, see Prop. 1.21. However, we will see in an example that the module $\Omega^1_{S,p}(\log D)$ is not equal to its reduced version $\Omega^1_{S,p}(\log D_{red})$.

**Lemma 1.20.** *Let $(S, D)$ be a manifold with $\dim S = n$ and its divisor. Suppose that $(D, p) = \bigcup_{i=1}^m (D_i, p)$ is the irreducible decomposition of $D$ at a point $p \in S$ and that $D$ is given locally at $p$ by a reduced equation $h = h_1 \cdots h_m$, where $h_i$ corresponds to the irreducible component $D_i$ for $i = 1, \ldots, m$. Then*

$$\mathrm{Der}_{S,p}(\log D) = \bigcap_{i=1}^m \mathrm{Der}_{S,p}(\log D_i).$$

*Proof.* Suppose that $\delta \in \mathrm{Der}_{S,p}(\log D)$. Then by definition of logarithmic vector fields we have

$$\delta(h) = \sum_{i=1}^m h_1 \cdots \widehat{h_i} \cdots h_m \delta(h_i) = gh$$

for a $g \in \mathcal{O}_{S,p}$. Dividing this equation by $h_j$ it follows that

$$h_1 \cdots \widehat{h_j} \cdots h_m \frac{\delta(h_j)}{h_j}$$

is holomorphic. Since $h_j$ is irreducible and does not divide any of the $h_i$ for $i \neq j$, one concludes that $\delta(h_j) \in (h_j)$. This argument applies to any $j = 1, \ldots, m$, so that $\delta$ is contained in $\bigcap_{i=1}^m \mathrm{Der}_{S,p}(\log D_i)$.
Conversely, suppose that $\delta(h_i) = g_i h_i$, $g_i \in \mathcal{O}_{S,p}$ for all $i = 1, \ldots, m$. Then

$$\delta(h_1 \cdots h_m) = h_1 \cdots h_m (\sum_{i=1}^m g_i),$$

which implies that $\delta(h) \in (h)$. □

Note that for the proof of Lemma 1.20 one does not need the irreducibility of the $h_i$, only that they are coprime. Hence the statement holds in general for $(D, p) = \bigcup_{i=1}^m (D_i, p)$, where the $(D_i, p)$ are defined by mutually coprime $h_i$.

**Proposition 1.21.** *Let $S$ be a complex manifold of dimension $n$ together with a divisor $D \subseteq S$. Suppose that at a point $p \in S$ the divisor $(D,p) = \bigcup_{i=1}^m (D_i,p)$ is given by a non-reduced equation $h^\alpha := h_1^{\alpha_1} \cdots h_m^{\alpha_m}$, where the $h_i$ are the equations defining the irreducible components at $p$ and $\alpha_i \in \mathbb{N}_{>0}$ are their multiplicities. Then*

$$\mathrm{Der}_{S,p}(\log D) = \mathrm{Der}_{S,p}(\log D_{red}),$$

*where $D_{red}$ denotes the reduced divisor defined by the equation $h = h_1 \cdots h_m$.*

*Proof.* Suppose that $\delta \in \mathrm{Der}_{S,p}(\log D_{red})$. By Lemma 1.20 we have $\delta(h_i) = g_i h_i, g_i \in \mathcal{O}_{S,p}$. Applying $\delta$ to $h^\alpha$ yields

$$\delta(h^\alpha) = \sum_{i=1}^m \alpha_i h_1^{\alpha_1} \cdots h_i^{\alpha_i - 1} \cdots h_m^{\alpha_m} \delta(h_i) = h^\alpha (\sum_{i=1}^m \alpha_i g_i).$$

Conversely, let $\delta$ be an element of $\mathrm{Der}_{S,p}(\log D)$. Then by definition we have $\delta(h^\alpha) = g h^\alpha$. Expanding $\delta$ yields

$$\delta(h^\alpha) = \sum_{i=1}^m \alpha_i h_1^{\alpha_1} \cdots h_i^{\alpha_i - 1} \cdots h_m^{\alpha_m} \delta(h_i).$$

Dividing through $h_j^{\alpha_j}$ shows that $\alpha_j h_1^{\alpha_1} \cdots \widehat{h_j} \cdots h_m^{\alpha_m} \frac{\delta(h_j)}{h_j}$ is holomorphic. Since all $h_i$ are irreducible and distinct from $h_j$, it follows that $\delta(h_j) \in (h_j)$ for any $j = 1, \ldots, m$. By Lemma 1.20 it holds that $\mathrm{Der}_{S,p}(\log D_{red}) = \bigcap_{i=1}^m \mathrm{Der}_{S,p}(\log (D_i)_{red})$, so this proves our assertion. □

For a non-reduced $D$ it is easy to see that $\Omega^1_{S,p}(\log D) \supsetneq \Omega^1_{S,p}(\log D_{red})$: if $\omega \in \Omega^1_{S,p}(\log D_{red})$ then $h^\alpha \omega = h^{\alpha-1}(h\omega)$ and $h^\alpha d\omega = h^{\alpha-1}(hd\omega)$ are clearly holomorphic. The meromorphic differential form $\omega = \frac{dh}{h^\alpha}$ is in $\Omega^1_{S,p}(\log D)$ since $\omega h^\alpha$ is holomorphic and $dh \wedge \omega = \frac{1}{h^\alpha} dh \wedge dh = 0$ is also holomorphic. But $\omega h$ is not holomorphic and thus $\omega$ is not in $\Omega^1_{S,p}(\log D_{red})$! Thus for a non-reduced divisor $D$ the logarithmic differentials and the logarithmic derivations are not dual to each other, that is, $\Omega^1(\log D)$ is not reflexive.

There have been a few approaches to overcome this obstruction. In [105] Ziegler defines generalized logarithmic vector fields resp. differential form modules of hyperplane arrangements with multiplicities, so-called

multiarrangements. In the theory of hyperplane arrangements this leads to interesting insights about the combinatorics and topology of multi-arrangements as well as simple arrangements (i.e., arrangements defined by a reduced polynomial). We refer to [85] for more references on multi-arrangements.
However, the geometry of a divisor does not change by introducing multiplicities, in particular, the normal crossings property only depends on the reduced equation of the divisor. For this reason only divisors given by reduced equations are considered in this thesis.

### Euler-homogeneous divisors

This section is about so-called Euler-homogeneous divisors. These divisors are of special interest, since they are a generalization of quasi-homogeneous divisors (for a divisor with only isolated singularities the two notions coincide). Often results about free divisors are much simpler to prove for free Euler-homogeneous divisors and can then be generalized. Here we list some properties of Euler-homogeneous divisors, which will be used in Chapter 2. In particular, we show a characterization of free Euler-homogeneous divisors in terms of logarithmic differential forms (Prop. 1.29).

**Definition 1.22.** Let $(S, D)$ be an $n$-dimensional complex manifold together with a divisor $D \subseteq S$. The divisor $D$ is called *Euler-homogeneous* at a point $p$ if for some (reduced) local defining equation $h$ of $D$ there exists a vector field $\eta \in \mathrm{Der}_{S,p}(\log D)$ such that $\eta h = uh$ with $u \in \mathcal{O}^*_{S,p}$ (we also say that $(D, p)$ is a *Euler-homogeneous* singularity). Such a vector field $\eta \in \mathrm{Der}_{S,p}(\log D)$ is called an *Euler vector field.* A divisor $D$ is called Euler-homogeneous if for any point $p$ in $D$ the singularity $(D, p)$ is Euler-homogeneous.

**Definition 1.23.** A divisor $D$ is said to be *quasi-homogeneous* at $p \in S$, where $S$ is locally isomorphic to $\mathbb{C}^n$, if there exist complex coordinates $(x_1, \cdots, x_n)$ locally at $p$ such that $D$ is defined by a polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$ and $P$ is quasi-homogeneous with weights $(w_1, \ldots, w_n)$, $0 < w_i \leq \frac{1}{2}$. This means that $P$ is a linear combination of monomials $x_1^{m_1} \cdots x_n^{m_n}$ with $\sum_{i=1}^n w_i m_i = 1$.

By a theorem of Saito [78], an isolated hypersurface singularity $(D, p)$, defined by $h$ for an $h \in \mathcal{O}_{S,p}$ is quasi-homogeneous at $p$ if and only if $h$ is contained in the Jacobian ideal of $h$, that is, in the ideal generated by the partial derivatives $\partial_{x_i} h$ (in Chapter 2 this ideal will be studied in more detail). But then there exist some $g_i \in \mathcal{O}_{S,p}$ such that $h = \sum_{i=1}^n g_i \partial_{x_i} h$. Thus the vector field $\eta := \sum_{i=1}^n g_i \partial_{x_i}$ is an Euler-vector field and $D$ is Euler-homogeneous at $p$. This shows that if $(D,p)$ is an isolated hypersurface singularity, then $(D,p)$ is quasi-homogeneous if and only if it is Euler-homogeneous. However, nonisolated Euler-homogeneous singularities need not be quasi-homogeneous:

*Example* 1.24. (The 4-lines) The divisor $D$ in $\mathbb{C}^3$ given at 0 by $h = xy(x+y)(xz+y)$ is not quasi-homogeneous but it has an Euler vector field, see e.g. [23].

*Remark* 1.25. One can show that if $(D,p)$ defined by $h \in \mathcal{O}_{S,p}$ is Euler-homogeneous then there exists a *formal* change of coordinates such that $h$ is transformed into a quasi-homogeneous polynomial, see [39].

In the next lemmata we look at logarithmic vector fields along Euler-homogeneous divisors: if $(D,p)$ is Euler-homogeneous we have a certain splitting of the basis of $\mathrm{Der}_{S,p}(\log D)$.

**Lemma 1.26.** *Let $(D,p)$ be an Euler-homogeneous divisor with defining equation $h$ and let $\eta$ be an Euler vector field. Write $\mathcal{M} = \{\delta \in \mathrm{Der}_{S,p}(\log D) : \delta(h) = 0\}$. Then we have the direct sum decomposition*

$$\mathrm{Der}(\log D)_{S,p} \cong \mathcal{O}_{S,p}\eta \oplus \mathcal{M}$$

*If $D$ is moreover free at $p$, then $\mathcal{M}$ is a free submodule of $\mathrm{Der}_{S,p}(\log D)$.*

*Proof.* Let $h = 0$ be the defining equation of $D$ at $p$. Since $(D,p)$ is Euler-homogeneous we have $\eta h = uh$ with $u \in \mathcal{O}^*_{S,p}$. Let $\delta \in \mathrm{Der}_{S,p}(\log D)$. Then by definition of logarithmic derivations we have $\delta h = gh$, $g \in \mathcal{O}_{S,p}$. It follows that $\delta(h) = gu^{-1}\eta(h)$, hence $\delta - gu^{-1}\eta \in \mathcal{M}$. Thus we can write any $\delta$ as a sum $\delta = gu^{-1}\eta + (\delta - gu^{-1}\eta)$. If $D$ is free, then $\mathcal{M}$ is a free submodule of $\mathrm{Der}_{S,p}(\log D)$ since it is a direct summand of a free module over a local ring, see the section about projective modules in Appendix A. □

**Lemma 1.27.** *Let $(D,p)$ be a free Euler-homogeneous divisor in $(S,p)$, given by $h(x_1,\ldots,x_n) \in \mathcal{O}_{S,p}$ and let $\mathcal{M}$ be defined as above. The submodule $\mathcal{M}$ of $\mathrm{Der}_{S,p}(\log D)$ is canonically isomorphic to the module of 1-cycles of the Koszul complex $K(\partial_{x_1}h,\ldots,\partial_{x_n}h)$. The modules of syzygies of order $\geq 2$ are trivial.*

*Proof.* The coefficients of any vector field $\delta \in \mathcal{M}$ give rise to a relation between the $\partial_{x_i} h$. Conversely any relation (or syzygy of first order) between the partial derivatives of $h$ gives an element in $\mathcal{M}$. Since $\mathcal{M}$ is free, the higher syzygies are trivial. □

**Lemma 1.28.** *Let $(D,p)$ be an Euler-homogeneous singularity, suppose that $D$ is free at $p$ and let $\delta_1,\ldots,\delta_{n-1}$ be a basis of the submodule $\mathcal{M} \subseteq \mathrm{Der}_{S,p}(\log D)$. Then every partial derivative $\partial_{x_1}h,\ldots,\partial_{x_n}h$ is given (up to a unit $u \in \mathcal{O}^*_{S,p}$) by one of the principal minors of the $(n-1)\times n$-matrix formed by the coefficients of the vector fields $\delta_1,\ldots,\delta_{n-1}$.*

*Proof.* From Lemma 1.26 it follows that the Euler vector field $\eta$ and the vector fields $\delta_1,\ldots,\delta_{n-1}$ are a basis of the $\mathcal{O}_{S,p}$-module $\mathrm{Der}_{S,p}(\log D)$. Consider the $n\times n$- matrix $M$ whose rows are formed by the coefficients of this basis, that is, the first row corresponds to the coefficients of $\eta$ and the last row to the coefficients of $\delta_{n-1}$. By Saito's criterion, $\det M = kh$ for some unit $k$ and from the definition of $\eta$ follows $\eta h = k'h$, for some $k' \in \mathcal{O}^*_{S,p}$. Hence $M\partial_{\mathbf{x}}h = (k'h,0,\ldots,0)$. Here $\partial_{\mathbf{x}}h$ denotes the column vector $(\partial_{x_1}h,\ldots,\partial_{x_n}h)^T$. Thus, by Cramer's rule, the partial derivatives $\partial_{x_i}h$ correspond to the $(n-1)\times(n-1)$ minors of the matrix $(M_{i,j})_{i=2,\ldots n, j=1,\ldots n}$. □

Now we give a characterization of free Euler-homogeneous divisors in terms of logarithmic differential forms.

**Proposition 1.29.** *Let $D$ in $S$ be a free divisor that is locally at a point $p$ of $S$ defined by a reduced $h \in \mathcal{O}_{S,p}$. Then $D$ is Euler-homogeneous at $p$ if and only if there exists a basis $\omega_1,\ldots,\omega_n$ of $\Omega^1_{S,p}(\log D)$ such that $\frac{dh}{h}$ can be chosen as $\omega_1$.*

*Proof.* Let $D$ be an Euler-homogeneous divisor and let $\eta \in \mathrm{Der}_{S,p}(\log D)$ be such that $\eta(h) = h$. Then we can find a basis of $\mathrm{Der}_{S,p}(\log D)$ consisting of $\eta$ and some $\delta_2,\ldots,\delta_n$ with $\delta_i(h) = 0$ (cf. Lemma 1.26). Denote by $A$ the $n\times n$ matrix with rows the coefficients $\eta_1,\ldots,\eta_n$ of $\eta$ and $\delta_{i1},\ldots,\delta_{in}$ of the $\delta_i$. By Saito's criterion, $\det(A) = uh$ for some unit $u \in \mathcal{O}_{S,p}$. Consider the system of equations

$$A(\partial_{x_1}h,\ldots,\partial_{x_n}h)^T = (h,0,\ldots,0)^T.$$

By Cramer's rule it follows that

$$\partial_{x_i}h = \frac{(-1)^{i+1}\det(A_i)}{u}, \tag{1.1}$$

where $A_i$ is the matrix formed by replacing the $i$-th column vector of $A$ by $(1,0,\ldots,0)^T$. Using the duality between logarithmic vector fields and logarithmic differential 1-forms we get: since $\eta, \delta_2, \dots, \delta_n$ are a basis of $\mathrm{Der}_{S,p}(\log D)$, there exists a unique basis $\omega_1, \dots, \omega_n$ of $\Omega^1_{S,p}(\log D)$ such that $\omega_i \cdot \eta = \delta_{i1}$ and $\omega_i \cdot \delta_j = 0$ for $i \neq j$, that is, $\omega_1, \dots, \omega_n$ are the dual basis of $\eta$ and the $\delta_i$'s. Hence we get the following system of equations for the coefficients $\omega_{11}, \dots, \omega_{1n}$ of $\omega_1$:

$$A(\omega_{11}, \dots, \omega_{1n})^T = (1, 0, \dots, 0)^T.$$

Using again Cramer's rule it follows that $\omega_{1i} = \frac{1}{uh}(-1)^{i+1}\det(A_i)$, which is by (1.1) equal to $\frac{\partial_{x_i} h}{h}$. Hence $\omega_1 = \frac{dh}{h}$, which is what had to be shown.
Conversely, suppose that $\Omega^1_{S,p}(\log D)$ has a basis $\frac{dh}{h}, \omega_2, \dots, \omega_n$. Then there exists a vector field $\delta \in \mathrm{Der}_{S,p}(\log D)$ with the property $\delta \cdot \frac{dh}{h} = 1$. But this is equivalent to $\frac{\delta(h)}{h} = 1$, which means that $\delta(h) = h$ and that $\delta$ is an Euler-vector field for $D$. □

**Question 1.30.** *What can we say if in the above Proposition we drop the freeness assumption on $D$? Does then "$D$ is Euler-homogeneous" mean that $\frac{dh}{h}$ can be chosen as a member of a minimal system of generators of $\Omega^1_{S,p}(\log D)$?*

One has to be careful with the notion of Euler homogenity. The next example exhibits a method how to obtain an Euler-homogeneous divisor from an arbitrary divisor.

*Example* 1.31. (This example is inspired by [31]) Let $(D, 0) \subseteq (\mathbb{C}^{n+1}, 0)$ and let $(x_1, \dots, x_n, y)$ be the complex coordinates at the origin. Suppose that $D = D' \times \mathbb{C}$ with $D'$ defined by $f(x)$ and that $f(x)$ defines a non-quasihomogeneous isolated singularity in $(\mathbb{C}^n, 0)$. Then $e^z f(x)$ is a defining equation for $D$. Hence $D$ is Euler-homogeneous and $\eta = \partial_z$ is an Euler vector field. However, the divisor $D'$ is not Euler-homogeneous.

The above example shows that Euler-homogenity of a divisor that has a smooth factor does not imply that all its irreducible components are Euler-homogeneous. Hence we shall assume that the divisor $D$ is *strongly Euler-homogeneous* at $p$, which means that $D$ has an Euler-vector field that vanishes at $p$. The notion of strong Euler–homogenity plays a role in connection with the logarithmic comparison theorem, see [17, 41].
For a strongly Euler-homogeneous singularity $(D, p) = \bigcup_{i=1}^m (D_i, p)$, where the $D_i$ denote the irreducible components of $D$, there exists an

Euler-vector field $\eta$ for $D$ and by the Leibniz rule it follows that $\eta$ has also to be an Euler-vector field for at least one of the $D_i$. This means, at least one irreducible component $D_i$ is also strongly Euler-homogeneous. In general not all components have this property:

*Example* 1.32. Consider the divisor $D \subseteq \mathbb{C}^3$ that is locally at 0 defined by $xf(y,z) = 0$, where $f$ is a reduced irreducible non-quasihomogeneous polynomial. Then $D$ is strongly Euler-homogeneous with Euler vector field $\eta = x\partial_x$. The irreducible component $(D_1, 0) = \{x = 0\}$ is also strongly Euler-homogeneous but the other irreducible component $(D_2, 0) = \{f(y,z) = 0\}$ certainly not.

The opposite implication is also not true:

*Example* 1.33. Let $D$ in $\mathbb{C}^2$ at $p = (x, y)$ be the union of the parabola $D_1 = \{x - y^2 = 0\}$ and the cusp $D_2 = \{x^3 - y^2\}$. Then $D$ is locally at $p$ given by $h = (x - y^2)(x^3 - y^2)$. Since in dimension 2 the singularities of divisors (= curves) are always isolated, Saito's result on quasi-homogeneous singularities, see [78], implies that a curve $C$ defined by a polynomial $f$ is quasi-homogeneous if and only if it is Euler-homogeneous if and only if $f$ is contained in its Jacobian ideal. Clearly $D_1$ and $D_2$ are quasi-homogeneous (with weights $(2, 1)$ and $(2, 3)$). But $D$ is not quasi-homogeneous, since $h \notin (\partial_x h, \partial_y h)$.

One can give more interesting examples of Euler-homogeneous divisors that appear in connection with the logarithmic meromorphic comparison theorem, see e.g. [71] for an overview. Therefore we need a bit of notation.

**Definition 1.34.** Let $R$ be a commutative ring and $I \subseteq R$ an ideal. One says that $I$ is of *linear type* if the canonical (surjective) map of graded algebras $\mathrm{Sym}_R(I) \to \mathcal{R}(I)$ is an isomorphism. Here $\mathrm{Sym}_R(I)$ denotes the symmetric algebra of the $R$-module $I$ and $\mathcal{R}(I) = \bigoplus_{i=0}^{\infty} I^d t^d \subseteq R[t]$ is the Rees algebra of $I$.

Denote by $(S, D)$ a complex manifold of $\dim S = n$ together with a divisor $D \subseteq S$. Denote $h$ the defining equation of $D$ at a point $p$. We say that $D$ is of *Jacobian linear type* at $p \in D$ if the stalk $(h, \partial_{x_1} h, \ldots, \partial_{x_n} h)$ of its Jacobian ideal plus the defining equation is of linear type. One says that $D$ is of *Jacobian linear type* if it is of Jacobian linear type at all $p \in D$.

Suppose that $((h) + J_h)$ is generated by some $f_1, \ldots, f_k$. Then being of Jacobian linear type means that the kernel of the morphism of graded algebras $\varphi : \mathcal{O}[X_1, \ldots, X_k] \to \mathcal{R}((h)+J_h)$ sending $X_i$ to $f_i t$ is generated by homogeneous elements of degree 1 (see [19]).

**Proposition 1.35.** *If $D$ is of Jacobian linear type at $p$ then $(D, p)$ is an Euler-homogeneous singularity.*

*Proof.* See Rmk. 1.6.6. of [20]. □

### Remarks and some applications

In [28] Deligne introduced the logarithmic de Rham complex, also see e.g. [29], [46] and [97]. The applications of the logarithmic complex mainly lie in cohomology theory. The logarithmic comparison theorem (LCT) states that one can compute the cohomology of the complement of a divisor just from the logarithmic complex: denote by $\Omega^\bullet_S(\log D)$ the logarithmic de Rham complex, which is naturally contained in $\Omega^\bullet(*D)$, the complex of meromorphic forms on $X$ with meromorphic poles (of arbitrary order) along $D$. The Grothendieck-comparison theorem states that for any divisor $D \subseteq S$ the natural morphism

$$\Omega^\bullet_S(*D) \longrightarrow \mathrm{R}j_*\mathbb{C}_U,$$

where $j : U = S\backslash D \hookrightarrow S$ is the natural inclusion, is a quasi-isomorphism, see [63]. Then one asks if the inclusion $\Omega^\bullet_S(\log D) \subseteq \Omega^\bullet_S(*D)$ also yields a quasi-isomorphism. If this is the case then one says that (LCT) holds for $D$. However, (LCT) is proven to hold only for special classes of free divisors and a general characterization is still open, see [22, 71]. Using $\mathcal{D}$-module theory some results for free divisors can be obtained, see e.g. [19]. In order to study meromorphic connections and partial differential equations one can also make use of the theory of logarithmic differential forms, see [7, 28, 73]. Many algebraic properties of free divisors and logarithmic vector fields were studied by A. G. Aleksandrov, see e.g. [4, 5]. We have already mentioned in some examples that free divisors appear in the theory of bifurcations and also in the theory of hyperplane arrangements. Another interesting field of study is the connection between linear free divisors and quiver representations [15].

It is noteworthy that the original definition of logarithmic differential form along a normal crossing divisor of Deligne is different from Saito's. However, we will see below that the two definitions coincide for normal crossing divisors (also see [28, II,3]).

**Definition 1.36** (Deligne, [28])**.** Let $D$ be a normal crossing divisor in a complex manifold $S$ and denote its complement $S\backslash D$ by $U$. Let $j : U \hookrightarrow S$ be the natural inclusion. One defines the *logarithmic de Rham* *complex* $\Omega^\bullet_S(\log D)$ as follows: $\Omega^\bullet_S(\log D)$ is the smallest subcomplex of $j_*\Omega^\bullet_U$ containing $\Omega^\bullet_S$ that is stable under the exterior product and such that for any local section $f \in j_*\mathcal{O}_U$ that is meromorphic along $D$ the differential form $\frac{df}{f}$ is a local section of $\Omega^1_S(\log D)$.

**Proposition 1.37.** *Let $D$ be a normal crossing divisor in a complex manifold $S$ with $\dim S = n$. A section $\omega$ of $j_*\Omega^p_U$ has a logarithmic pole along $D$ if and only if $\omega$ and $d\omega$ have at most a simple pole along $D$. Moreover, the sheaf $\Omega^1_S(\log D)$ is locally free and $\Omega^q_S(\log D) = \bigwedge^q \Omega^1_S(\log D)$.*

*Proof.* At a point $p \in S$ with coordinates $(x_1, \ldots, x_n)$ we may suppose that $D$ is given by the holomorphic function $x_1 \cdots x_k$. Then a section of $(j_*\mathcal{O}_U)$ meromorphic along $D$ is locally at $p$ of the form $f = g\prod_{i=1}^k x_i^{k_i}$ with $g \in \mathcal{O}^*_{S,p}$ and $k_i \in \mathbb{Z}$. Then compute

$$\frac{df}{f} = \frac{dg}{g} + \sum_{i=1}^{k} k_i \frac{dx_i}{x_i},$$

where $\frac{dg}{g} \in \Omega^1_{S,p}$. It follows that any section of $\Omega^1_{S,p}(\log D)$ can be written as an $\mathcal{O}_{S,p}$-linear combination of the $\frac{dx_i}{x_i}, i = 1, \ldots, k$, and the $dx_i$, $i = k+1, \ldots, n$. Thus one sees that $\Omega^1_{S,p}(\log D)$ is locally free with the above basis and it is clear that $\Omega^q_S(\log D) = \bigwedge^q \Omega^1_S(\log D)$. With the help of the explicit basis it follows that a section $\omega \in \Omega^q_{S,p}(\log D)$ and also $d\omega$ have at most a simple pole along $D$. Now it remains to show that a $q$-form $\omega \in j_*\Omega^q_{U,p}$, with $x_1 \cdots x_k\omega$ and $x_1 \cdots x_k d\omega$ holomorphic is also a section of $\Omega^q_{S,p}(\log D)$. Since $\Omega^1_S(\log D)$ is a locally free sheaf of analytic modules, it is enough to show the assertion for a subset of $S$ of codimension $\geq 2$, that is, it is enough to show the assertion for germs $\omega$ of $\Omega^q_{S,p}(\log D)$, where $p$ is a smooth point of $D$. Therefore we may assume that locally at $p$ the divisor is given by $x_1 = 0$. Then take an $\omega \in j_*\Omega^q_{U,p}$ such that also $x_1\omega$ and $x_1 d\omega$ (and equivalently $x_1\omega$ and $\omega \wedge dx_1$) are holomorphic. The form $\omega$ can be written uniquely as $\omega = \omega_1 + \omega_2 \wedge dx_1/x_1$, where $\omega_1$ and $\omega_2$ are meromorphic $q$- and $(q-1)$-forms not containing $dx_1$. Since $\omega x_1$ is holomorphic, this implies that $\omega_2$ and consequently $x_1\omega_1$ are holomorphic. From $dx_1 \wedge \omega = dx_1 \wedge \omega_1 + 0$ holomorphic follows that $\omega_1$ also has to be holomorphic. Hence $\omega$ is a $\mathcal{O}_{S,p}$-linear combination of elements of the form $\frac{1}{x_1}dx_1 \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_q}$, and $dx_{i_1} \wedge \cdots \wedge dx_{i_q}$ for $i_j \in \{2, \ldots, n\}, j = 1, \ldots, q$, which are easily seen to be generators of $\bigwedge^q \Omega^1_{S,p}(\log D)$. □

### 1.1.2 Residue of logarithmic forms

Historically, the study of residues of differential forms was initiated by A. Cauchy in 1825: he considered residues of holomorphic functions in one variable. Later, in 1887, H. Poincaré introduced the notion of the residue of a rational 2-form in $\mathbb{C}^2$. This was generalized by G. de Rham and J. Leray to the class of $d$-closed meromorphic $q$-forms with poles of first order along a smooth divisor. The modern algebraic treatment of residues in duality theory is due to Leray and Grothendieck, see for example [48]. We will study the *logarithmic* residue as introduced by K. Saito.

The residue of logarithmic forms is a tool to study the structure of the module of logarithmic differential forms along $D$. It is tightly connected to the normalization of $D$. Locally, the residue of $\Omega^1_{S,p}(\log D)$ is contained in the ring of meromorphic functions $\mathcal{M}_{D,p}$ on $D$. In some way it measures how far away a logarithmic $q$-form is from being holomorphic.

In this section we give the definition of the logarithmic residue and list some of its properties, which will be used in the sequel. For a complete treatment and all proofs of our assertions see [81, §2].
Let $S$ be an $n$-dimensional complex manifold and $D$ a divisor in $S$ given locally at a point $p \in S$ by a reduced equation $h \in \mathcal{O}_{S,p}$ and denote by $\pi : \widetilde{D} \to D$ the normalization of $D$. Let $\mathcal{O}_D$ and $\mathcal{M}_D$ (resp. $\mathcal{O}_{\widetilde{D}}$ and $\mathcal{M}_{\widetilde{D}}$) be the sheaves of germs of holomorphic and meromorphic functions on $D$ (resp. $\widetilde{D}$). Further denote by $\Omega^q_D$ (resp. $\Omega^q_{\widetilde{D}}$) the sheaf of germs of holomorphic $q$-forms on $D$ (resp. $\widetilde{D}$). One has $\mathcal{O}_{D,p} = \mathcal{O}_{S,p}/(h)\mathcal{O}_{S,p}$ and $\Omega^q_{D,p} = \Omega^q_{S,p}/(h\Omega^q_{S,p} + dh \wedge \Omega^{q-1}_{S,p})$ and also $\mathcal{M}_D \otimes_{\mathcal{O}_D} \Omega^q_D = \pi_*(\mathcal{M}_{\widetilde{D}} \otimes_{\mathcal{O}_{\widetilde{D}}} \Omega^q_{\widetilde{D}})$. In particular for $q = 0$ we have $\pi_*(\mathcal{M}_{\widetilde{D}}) = \mathcal{M}_D$ since $\pi$ is birational.

**Definition 1.38.** Let $(S, D)$, $p$ and $h$ be defined as in Lemma 1.2. Let $\omega$ be any element in $\Omega^q_{S,p}(\log D)$. Then by Lemma 1.2 one can find a presentation

$$g\omega = \frac{dh}{h} \wedge \xi + \eta,$$

with $g$ holomorphic and $\dim \mathcal{O}_{D,p}/(g)\mathcal{O}_{D,p} \leq n-2$, $\xi \in \Omega^{q-1}_{S,p}$ and $\eta \in \Omega^q_{S,p}$. The *residue homomorphism* $\rho$ is defined as the $\mathcal{O}_{S,p}$-linear homomorphism of sheaves

$$\rho : \Omega^q_S(\log D) \longrightarrow \mathcal{M}_D \otimes_{\mathcal{O}_D} \Omega^{q-1}_D$$
$$\omega \longmapsto \rho(\omega) = \frac{\xi}{g}.$$

We often call $\rho(\Omega^1_{S,p}(\log D))$ the *logarithmic residue* (of $D$ at $p$).

**Lemma 1.39.** *The residue homomorphism $\rho$ is well defined.*

For the proof of this lemma we use the following generalization of the de Rham lemma, due to K. Saito:

**Lemma 1.40** (Generalized de Rham lemma)**.** *Let $h$ be a non-constant element in $\mathcal{O}_{S,p}$. Then there exists an integer $k$ such that for any $\omega \in \Omega^q_{S,p}$ with $\omega \wedge dh = 0$ one has*

$$(\partial_{x_i} h)^k \omega = \zeta_i \wedge dh$$

*for some $\zeta_i \in \Omega^{q-1}_{S,p}$ and any $i = 1, \dots, n$.*

*Proof.* See [79]. □

*Proof of Lemma 1.39.* Since $\rho$ is a homomorphism, it is sufficient to show the assertion for $\omega \equiv 0$. Suppose that there are two presentations of $\omega$:

$$g\omega = 0 \wedge \frac{dh}{h} + 0 = \xi \wedge \frac{dh}{h} + \eta,$$

where $\xi \in \Omega^{q-1}_{S,p}$ and $\eta \in \Omega^q_{S,p}$. Thus we have to show that $\xi$ restricted to $D$ is 0. The above equation implies that $\xi \wedge \frac{dh}{h} = -\eta$. Wedging this equation with $dh$ we obtain $\eta \wedge dh = 0$. By the generalized de Rham lemma there exists an integer $k$ such that for each $i = 1, \dots, n$ one has $(\partial_{x_i} h)^k \eta = \zeta \wedge dh$, for some $\zeta \in \Omega^{q-1}_{S,p}$. It follows that $(\xi(\partial_{x_i} h)^k + h\zeta) \wedge dh = 0$. Another application of the de Rham lemma yields that $(\partial_{x_i} h)^l \xi \in h\Omega^{q-1}_{S,p} + dh \wedge \Omega^{q-2}_{S,p}$ for some natural number $l$, that is, $(\partial_{x_i} h)^l \xi$ is 0 in $\Omega^{q-1}_{D,p}$. However, $dh$ is locally on $D \setminus \{g = 0\}$ not equal to zero, because the singular locus is a proper analytic subset of $D$. Since $\partial_{x_i} h$ is a nonzerodivisor in $\mathcal{O}_{D,p} = \mathcal{O}_{S,p}/(h)$ for a suitable $i$ (after a possible change of coordinates), the equation

$$\xi(\partial_{x_i} h)^k = -h\xi$$

restricted to $D = \{h = 0\}$ yields that $\xi = 0 \in \mathcal{O}_{D,p}$. □

**Lemma 1.41.** *(i) Let $\omega \in \Omega^q_{S,p}(\log D)$. Then the residue $\rho(\omega)$ equals 0 if and only if $\omega$ is holomorphic.*
*(ii) The sequence*

$$0 \longrightarrow \Omega^q_{S,p} \longrightarrow \Omega^q_{S,p}(\log D) \xrightarrow{\rho} \pi_*(\mathcal{M}_{\widetilde{D},p} \otimes \Omega^{q-1}_{\widetilde{D},p}) \tag{1.2}$$

*is exact, $p \in D$. Since a sequence of sheaves is exact if and only if the corresponding sequence of stalks is exact, also*

$$0 \longrightarrow \Omega^q_S \longrightarrow \Omega^q_S(\log D) \xrightarrow{\rho} \pi_*(\mathcal{M}_{\widetilde{D}} \otimes \Omega^{q-1}_{\widetilde{D}}) \tag{1.3}$$

*is exact.*
*(iii) The following diagram is commutative*

$$\begin{array}{ccc} \Omega^q_S(\log D) & \xrightarrow{\rho} & \pi_*(\mathcal{M}_{\widetilde{D}} \otimes \Omega^{q-1}_{\widetilde{D}}) \\ \downarrow d & & \downarrow d \\ \Omega^{q+1}_S(\log D) & \xrightarrow{\rho} & \pi_*(\mathcal{M}_{\widetilde{D}} \otimes \Omega^{q}_{\widetilde{D}}). \end{array} \tag{1.4}$$

*(iv) $\rho(\Omega^q_S(\log D))$ is an $\mathcal{O}_{\widetilde{D}}$-coherent submodule of $\mathcal{M}_{\widetilde{D}} \otimes \Omega^{q-1}_{\widetilde{D}}$.*
*(v) The logarithmic residue $\rho(\Omega^1_S(\log D))$ contains $\pi_*\mathcal{O}_{\widetilde{D}}$.*

*Proof.* (i): The representation $g\omega = \frac{dh}{h} \wedge \xi + \eta$ yields that $\rho(\omega) = 0$ is equivalent to $g\omega \in \Omega^q_{S,p}$. By definition $h\omega$ is also holomorphic. Since $\omega = \frac{\eta}{g}$, this implies that $\frac{h\eta}{g} \in \Omega^q_{S,p}$. But $h$ and $g$ must not have a common prime factor because by assumption on $g$, the dimension of $\{g = 0\} \cap D$ at $p$ is $\leq n-2$. Thus $g$ has to divide $\eta$. Hence $\omega = \frac{\eta}{g}$ is holomorphic. The other implication is trivial.
(ii): The exactness of the sequence (1.2) follows from (i).
(iii): Direct computation.
(iv): Let $D$ be defined in an open set $U \subseteq S$ by $h(x) = 0$. By the construction of the logarithmic residue and Lemma 1.2 we have

$$\partial_{x_i} h \cdot \rho(\Omega^q_S(\log D))|_U \subseteq \Omega^{q-1}_D|_{D \cap U}.$$

Since $\Omega^q_S(\log D)$ is a coherent $\mathcal{O}_S$-sheaf, $\rho : \mathcal{O}_S \to \mathcal{O}_D$ is a homomorphism and $\mathcal{O}_D$ is contained in the coherent sheaf $\mathcal{O}_{\widetilde{D}}$, it follows that $\rho(\Omega^q_S(\log D))|_U$ is a coherent $\mathcal{O}_{\widetilde{D}}$-sheaf.
(v): Let $\alpha$ be an element of $\pi_*\mathcal{O}_{\widetilde{D},p}$. Since each $\partial_{x_i} h$ is a universal

denominator[1] (after a possible change of coordinates) for $\pi_*\mathcal{O}_{\widetilde{D},p}$ (see Appendix A), it follows that $(\partial_{x_i}h)\alpha$ is in $\mathcal{O}_{D,p}$ and can be represented by some $a_i \in \mathcal{O}_{S,p}$. Thus

$$(\partial_{x_i}h)a_j - (\partial_{x_j}h)a_i = b_{ij}h$$

for some $b_{ij} \in \mathcal{O}_{S,p}$. Now inspired by the proof of Lemma 1.2 we set

$$\omega = \frac{1}{h}\sum_{i=1}^{n} a_i dx_i.$$

It is clear that $\omega \in \Omega^1_{S,p}(\log D)$ and that $(\partial_{x_j}h)\omega = a_j\frac{dh}{h} + \sum_{i=1}^n b_{ji}dx_i$. Thus, for a suitable $j$, the residue of $\omega$ is $\frac{a_j}{\partial_{x_j}(h)}|_D = \alpha$. Hence $\alpha$ is contained in $\rho(\Omega^1_{S,p}(\log D))$. $\square$

The next theorem of Saito, see [81, (2.9)], is of crucial importance for our characterizations of normal crossing divisors. Here we give the original statement and in section 1.2 we show how to modify the theorem for our purposes.

**Theorem 1.42.** *Let $(S,D)$ be a pair of a complex $n$-dimensional manifold and a divisor $D \subseteq S$. Suppose that locally at a point $p$ the divisor $D$ decomposes into irreducible components $(D,p) = (D_1,p)\cup\ldots\cup(D_m,p)$. Let $h = h_1\cdots h_m$ be the corresponding decomposition of the local equation of $D$, each irreducible factor $h_i$ corresponding to $D_i$. Then the following conditions are equivalent:*
*(i)* $\Omega^1_{S,p}(\log D) = \sum_{i=1}^m \mathcal{O}_{S,p}\frac{dh_i}{h_i} + \Omega^1_{S,p}$.
*(ii)* $\Omega^1_{S,p}(\log D)$ *is generated by closed forms.*
*(iii)* $\rho(\Omega^1_{S,p}(\log D)) = \bigoplus_{i=1}^m \mathcal{O}_{D_i,p}$.
*(iv) (a) For each $i = 1,\ldots,m$ the component $D_i$ is normal (i.e., $\dim \operatorname{Sing} D_i \leq n-3$),*
*(b) $D_i$ intersects $D_j$ transversally for $i \neq j$ and $i,j = 1,\ldots,m$,*
*(c) $\dim(D_i \cap D_j \cap D_k) \leq n-3$ for all $i,j,k$ distinct and $i,j,k = 1,\ldots,m$.*

Note that $(iv)$ implies that $D_i$ and $D_j$ have normal crossings outside an $(n-3)$-dimensional subset of $D$. The following example shows that for $(S,D)$ and $\dim S \geq 3$ the module $\Omega^1_{S,p}(\log D)$ may be generated by

[1] Here we tacitly assume that $\partial_{x_i}h \neq 0$ for all $i$. If one $\partial_{x_i}h$ were equal to 0, then $h$ would be independent from $x_i$ and locally $(D,p) \cong (D' \times \mathbb{C}, (p',0))$ for some $(D',p') \subseteq (\mathbb{C}^{n-1},0)$. Then one may consider $D'$ instead of $D$.

closed forms as in Theorem 1.42, but does not need to be free for all $p$ and $D$ does not necessarily have normal crossings everywhere.

*Example* 1.43. Let $D$ be the divisor in $\mathbb{C}^3$ defined by $h = xz(x+z-y^2)$. This divisor is called *Tülle* and is studied in more detail in [33]. Tülle consists of three components, which are smooth, intersect pairwise transversally and whose triple intersection is a point. Thus it fulfills the assumption (iv) of Theorem 1.42. One can apply Aleksandrov's theorem (Theorem 2.6) to show that $D$ is not free at the origin, namely, the local ring $\mathcal{O}_{\operatorname{Sing} D,0}$ defining the singular locus $(\operatorname{Sing} D, 0)$ is not Cohen–Macaulay. Hence $\Omega^1_{\mathbb{C}^3,0}(\log D)$ can be generated by closed forms but it is not free. Note that Tülle does not have normal crossings at the origin.

*Proof.* $(i) \Rightarrow (ii)$: This is clear since $d\left(\frac{dh_i}{h_i}\right) = 0$.
$(ii) \Rightarrow (iii)$: If $\omega$ is a closed logarithmic 1-form, then from the commutativity of diagram (1.4) it follows that $\rho(\omega)$ is a constant $c_i \in \mathbb{C}$ on each branch $D_i$. By the exactness of the sequence (1.2) it follows that $\omega = \sum_{i=1}^m c_i \frac{dh_i}{h_i} + \eta$ for some $\eta \in \Omega^1_{S,p}$. So if $\Omega^1_{S,p}(\log D)$ is generated by closed forms $\omega_i = \sum_{j=1}^m c_{ij} \frac{dh_j}{h_j} + \eta_i$ with $c_{ij} \in \mathbb{C}$, $i = 1, \ldots, k$ and $k \geq n$, then each $\omega \in \Omega^1_{S,p}(\log D)$ is of the form

$$\omega = \sum_{i=1}^k a_i \omega_i = \sum_{i=1}^k \sum_{j=1}^m a_i c_{ij} \frac{dh_j}{h_j} + \sum_{i=1}^k a_i \eta_i.$$

Then $\rho(\omega) = \sum_{j=1}^m (\sum_{i=1}^k a_i c_{ij}) 1_{D_j,p}$ is contained in $\bigoplus_{i=1}^m \mathcal{O}_{D_i,p}$. Conversely, by Lemma 1.41 (v), $\rho(\Omega^1_S(\log D))$ contains $\pi_*\mathcal{O}_{\widetilde{D}} = \bigoplus_{i=1}^m \pi_*\mathcal{O}_{\widetilde{D}_i}$, which contains $\bigoplus_{i=1}^m \mathcal{O}_{D_i}$.
$(iii) \Rightarrow (i)$ : From the sequence (1.2) we get an exact sequence

$$0 \longrightarrow \Omega^1_{S,p} \longrightarrow \Omega^1_{S,p}(\log D) \xrightarrow{\ \rho\ } \bigoplus_{i=1}^m \mathcal{O}_{D_i,p} \longrightarrow 0. \qquad (1.5)$$

Hence $\Omega^1_{S,p}(\log D) = \sum_{i=1}^m \mathcal{O}_{D_i,p} \frac{dh_i}{h_i} + \Omega^1_{S,p}$.
$(iii) \Rightarrow (iv)$ : From (v) of Lemma 1.41 one gets

$$\rho(\Omega^1_{S,p}(\log D)) = \bigoplus_{i=1}^m \mathcal{O}_{D_i,p} \subseteq \bigoplus_{i=1}^m \mathcal{O}_{\widetilde{D}_i,p} = \widetilde{\mathcal{O}}_{D,p} \subseteq \rho(\Omega^1_{S,p}(\log D)).$$

From this it follows that $\mathcal{O}_{D_i,p} = \mathcal{O}_{\widetilde{D}_i,p}$ for all $i = 1, \ldots, m$, that is, each $D_i$ is normal at $p$. Thus the singular locus of any $D_i$ is of codimension

$\geq 3$ in $S$. Next suppose that two $D_i$, $D_j$ intersect tangentially along an $(n-2)$-dimensional subset of $S$. At a general point $q$ of their intersection both $D_i$ and $D_j$ are smooth and so one can choose local coordinates $(x_1, \dots, x_n)$ such that $D_i = \{x_1 = 0\}$ and $D_j = \{x_1 + x_2^t = 0\}$ for some $t \geq 2$. Then one easily computes that

$$\omega = \frac{x_2 dx_1 - t x_1 dx_2}{x_1(x_1 + x_2^t)}$$

is an element of $\Omega^1_{S,q}(\log D)$ and that $\rho(\omega)|_{D_i} = (-1)^i x_2^{-t+1}$. Thus $\rho(\omega)|_{D_i}$ is meromorphic with a pole along $D_i \cap D_j$. From the coherence of $\rho(\Omega^1_S(\log D))$ and condition (iii) it follows that $\rho(\Omega^1_S(\log D)) = \bigoplus_{i=1}^m \mathcal{O}_{D_i}$ in a neighbourhood of $p$. Since we can choose $q$ arbitrarily close to $p$, this yields a contradiction, and $D_i$ and $D_j$ have to be transversal. Finally suppose the opposite of (c), namely that there are three components $D_i, D_j, D_k$, whose triple intersection has dimension $(n-2)$. At a general point $q$ of $D_i \cap D_j \cap D_k$ all three components are smooth and any two of them intersect transversally. This implies that we can find local coordinates $(x_1, \dots, x_n)$ at $q$ such that $D_i = \{x_1 = 0\}$, $D_j = \{x_2 = 0\}$ and $D_k = \{x_1 - x_2 = 0\}$. The form

$$\omega = \frac{x_2 dx_1 - x_1 dx_2}{x_1 x_2 (x_1 - x_2)}$$

is an element of $\Omega^1_{S,q}(\log D)$ and $\rho(\omega)|_{D_1} = x_2^{-1}$, $\rho(\omega)|_{D_2} = \frac{1}{2} x_2^{-1}$, $\rho(\omega)|_{D_3} = x_1^{-1}$ which is meromorphic with a pole along $D_i \cap D_j \cap D_k$. Again $q$ can be chosen arbitrarily close to $p$, so we get a contradiction. This shows that $\dim(D_i \cap D_j \cap D_k) \leq n - 3$ for $i \neq j \neq k$.
*(iv)* $\Rightarrow$ *(iii)* : Suppose that $(D, p) = \bigcup_{i=1}^m (D_i, p)$ is the decomposition of $D$ into irreducible components that are normal (condition (iv) (a)). At a smooth point $p$ of $D$ one can easily compute the residue: we can assume that locally at $p$ the equation of $h$ is $\{x_1 = 0\}$. From the proof of Lemma 1.2 it follows that any $\omega \in \Omega^1_{S,p}(\log D)$ can be written as $(\partial_{x_i} h)\omega = \xi \frac{dh}{h} + \eta$ with $\xi \in \mathcal{O}_{S,p}$ and $\eta \in \Omega^1_{S,p}$ and some suitable derivative $\partial_{x_i} h$. In the case of $h = x_1$ one can take $\partial_{x_1} h = 1$ and thus any $\omega$ is of the form

$$\omega = \xi \frac{dh}{h} + \eta.$$

Hence $\rho(\omega) = \xi|_D$ is contained in $\mathcal{O}_{D,p}$. Suppose now that $p$ is contained in some intersection $(D_1 \cap D_2) \setminus (\operatorname{Sing} D_1 \cup \operatorname{Sing} D_2 \cup \bigcup_{i=3}^m D_i)$. By condition (iv)(b) the two components $D_1$ and $D_2$ intersect transversally in $p$. Thus we may assume that $D_1$ is given locally at $p$ by the equation $h_1 = x_1 - x_2$ and $D_2$ by $h_2 = x_1 + x_2$. In particular, $\partial_{x_1}(h_1 h_2) = 2x_1$ and $\partial_{x_2}(h_1 h_2) = -2x_2$. Then for a form $\omega \in \Omega^1_{S,p}(\log D)$ there are two possible representations, namely

$$2x_1\omega = \xi \frac{d(h_1h_2)}{h_1h_2} + \eta, \quad \text{or} \quad -2x_2\omega = \xi' \frac{d(h_1h_2)}{h_1h_2} + \eta',$$

such that $h$ neither divides $\xi$ nor $\xi'$.The residue of $\omega$ is $\frac{1}{2x_1}\xi = \frac{1}{-2x_2}\xi'$. Since $\xi, \xi'$ are holomorphic, it follows from this equation that $\xi|_D$ is a multiple of $x_1$ and that $\xi'|_D$ is a multiple of $x_2$, which implies that $\rho(\omega)$ is holomorphic. This argument and the assumption (iv) show that outside the set $\operatorname{Sing} D \cap (\bigcup_{i=1}^m \operatorname{Sing} D_i) \cap (\bigcup_{i,j,k}(D_i \cap D_j \cap D_k))$, which is of dimension less than or equal to $(n-3)$, the residue $\rho(\omega)$ is holomorphic and contained in $\mathcal{O}_{D,p}$. We apply Hartogs' theorem (Thm. A.21) to extend $\rho(\omega)$ holomorphically to $\mathcal{O}_{D,p}$. □

The following case of Thm. 1.42 is particularly interesting, since it determines an explicit minimal system of generators of $\Omega^1_{S,p}(\log D)$:

**Corollary.** *Let $(S, D)$ be as in the theorem and suppose that $D$ is irreducible with local equation $h = 0$. Then the following conditions are equivalent:*
*(i) $D$ is normal at $p$.*
*(ii) $\rho(\Omega^1_{S,p}(\log D)) = \mathcal{O}_{D,p}$*
*(iii) $\Omega^1_{S,p}(\log D)$ is generated by $\frac{dh}{h}, dx_1, \ldots, dx_n$. In particular, if $D$ is not smooth at $p$, then this is a minimal system of generators of $\Omega^1_{S,p}(\log D)$.*

*Proof.* This is just Thm. 1.42 for $m = 1$. For second part of (*iii*) let us suppose that $D$ is normal but not smooth at $p$ and that $\Omega^1_{S,p}(\log D)$ has a minimal system of generators consisting of $n$ elements. This implies that $\Omega^1_{S,p}(\log D)$ is free. But by Prop. 2.5 a free divisor is either smooth or non-normal. Contradiction. □

## 1.2 A characterization of normal crossings by logarithmic forms and vector fields

In this section we give a characterization of a normal crossing divisor in terms of generators of its module of logarithmic differential forms resp. vector fields (Thm. 1.52). Namely, a divisor $D \subseteq S$ has normal

crossings at a point $p$ if and only if $\Omega^1_{S,p}(\log D)$ is a free $\mathcal{O}_{S,p}$-module and has a basis of closed forms or if and only if $\mathrm{Der}_{S,p}(\log D)$ is a free $\mathcal{O}_{S,p}$-module and has a basis of commuting vector fields (this means that there exist logarithmic derivations $\delta_1, \ldots, \delta_n$ such that the $\delta_i$ form a basis of $\mathrm{Der}_{S,p}(\log D)$ and $[\delta_i, \delta_j] = 0$ for all $i, j = 1, \ldots, n$). We remark that we only show the *existence* of bases with these properties of $\Omega^1_{S,p}(\log D)$ and $\mathrm{Der}_{S,p}(\log D)$ in case $D$ has normal crossings at $p$. We do not have a procedure to *construct* such bases.
The section is organized as follows: first we consider the two-dimensional case in Proposition 1.44. Then we state a few lemmata in order to prove the assertion for logarithmic differential forms. We prove the equivalence of the characterizations in terms of logarithmic differential forms and vector fields in Prop. 1.54.

In (2.11) of [81] Saito uses Theorem 1.42 to study the case of $\dim S = 2$. Here we give a more elementary proof.

**Proposition 1.44.** *Let $(S, D)$ be as usual a manifold together with a divisor $D \subseteq S$ and suppose that $\dim S = 2$. Let $(D, p) = \bigcup_{i=1}^m (D_i, p)$ be the decomposition of $D$ into irreducible components at a point $p$ with the corresponding (reduced) equation $D = \{h = h_1 \cdots h_m = 0\}$, where $h \in \mathcal{O}_{S,p}$. Then the following are equivalent:*
*(i) $\Omega^1_{S,p}(\log D)$ has a basis of closed forms.*
*(ii) Either $m = 2$ and the components $D_1$ and $D_2$ of $D$ are smooth and meet transversally at $p$ (i.e., $D$ has normal crossings at $p$) or $D$ is smooth at $p$.*

In order to prove the proposition we need a lemma to connect bases of the dual modules $\mathrm{Der}_{S,p}(\log D)$ and $\Omega^1_{S,p}(\log D)$.

**Lemma 1.45.** *Let $(S, D)$ be defined as in Prop. 1.44 and let $(x, y)$ be local coordinates at $p$. Let $\delta_1 = a\partial_x + b\partial_y$ and $\delta_2 = c\partial_x + d\partial_y$, with $a, b, c, d \in \mathcal{O}_{S,p}$, be a basis of $\mathrm{Der}_{S,p}(\log D)$ and suppose that $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = h$. Then the corresponding dual basis $\omega_1 = \delta_1^*, \omega_2 = \delta_2^*$ of $\Omega^1_{S,p}(\log D)$ is closed if and only if*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \partial_x h \\ \partial_y h \end{pmatrix} = h \begin{pmatrix} \partial_x a + \partial_y b \\ \partial_x c + \partial_y d \end{pmatrix}. \tag{1.6}$$

*Proof.* The proof is done by direct calculation: from Lemma 1.9 and its corollary follows that in dimension 2 any reduced divisor $D$ is free and that the modules of logarithmic derivations and logarithmic differential forms are dual to each other. Linear algebra says that the dual basis to $(\delta_1, \delta_2)$ of $\Omega^1_{S,p}(\log D)$ is

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \frac{1}{h} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}^T \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

Plugging in the conditions for closedness of the $\omega_i$, equation (1.6) follows. □

*Proof of Prop. 1.44.* First suppose that (ii) holds. Then locally at $p$, the divisor $D$ has at most 2 irreducible components, that is, $(D,p) = (D_1,p) \cup (D_2,p)$ or $(D,p)$ is smooth and irreducible. We have already seen in example 1.10 that $D$ has a basis consisting of closed forms.
Conversely, suppose $(i)$ holds. Then the module $\Omega^1_{S,p}(\log D)$ is free and there exists a basis of closed forms $\omega_1, \omega_2$. Note that $dh/h \in \Omega^1_{S,p}(\log D)$ is closed. We can express it in terms of the basis, that is, $dh/h = a\omega_1 + b\omega_2$ with $a, b \in \mathcal{O}_{S,p}$. To show that either $a$ or $b \in \mathcal{O}_{S,p}$ is invertible we use the logarithmic residue. By linearity of the residue homomorphism, the following identity holds:

$$\rho(dh/h) = 1 = a|_D \cdot \rho(\omega_1) + b|_D \cdot \rho(\omega_2). \tag{1.7}$$

Like in the proof of Thm. 1.42, one obtains that $\rho(\omega_i)|_{D_j} = c_{ij} \in \mathbb{C}$ is constant on each component $D_j$. From (1.7) it follows that $a(p) \neq 0$ or $b(p) \neq 0$. Hence we can assume $b(p) \neq 0$ so that $b$ is locally invertible. Then $dh/h$ and $\omega_1$ are a basis of $\Omega^1_{S,p}(\log D)$. By duality of logarithmic forms and derivations there exists a derivation $\delta \in \mathrm{Der}_{S,p}(\log D)$ such that $\delta \cdot (dh/h) = 1$ and hence $\delta h = h$. Since $D$ is a reduced curve, the singularity at $p$ is isolated. By a Theorem of Saito [78], $h$ is quasihomogeneous. This means that in suitable coordinates there exists an Euler vector field $\eta = \alpha x\partial_x + \beta y \partial_y$ in $\mathrm{Der}_{S,p}(\log D)$ with $\alpha, \beta \in \mathbb{C}$. Then $(\partial_y h)\partial_x - (\partial_x h)\partial_y$ and $\eta$ form a basis of $\mathrm{Der}_{S,p}(\log D)$. The corresponding basis of $\Omega^1_{S,p}(\log D)$ is closed if and only if equation (1.6) is satisfied, in particular $\eta h = (\alpha+\beta)h$. Again by a result of Saito (Satz 4.1 and Lemma 2.3. of [78]) one can find a holomorphic coordinate transformation such that $\alpha, \beta$ are positive rational numbers $\leq 1/2$ and such that $\eta h = h$. Hence we must have $\alpha = \beta = 1/2$. From this and the well-known Euler relation for homogeneous polynomials follows that $h$ is homogeneous of degree 2. Since $h$ is assumed to be reduced, the only possibility is $h = xy$, that is, $D$ has normal crossings at $p$. □

*Remark* 1.46. In section 1.4 we comment on a question of Saito about the connection between the logarithmic residue and the topology of a divisor. We will see that the proposition above settles the two-dimensional case.

The following lemmata are used to prove Theorem 1.52, which generalizes Proposition 1.44 to the higher dimensional case.

**Lemma 1.47.** *Denote by $(S, D)$ a complex manifold of dimension $n$ together with a divisor $D \subseteq S$, and let $(D, p) = \bigcup_{i=1}^{m}(D_i, p)$ be the decomposition of $D$ into irreducible components at a point $p$ in $S$. Suppose that $h = h_1 \cdots h_m$ is the local equation of $D$ at $p$, where each $h_i$ corresponds to $D_i$. Then $D$ has normal crossings at $p$ if and only if the $dh_i/h_i$ are part of a basis, whose elements are closed, of the form $\omega_1 = dh_1/h_1, \ldots, \omega_m = dh_m/h_m, \omega_{m+1} = df_{m+1}, \ldots, \omega_n = df_n$ of $\Omega^1_{S,p}(\log D)$, that is,*

$$\frac{dh_1}{h_1} \wedge \cdots \wedge \frac{dh_m}{h_m} \wedge df_{m+1} \wedge \cdots \wedge df_n = \frac{c}{h} \cdot dx_1 \wedge \cdots \wedge x_n,$$

*where the $f_i$ are some suitable elements in $\mathcal{O}_{S,p}$ and $c \in \mathcal{O}^*_{S,p}$.*

*Proof.* If $D$ has normal crossings at $p$ then one can find coordinates $x = (x_1, \ldots, x_n)$ such that $h = x_1 \cdots x_m$ is the defining equation of $D$ at $p$. Then clearly

$$\frac{dx_1}{x_1}, \ldots, \frac{dx_m}{x_m}, dx_{m+1}, \ldots, dx_n$$

form a basis of $\Omega^1_{S,p}(\log D)$.
Conversely, suppose that $\frac{dh_1}{h_1} \wedge \ldots \wedge \frac{dh_m}{h_m} \wedge df_{m+1} \wedge \cdots \wedge df_n = c/h \cdot dx_1 \wedge \ldots \wedge x_n$. This means that the Jacobian matrix of the $h_1, \ldots, h_m$, $f_{m+1}, \ldots, f_n$ has determinant $c \in \mathcal{O}^*_{S,p}$. By the implicit function theorem the $h_i$ and the $f_i$ are complex coordinates at $p$. Then, by definition $D$ has normal crossings at $p$. □

**Lemma 1.48.** *Let $D \subseteq S$ be a divisor in a complex manifold $S$ with* $\dim S = n$. *Suppose that $D$ is free at a point $p \in S$ and $\Omega^1_{S,p}(\log D)$ has a basis $\omega_1, \ldots, \omega_n$ such that $\omega_1, \ldots, \omega_k$, $k < n$ are in $\Omega^1_{S,p}$. Then one can find a local isomorphism $(D, p) \cong (D', p') \times (\mathbb{C}^k, 0)$, where $(D', p')$ is in $(\mathbb{C}^{n-k}, p')$.*

*Proof.* Since $\Omega^1_{S,p}(\log D)$ is free with basis $\omega_1, \dots, \omega_n$, there is a unique basis $\delta_1, \dots, \delta_n$ of $\mathrm{Der}_{S,p}(\log D)$ satisfying $\omega_i \cdot \delta_j = \delta_{ij}$. For any $\omega_i$, $i = 1, \dots, k$, one thus has $\omega_i \cdot \delta_i = 1$. For all coefficients of $\omega_i = \sum_{j=1}^n w_{ij} dx_j$ and $\delta_i = \sum_{j=1}^n d_{ij} \partial_{x_j}$ are holomorphic, this yields the equation

$$1 = \sum_{j=1}^n w_{ij} d_{ij}.$$

Since $\mathcal{O}_{S,p}$ is a local ring, at least one $w_{ij} d_{ij}$, w.l.o.g., for $j = 1$, is invertible in $\mathcal{O}_{S,p}$, which implies $d_{i1} \in \mathcal{O}^*_{S,p}$. Applying $\delta_i$ to $h$, the local defining equation of $D$ gives $d_{i1} \partial_{x_1} h \in (h, \partial_{x_2} h, \dots, \partial_{x_n} h)$. With the triviality lemma A.44 one can find a biholomorphic map $\varphi_i$ such that $h \circ \varphi_i = v h(0, x_2, \dots, x_n)$, where $v \in \mathcal{O}^*_{S,p}$, also defining $D$. Hence $D$ is locally isomorphic to some $D' \times \mathbb{C}$. Applying this construction for the remaining $\omega_i$, one arrives at $(D, p) \cong (D', p') \times (\mathbb{C}^k, 0)$. □

**Lemma 1.49.** *Let $(D, p) = \bigcup_{i=1}^m (D_i, p)$ be given by the reduced equation $h = h_1 \cdots h_m$ and let $\omega \in \Omega^1_{S,p}(\log D)$ be a closed form. Then:*
*(i) The residue of $\omega$ along each branch $D_i$ is constant, that is, $\rho(\omega)|_{D_i} = c_i$ with $c_i \in \mathbb{C}$ for $i = 1, \dots, m$.*
*(ii) $\omega$ can be represented as $\omega = \sum_{i=1}^m c_i dh_i / h_i + \xi$, where $c_i \in \mathbb{C}$ and $\xi \in \Omega^1_{S,p}$ is closed.*
*(iii) If the residue of $\omega$ along at least one branch $D_i$ is non-zero, then $\omega$ can be represented as*

$$\omega = \sum_{i=1}^m c_i \frac{dh'_i}{h'_i}, \quad c_i \in \mathbb{C},$$

*with $h'_i = u_i h_i$ and $u_i \in \mathcal{O}^*_{S,p}$. Note that $h'_i$ also define s $D_i$ and that $h' = h'_1 \cdots h'_m$ also defines $D$ near $p$.*

*Proof.* (i): Let $\omega$ be a closed logarithmic form for $D = \bigcup_{i=1}^m D_i$. Since diagram (1.4) is commutative, one has $d(\rho(\omega)) = \rho(d\omega) = \rho(0) = 0$. Hence $\rho(\omega)$ is locally a constant $c_i \in \mathbb{C}$ on each branch $D_i$ of $D$.
(ii): By (i) and the exactness of the sequence (1.2) $\omega$ can be represented as $\omega = \sum_{i=1}^m c_i dh_i / h_i + \xi$ with $\xi \in \Omega^1_{S,p}$. Since $d\omega = 0$, differentiating $\omega$ yields that $d\xi = 0$, that is, $\xi$ is closed.
(iii): Suppose that $\omega = \sum_{i=1}^m c_i \frac{dh_i}{h_i} + \xi$, with $c_i \in \mathbb{C}$, is a closed logarithmic form. Since we consider germs of differential forms, one can assume (Poincaré's lemma) that $\xi$ is exact and hence that $\xi = df$ for some $f \in \mathcal{O}_{S,p}$. Now assume that the residue $\rho(\omega)|_{D_1}$ is non-zero.

Define $h_1' := h_1 \exp(f/c_1)$. Then $h_1' h_2 \cdots h_m$ also defines $D$ because multiplying with a unit does not change the zero-set locally at $p$. The following holds:

$$c_1 \frac{dh_1'}{h_1'} = c_1 \frac{dh_1}{h_1} + df = c_1 \frac{dh_1}{h_1} + \xi.$$

Hence we have $\omega = c_1 dh_1'/h_1' + \sum_{i=2}^m c_i dh_i/h_i$. □

**Lemma 1.50.** *Let $(D,p) = \bigcup_{i=1}^m (D_i,p)$ be free at $p$ and let $\Omega^1_{S,p}(\log D)$ have a basis $\omega_1, \ldots, \omega_n$ consisting of closed forms. Then $m \leq n$ and maximally $n - m$ elements $\omega_i$ of this basis are holomorphic forms.*

*Proof.* From Lemma 1.49 it follows that each closed basis element $\omega_i$ can be represented as $\omega_i = \sum_{j=1}^m c_{ij} dh_j/h_j + df_i$ with $df_i \in \Omega^1_{S,p}$ and $c_{ij} \in \mathbb{C}$ for $j = 1, \ldots, m$. First suppose that $m > n$. Then by Saito's criterion one knows that $\bigwedge_{i=1}^n \omega_i = \frac{c}{h_1 \cdots h_m} \cdot dx_1 \wedge \ldots \wedge dx_n$ with $c \in \mathcal{O}^*_{S,p}$. This means that the $n$-form $\bigwedge_{i=1}^n \omega_i$ has a simple pole at $h_1 \cdots h_m$. But forming the wedge product of the $\omega_i$ we obtain (by a simple computation) $\bigwedge_{i=1}^n \omega_i = \frac{g}{h_1 \cdots h_m} \cdot dx_1 \wedge \ldots \wedge dx_n$ with $g \in (h_1, \ldots, h_m) \subseteq \mathfrak{m}$. Thus $g$ is not invertible, which is a contradiction to Saito's criterion.
For the second assertion suppose that $\omega_i = df_i$, $f_i \in \mathcal{O}_{S,p}$ for $i = m, \ldots, n$ are holomorphic, that is, the basis contains $n - m + 1$ closed holomorphic elements. An application of Lemma 1.48 yields an isomorphism $(D,p) \cong (D',0) \times (\mathbb{C}^{n-m+1}, 0)$ with $(D',0) \subseteq (\mathbb{C}^{m-1}, 0)$. This means that $D'$ would be a free divisor with $m$ irreducible components and with a basis of closed forms in an $m-1$ dimensional manifold. Contradiction to the first assertion of this lemma.

□

**Proposition 1.51.** *Let $(D,p) = \bigcup_{i=1}^m (D_i,p)$ be free at $p$ and let $\Omega^1_{S,p}(\log D)$ have a basis consisting of closed forms $\omega_1, \ldots, \omega_n$. Then $m \leq n$ and $\omega_i$ can be chosen as $\omega_i = dh_i'/h_i'$ where $h_i' = f_i h_i$ with $f_i \in \mathcal{O}^*_{S,p}$ for $i = 1, \ldots, m$ and $\omega_i = df_i$ with $f_i \in \mathcal{O}_{S,p}$ holomorphic for $i = m+1, \ldots, n$. In particular, one can find defining equations $h_i'$ of $D$ such that the $dh_i'/h_i'$ form part of a basis of $\Omega^1_{S,p}(\log D)$.*

*Proof.* From Lemma 1.50 it follows that $m \leq n$ and from Lemma 1.49 it follows that $(\omega_1, \ldots, \omega_n)$ can be represented as

$$(\omega_1, \ldots, \omega_n)^T = \begin{pmatrix} C & 0 \\ 0 & I_{n-m} \end{pmatrix} \begin{pmatrix} \frac{dh}{h} \\ 0 \end{pmatrix} + \begin{pmatrix} \underline{\xi} \\ \underline{df} \end{pmatrix}$$

with $C$ an $m \times m$-matrix with entries in $\mathbb{C}$, $\frac{d\underline{h}}{\underline{h}} = (\frac{dh_1}{h_1}, \ldots, \frac{dh_m}{h_m})^T$, $\underline{\xi} = (\xi_1, \ldots, \xi_m)^T$ with $\xi_i \in \Omega^1_{S,p}$ and $\underline{df} = (df_{m+1}, \ldots, df_n)^T$ with $f_i \in \mathcal{O}_{S,p}$. With the Gauss–Algorithm one can find a matrix $\begin{pmatrix} M & 0 \\ 0 & I_{n-m} \end{pmatrix} \in GL_n(\mathbb{C})$, with $M$ an $m \times m$ sub-matrix, such that $MC$ is in row echelon form, that is, the last $m-k$ rows of $MC$ are zero for $k = \operatorname{rank}(C)$ and the first $k$ rows form the $k \times k$ identity matrix. Then

$$(\tilde{\omega}_1, \ldots \tilde{\omega}_n)^T = \begin{pmatrix} M & 0 \\ 0 & I_{n-m} \end{pmatrix} (\omega_1, \ldots, \omega_n)^T = \begin{pmatrix} MC\frac{d\underline{h}}{\underline{h}} + M\underline{\xi} \\ \underline{df} \end{pmatrix}$$

is also a closed basis $\underline{\tilde{\omega}}$ of $\Omega^1_{S,p}(\log D)$. If $\operatorname{rank}(C) = k < m$, then the forms $\tilde{\omega}_{m-k}, \ldots, \tilde{\omega}_n$ would be holomorphic. But this is a contradiction to Lemma 1.50, hence it follows that $C \in GL_m(\mathbb{C})$. Thus one can assume that $(\omega_1, \ldots, \omega_m)$ is of the form $(\frac{dh_1}{h_1} + \xi'_1, \ldots, \frac{dh_m}{h_m} + \xi'_m)$, where $\xi'_i = M\xi_i$. As in Lemma 1.49 write $\omega_i = dh'_i/h'_i$, where for $\xi'_i = df_i/f_i$, $f_i \in \mathcal{O}^*_{S,p}$ one has $h'_i = f_i h_i$. The change of one $h_i$ does not affect the others since $h_i$ is assumed to be irreducible. The functions $h'_i$ also define the divisor $D$ at $p$ since $\exp(\log f_i)$ is in $\mathcal{O}^*_{S,p}$. The assertion of the proposition follows. □

**Theorem 1.52.** *Denote by $(S, D)$ a complex manifold with $\dim S = n \geq 2$ together with a divisor $D \subseteq S$ and let $p \in S$ be a point. The following conditions are equivalent:*
*(i) $\Omega^1_{S,p}(\log D)$ is free and has a basis of closed forms.*
*(ii) $D$ has normal crossings at $p$.*

*Proof.* (ii) ⇒ (i) is a simple computation (cf. Example 1.10).
Conversely, suppose that $\Omega^1_{S,p}(\log D)$ has a basis of closed forms. By Proposition 1.51 we can assume that $(D, p) = \bigcup_{i=1}^m (D_i, p)$ has $m \leq n$ irreducible components and that the closed basis of $\Omega_{S,p}(\log D)$ is of the form $dh_1/h_1, \ldots, dh_m/h_m, df_{m+1}, \ldots, df_n$, where $h_i$ is the reduced function corresponding to the component $(D_i, p)$. By Lemma 1.47 the existence of a closed basis of this form is equivalent to $(D, p)$ having normal crossings. □

The following lemma will be useful in a few occasions in Chapter 2:

**Lemma 1.53.** *Let $(S, D)$ be a manifold together with a divisor $D \subseteq S$ and $p$ a point in $S$. If $\Omega^1_{S,p}(\log D)$ is free and generated by closed forms, it has a basis of closed forms.*

*Proof.* Let $(\omega_1, \dots, \omega_n)$ be a basis of $\Omega^1_{S,p}(\log D)$. Suppose that the module $\Omega^1_{S,p}(\log D)$ may also be generated by *closed* logarithmic forms $(\xi_1, \dots, \xi_m)$. It is clear that $m \geq n$. Then, using Nakayama's lemma, it follows that $(\overline{\omega}_1, \dots, \overline{\omega}_n)$ is a basis of the $\mathcal{O}/\mathfrak{m}\mathcal{O} = \mathbb{C}$-vector space $\Omega^1_{S,p}(\log D)/\mathfrak{m}\Omega^1_{S,p}(\log D)$ and $(\bar{\xi}_1, \dots, \bar{\xi}_m)$ also generate this vector space. Then there exists a matrix $\bar{A} \in M_{n,m}(\mathbb{C})$ of rank $n$ such that

$$\bar{A}(\bar{\xi}_1, \dots, \bar{\xi}_m)^T = (\bar{\omega}_1, \dots, \bar{\omega}_n)^T.$$

By standard linear algebra wlog. $\bar{\xi}_1, \dots, \bar{\xi}_n$ form a basis of the $\mathbb{C}$-vector space $\Omega^1_{S,p}(\log D)/\mathfrak{m}\Omega^1_{S,p}(\log D)$. Again applying of Nakayama's lemma yields that $\xi_1, \dots, \xi_n$ are a basis of $\Omega^1_{S,p}(\log D)$. □

## 1.2.1 Logarithmic derivations vs. differential forms

Here we state an equivalent formulation of Theorem 1.52 in terms of logarithmic vector fields. Furthermore we will also pose some questions about the relationship between $\Omega^1_{S,p}(\log D)$ and $\mathrm{Der}_{S,p}(\log D)$.

As usual denote by $(S, D)$ a complex manifold of complex dimension $n$ together with a divisor $D \subseteq S$. Let $(x_1, \dots, x_n)$ be complex coordinates of $S$ at a point $p$.It was already shown that $\mathrm{Der}_{S,p}(\log D)$ is closed under the Lie bracket $[\cdot, \cdot]$.

**Proposition 1.54.** *Suppose that* $\delta^1, \dots, \delta^n$ *form a basis of* $\mathrm{Der}_{S,p}(\log D)$. *Then* $[\delta^i, \delta^j] = 0$ *for all* $i, j \in \{1, \dots, n\}$ *if and only if the basis* $\omega_1, \dots, \omega_n$ *of* $\Omega^1_{S,p}(\log D)$ *satisfying* $\omega_i \cdot \delta^j = \delta_{ij}$ *consists of closed forms.*

*Proof.* We have

$$d\omega(\xi^1, \xi^2) = \xi^1(\omega(\xi^2)) - \xi^2(\omega(\xi^1)) - \omega([\xi^1, \xi^2]), \tag{1.8}$$

where $\omega$ is a differential 1-form and $\xi^1, \xi^2$ are vector fields (see e.g. [21, Def. 4.4.]). First, suppose that $[\delta^i, \delta^j] = 0$ for all pairs $(i, j)$. Plugging $\delta^i, \delta^j$ into a basis element $\omega_k$ yields $d\omega_k(\delta^i, \delta^j) = \delta^i(\delta_{jk}) - \delta^j(\delta_{ik}) - \omega(0) = 0$. Hence any basis element $\omega_k$, is closed.
Conversely, if each $\omega_k$ is closed, it follows from (1.8) that $\omega_k([\delta^i, \delta^j]) = 0$. Since $\mathrm{Der}_{S,p}(\log D)$ is closed under $[\cdot, \cdot]$ and the $\delta$'s form a basis of $\mathrm{Der}_{S,p}(\log D)$, the equation $[\delta^i, \delta^j] = \sum_{k=1} g_k \delta^k$ holds for some $g_k \in \mathcal{O}_{S,p}$. Using the $\mathcal{O}_{S,p}$-linearity of $\omega_k$ we obtain

$$0 = \omega_k([\delta^i, \delta^j]) = \sum_{l=1}^{n} g_l \omega_k(\delta^l) = g_k.$$

Since this equality holds for any $i, j, k$ it follows that $[\delta^i, \delta^j] = 0$ for all pairs $(i, j)$. □

*Remark* 1.55. The Lie bracket is stable under coordinate changes: A basis of commuting logarithmic derivations of $\mathrm{Der}_{S,p}(\log D)$ commutes after a coordinate transformation.

**Question 1.56.** *1. Construct special bases: we ask for a constructive algorithm for a closed basis of* $\Omega^1_{S,p}(\log D)$ *(resp. a basis of commuting fields of* $\mathrm{Der}_{S,p}(\log D)$*), which in the first place determines if there exists such a basis.*

*2. Construct a minimal system of generators of* $\Omega_{S,p}(\log D)$*, in particular in the case where* $(D, p)$ *is not free.*

## 1.3 Normal crossings and the logarithmic residue

In this section we give a characterization of normal crossing divisors by their logarithmic residue $\rho(\Omega^1_{S,p}(\log D))$. This characterization also leads to an answer to a question of K. Saito concerning the logarithmic residue, which will be considered in section 1.4. These results are due to Granger and Schulze [43].
It was already shown that the logarithmic residue of $\Omega^1_{S,p}(\log D)$ always contains the ring of weakly holomorphic functions on $D$. So it is quite natural to ask when the two rings are the same. We will see that for free divisors the answer is surprisingly simple (under the mild additional condition that the normalization of $D$ is Gorenstein): $\rho(\Omega^1_{S,p}(\log D)) = \pi_*\mathcal{O}_{\widetilde{D},p}$ if and only if $(D, p)$ has normal crossings. Note that this fact yields a second characterization of normal crossing divisors. In general the equality is equivalent to saying that $(D, p)$ has normal crossings in codimension 1 (see Thm. 1.82).
This section is organized as follows: first we consider examples of divisors $(D, p)$ with weakly holomorphic logarithmic residue which lead the way to the formulation of theorem 1.63. Then some properties of divisors with weakly holomorphic residues are studied. Finally we introduce the dual logarithmic residue in order to prove the theorem.

Suppose that $D$ is a free divisor whose logarithmic residue $\rho(\Omega^1_S(\log D))$ is equal to $\pi_*\mathcal{O}_{\widetilde{D}}$. Recall that $\pi_*\mathcal{O}_{\widetilde{D}}$ is equal to the normalization $\widetilde{\mathcal{O}}_D$

and also to the ring of weakly holomorphic functions (see Appendix A). Since we consider free divisors, it is possible to compute $\rho(\Omega^1_S(\log D))$ and $\pi_*\mathcal{O}_{\widetilde{D}}$ explicitly with a computer algebra system: from a basis of $\Omega^1_{S,p}(\log D)$ the residue $\rho(\Omega^1_{S,p}(\log D))$ can be computed, and it is also possible to compute the normalization of $D$. However, computing normalizations is of high complexity, so we are confined to low dimensional examples.

*Example* 1.57. Let $D \subseteq S$ with $\dim S = n$ be smooth at a point $p$. Then locally at $p$ we can find coordinates $(x_1, \ldots, x_n)$ such that $D = \{x_1 = 0\}$. Since $\Omega^1_{S,p}(\log D)$ is generated by $\frac{dx_1}{x_1}, dx_2, \ldots, dx_n$, the residue of a logarithmic form $\omega = a_1\frac{dx_1}{x_1} + \sum_{i=2}^n a_i dx_i$ is just $a_1|_D$ and hence $\rho(\Omega^1_{S,p}(\log D) = \mathcal{O}_{D,p}$, also cf. Thm. 1.42.

*Example* 1.58. Consider the cusp $D$ in $\mathbb{C}^2$, given by $h = x^3 - y^2$ with coordinate ring $\mathcal{O}_{D,0} = \mathbb{C}\{x, y\}/(x^3 - y^2)$. In Appendix A we will see that $\widetilde{\mathcal{O}}_D = \mathbb{C}\{t\}$ with $t = \frac{y}{x}$. A basis of $\Omega^1_{\mathbb{C}^2,0}(\log D)$ is $\omega_1 = \frac{dh}{h}$ and $\omega_2 = \frac{1}{h}(3ydx + 2xdy)$. Thus the residue of a logarithmic form $\omega = a\omega_1 + b\omega_2$, where $a, b \in \mathcal{O}_{\mathbb{C}^2,0}$, is $\rho(\omega) = a|_D + b|_D\rho(\omega_2)$. But $\rho(\omega_2) = \frac{x}{y} = t^{-1}$ is clearly not in $\mathbb{C}\{t\}$. Thus it follows that $\Omega^1(\log D) \supsetneq \pi_*\mathcal{O}_{\widetilde{D}}$.

*Example* 1.59. Let $(D, p) \subseteq (\mathbb{C}^3, 0)$ be an $E_8$-singularity of local equation $x^2 + y^3 + z^5 = 0$. Then $D$ is normal and by the corollary of Thm. 1.42 the logarithmic residue $\rho(\Omega^1_{S,p}(\log D))$ is $\mathcal{O}_{D,p}$. Note that $D$ is not free at the origin, since it is normal.

*Example* 1.60. (The *4-lines*) In this example, the divisor $D$ is free but does not have normal crossings outside an $(n-3)$-dimensional subset of $D$. Let $D$ be the divisor in $\mathbb{C}^3$ given at $p = (x, y, z)$ by $h = (x+y)y(x+2y)(x+y+yz)$. Note that $D$ is just the 4-lines divisor from Example 1.16 in different coordinates, because in order to compute the residue of a logarithmic form, at least one partial derivative $\partial_{x_i} h$ must not have a common factor with $h$. The divisor $D$ is free, thus one can compute a basis of $\Omega^1_{\mathbb{C}^3,p}(\log D)$, namely

$$\omega_1 = \frac{dh}{h}$$
$$\omega_2 = \frac{1}{4h}(y(zx + 9yz + 7x + 7y)dx - x(zx + 9yz + 7x + 7y)dy - (x + y))y(2y + x)dz)$$
$$\omega_3 = \frac{1}{4h}(y(x + y + yz)dx - x(x + y + yz)dy)$$

This basis is the dual to the basis of $\mathrm{Der}_{\mathbb{C}^3,p}(\log D)$ given in Example 1.16. The direct image of the normalization of $D$ $\pi_*\mathcal{O}_{\widetilde{D},p} \cong \widetilde{\mathcal{O}}_{D,p}$ is isomorphic to

$\mathbb{C}\{x,y,z\}/(x+y)\oplus\mathbb{C}\{x,y,z\}/(y)\oplus\mathbb{C}\{x,y,z\}/(x+2y)\oplus\mathbb{C}\{x,y,z\}/(x+y+yz)$.

Since $\dim(\{h = \partial_y h = 0\}) = 1$, we have $\rho(\omega_i) = \frac{a_{i2}}{\partial_y h}$, where $\omega_i = \frac{1}{h}(a_{i1}dx + a_{i2}dy + a_{i3}dz)$ for $i = 1,2,3$. For example the computation of $\rho(\omega_3)|_{D_1} = -\frac{1}{4x}$ (here we use the relation $x = -y$ in $\mathcal{O}_{D_1} = \mathbb{C}\{x,y,z\}/(x+y)$) shows that the residue of $\omega_3$ is not holomorphic in $\pi_*\mathcal{O}_{\widetilde{D}_1,p}$. Hence the inclusion $\pi_*\mathcal{O}_{\widetilde{D},p} \supsetneq \rho(\Omega^1_{S,p}(\log D))$ is strict.

*Example* 1.61. This is an example of a free reducible divisor $D$, but whose irreducible components are not all free. Here $D$ does not have normal crossings outside an $(n-3)$-dimensional subset and we will see that $\pi_*\mathcal{O}_{\widetilde{D}}$ is strictly contained in the logarithmic residue.
Let $(D,0) = (D_1,0)\cup(D_2,0)$ be the divisor in $(\mathbb{C}^3,0)$, defined by $h = h_1h_2 = z(x^2-y^2z)$, that is, $D_1$ is the $\{z=0\}$-plane and $D_2$ is the Whitney Umbrella. In Example 1.13 we have already seen that $D_2$ is not free at the origin, whereas the $\{z=0\}$-plane is smooth and hence $D_1$ is free everywhere. For $D$ we can compute a basis of $\Omega^1_{\mathbb{C}^3,0}(\log D)$ (by computing a basis of $\mathrm{Der}_{\mathbb{C}^3,0}(\log D)$ with Singular [98] and taking the dual basis of this basis):

$$\begin{aligned}\omega_1 &= \frac{1}{h}(2xzdx - 2yz^2dy + (x^2-2y^2z)dz)\\ \omega_2 &= \frac{1}{h}(-2xzdx + 2yz^2dy + y^2zdz)\\ \omega_3 &= \frac{1}{h}(2yzdx - 2xzdy - xydz).\end{aligned}$$

In order to verify $\rho(\Omega^1_{\mathbb{C}^3,0}) \supsetneq \pi_*\mathcal{O}_{\widetilde{D},0}$, it has to be shown that the residue of at least one basis element $\omega_i$ is not holomorphic on the normalization of $D$. First we remark that $\pi_*\mathcal{O}_{\widetilde{D},0} \cong \mathbb{C}\{x,y\}\oplus\mathbb{C}\{y,\frac{x}{y}\}$ and that $\partial_z h = x^2-2y^2z$ is a suitable universal denominator for the computation of the residues. The residues of $\omega_1$ and $\omega_2$ are holomorphic on the normalization (they are $1\oplus 1$ resp. $0\oplus 1$ in $\pi_*\mathcal{O}_{\widetilde{D},p}$). However, $\rho(\omega_3) = -\frac{xy}{x^2-2y^2z}|_D$ is $-\frac{y}{x}$ in $\mathbb{C}\{x,y\}$, which is clearly not holomorphic there and $\frac{xy}{y^2z} = \frac{y}{x}$ in $\mathbb{C}\{y,\frac{x}{y}\}$ which is also not holomorphic in this ring.

*Example* 1.62. Consider the Whitney Umbrella $D$ given by $h = x^2-y^2z$ from example 1.13. The normalization $\widetilde{D}$ is smooth at the origin and has coordinate ring $\pi_*\mathcal{O}_{\widetilde{D},0} = \mathbb{C}\{x,y,z,t\}/(x^2-y^2z, yt-x, z-t^2) \cong \mathbb{C}\{y,t\}$. One can show that $\Omega^1_{\mathbb{C}^3,0}(\log D)$ is generated by $dh/h, \omega =$

$(yzdx - xzdy - 1/2xydz)/h$ and $dx, dy, dz$. Since $\rho(\omega) = yz/2x = t/2$ it follows that $\rho(\Omega^1_{\mathbb{C}^3,0}(\log D))$ is holomorphic on the normalization. Note that $D$ is not free.

These examples lead to the following

**Theorem 1.63.** *Let $(S, D)$ be a manifold of complex dimension $n$ together with a divisor $D \subseteq S$. Suppose that $D$ is a free divisor, that*

$$\rho(\Omega^1_S(\log D)) = \pi_* \mathcal{O}_{\widetilde{D}}$$

*and that the multi-germ $(\widetilde{D}, \pi^{-1}(p))$ is Gorenstein for all $p \in D$. Then $D$ has normal crossings.*

Geometrically this theorem means that a free divisor with a "nice" residue of logarithmic forms (and whose normalization satisfies a mild technical condition) is a normal crossing divisor. The proof of this theorem uses of the dual logarithmic residue, a notion introduced by Granger and Schulze in [43], and a theorem of R. Piene (see Theorem A.42) about the relationship of the Jacobian ideal and the conductor ideal in the normalization.
First we consider some general properties of divisors with weakly holomorphic residue, in particular we show that if $D$ is a free divisor in a complex manifold $S$ of dimension $n$, having $n$ irreducible components $D_i$ at a point $p$ and satisfying $\rho(\Omega^1_{S,p}(\log D)) = \pi_* \mathcal{O}_{\widetilde{D},p}$, then $D$ has normal crossings at $p$ (Corollary to Lemma 1.67). Then we introduce the dual logarithmic residue and prove Theorem 1.63 (following Granger and Schulze [43]).

## 1.3.1 Divisors with weakly holomorphic residue

Here we show first an analogue of Theorem 1.42 $(i) \Leftrightarrow (iii)$. Then some properties of $\pi_* \mathcal{O}_{\widetilde{D},p}$ are considered (Cohen–Macaulayness). Finally we show how to choose "good" generators for $\Omega^1_{S,p}(\log D)$ if $\rho(\Omega^1_{S,p}(\log D)) = \pi_* \mathcal{O}_{\widetilde{D},p}$ and that $D$ is Euler–homogeneous in this case (Lemma 1.67).

**Proposition 1.64.** *Let $(S, D)$ be a divisor $D$ in a complex manifold $S$ of dimension $n$. Then the following are equivalent:*
*(i)* $\Omega^1_{S,p}(\log D) =_{\mathcal{O}_{S,p}} \langle \omega_1, \dots \omega_k \rangle + \Omega^1_{S,p}$, *such that* $\rho(\omega_1), \dots, \rho(\omega_k) \in \pi_* \mathcal{O}_{\widetilde{D},p}$ *generate* $\pi_* \mathcal{O}_{\widetilde{D},p}$ *as* $\mathcal{O}_{D,p}$*-module.*
*(ii)* $\rho(\Omega^1_{S,p}(\log D)) = \pi_* \mathcal{O}_{\widetilde{D},p}$.

*Proof.* The implication $(i) \Rightarrow (ii)$ is clear, since $\rho$ is a sheaf homomorphism and $\rho(\Omega^1_{S,p}) = 0$. Suppose now that $\rho(\Omega^1_{S,p}(\log D)) = \pi_*\mathcal{O}_{\widetilde{D},p}$. The normalization is a finitely generated $\mathcal{O}_{D,p}$-module, i.e., $\pi_*\mathcal{O}_{\widetilde{D},p} = \sum_{i=1}^k \mathcal{O}_{D,p}\alpha_i$ for some $\alpha_i \in \pi_*\mathcal{O}_{\widetilde{D},p}$. By the exact sequence

$$0 \longrightarrow \Omega^1_{S,p} \longrightarrow \Omega^1_{S,p}(\log D) \xrightarrow{\rho} \pi_*\mathcal{O}_{\widetilde{D},p} \longrightarrow 0 \tag{1.9}$$

(obtained from the sequence (1.2)) there exist some $\omega_i \in \Omega^1_{S,p}(\log D)$ such that $\rho(\omega_i) = \alpha_i$ for each $i = 1, \ldots, k$. Now take any $\omega \in \Omega^1_{S,p}(\log D)$. Then $\rho(\omega) = \sum_{i=1}^k a_i\rho(\omega_i)$ for some $a_i \in \mathcal{O}_{D,p}$. Choose some representatives of the $a_i \in \mathcal{O}_{S,p}$ and define $\omega' := \sum_{i=1}^k a_i\omega_i$. Clearly $\omega' \in \Omega^1_{S,p}(\log D)$ as well as $\omega - \omega'$. But $\rho(\omega - \omega') = 0$, so $\omega - \omega'$ is holomorphic by Lemma 1.41. This shows that any $\omega \in \Omega^1_{S,p}(\log D)$ can be written as an $\mathcal{O}_{S,p}$-linear combination of the $\omega_i$ and some holomorphic form. □

**Lemma 1.65.** *Let $(S, D)$ be a divisor $D$ in a complex manifold $S$ of dimension $n$. Suppose that at a point $p$ the divisor is free and and $\rho(\Omega^1_{S,p}(\log D)) = \pi_*\mathcal{O}_{\widetilde{D},p}$.*
*(i) The ring $\pi_*\mathcal{O}_{\widetilde{D},p}$ is Cohen–Macaulay.*
*(ii) If $D$ additionally is not smooth and does not contain a smooth factor at $p$, i.e., is not locally isomorphic to some Cartesian product $(D', p') \times (\mathbb{C}^k, 0)$ for some $0 < k < n$, one may assume that $\pi_*\mathcal{O}_{\widetilde{D},p}$ is minimally generated by $n$ elements $\alpha_i$, where $\alpha_1 = 1$ and $\alpha_i \in \pi_*\mathcal{O}_{\widetilde{D},p} \backslash \mathcal{O}_{D,p}$.*

*Proof.* (i): Under our assumptions, the exact sequence

$$0 \longrightarrow \Omega^1_{S,p} \longrightarrow \Omega^1_{S,p}(\log D) \xrightarrow{\rho} \pi_*\mathcal{O}_{\widetilde{D},p} \longrightarrow 0 \tag{1.10}$$

yields a free resolution of $\pi_*\mathcal{O}_{\widetilde{D},p}$ (as $\mathcal{O}_{S,p}$-module). Since we are working over a regular local ring, it follows that $\operatorname{projdim}_{\mathcal{O}_{S,p}}(\pi_*\mathcal{O}_{\widetilde{D},p}) \leq 1$. With the Auslander–Buchsbaum formula follows $\operatorname{depth}(\mathfrak{m}_S, \pi_*\mathcal{O}_{\widetilde{D},p}) \geq n - 1$ (where $\mathfrak{m}_S$ denotes the maximal ideal of $\mathcal{O}_{S,p}$). Since the depth is stable under local homomorphisms, the depth of $\pi_*\mathcal{O}_{\widetilde{D},p}$ in $\mathcal{O}_{D,p}$ is greater than or equal to $n - 1$, that is, $\operatorname{depth}(\mathfrak{m}_D, \pi_*\mathcal{O}_{\widetilde{D},p}) \geq n - 1$. First suppose that $(D, p)$ is irreducible, then $\pi_*\mathcal{O}_{\widetilde{D},p}$ is a local ring. Since then $\mathcal{O}_{D,p} \subseteq \pi_*\mathcal{O}_{\widetilde{D},p}$ is a finite ring extension it follows e.g.

by [27, 6.5.29] that the depth of $\pi_*\mathcal{O}_{\widetilde{D},p}$ as an $\pi_*\mathcal{O}_{\widetilde{D},p}$-module is also greater than or equal to $n-1$. Clearly, $\dim(\pi_*\mathcal{O}_{\widetilde{D},p}) = n-1$ and so the assertion follows from the height-depth inequality.
If $(D,p) = \bigcup_{i=1}^m (D_i,p)$, where $(D_i,p)$ denote the irreducible components, then $\pi_*\mathcal{O}_{\widetilde{D},p} = \bigoplus_{i=1}^m \pi_*\mathcal{O}_{\widetilde{D}_i,p}$ is a semi-local ring with $m$ maximal ideals $\mathfrak{m}_{\widetilde{D}_i}$, $i=1,\ldots,m$, cf. Thm. A.12. Then $\pi_*\mathcal{O}_{\widetilde{D},p}$ is Cohen–Macaulay if $(\pi_*\mathcal{O}_{\widetilde{D},p})_{\mathfrak{m}_{\widetilde{D}_i}} \cong \pi_*\mathcal{O}_{\widetilde{D}_i,p}$ is Cohen–Macaulay for all $i=1,\ldots,m$. But this follows from the irreducible case since $\operatorname{depth}(\mathfrak{m}_S, \pi_*\mathcal{O}_{\widetilde{D},p}) = \operatorname{depth}(\mathfrak{m}_S, \pi_*\mathcal{O}_{\widetilde{D}_i,p})$ for all $i=1,\ldots,m$.
(ii): follows from lemmata 1.64 and 1.48 and an application of the lemma of Nakayama. □

**Lemma 1.66.** *Let $D \subseteq S$ be a divisor in a complex manifold $S$. If $b$ is an element in $\mathcal{O}_{D,p}$ that is invertible in $\pi_*\mathcal{O}_{\widetilde{D},p}$ then $b$ is already invertible in $\mathcal{O}_{D,p}$.*

*Proof.* From Appendix A we know that $\widetilde{\mathcal{O}}_{D,p} = \pi_*\mathcal{O}_{\widetilde{D},p}$, so $\frac{1}{b}$ is integral over $\mathcal{O}_{D,p}$. Hence it satisfies a monic polynomial equation of the form

$$(\frac{1}{b})^k + c_{k-1}(\frac{1}{b})^{k-1} + \cdots + c_0 = 0,$$

with coefficients $c_i \in \mathcal{O}_{D,p}$. Multiplying this equation with $b^k$ yields

$$1 = b(-c_{k-1} - \cdots - c_0 b^{k-1}),$$

that is, $b$ is invertible in $\mathcal{O}_D$. □

**Lemma 1.67.** *Let $D \subseteq S$ be a divisor in a complex manifold $S$ of dimension $n$. Suppose that $\rho(\Omega^1_{S,p}(\log D)) = \pi_*\mathcal{O}_{\widetilde{D},p}$. Then $\frac{dh}{h} \in \Omega^1_{S,p}(\log D)$ can be chosen as an element of a minimal system of generators of $\Omega^1_{S,p}(\log D)$. If $(D,p) = \bigcup_{i=1}^m (D_i,p)$, defined by $h = h_1 \cdots h_m$ in $\mathcal{O}_{S,p}$ then the $\frac{dh_i}{h_i}$ form part of a minimal system of generators of $\Omega^1_{S,p}(\log D)$.*

*Proof.* Clearly $\frac{dh}{h}$ is an element of $\Omega^1_{S,p}(\log D)$. Since $\Omega^1_S(\log D)$ is a coherent analytic sheaf, the stalk $\Omega^1_{S,p}(\log D)$ has a finite minimal system of generators $\omega_1,\ldots,\omega_k$ with $k \geq n$. One can write

$$\frac{dh}{h} = \sum_{i=1}^k a_i \omega_i,$$

for some $a_i \in \mathcal{O}_{S,p}$. Taking residues one gets

$$1_{\pi_*\mathcal{O}_{\widetilde{D},p}} = \sum_{i=1}^{n} a_i|_D \rho(\omega_i). \tag{1.11}$$

First assume that $D$ is irreducible at $p$. Then $\pi_*\mathcal{O}_{\widetilde{D},p}$ is a local ring, see Appendix A, and at least one $a_i|_D$ has to be invertible in $\pi_*\mathcal{O}_{\widetilde{D},p}$. By Lemma 1.66 this $a_i|_D$ is already invertible in $\mathcal{O}_{D,p}$. Thus $a_i(0) \neq 0$ and hence $a_i$ is contained in $\mathcal{O}^*_{S,p}$. This implies that $\frac{dh}{h}$ can be chosen as an element of a minimal system of generators of $\Omega^1_{S,p}(\log D)$ instead of $\omega_i$.
If $(D,p) = \bigcup_{i=1}^m (D_i,p)$ is the decomposition into irreducible components, equation (1.11) reads as follows:

$$1_{\pi_*\mathcal{O}_{\widetilde{D}},p} = \sum_{i=1}^{k} a_i|_D \rho(\omega_i) = \sum_{j=1}^{m} \left( \sum_{i=1}^{k} a_i|_{D_j} \rho(\omega_i)|_{D_j} \right).$$

Since the sum of the $\pi_*\mathcal{O}_{\widetilde{D}_j,p}$ is direct,

$$1_{\pi_*\mathcal{O}_{\widetilde{D}_1},p} = \sum_{i=1}^{k} a_i|_{D_1} \rho(\omega_i)|_{D_1}.$$

Like in the irreducible case, it follows that $a_i|_{D_1}$, wlog. for $i=1$, has to be invertible in $\pi_*\mathcal{O}_{\widetilde{D}_1,p}$. Also, it follows that a representative of $a_1|_{D_1}$ in $\mathcal{O}_{S,p}$, namely $a_1$, is invertible in $\mathcal{O}_{S,p}$, so we may exchange $\omega_1$ and $\frac{dh_1}{h_1}$. For $\frac{dh_2}{h_2}$ a similar argument is used: we can now write

$$\frac{dh_2}{h_2} = b_1 \frac{dh_1}{h_1} + \sum_{i=2}^{k} b_i \omega_i$$

for some $b_i \in \mathcal{O}_{S,p}$. Taking residues yields

$$1_{\pi_*\mathcal{O}_{\widetilde{D}_2,p}} = \sum_{j=1}^{m} \left( b_1|_{D_j} \delta_{1j} + \sum_{i=2}^{k} b_i|_{D_j} \rho(\omega_i)|_{D_j} \right).$$

The choice of $\frac{dh_1}{h_1}$ as an element of the minimal system of generators of $\Omega^1_{S,p}(\log D)$ does not affect this equation, since

$$1_{\pi_*\mathcal{O}_{\widetilde{D}_2},p} = \sum_{i=2}^{k} b_i|_{D_2} \rho(\omega_i)|_{D_2}.$$

Again with Lemma 1.66 we find that wlog. $b_2$ is invertible in $\mathcal{O}_{S,p}$ and we may choose $\frac{dh_2}{h_2}$ as an element of the minimal system of generators of $\Omega^1_{S,p}(\log D)$ instead of $\omega_2$. We continue in this way until all $\frac{dh_i}{h_i}$ are part of the minimal system of generators. Clearly, also $\frac{dh}{h}, \frac{dh_2}{h_2}, \ldots, \frac{dh_m}{h_m}$ are also part of any minimal system of generators. Thus we have shown our claim. □

*Remark* 1.68. Consider $D$ with the assumptions of Lemma 1.67 and suppose further that $D$ is free. Then the element $\frac{dh}{h}$ can be chosen as an element of a basis of $\Omega^1_{S,p}(\log D)$. This property is by Proposition 1.29 equivalent to saying that $D$ is a free Euler-homogeneous divisor. Hence we have shown that free divisors $D$ with $\pi_*\mathcal{O}_{\widetilde{D},p} = \rho(\Omega^1_{S,p}(\log D))$ are Euler-homogeneous at $p$.
Lemma 1.67 shows in particular that a minimal system of generators of $\Omega^1_{S,p}(\log D)$ must consist of at least $m$ elements, where $m$ is the number of irreducible components of $D$ at $p$. Hence, if $D$ is free and has more than $n$ irreducible components at $p$, it follows from the lemma that the logarithmic residue is not holomorphic on the normalization of $D$.

**Corollary.** *Let $D$ be a divisor in a complex manifold $S$ of dimension $n$ and suppose that at a point $p$, $D$ has $n$ irreducible components $(D_i, p)$. If $D$ has weakly holomorphic residue and is free at $p$, then $D$ has normal crossings at $p$.*

*Proof.* Suppose that the equation of $D$ at $p$ is $h = h_1 \cdots h_n$, where the $h_i$ correspond to the $D_i$. By lemma 1.67 the logarithmic forms $\omega_1 = dh_1/h_1, \ldots, \omega_n = dh_n/h_n$ form a basis of $\Omega^1_{S,p}(\log D)$. By Saito's criterion it follows that

$$dh_1 \wedge \cdots \wedge dh_n = u dx_1 \wedge \cdots \wedge dx_n,$$

for some unit $u \in \mathcal{O}_{S,p}$. By lemma 1.47 the $h_i$ are coordinates at $p$ and hence $D$ has normal crossings at $p$. □

The next lemma about the relationship of the conductor ideal $C_D$ (see appendix A) and the Jacobian ideal of the divisor will be useful in chapter 2.

**Lemma 1.69.** *Let $D$ be a divisor in a complex manifold $S$ of dimension $n$ and let $D = \{h = 0\}$ at a point $p$, where $h \in \mathcal{O}_{S,p}$ is reduced. Denote by $\widetilde{J}_h$ the Jacobian ideal*[2] *of $D$ at $p$, that is, the ideal generated by the*

[2] This notation is explained in Chapter 2.

*partial derivatives of $h$ in the ring $\mathcal{O}_{D,p} = \mathcal{O}_{S,p}/(h)$. Then*
*(i): $\widetilde{J}_h \subseteq C_{D,p}$.*
*(ii): If $D$ is free at $p$ and $\rho(\Omega^1_{S,p}(\log D)) = \pi_*\mathcal{O}_{\widetilde{D},p}$, then a power of $C_{D,p}$ is contained in $\widetilde{J}_h$. In particular, if $\widetilde{J}_h = \sqrt{\widetilde{J}_h}$, then $\widetilde{J}_h = C_{D,p}$.*

*Proof.* (i): Since $h$ is reduced, by Tsikh's theorem (Thm. A.32) all $\partial_{x_i} h$ are universal denominators for $\pi_*\mathcal{O}_{\widetilde{D},p}$. Hence $(\partial_{x_i} h)\pi_*\mathcal{O}_{\widetilde{D},p} \subseteq \mathcal{O}_{D,p}$. By definition of the conductor, $\partial_{x_i} h \in C_{D,p}$.
(ii): If $\rho(\Omega^1_{S,p}(\log D)) = \pi_*\mathcal{O}_{\widetilde{D},p}$, one can find for any $\alpha \in \pi_*\mathcal{O}_{\widetilde{D},p}$ a logarithmic 1-form $\omega$ such that $\rho(\omega) = \alpha$. Suppose that $g \neq 0 \in C_{D,p}$. Then $\alpha$ has a presentation $\alpha = \xi/g = a_1/\partial_{x_1} h = \cdots = a_n/\partial_{x_n} h$ for some $\xi, a_i \in \mathcal{O}_{S,p}$. Thus one can take $\omega = 1/h \sum_{i=1}^n a_i dx_i$ and a computation shows that $\omega$ also has a presentation $\omega = \frac{\xi}{g}\frac{dh}{h} + \frac{\eta}{g}$ for some $\eta \in \Omega^1_{S,p}$.
By Lemma 1.67 one may find a basis $\omega_1 = \frac{dh}{h}, \omega_i = \frac{\xi_i}{g}\frac{dh}{h} + \frac{\eta_i}{g}$ for $i = 2, \ldots, n$. Saito's criterion yields

$$dh \wedge (\eta_2 \wedge \cdots \wedge \eta_n) = ug^{n-1} \bigwedge_{i=1}^n dx_i,$$

where $u \in \mathcal{O}^*_{S,p}$. Hence $g^{n-1}$ is contained in $(\partial_{x_1} h, \ldots, \partial_{x_n} h)$ for any $g \in C_{D,p}$. If $\widetilde{J}_h$ is radical, it follows that even $g \in J_h$, that is, $C_{D,p} \subseteq \widetilde{J}_h$. □

*Remark* 1.70. Note that assertion (i) of Lemma 1.69 holds for any reduced divisor $D$ with no assumptions on the logarithmic residue.

### 1.3.2 The dual logarithmic residue

The dual logarithmic residue was introduced by Granger and Schulze in [43]. In some sense, it relates the Jacobian ideal of a divisor with its conductor into the normalization. Here, it will be introduced in order to show Thm. 1.63. The proof of this theorem will also make use of a result by R. Piene about ideals in the normalization.
In the next section, section 1.4, we will indicate how to use Thm. 1.63 to answer a question by K. Saito about the logarithmic residue.

Let $(S, D)$ be a complex manifold $S$ of dimension $n$ together with a divisor $D$ that is locally at a point $p \in S$ given by $\{h = 0\}$. Denote by

$\pi : \widetilde{D} \to D$ the normalization of $D$. Here we will abbreviate $\mathcal{O}_{S,p}$ to $\mathcal{O}_S$ etc. From sequence (1.2) one gets an exact sequence

$$0 \longrightarrow \Omega^1_S \longrightarrow \Omega^1_S(\log D) \xrightarrow{\rho} \rho(\Omega^1_S(\log D)) \longrightarrow 0. \tag{1.12}$$

By applying the functor $\mathrm{Hom}_{\mathcal{O}_S}(-, \mathcal{O}_S)$ to (1.12) one obtains

$$0 \longrightarrow \mathrm{Der}_S(\log D) \longrightarrow \mathrm{Der}_S \xrightarrow{\sigma} \cdots$$

$$\cdots \xrightarrow{\sigma} \rho(\Omega^1_S(\log D))^\vee \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_S}(\Omega^1_S(\log D), \mathcal{O}_S) \longrightarrow 0. \tag{1.13}$$

Here $-^\vee$ denotes $\mathrm{Hom}_{\mathcal{O}_D}(-, \mathcal{O}_D)$. By Lemma 4.5 of [31] one has

$$\mathrm{Ext}^1_{\mathcal{O}_S}(\rho(\Omega^1_S(\log D)), \mathcal{O}_S) = \mathrm{Hom}_{\mathcal{O}_D}(\rho(\Omega^1_S(\log D)), \mathcal{O}_D) = \rho(\Omega^1_S(\log D))^\vee,$$

which explains the third term on the right in (1.13). Then we call $\rho(\Omega^1_S(\log D))^\vee$ the *dual logarithmic residue* and denote it shortly by $\mathscr{R}^\vee_D$.
One can show (see [43]) that $\rho(\Omega^1_S(\log D)) = \widetilde{J}_h^\vee$, where $\widetilde{J}_h$ is the ideal generated by $(\partial_{x_1} h, \ldots, \partial_{x_n} h) \subseteq \mathcal{O}_D$, that is, the Jacobian ideal of $D$. We will need the following

**Proposition 1.71.** *Let $D \subseteq S$ be free. If the logarithmic residue is weakly holomorphic, i.e., $\rho(\Omega^1_{S,p}(\log D)) = \pi_* \mathcal{O}_{\widetilde{D},p}$, then $\widetilde{J}_h \subseteq \mathcal{O}_D$ is equal to the conductor ideal $C_D$ (as defined in the appendix). Conversely, if $\widetilde{D}$ is Cohen–Macaulay at $p$ and $\widetilde{J}_h = C_D$, then*

$$\rho(\Omega^1_{S,p}(\log D)) = \pi_* \mathcal{O}_{\widetilde{D},p}.$$

*Proof.* See [43] Cor. 3.5. □

Now we are nearly ready for the proof of Thm. 1.63, which will make use of Thm. A.42 of the appendix. It was pointed out by D. Mond to use Piene's theorem in order to prove the assertion. Here we also remark that we need Thm. A.42 to prove our main result Thm. 2.1 but there the (dual) logarithmic residue will not appear.

**Lemma 1.72.** *Let $D \subseteq S$ be a divisor in a complex manifold of dimension $n$. Suppose that $D$ is free at $p$, $\rho(\Omega^1_{S,p}(\log D)) = \pi_* \mathcal{O}_{\widetilde{D},p}$ and that $(D,p) = \bigcup_{i=1}^m (D_i, p)$, where each irreducible component $D_i$ is normal. Then all $(D_i, p)$ are smooth and $(D,p)$ has normal crossings.*

*Proof.* Since all irreducible components are normal, it follows that $\rho(\Omega^1_{S,p}(\log D)) = \bigoplus_{i=1}^m \mathcal{O}_{D_i,p}$. By Theorems 1.42 and 1.52 $(D,p)$ is a normal crossing singularity. □

*Proof of Thm. 1.63.* By our hypothesis, Piene's Theorem A.42 yields the equality of ideals

$$C_D I_\pi \mathcal{O}_{\widetilde{D},p} = \widetilde{J}_h \mathcal{O}_{\widetilde{D},p}.$$

Here $I_\pi$ denotes the ramification ideal of the normalization, that is, $I_\pi = F^0_{\widetilde{D}}(\Omega^1_{\widetilde{D}/D})$. By Prop. 1.71 this implies the equality of the ideals $C_D = C_D I_\pi$ in $\pi_*\mathcal{O}_{\widetilde{D},p}$. By Nakayama's lemma, it follows that $I_\pi = \pi_*\mathcal{O}_{\widetilde{D},p}$. Hence $\Omega^1_{\widetilde{D}/D} = 0$. By [6, VI, Prop. 1.18, Prop. 1.20] (localization to an irreducible component $D_i$ and base change) it follows that $\Omega^1_{\widetilde{D}_i/D_i} = 0$ for all $i = 1, \ldots, m$. Suppose that $\widetilde{D}_i$ is smooth at $\tilde{p}_i = \pi^{-1}(p)$ on $\widetilde{D}_i$, then $\mathcal{O}_{\widetilde{D}_i,\tilde{p}_i} \cong \mathbb{C}\{z_1, \ldots, z_{n-1}\}$ for some independent variables $z_1, \ldots, z_{n-1}$. Hence one has an inclusion of rings

$$\mathcal{O}_{D_i,p} = \mathbb{C}\{f_1, \ldots, f_r\} \subseteq \mathbb{C}\{z_1, \ldots, z_{n-1}\},$$

where $f_1, \ldots, f_r \in \mathcal{O}_{\widetilde{D}_i,\tilde{p}_i}$ and $r \geq n-1$. By definition one can write

$$0 = \Omega^1_{\widetilde{D}_i/D_i} = \bigoplus_{j=1}^{n-1} \mathcal{O}_{\widetilde{D}_i,\tilde{p}_i} dz_j / \sum_{k=1}^{r} \mathcal{O}_{\widetilde{D}_i,\tilde{p}_i} df_k.$$

By Nakayama's lemma one finds $n-1$ generators of $\mathcal{O}_{D_i,p}$, w.l.o.g., $f_1, \ldots f_{n-1}$ such that the Jacobian determinant $\frac{\partial(f_1,\ldots,f_{n-1})}{\partial(z_1,\ldots,z_{n-1})} \neq 0$. By the implicit function theorem, $f_1, \ldots, f_{n-1}$ are independent variables and hence $\mathcal{O}_{D_i,p} \cong \mathcal{O}_{\widetilde{D}_i,\tilde{p}_i}$ is smooth. Since $\pi$ is a finite map and $\operatorname{codim}(\operatorname{Sing} \widetilde{D}_i, \widetilde{D}) \geq 2$ it follows that $\operatorname{codim}(\operatorname{Sing} D_i, D_i) \geq 2$. By Thm.A.24 $D_i$ is normal for all $i = 1, \ldots, m$. Thus $(D,p)$ a union of normal components. By definition $(D,p)$ is free and by Lemma 1.72 it has normal crossings. □

## 1.4 On a question by K. Saito

Theorem 1.42 suggests that $\rho(\Omega^1_S(\log D))$, the residue of logarithmic 1-forms, is directly related to the geometry of the divisor $D$. Kyoji Saito has considered the relationship between the logarithmic residue and the local fundamental group of the complement of the divisor. Based on the two-dimensional case, see Prop. 1.44, Saito asked the following, cf. [81, (2.12)]:

**Question 1.73** (K. Saito). *Let $(S, D)$ be a manifold with $\dim S = n$ together with a divisor $D \subseteq S$ and let $p$ be a point on $D$. Are the following equivalent?*
*(i) The local fundamental group $\pi_{1,q}(S \backslash D)$ for $q$ near $p$ is abelian.*
*(ii) There exists an $(n-3)$-dimensional analytic subset $Z$ of $D$, such that $D \backslash Z$ has only normal crossing singularities in a neighbourhood of $p$.*
*(iii) $\rho(\Omega^1_{S,p}(\log D)) = \pi_* \mathcal{O}_{\widetilde{D},p}$.*

The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ were proven by Saito in [81]. In 1985 Lê and Saito gave a topological proof of the equivalence of $(i)$ and $(ii)$. The implication $(iii) \Rightarrow (ii)$ was only recently proven by Granger and Schulze [43]. Hence all three conditions are equivalent. There seems to be no obvious link between the residue and the fundamental group, and nobody seems to have studied how to prove directly that $(i)$ is equivalent to $(iii)$.
We make a short excursion to fundamental groups in order to understand the equivalence $(i) \Leftrightarrow (ii)$. Then, following Granger and Schulze [43, Thm. 4.2], we also prove the implication $(iii) \Rightarrow (ii)$.

### 1.4.1 The local fundamental group of the complement of a hypersurface

In this section we discuss the first equivalence of Saito's question, namely a topological characterization of divisors with normal crossings in codimension 1. Now consider small *balls* $B^{2n}_\epsilon$ centered in $p \in S$ and defined by

$$B^{2n}_\epsilon = \{x \in S : \|x - p\| \leq \epsilon\}.$$

For $\epsilon > 0$ sufficiently small these balls make up a fundamental system of good neighbourhoods of $p \in S$, see [92]. Then the local fundamental group of the complement of $(D, p) \subseteq (S, p)$ is defined as the fundamental group $\pi_{1,q}(B^{2n}_\epsilon \backslash D)$, for $\epsilon > 0$ sufficiently small and $q \in B^{2n}_\epsilon \backslash D$.

**Theorem 1.74** (Lê–Saito). *Let $D$ be a divisor in a complex manifold $S$ of dimension $n$. Then $D$ has normal crossings in codimension 1 at a point $p$ if and only if the local fundamental group $\pi_{1,q}(S \backslash D)$ for $q$ in a neighbourhood of $p$ is abelian.*

In [92] Lê and Saito first showed Thm. 1.74 for irreducible $D$ and then for the general case. They reduce the problem to $\dim S = 2$ and use topological methods. We will shortly discuss the (easy) implication $(i) \Rightarrow (ii)$. Before we start, we give a few examples of divisors and the fundamental groups of their complements.

*Example* 1.75. (The line minus a point) Let $S = \mathbb{C}^1$ with coordinate $x$ and the divisor $D$ be given by $\{x = 0\}$. We compute the local fundamental group $\pi_{1,q}(B^2_\epsilon \backslash \{0\})$ via the universal cover of $S\backslash\{0\}$. Therefore denote by $X$ the universal cover of $S\backslash\{0\}$. One can prove that $\pi_{1,q}(B^2_\epsilon \backslash \{0\})$ is isomorphic to $\mathrm{Aut}_{B^2_\epsilon\backslash\{0\}}(X)$, that is, the group of deck transformations of $X$. Here $X = S$ via the map

$$S \xrightarrow{\exp} S\backslash\{0\}$$

and the deck transformations of $X$ are given by $z \mapsto z + 2\pi i k$, for some $k \in \mathbb{Z}$. It follows that $\pi_{1,q}(B^2_\epsilon \backslash \{0\}) = \mathbb{Z}$. This can also be interpreted by saying that the local fundamental group is generated by a small loop around $x = 0$.

*Example* 1.76. (The normal crossing divisor) Suppose that $S = \mathbb{C}^n$ and that at a point $p$ with complex coordinates $(x_1, \ldots, x_n)$ the divisor is $D = \{x_1 \cdots x_d = 0\}$, $d \leq n$. The complement $B^{2n}_\epsilon \backslash D = \{0 < |x_i| < \epsilon, i = 1, \ldots, d\}$ can be contracted on the $d$-Torus, which is given by $\prod_{i=1}^d \{|x_i| = \epsilon\}$. Hence $\pi_{1,q}(B^{2n}\backslash D) \cong \mathbb{Z}^d$ for $q \in B^{2n}_\epsilon$. The generators of this fundamental group correspond to small loops around the components $D_i = \{x_i = 0\}$.

*Example* 1.77. (The Cusp) Let $S = \mathbb{C}^2$ and $D = \{4x_1^3 = 27x_2^2\}$. The local fundamental group of the complement of $D$ is not not abelian, which can be seen as follows (here the example is only sketched, see [36, §22] for details): the curve $D$ is the branch locus of the map $f : \mathbb{C}^2 \to \mathbb{C}^2, (u, v) \mapsto (u, v^3 + uv)$. Consider $K = D \cap S^3$, which is a knot in real three-space. One finds that the restriction of $f : \mathbb{C}^2 \backslash f^{-1}(D) \longrightarrow S\backslash D$, namely

$$\pi : f^{-1}(S^3 \backslash K) \longrightarrow S^3 \backslash K$$

is a three-sheeted covering that is not regular. Taking as base point $x = (1, 0)$ one gets that $\pi_1(S^3\backslash K)$ is not abelian, since any connected covering of a manifold with abelian fundamental group is regular.

In 1929, Zariski considered in [102] the question of finding a covering of $\mathbb{P}^2$ branched along a given projective plane curve $C$. This problem can be phrased in terms of fundamental groups. In his paper Zariski states the following

**Theorem 1.78.** *Let $C$ be an algebraic curve in the projective plane $\mathbb{P}^2(k)$, where $k$ is any algebraically closed field. If $C$ has only nodes as singularities, then the étale fundamental group $\pi_1(\mathbb{P}^2(k)\backslash C)$ is abelian.*

Zariski showed this theorem using a result by Enriques–Severi, namely that any curve with only nodal singularities can be degenerated to lines in general position. However, Severi's proof of this result (see Vorlesungen über algebraische Geometrie, 1921, Anhang F) was found to be erroneous, so Zariski's proof of Thm. 1.78 was not complete at that time (Severi's result was established by Harris [47] only in 1986). It took some years until 1980 when Fulton [35] was able to give the first correct proof of Zariski's theorem: he used methods introduced by Abhyankar [1, 2], who showed some special cases of Theorem 1.78, as well as a strong version of the Bertini connectedness theorem, see the paper by Fulton and Hansen [37]. Also in 1980, Deligne [30] gave an account of Fulton's work in Séminaire Bourbaki, where he strengthened Fulton's result in the complex case:

**Theorem 1.79.** *Let $C$ be a plane projective curve in $\mathbb{P}^2(\mathbb{C})$, which only has node singularities. Then the (topological) fundamental group $\pi_1(\mathbb{P}^2(\mathbb{C})\backslash C)$ is abelian.*

Note here that if one replaces $\mathbb{C}$ by an algebraically closed field of characteristic 0 then the same assertion is true with the algebraic (= étale) fundamental group instead of the topological fundamental group. Now we are ready to prove the "only if" part of Thm. 1.74, namely: Let $(S, D)$ be a divisor and its manifold. If the local fundamental group $\pi_{1,q}(S\backslash D)$ for $q$ in a suitable neighbourhood of $p$ is abelian, then there exists an $(n-3)$-dimensional analytic set $Z$ in $S$ such that the complement of $Z$ in $D$ has only normal crossing singularities in a neighbourhood of $p$.

*Proof of $\Leftarrow$ of Thm. 1.74.* Suppose there exists an $(n-2)$-dimensional subset $Z$ of $D$ such that $D\backslash Z$ does not have normal crossings. Then $Z$ must be the union of irreducible branches of $\operatorname{Sing} D$. We may suppose that $Z$ is irreducible. We consider $D$ as a family of plane curve germs along $Z$. At a general point of $Z$, $D$ is equisingular (see Zariski's equisingularity theory [103]), and hence topologically trivial along $Z$. Thus the local fundamental group $\pi_{1,q}(S\backslash D)$ is isomorphic to the fundamental group of the complement of a plane curve. But the generic member of a family of germs of plane curves along $Z$ does not at most have node singularities because by assumption, $D$ does not have normal crossings along $Z$. Hence $\pi_{1,q}(S\backslash D)$ is not abelian, which follows from Theorem 1.79 and the computations of Zariski in [102], in which he shows that curves that have more complicated singularities than nodes, have a non-abelian fundamental group. □

### 1.4.2 Answer to Saito's question

Now we consider the equivalence of *(ii)* and *(iii)* of Saito's question. As already shown in [81, Lemma 2.13], the implication *(ii)* ⇒ *(iii)* always holds:

**Lemma 1.80.** *Let* $(S, D)$ *be as usual,* $\dim S = n$ *and suppose that* $D$ *has normal crossings outside a set* $Y$ *with* $\operatorname{codim}(Y, S) \geq 3$. *Denote by* $\pi : \widetilde{D} \to D$ *the normalization of* $D$. *Then we have*

$$\rho(\Omega^1_{S,p}(\log D)) = \pi_* \mathcal{O}_{\widetilde{D},p},$$

*that is, the residue of* $\Omega^1(\log D)$ *are the weakly holomorphic functions on* $D$.

*Proof.* Since $D\backslash Y$ has normal crossings, any point $p$ in this set satisfies the condition (iv) of Theorem 1.42. But this implies that $\rho(\omega)$ is holomorphic on $D\backslash Y$ for any $\omega \in \Omega^1_{S,p}(\log D)$. Hence $\rho(\omega)$ is also holomorphic on $\widetilde{D\backslash Y}$. The codimension of $Y$ in $D$ is greater than or equal to 2, so the codimension of its normalization $\pi^{-1}(Y)$ in $\widetilde{D}$ is also greater than or equal to 2 and we have $\widetilde{D}\backslash\pi^{-1}(Y) = \widetilde{D\backslash Y}$. But $\widetilde{D}$ is a normal variety, so we may apply the Extension theorem of Hartogs, Thm. A.21, to conclude that $\rho(\omega)$ is holomorphic on whole $\widetilde{D}$. □

The other implication follows from Theorem 1.63 and the following proposition about freeness in codimension one. This proposition makes use of Aleksandrov's algebraic characterization of free divisors, which will be discussed in Chapter 2. For notation and definition of the Jacobian ideal sheaf $\mathcal{J}$ and $\mathcal{O}_{\operatorname{Sing} D}$ see Section 2.2.

**Proposition 1.81.** *Let* $(S, D)$ *be a complex* $n$*-dimensional manifold together with a divisor* $D \subseteq S$. *Then the set* $Z := \{p \in S : D \text{ is not free at } p\}$ *is an analytic subset of* $S$ *of dimension at most* $n - 3$. *In particular,* $D$ *is free in codimension 1.*

*Proof.* Recall that $\mathcal{O}_{\operatorname{Sing} D}$ is defined as $\mathcal{O}_D/\widetilde{\mathcal{J}}$, where $\widetilde{\mathcal{J}}$ is the Jacobian ideal restricted to $D$. By Aleksandrov's theorem (Thm. 2.6) $Z$ is equal to the set

$$\{p \in D \colon \mathcal{O}_{\operatorname{Sing} D,p} \text{ not Cohen–Macaulay of dimension } n-2 \text{ or } \mathcal{O}_{\operatorname{Sing} D,p} \neq 0\}.$$

It is easy to see that $\mathcal{O}_{\operatorname{Sing} D}$ is a coherent analytic $\mathcal{O}_S$ sheaf ($\mathcal{O}_D$ is coherent by Cartan's coherence theorem and the Jacobian ideal sheaf $\mathcal{J}$ is coherent, since it is finitely generated at any stalk and the syzygies between partial derivatives are also finitely generated. By the Metatheorem of coherent sheaves A.15, $\mathcal{O}_D/\widetilde{\mathcal{J}} = \mathcal{O}_{\operatorname{Sing} D}$ is also a coherent $\mathcal{O}_S$-sheaf). Then consider the singular set (as defined in Appendix A)

$$S_m(\mathcal{O}_{\operatorname{Sing} D}) = \{p \in S : \operatorname{depth}_p \mathcal{O}_{\operatorname{Sing} D} \leq m\}.$$

Since $Z = S_{n-3}(\mathcal{O}_{\operatorname{Sing} D})$, it follows from Scheja's theorem (Thm. A.16) that $Z$ is an analytic subset of dimension at most $n-3$. □

**Theorem 1.82.** *Let $(S, D)$ be a complex manifold together with a divisor $D \subseteq S$. If the logarithmic residue $\rho(\Omega^1_S(\log D)) = \pi_* \mathcal{O}_{\widetilde{D}}$, then $D$ has normal crossings in codimension 1.*

*Proof.* By Prop. 1.81 $D$ is free outside an analytic subset $Z \subseteq S$ of codimension at least 2 in $D$. Since $\rho(\Omega^1_{S,p}(\log D)) = \pi_* \mathcal{O}_{\widetilde{D},p}$ for all $p \in S$ and $\widetilde{D}$ is by definition smooth in codimension 1 it follows from Thm. 1.63 that $D$ has normal crossings outside an analytic set of codimension 2 in $D$. □

Theorem 1.82 proves the missing implication $(iii) \Rightarrow (ii)$ of Saito's question. Hence the answer to Saito's question is positive.

# Chapter 2

# Algebraic characterization of normal crossing divisors

Here a characterization of a normal crossing divisor is given in terms of the Jacobian ideal defining the singular locus of the divisor. Our result is the following: a divisor $D$ in a complex manifold $S$ of complex dimension $n$ has normal crossings at a point $p \in S$ if and only if the local ring $\mathcal{O}_{\operatorname{Sing} D,p} = \mathcal{O}_{S,p}/(h, J_h)$, where $J_h$ denotes the Jacobian ideal of $D = \{h = 0\}$, is Cohen–Macaulay of dimension $n-2$, $J_h$ is a radical ideal and moreover, the normalization $\widetilde{D}$ of $D$ is Gorenstein (Thm. 2.1). This criterion makes it possible to determine whether a divisor has normal crossings at a point without knowing its decomposition into irreducible components.
This chapter is devoted to prove the above characterization of normal crossing divisors: first the theorem about the singularities of normal crossing divisors is stated and motivated by some examples. Then we consider the algebraic characterization of free divisors (due to A. G. Aleksandrov, A. Simis and H. Terao[2]): a divisor $D$ is free at a point $p$ if and only if it is either smooth at $p$ or $\mathcal{O}_{\operatorname{Sing} D,p}$ is Cohen–Macaulay of dimension $n-2$. The rest of the chapter is used to prove Theorem 2.1: we first pass by some special cases, for which no condition on the normalization of $D$ is required. Then we introduce the notion of splayed

[2] In this text this result will always be referred to as *Aleksandrov's Theorem* because the author has learned it from [3,4]. As pointed out by A. Simis, the same result was also independently proven by H. Terao in [94] (algebraic case) and in general in [95] and later (in the algebraic case) by A. Simis in [89].

divisor, which is needed to reduce the problem to an irreducible divisor. Finally the assertion of the theorem is shown similarly to the results from Chapter 1 on the logarithmic residue. Most of the algebra used in this chapter is explained in Appendix A.

## 2.1 The main theorem

Let $D$ be a divisor in a complex manifold $S$ with $\dim S = n$ and suppose that $D$ is given at a point $p = (x_1, \dots, x_n)$ by the reduced equation $h(x) = 0$, $h \in \mathcal{O}_{S,p}$. Recall that the Jacobian ideal of $h$ is the ideal generated by the partial derivatives of $h$. It is denoted by $J_{h,p} = (\partial_{x_1} h, \dots, \partial_{x_n} h)\mathcal{O}_{S,p}$. We will often simply write $J_h$ instead of $J_{h,p}$. There is a canonical epimorphism sending $\mathcal{O}_{S,p}$ to $\mathcal{O}_{D,p} = \mathcal{O}_{S,p}/(h)$. We denote by $\widetilde{J_h}$ the Jacobian ideal in $\mathcal{O}_{D,p}$ (most of the time $\widetilde{J_h}$ is also simply denoted by $J_h$). The associated analytic coherent ideal sheaves are denoted by $\mathcal{J} \subseteq \mathcal{O}_S$ and $\widetilde{\mathcal{J}}$ in $\mathcal{O}_D$. The *singular locus* of $D$ is denoted by $\operatorname{Sing} D$ and is defined by the ideal sheaf $\widetilde{\mathcal{J}} \subseteq \mathcal{O}_D$. The local ring of $\operatorname{Sing} D$ at a point $p$ is denoted by

$$\mathcal{O}_{\operatorname{Sing} D,p} = \mathcal{O}_{S,p}/((h) + J_h) = \mathcal{O}_{D,p}/\widetilde{J_h}.$$

Sometimes $\mathcal{O}_{\operatorname{Sing} D,p} = \mathbb{C}\{x_1, \dots, x_n\}/(h, \partial_{x_1} h, \dots, \partial_{x_n} h)$ is also called the *Tjurina algebra*, see e.g. [27]. Note that we *always* consider $\operatorname{Sing} D$ with the (possibly non-reduced) structure given by the Jacobian ideal of $D$. Hence in general $(\operatorname{Sing} D, p)$ is a complex space germ and not necessarily reduced. We often say that $\operatorname{Sing} D$ *Cohen–Macaulay*, which means that $\mathcal{O}_{\operatorname{Sing} D,p}$ is Cohen–Macaulay for all points $p \in \operatorname{Sing} D$. The definition of Cohen–Macaulay modules and further properties of them can be found in Appendix A. If $D$ is an Euler–homogeneous divisor, then the $\mathcal{O}_{S,p}$-modules $\mathcal{O}_{S,p}/J_h$ and $\mathcal{O}_{D,p}/\widetilde{J_h}$ are equal.

In chapter 1 it was shown that a normal crossing divisor is free. Therefore our idea is to impose additional conditions in order to single out the normal crossing divisors. By Aleksandrov's theorem in the next section (see Thm. 2.6) free divisors can be completely described by their Jacobian ideal. So the right additional requirement turns out to be radicality of the Jacobian ideal. Hence a purely algebraic criterion is obtained, which allows to determine whether a divisor has normal crossings at a point $p$, even without knowing its decomposition into irreducible components.

**Theorem 2.1.** *Let $D = \{h = 0\}$ be a divisor in a complex manifold $S$, $\dim S = n$. Denote by $\pi : \widetilde{D} \to D$ the normalization of $D$. Then the following are equivalent:*
*(1) $D$ has normal crossings at any point $p$ in $D$.*
*(2) $D$ is free at any point $p$, $J_{h,p}$ is radical and $(\widetilde{D}, \pi^{-1}(p))$ is Gorenstein.*

*Remark* 2.2. Using Aleksandrov's algebraic characterization of free divisors (Thm. 2.6), condition (2) of the above theorem can also be phrased as:
(2') At any point $p \in D$ the Tjurina algebra $\mathcal{O}_{\mathrm{Sing}\,D,p}$ is reduced and either 0 or Cohen–Macaulay of dimension $n-2$ and $\pi_*\mathcal{O}_{\widetilde{D},p}$ is a Gorenstein ring.
Another equivalent formulation is:
(2") At any point $p \in D$, where $D = \{h = 0\}$, the Jacobian ideal $J_h$ is either equal to $\mathcal{O}_{S,p}$ or it is radical, perfect and with $\mathrm{depth}(I, \mathcal{O}_{S,p}) = 2$ and $\pi_*\mathcal{O}_{\widetilde{D},p}$ is Gorenstein.

*Remark* 2.3. The condition $\widetilde{D}$ Gorenstein is technical and only needed to apply Piene's theorem in our proof of Thm. 2.1. In some special cases (see section 2.3) it can be omitted. We do not know if this condition is necessary in general (cf. Remark 2.50).

Before commenting on the proof of Thm. 2.1, let us consider some examples:

*Example* 2.4. (1) Let $D$ be the cone in $\mathbb{C}^3$, given by the equation $z^2 = xy$. It does not have normal crossings at the origin but the Jacobian ideal $J_{h,0} = (z, x, y)$ is clearly radical and $\mathcal{O}_{\mathbb{C}^3,0}/(x, y, z) \cong \mathbb{C}$ is Cohen–Macaulay. However, the depth of $\mathcal{O}_{\mathbb{C}^3,0}/J_{h,0}$ is 0 and thus too small.
(2) The divisor in $\mathbb{C}^3$ given by the equation $h = xy(x - z^2)$ is free (by Aleksandrov's theorem) but does not have normal crossings. Note that its singular locus is a Cohen–Macaulay curve. The Jacobian ideal is

$$J_h = (y, x - z^2) \cap (x, y) \cap (z^2 - 2x, xz, x^2),$$

which is not radical.
(3) Let $S = \mathbb{C}^3$ and $D$ be the "4-lines" defined by $h = xy(x+y)(x+yz)$. This divisor $D$ is free, and a basis of $\mathrm{Der}_{S,p}(\log D)$ is given by $\delta^1 = x\partial_x + y\partial_y, \delta^2 = (zy + x)\partial_z$ and $\delta^3 = -x^2\partial_x + y^2\partial_y - (x + yz)\partial_z$. Its Jacobian ideal is the intersection of the three primary ideals $(x+y, z-1)$, $(x, z)$ and $(y^4, 2xy^2z + y^3z + 3x^2y + 2xy^2, 4x^2yz - 3y^3z + 2x^3 - 5x^2y - 6xy^2)$, which is not radical (the radical $\sqrt{J_h}$ is $(x + y, z - 1) \cap (x, z) \cap$

$(x, y)$) and $D$ does not have normal crossings at the origin.
(4) The divisor $D$ in $\mathbb{C}^3$ defined by $h = -x^4y^2 - xy^3 + x^4z + xyz + x^3y^3z + y^4z - x^3yz^2 - y^2z^2 - xy^2z^2 + xz^3 + x^4z^3 + xyz^3 + y^3z^3 - yz^4 - x^3yz^4 - y^2z^4 + xz^5 - yz^6$ has normal crossings at 0: its Jacobian ideal is of height 2, radical and $\mathbb{C}[x, y, z]/J_h$ is Cohen–Macaulay. A basis of $\Omega^1_{S,p}(\log D)$ is

$$\omega_1 = \frac{3x^2dx + dy + 2zdz}{x^3 + z^2 + y}, \ \omega_2 = \frac{dx - zdy - ydz}{x - yz}$$

and

$$\omega_3 = \frac{2ydy + (-1 - 3z^2)dz}{y^2 - z - z^3}.$$

Since $D$ is the union of three smooth surfaces, the normalization $\widetilde{D}$ is smooth and hence Gorenstein.

### 2.1.1 Structure of the proof of Thm. 2.1

The implication (1) $\Rightarrow$ (2) is a straightforward computation. The other direction occupies the rest of the chapter. Since the freeness of a divisor is a necessary condition in (1), we show in section 2.2 the algebraic characterization of free divisors due to A. G. Aleksandrov. This is followed by showing (2) $\Rightarrow$ (1) of Thm. 2.1 for some special cases, namely for divisors in manifolds $S$ of dimension 2 (Prop. 2.15), for $\operatorname{Sing} D$ smooth (Lemma 2.17), for $\operatorname{Sing} D$ Gorenstein (Prop. 2.18) and for hyperplane arrangements and generalizations thereof (Prop. 2.32). For these cases, the assumption $\widetilde{D}$ Gorenstein is not needed.

However, the ideas to show the special cases do not lead to a proof in general. Therefore, our strategy to prove the general case is the following:
(i) If $(D, p) = \bigcup_{i=1}^m (D_i, p)$ is free and a union of irreducible components and has radical Jacobian ideal, then we show that each $D_i$ is also free and has radical Jacobian ideal.
(ii) If $D$ is free, irreducible, has radical Jacobian ideal at $p$ and the normalization $\widetilde{D}$ is Gorenstein, then $D$ is already smooth at $p$.
(iii) A free divisor $D$, which is a union of smooth irreducible hypersurfaces and has a radical Jacobian ideal, is already a normal crossing divisor.

In order to obtain (i) we introduce a generalization of normal crossing divisors, so-called splayed divisors. A splayed divisor $D$ is a union of transversally meeting hypersurfaces that are possibly singular. First it is shown that (i) holds for splayed divisors (Lemma 2.38). Then we prove that a divisor $(D,p) = \bigcup_{i=1}^m (D_i,p)$ with radical Jacobian ideal is splayed (Prop. 2.48). Therefore a characterization of splayed divisors via their Jacobian ideals is shown (the Leibniz property - see Thm. 2.43). All this is explained in sections 2.4.1 and 2.4.2.
Claim (ii) then follows from Piene's Theorem (Thm. A.42), similarly like the results on the logarithmic residue of chapter 1. Note that here we do not need the dual residue because if $J_h$ is radical, one can show that it is equal to the conductor $C_D$. Finally, claim (iii) follows from different previous results, namely, either from the hyperplane arrangement case (Prop. 2.32) or from the second corollary of proposition 2.48.

## 2.2 Algebraic characterization of free divisors

We have defined free divisors via the modules of logarithmic vector fields or logarithmic differential forms. However, there also exists a characterization of free divisors by their singularities, which is due to A. G. Aleksandrov [4] (cf. footnote 2). Namely, a divisor $D$ in a complex manifold is free if and only if it is smooth or its singular locus defined by the Jacobian ideal is Cohen–Macaulay of codimension 1 in $D$. The first result in this direction was obtained by H. Terao [94], who characterized a free hyperplane arrangement in an algebraic manifold by the corresponding property. This has also been discovered independently by Simis [89]. In 1986, Aleksandrov proved the Cohen–Macaulayness of the singular locus for Euler–homogeneous free divisors, see [3]. Eventually, in his 1990 paper [4] he was able to extend his result to arbitrary free divisors in complex manifolds.
The characterization of free divisors by their Jacobian ideals can be used to obtain a simple proof that the discriminant of a miniversal deformation of a complete intersection with an isolated singularity is a free divisor, see [4,60,81]. Consequently, this algebraic characterization is useful whenever free divisors appear in the theory of discriminants and bifurcations, see [13,25,68]. Moreover, Aleksandrov's freeness criterion is effective: one can check Cohen–Macaulayness with a computer algebra system like Singular [98].

In this section a proof of Aleksandrov's theorem is given. As a corollary we regain that all reduced divisors in a complex manifold $S$ with $\dim S = 2$ are free.

**Proposition 2.5.** *Let $D$ be a free divisor in $S$. Then either $D$ is smooth at a point $p \in D$ or* $\operatorname{codim}_p(\operatorname{Sing} D, S) = 2$, *which is equivalent to* $\operatorname{codim}_p(\operatorname{Sing} D, D) = 1$. *Thus it follows that $D$ is not normal at its singular points. Moreover, $\mathcal{O}_{\operatorname{Sing} D,p}$ is a Cohen–Macaulay ring.*

*Proof.* Let $h = 0$ be the equation for $D$ at a point $p \in \operatorname{Sing} D$. Then there is an exact sequence

$$\operatorname{Syz}(\partial_{x_1} h, \ldots, \partial_{x_n} h, h) \longrightarrow \mathcal{O}_{S,p}^{n+1} \xrightarrow{\varphi} \mathcal{O}_{S,p} \longrightarrow \mathcal{O}_{S,p}/(h, J_h) \longrightarrow 0$$

of $\mathcal{O}_{S,p}$-modules. Here $\varphi$ denotes the map sending $(a_1, \ldots, a_{n+1}) \in \mathcal{O}_{S,p}^{n+1}$ to $\sum_{i=1}^{n} a_i \partial_{x_i} h + a_{n+1} h$ and $\operatorname{Syz}(\partial_{x_1} h, \ldots, \partial_{x_n} h, h)$ denotes the first syzygy module of $((h) + J_h)$. As explained in the proof of Lemma 1.7, $\operatorname{Der}_{S,p}(\log D)$ is canonically isomorphic to $\operatorname{Syz}(\partial_{x_1} h, \ldots, \partial_{x_n} h, h)$. Since $\operatorname{Der}_{S,p}(\log D)$ is by assumption a free $\mathcal{O}_{S,p}$-module of rank $n$, it follows that

$$0 \longrightarrow \operatorname{Syz}(\partial_{x_1} h, \ldots, \partial_{x_n} h, h) \longrightarrow \mathcal{O}_{S,p}^{n+1} \xrightarrow{\varphi} \mathcal{O}_{S,p} \longrightarrow \mathcal{O}_{S,p}/(h, J_h) \longrightarrow 0$$

is a free resolution of $\mathcal{O}_{S,p}/((h) + J_h) = \mathcal{O}_{\operatorname{Sing} D,p}$. This means that

$$\operatorname{projdim}_{\mathcal{O}_{S,p}} \mathcal{O}_{\operatorname{Sing} D,p} = 2,$$

and by the Auslander–Buchsbaum formula $\operatorname{depth}(\mathcal{O}_{\operatorname{Sing} D,p}) = n - 2$. Since $D$ is a reduced divisor, its singular locus must be a proper analytic subset of $D$, that is, $\operatorname{codim}_p(\operatorname{Sing} D, D) \geq 1$, or equivalently $\operatorname{codim}_p(\operatorname{Sing} D, S) \geq 2$. The well-known dimension-depth inequality yields

$$n - 2 = \operatorname{depth}(\mathcal{O}_{\operatorname{Sing} D,p}) \leq \dim(\mathcal{O}_{\operatorname{Sing} D,p}) \leq n - 2.$$

Hence $\mathcal{O}_{\operatorname{Sing} D,p}$ is Cohen–Macaulay of (Krull-)dimension $n - 2$. □

**Theorem 2.6** (Aleksandrov)**.** *Let $D \subseteq S$ be a non-normal divisor, that is,* $\operatorname{codim}_p(\operatorname{Sing} D, D) = 1$ *for any $p \in \operatorname{Sing} D$. Then the following conditions are equivalent:*
*(i) $D$ is a free divisor,*
*(ii)* $\operatorname{Sing} D$ *is Cohen–Macaulay, that is, for every $p \in D$ the local ring $\mathcal{O}_{\operatorname{Sing} D,p}$ is Cohen–Macaulay.*

*Proof.* The statement is local, so we choose a point $p \in \operatorname{Sing} D$ and consider $D$ locally at $p$. Let $D$ at $p$ be defined by a reduced $h \in \mathcal{O}_{S,p}$. The singular locus $\operatorname{Sing} D$ is defined by the ideal $((h) + J_h) \subseteq \mathcal{O}_{S,p}$.
The implication (i) $\Rightarrow$ (ii) was already shown in Prop. 2.5. It remains to prove that if $(\operatorname{Sing} D, p)$ is Cohen–Macaulay then $D$ is free at $p$. First suppose that the $\partial_{x_i} h$ form a minimal basis of $J_h$ and that $D$ is Euler–homogeneous at $p$. Hence there exists a vector field $\eta \in \operatorname{Der}_{S,p}(\log D)$ such that $\eta(h) = h$ and thus $\mathcal{O}_{\operatorname{Sing} D,p} = \mathcal{O}/J_h$. The depth of $J_h$ on $\mathcal{O}_{S,p}$ is 2 and hence $\dim \mathcal{O}_{S,p} = 2 + \dim(\mathcal{O}_{S,p}/J_h) = 2 + \operatorname{depth}(\mathcal{O}_{S,p}/J_h)$. By the Auslander–Buchsbaum formula (Thm. A.7) it follows that $\operatorname{projdim}_{\mathcal{O}_{S,p}}(\mathcal{O}_{S,p}/J_h) = n - (n-2) = 2$. Therefore the theorem of Hilbert–Burch, see Appendix A, can be applied. Hence any generator $\partial_{x_i} h$, $i = 1, \ldots n$ of $J_h$ is given as the $i$-th principal minor of some $(n-1) \times n$-matrix $M$ in $M_{n-1,n}(\mathcal{O}_{S,p})$, in other words, $\partial_{x_i} h = \det((e_i, M))$, with $e_i$ the $i$-th standard basis column vector. The rows of $M$ define logarithmic vector fields, since $M \partial_{\mathbf{x}} h = 0$. Taking the coefficients of the Euler–vector field $\eta$ as first row an $n \times n$-matrix $\begin{pmatrix} \eta \\ M \end{pmatrix}$ is obtained. The determinant of this matrix is (cofactor expansion of the first row)

$$\sum_{i=1}^{n} \eta_i \partial_{x_i} h = \eta h = h,$$

thus by Saito's criterion the $n$ rows of this matrix form a basis of $\operatorname{Der}_{S,p}(\log D)$. If the $\partial_{x_i} h$ do not form a minimal basis of $J_h$, that is, some

$$\partial_{x_j} h \in (h, \partial_{x_1} h, \ldots, \widehat{\partial_{x_j} h}, \ldots, \partial_{x_n} h),$$

then one can apply the triviality lemma A.44. This lemma yields that $(D, p) \cong (D' \times \mathbb{C}, (p', 0))$ with $(D', p') \subseteq (\mathbb{C}^{n-1}, 0)$. Then the smooth factor of $D$ can be neglected and it is enough to consider $(D', p') \subseteq (\mathbb{C}^{n-1}, 0)$. So we are back to the case already considered. If $D$ is not Euler–homogeneous at $p$, we may suppose that the ideal defining $(\operatorname{Sing} D, p)$, namely $(h) + J_h$, is minimally generated by $h$ and $\partial_{x_1} h, \ldots, \partial_{x_n} h$. By an argument of Schaps [83, proof of Thm. 1], one finds that $h$ and its partial derivatives are the maximal minors of some $n \times (n+1)$ matrix $M$. Then the assertion follows again from Saito's criterion. □

*Remark* 2.7. The statement that any element of the matrix whose principal minors are the $\partial_{x_i} h$ can be chosen in $\mathfrak{m}$ can also be expressed with logarithmic stratifications, see [81, §3]: if some $\partial_{x_1}h, \dots \partial_{x_k}h$ are already contained in the ideal generated by $h, \partial_{x_{k+1}}h, \dots, \partial_{x_n}h$, then this means that the point $p$ is contained in a $k$-dimensional logarithmic stratum $D_\alpha$ of $D$.

*Remark* 2.8. Aleksandrov also proved the following equivalence in [4]: $D$ is free at $p$ if and only if $\operatorname{Sing} D$ is a locally determinantal variety, i.e., $(\operatorname{Sing} D, p)$ is given by the determinants of the maximal minors of a matrix with entries in $\mathcal{O}_{S,p}$. In order to prove this equivalence for the 'only if' part one uses the Hilbert–Burch matrix $M$ of the above proof and for the other implication one shows that $((h)+J_h)$ is a perfect ideal. Then by remark A.10 the ring $\mathcal{O}_{S,p}/((h)+J_h)$ is Cohen–Macaulay.
So if $D$ is free with basis $\delta_i = \sum_{j=1}^n a_{ij}\partial_{x_j}$ with $\delta_i(h) = f_i h$, $i = 1, \dots, n$ of $\operatorname{Der}_{S,p}(\log D)$, then one can explicitly write down the $(n+1)\times n$ matrix $M$: take the matrix $(a_{ij})$ and as last row $(f_1, \dots, f_n)$.

**Corollary.** *Let $D$ be a divisor in a complex manifold $S$ with $\dim S = 2$. Then $\operatorname{Sing}(D)$ is Cohen–Macaulay. In particular, any divisor in a 2-dimensional manifold is free.*

*Proof.* Locally at a point $p \in S$, the divisor $D$ is given by a reduced holomorphic $h \in \mathcal{O}_{S,p}$. If $D$ is smooth at $p$, then $\mathcal{O}_{\operatorname{Sing} D,p} = 0$ and by definition Cohen–Macaulay. If $D$ is singular at $p$, then $p$ must be an isolated singular point, which follows from $h$ reduced. Hence the Krull dimension of the ring $\mathcal{O}_{\operatorname{Sing} D,p}$ is equal to 0. Since the depth of a local ring is always less or equal than its dimension, it follows that $\operatorname{depth}(\mathcal{O}_{\operatorname{Sing} D,p}) = 0$ and thus $\dim(\mathcal{O}_{\operatorname{Sing} D,p}) = \operatorname{depth}(\mathcal{O}_{\operatorname{Sing} D,p}) = 0$, which means that the local ring $\mathcal{O}_{\operatorname{Sing} D,p}$ is Cohen–Macaulay of codimension 2. By Thm. 2.6 $D$ is free at $p$. $\square$

*Example* 2.9. (1) (Whitney Umbrella) Let $D \subseteq \mathbb{C}^3$ be given at the origin by $h = x^2 - y^2 z = 0$. The Jacobian ideal $J_{h,0}$ is $(x, yz, y^2)$. An easy computation shows that an irredundant primary decomposition of $J_{h,0}$ is $(x,y)\cap(x,y^2,z)$. Then $J_{h,0}$ has an embedded primary component and hence $\mathcal{O}_{\operatorname{Sing} D,0} = \mathbb{C}\{x,y,z\}/J_{h,0}$ is not Cohen–Macaulay. However, at any point $p = (0,0,t)$, $t \neq 0$ in the $z$-axis different from 0 the divisor $D$ is defined by $h_p = x^2 - y^2(z+t)$ and $(h_p)+J_{h,p} = (x, yz+yt, y^2) = (x,y)$. This yields that $\mathcal{O}_{\operatorname{Sing} D,p} = \mathbb{C}\{z\}$ is Cohen–Macaulay of dimension 1. Thus $D$ is free at all points $p \in S, p \neq 0$.
(2) (This example is taken from [86], where it is denoted by $F_{B,1}$) Let $(D,0) \subseteq (\mathbb{C}^3, 0)$ be the divisor defined by $h = z(x^2y^2 - 4y^3 - 4x^3z + 18xyz - 27z^2)$. The Jacobian ideal $J_h$ is of height 2 in $\mathbb{C}\{x,y,z\}$ and

$\mathcal{O}_{\mathrm{Sing}\,D,0}$ is Cohen–Macaulay (use e.g. the Auslander–Buchsbaum formula to show this). Hence $D$ is free at 0. Note that the radical of the Jacobian ideal is $(y,z)\cap(4y-x^2,z)\cap(3y-x^2,27z-x^3)$, the union of three smooth curves.
(3) The hyperplane arrangement $H$ in $\mathbb{C}^4$ given by $h = xy(z+w)(x+w)(x+w+z)$ is free, since $\mathcal{O}_{\mathrm{Sing}\,D,p}$ is Cohen–Macaulay. The singular locus of $H$ consists of 8 planes in $\mathbb{C}^4$.

## 2.3 Special cases

We start this sections with a few general remarks about radical Jacobian ideals. As explained in 2.1.1 here some special cases of the implication (2) ⇒ (1) of Theorem 2.1 are proven. Note that we do not need any requirements on the normalization of $D$ for these. First we consider a curve $D$ in a two-dimensional manifold $S$. Since then the singularities of $D$ are isolated, the proof of Thm. 2.1 is straightforward in this case. The theorem can be proved similarly if $\dim S = n \geq 2$ and $(\mathrm{Sing}\,D, p)$ is smooth at $p$. We also show a characterization of the Gorenstein case, namely, if the Jacobian ideal of $D$ at $p$ is radical then the ring $\mathcal{O}_{\mathrm{Sing}\,D,p}$ is Gorenstein of dimension $(n-2)$ if and only if $(\mathrm{Sing}\,D, p)$ is smooth. For this result we have two different proofs, the first one using Rossi's theorem and the second one using the theory of primitive ideals of Pellikaan and Siersma, see [75]. Then we turn to hyperplane arrangements and generalizations thereof.

The following lemma is nearly obvious: if two divisors $D$ and $D'$ are locally isomorphic at a point $p$, then their Tjurina algebras are locally isomorphic, that is, $\mathcal{O}_{\mathrm{Sing}\,D,p} \cong \mathcal{O}_{\mathrm{Sing}\,D',p}$.

*Remark* 2.10. Note here that the other implication does not hold in general, that is, if the singular loci of two divisors are isomorphic, the divisors themselves need not be isomorphic. However, in the case of isolated singularities this assertion is true (this is the content of the theorem of Mather–Yau [61]). The general case has been studied by Gaffney and Hauser and we refer to [39] for their results.

**Lemma 2.11.** *Let $f$ and $g$ be in $\mathcal{O}_{S,p}$ and suppose that the divisors $D = \{g = 0\}$ and $D' = \{f = 0\}$ are locally isomorphic. Then their singular loci are also isomorphic, that is,*

$$\mathcal{O}_{S,p}/((f) + J_f) \cong \mathcal{O}_{S,p}/((g) + J_g).$$

*Proof.* Let $\varphi$ be the isomorphism of $(S,p)$ sending $D'$ to $D$. Then $\varphi^*: \mathcal{O}_{S,p} \to \mathcal{O}_{S,p}$ is an algebra isomorphism sending $f$ to $\varphi^*(f) = f \circ \varphi$. We can suppose that $f \circ \varphi = g$ (otherwise $f \circ \varphi = ug$, with $u \in \mathcal{O}^*_{S,p}$, but their Jacobian ideals are the same: $((g)+J_g) = ((ug)+J_{ug})$). With the chain rule follows

$$\frac{\partial g}{\partial x_i} = \sum_{j=1}^{n} (\frac{\partial f}{\partial x_j} \circ \varphi) \frac{\partial \varphi_j}{\partial x_i} = \sum_{j=1}^{n} \varphi^*(\frac{\partial f}{\partial x_j}) \frac{\partial \varphi_j}{\partial x_i}.$$

Thus $J_g$ is contained in $\varphi^*(J_f)$. Since $\varphi^*$ is an isomorphism we also get $J_f \subseteq (\varphi^*)^{-1}(J_g)$. From this and $\varphi^*(f) = g$ follows $\varphi^*(J_f + (f)) \subseteq ((g) + J_g)$ and by symmetry we get $\varphi^*(J_f + (f)) = ((g) + J_g)$. □

In Proposition 2.13 it is shown that for radical Jacobian ideals the local ring $\mathcal{O}_{\mathrm{Sing}\,D,p}$ is already determined by the Jacobian ideal $J_h$, that is, $h \in J_h$. In particular, this implies that a divisor with radical Jacobian ideal is Euler-homogeneous.

**Lemma 2.12.** *Let $(S,D)$ be a pair of an $n$-dimensional complex manifold $S$ together with a divisor $D \subseteq S$ and let $D$ be defined at the point $p = (x_1, \ldots, x_n)$ by $h(x) \in \mathcal{O}_{S,p}$. Let $J_h \subseteq \mathcal{O}_{S,p}$ be the Jacobian ideal of $D$. Then $h$ belongs to the integral closure $\overline{J_h}$ of $J_h$.*

*Proof.* We show the statement with the complex-analytic criterion for integral dependence, see [59, (1.3)] or [11]: an element $f$ of $\mathcal{O}_{S,p}$ is in the integral closure of the ideal $I = (g_1, \ldots, g_m)$ if and only if for all analytic germs $\gamma : (\mathbb{C}, 0) \to (S,p)$ one has

$$(f \circ \gamma) \in \gamma^* I, \text{ where } \gamma^* I = (g_1 \circ \gamma, \ldots, g_m \circ \gamma) \subseteq \mathbb{C}\{t\}.$$

In our case we have to show that $h(\gamma(t)) \in (\partial_{x_1} h(\gamma(t)), \ldots, \partial_{x_n} h(\gamma(t)))$. Set $\mathrm{ord}(h(\gamma(t)) = k(\gamma)$. Then $\mathrm{ord}(\partial_t(h(\gamma(t)))) = k(\gamma) - 1$. Using the chain rule it follows that

$$\mathrm{ord}(\partial_t(h(\gamma(t))) = \mathrm{ord}(\sum_{i=1}^{n} \partial_t \gamma_i(t) \cdot \partial_{x_i} h \circ \gamma(t)) = k(\gamma) - 1.$$

Thus there exists an $i$ such that $\mathrm{ord}(\partial_{x_i} h \circ \gamma(t)) \leq k(\gamma) - 1$. This implies that $h(\gamma(t)) \in (\partial_{x_i} h \circ \gamma(t))$. □

**Proposition 2.13.** *Let $(S,D)$ be as in the previous lemma. If $J_h$ is radical, then $h \in J_h$, which implies $\mathcal{O}_{\mathrm{Sing}\,D,p} = \mathcal{O}_{S,p}/J_h$.*

*Proof.* Since $J_h$ is generated by $n$ elements, it follows from the theorem of Briançon–Skoda that $\overline{J_h^n} \subseteq J_h$, see [59]. Since $(\overline{J_h})^n \subseteq \overline{J_h^n}$ (see for example [57]), the $n$-th power of $h$ is contained in $J_h$ and since $J_h$ is radical it already contains $h$. □

*Remark* 2.14. The above proposition shows in particular that if $J_h$ is radical then also $J_h = \overline{J_h}$. The blowup of $D \subseteq S$ with center $\overline{J_h}$ is the Nash blowup of $(D,p)$, see e.g. [72]. It is an interesting question whether in the case of a radical Jacobian ideal this blowup is equal to the *normalized Nash blowup* (for details and notation see [57, Section 3]): the normalized Nash blowup of $D$ is the Nash blowup followed by normalization and determined by Projan $\bigoplus_{n\in\mathbb{N}} \overline{\mathcal{J}^n}$, where $\overline{\mathcal{J}}$ denotes the integral closure of the Jacobian ideal sheaf in $\mathcal{O}_S$. In order to obtain equalities of the two blowups it is necessary and sufficient that $\overline{\mathcal{J}^n} = (\overline{\mathcal{J}})^n$ for $n$ big enough. However, it is not known whether $\overline{J_h^n} = (\overline{J_h})^n$ if $J_h = \sqrt{J_h}$.

**Proposition 2.15.** *Let* $\dim S = 2$ *and the divisor* $D$ *be defined at a point* $p$ *by a reduced* $h \in \mathcal{O}_{S,p}$. *Then* $D$ *has normal crossings at* $p$ *if and only if* $D$ *is free at* $p$ *and* $J_h$ *is radical of depth 2 on* $\mathcal{O}_{S,p}$.

*Proof.* If $J_h$ has depth 2 on a two-dimensional regular local ring, then the singularity of $D$ at $p$ is isolated. Since $J_h$ is radical, it has an irredundant primary decomposition $\bigcap \mathfrak{p}$, with prime ideals $\mathfrak{p}$ that are all of height 2 (by the Cohen–Macaulay property of $\mathcal{O}_{S,p}/J_h$). If one $\mathfrak{p}$ were not equal to the maximal ideal $\mathfrak{m}$, then it would be strictly contained in $\mathfrak{m}$. However, then the height of $\mathfrak{m}$ would be greater than or equal to 3, which is a contradiction to $\text{height}(\mathfrak{m}) = \dim \mathcal{O}_{S,p} = 2$. This means that $\mathcal{O}_{\text{Sing}\, D,p} = \mathcal{O}_{S,p}/\mathfrak{m} \cong \mathbb{C}$ at $p$. Now one can use either a direct computation (see Remark 2.16) or apply the theorem of Mather–Yau [61] (also see [27]) for isolated singularities: two germs $(X,p)$ and $(Y,q)$ in $S$ with isolated singularity at $p$ resp. $q$ are isomorphic if and only if $\mathcal{O}_{\text{Sing}\, X,p}$ and $\mathcal{O}_{\text{Sing}\, Y,q}$ are isomorphic as local algebras. In our case let $(D,p)$ the germ of $D$ at $p$. The theorem of Mather–Yau means that $(D,p)$ is isomorphic to the normal crossings divisor $(N,p)$ (defined locally at $p = (x_1,x_2)$ by the equation $\{x_1x_2 = 0\}$) if and only if their singular loci are isomorphic. But $\mathcal{O}_{\text{Sing}\, N,p} = \mathcal{O}_{S,p}/(x_1,x_2) \cong \mathbb{C}$ is clearly isomorphic to $\mathcal{O}_{\text{Sing}\, D,p}$. The application of the theorem of Mather–Yau proves the proposition. □

*Remark* 2.16. Also a direct computation can be used to prove the previous proposition: therefore write $h = a_1x_1^2 + a_2x_1x_2 + a_3x_2^2$ for some

$a_i \in \mathbb{C}\{x_1, x_2\}$. If $J_h = (x_1, x_2)$, it follows that one of the $a_i$ is invertible in $\mathbb{C}\{x_1, x_2\}$, w.l.o.g. $a_1$ (possibly after a linear change of coordinates). Hence we may assume that $a_1 = 1$ and $h = x_1^2 + a_2 x_1 x_2 + a_3 x_2^2$. Consider the change of coordinates $\varphi(x_1, x_2) = (x_1 - \frac{a_2 x_2}{2}, x_2)$, which transforms $h$ into $h^* := \varphi^*(h) = x_1^2 - \tilde{a}_3 x_2^2$ with $\tilde{a}_3 = a_3 - 1/4a_2^2$. By Lemma 2.11 one has $J_h = J_{h^*}$, which implies that $\tilde{a}_3$ is also invertible and hence $\sqrt{\tilde{a}_3} \in \mathbb{C}\{x_1, x_2\}$. Therefore $h^* = (x_1 + \sqrt{\tilde{a}_3} x_2)(x_1 - \sqrt{\tilde{a}_3} x_2)$ defines a normal crossing divisor.
In Chapter 3 we will obtain yet another proof of the preceding proposition via mikado curves.

**Lemma 2.17.** *Let $S$ be a complex manifold of dimension $n$ together with a divisor $D \subseteq S$, which is defined at a point $p \in S$ by $h \in \mathcal{O}_{S,p}$. Suppose that $D$ is free at $p$ and that the Jacobian ideal $J_h$ is radical. Further suppose that $p$ is a non-singular point of* $\operatorname{Sing} D$. *Then locally at $p$ the divisor $D$ has normal crossings, more precisely, it is locally isomorphic to the union of two transversally intersecting hyperplanes.*

*Proof.* Let $(x_1, \ldots, x_n)$ be complex coordinates of $S$ around $p$. By Proposition 2.13 the defining ideal of $(\operatorname{Sing} D, p)$ is $J_h$. Since $(\operatorname{Sing} D, p)$ is smooth and of codimension 2 in $(S, p)$, wlog. $J_h = (x_1, x_2)$ can be assumed. We use the analytic triviality criterion A.44 to show that locally at $p$ the divisor $D$ is trivial along the subspace $\{x_3 = 0, \ldots, x_n = 0\}$, such that the defining equation $h$ can be chosen depending only on $x_1, x_2$. Therefore it must be shown that for all $3 \le i \le n$ one has $\partial_{x_i} h \in \mathfrak{m}(x_1, x_2)$ and that $(\partial_{x_1} h, \partial_{x_2} h) = (x_1, x_2)$: since $h$ is contained in $J_h$, it can be written as $h = f x_1 + g x_2$ for some $f, g \in \mathcal{O}_{S,p}$. Then taking the partial derivative $\partial_{x_1} h$ it follows that $f = \partial_{x_1} h - x_1 \partial_{x_1} f - x_2 \partial_{x_1} g$ is also contained in $J_h = (x_1, x_2)$. Taking the partial derivative $\partial_{x_2} h$ yields that $g$ is also contained in $J_h$. But then $h \in (x_1, x_2)^2$ and it is of the form $h = a x_1^2 + b x_1 x_2 + c x_2^2$ for some $a, b, c \in \mathcal{O}_{S,p}$. Then the partial derivative $\partial_{x_i} h$ for $3 \le i \le n$ is

$$\partial_{x_i} h = (\partial_{x_i} a) x_1^2 + (\partial_{x_i} b) x_1 x_2 + (\partial_{x_i} c) x_2^2 \in \mathfrak{m}(x_1, x_2).$$

With Nakayama's lemma, applied to the $\mathcal{O}_{S,p}/\mathfrak{m} = \mathbb{C}$-vector space $J_h/\mathfrak{m}J_h$, it follows that $J_h$ is minimally generated by $\partial_{x_1} h, \partial_{x_2} h$. The triviality lemma implies that one can find locally at $p$ a biholomorphic map $\varphi$ such that $h \circ \varphi(x_1, \ldots, x_n) = h(x_1, x_2, 0, \ldots, 0)$ defines a divisor isomorphic to $D$ and the germ $(D, p)$ is locally isomorphic to some $(D' \times \mathbb{C}^{n-2}, (0, 0))$, where $D' = \{h \circ \varphi(x_1, \ldots, x_n) = 0\}$. Hence we can consider the problem in dimension 2 and $p$ with coordinates $(x_1, x_2)$.

Now Proposition 2.15 tells us that $D' = \{h \circ \varphi = 0\}$ has normal crossings at $p$, that is, one can find coordinates $(y_1, \ldots, y_n)$ at $p$ such that $h = h(y_1, y_2, \ldots, y_n) = y_1 y_2$. $\square$

### 2.3.1 Gorenstein singularities

A particular class of Cohen–Macaulay rings are the so-called Gorenstein rings. We prove here Thm. 2.1 for $\mathcal{O}_{\mathrm{Sing}\, D,p}$ Gorenstein of dimension $(n-2)$. In general, Gorenstein rings lie between complete intersections and Cohen–Macaulay rings. However, in our situation, where the Jacobian ideal defining $\mathcal{O}_{\mathrm{Sing}\, D,p}$ has depth two on $\mathcal{O}_{S,p}$, one sees that Gorenstein rings are complete intersection rings, that is, the Jacobian ideal can be minimally generated by two elements. Then we can generalize the methods from the preceding section to show the following:

**Proposition 2.18.** *Let $(S, D)$ be the pair of an $n$-dimensional complex manifold together with a divisor $D \subseteq S$ and $D = \{h = 0\}$ at a point $p$. Suppose that $J_h$ is radical and $\mathcal{O}_{\mathrm{Sing}\, D,p}$ is a Gorenstein ring of Krull-dimension $n-2$. Then $(\mathrm{Sing}\, D, p)$ is smooth and $D$ has locally at $p$ normal crossings.*

First let us consider a possible counter-example to this proposition:

*Example* 2.19. (The cusp) The (reduced) cusp in $(\mathbb{C}^3, 0)$ cannot be the singular locus of a divisor $(D, 0)$: the cusp is defined by $I = (x_1^3 - x_2^2, x_3)$. By Serre's theorem (Thm. 2.21) below $\mathcal{O}/I$ is Gorenstein but clearly not regular. In order that $I$ equals $J_h$ for some $h \in \mathcal{O}$ one must have $\partial_{x_i} h = a_{i1}(x_1^3 - x_2^2) + a_{i2} x_3$, for $i = 1, 2, 3$. Now consider the $\mathbb{C}$-vector space $I/\mathfrak{m}I$. Since $\mathcal{O}$ is a local ring, Nakayama's lemma yields that $\overline{x_1^3 - x_2^2}, \overline{x_3}$ form a basis of this vector space. From the Poincaré lemma (see Lemma 3.5) it follows that the three functions $f_1, f_2, f_3$ are partial derivatives $\partial_{x_1} h, \partial_{x_2} h, \partial_{x_3} h$ if and only if $\partial_{x_2} f_1 = \partial_{x_1} f_2, \partial_{x_1} f_3 = \partial_{x_3} f_1, \partial_{x_3} f_2 = \partial_{x_2} f_3$. Writing out these conditions for the three functions $a_{i1}(x_1^3 - x_2^2) + a_{i2} x_3$ it follows that $a_{11}(0) = a_{21}(0) = a_{12}(0) = a_{22}(0) = 0$. Hence modulo $\mathfrak{m}$ the system of equations for the $\partial_{x_i} h$ looks as follows:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ a_{31}(0) & a_{32}(0) \end{pmatrix} (\overline{x_1^3 - x_2^2}, \overline{x_3})^T = (\overline{\partial_{x_1} h}, \overline{\partial_{x_2} h}, \overline{\partial_{x_3} h})^T.$$

But this contradicts the fact that the $\partial_{x_i} h$ generate $I$. Hence $I$ cannot be the Jacobian ideal $J_h$ of some reduced $h$.

We need some terminology concerning Gorenstein rings. Good references for the use and properties of Gorenstein rings are [8, 32, 56].

**Definition 2.20.** Let $R$ be a zero-dimensional local ring. Then $R$ is said to be *Gorenstein* if $R$ is injective as an $R$-module. A local ring $(R, \mathfrak{m})$ of depth $R = d$ is Gorenstein if for some maximal regular sequence $x_1, \ldots, x_d \in \mathfrak{m}$ the ring $R/(x_1, \ldots, x_d)$ is Gorenstein.

**Theorem 2.21** (Serre)**.** *Let $R$ be a regular local ring and $I \subseteq R$ an ideal with* $\operatorname{depth}(I, R) = 2$. *Then $R/I$ is Gorenstein if and only if $I$ is generated by a regular sequence of length 2.*

*Proof.* See [32, Cor. 21.20]. □

**Lemma 2.22.** *Let $(S, D)$ be as before, with* $\dim S = n$ *and $D = \{h = 0\}$ at a point $p = (x_1, \ldots, x_n)$. Suppose that the Jacobian ideal $J_h = (\partial_{x_1} h, \ldots, \partial_{x_n} h)$ is radical and $\mathcal{O}_{\operatorname{Sing} D, p}$ is Gorenstein of dimension $(n-2)$. Then $J_h$ can be generated by two derivatives $\partial_{x_i} h, \partial_{x_j} h$.*

*Proof.* Since $\mathcal{O}_{S,p}/J_h$ is Gorenstein, Thm. 2.21 yields that $J_h$ is generated by a regular sequence $f, g$ in $\mathfrak{m}$. Then there exists an $(n \times 2)$-matrix $A \in M_{n,2}(\mathcal{O}_{S,p})$ such that

$$A(f, g)^T = (\partial_{x_1} h, \ldots, \partial_{x_n} h)^T.$$

Consider the $\mathcal{O}_{S,p}/\mathfrak{m}$-module $J_h/\mathfrak{m}J_h$. The above equation reads as

$$\overline{A}(\overline{f}, \overline{g})^T = (\overline{\partial_{x_1} h}, \ldots, \overline{\partial_{x_n} h})^T$$

with $\overline{A} \in M_{n,2}(\mathbb{C})$, since $\mathcal{O}_{S,p}/\mathfrak{m} = \mathbb{C}$. Then we have a solvable linear system of equations with coefficients in $\mathbb{C}$. Thus $\overline{A}$ must have rank 2, that is, it has two linearly independent rows. Suppose that the first two rows are linearly independent. Then they can be transformed into the identity matrix $Id_2$ by elementary row operations and the other rows can be made equal to zero. Thus $f$ and $g$ are $\mathbb{C}$-linear combinations of $\partial_{x_1} h$ and $\partial_{x_2} h$ modulo $\mathfrak{m}J_h$, that is, $\partial_{x_1} h$ and $\partial_{x_2} h$ generate $J_h/\mathfrak{m}J_h$. This means $J_h = (\partial_{x_1} h, \partial_{x_2} h) + \mathfrak{m}J_h$ (as $\mathcal{O}_{S,p}$-modules). Applying Nakayama's lemma to the local ring $(\mathcal{O}_{S,p}, \mathfrak{m})$ yields $J_h = (\partial_{x_1} h, \partial_{x_2} h)$. □

*First proof of Prop. 2.18.* From Lemma 2.22 it follows that $J_h$ can be generated by two derivatives of $h$, wlog. $J_h = (\partial_{x_1} h, \partial_{x_2} h)$. Hence one has $\partial_{x_i} h = a_i \partial_{x_1}(h) + b_i \partial_{x_2}(h)$, $a_i, b_i \in \mathcal{O}_{S,p}$, for $3 \leq i \leq n$. Consider vector fields $\delta_i = \partial_{x_i} - a_i\partial_{x_1} - b_i\partial_{x_2}$ for $3 \leq i \leq n$. Since $\delta_i(h) = 0$, it follows that $\delta_i \in \mathrm{Der}_{S,p}(\log D)$. Evaluation of these $n-2$ vector fields at 0 shows that $\delta_3(0), \dots, \delta_n(0)$ are $\mathbb{C}$-linearly independent vectors in $(S,p) \cong (\mathbb{C}^n, 0)$. Thus Rossi's theorem can be applied (see [77]): locally at $p$ the germ $(D,p)$ is isomorphic to $(D' \times \mathbb{C}^{n-2}, (0,0))$, where $D'$ is locally contained in $\mathbb{C}^2$. Hence the problem has been reduced to $\dim_{\mathbb{C}} S = 2$. Then Prop. 2.15 shows that locally at $p$ the divisor $D$ is isomorphic to the union of two transversally intersecting hyperplanes. Hence the proof is complete. □

*Remark* 2.23. Instead of using Rossi's theorem in the above proof, we could use the argument in Lemma 2.3 of [22] and apply induction.

For the second proof we use the notion of primitive ideal: it was introduced by Pellikaan and Siersma (see [75] and references therein) in order to study analytic functions with given singular locus of dimension greater than 0.

**Definition 2.24.** Let $\mathcal{O}_{S,p}$ be the local ring at a point $p$ in a complex manifold $S$ of dimension $n$, $f \in \mathcal{O}_{S,p}$ define a divisor $\{f = 0\}$ and $I \subseteq \mathcal{O}_{S,p}$ an ideal. Denote by $J_f = (\partial_{x_1} f, \dots, \partial_{x_n} f)$ the Jacobian ideal of $f$. The *primitive ideal* $\int I$ of $I$ in $\mathcal{O}_{S,p}$ is defined as

$$\int I = \{f \in \mathcal{O}_{S,p} : (f) + J_f \subseteq I\}.$$

The primitive ideal $\int I$ is again an ideal: if $f, g \in \int I$ then the ideal $(f+g) + J_{f+g}$ is contained in $(f+g) + J_f + J_g \subseteq I$ and if $f$ is in $\int I$ and $g \in \mathcal{O}_{S,p}$ then $(fg) + J_{fg} \subseteq (fg) + fJ_g + gJ_f \subseteq f + J_f \subseteq I$.
One has the inclusion of ideals $I^2 \subseteq \int I \subseteq I$. In general, $\int I$ is hard to determine but if $I$ is radical, then it is characterized by (cf. [75, Prop. 1.6.]):

$$\int I = I^{(2)}.$$

Here $I^{(k)}$ denotes the $k$-th symbolic power of $I$: for a prime ideal $\mathfrak{p}$, the $k$-th symbolic power is defined as $\mathfrak{p}^{(k)} := \mathcal{O} \cap (\mathfrak{p}^k \mathcal{O}_{\mathfrak{p}})$ and for a radical ideal $I$ as $I^{(k)} = \mathfrak{p}_1^{(k)} \cap \dots \cap \mathfrak{p}_m^{(k)}$, where $I = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_m$ is the irredundant prime decomposition of $I$. In the case of Gorenstein singularities the primitive ideal can be described quite explicitly (also see [75]):

**Proposition 2.25.** *Let $I$ be a radical ideal in $\mathcal{O}_{S,p}$ that defines a Gorenstein singularity $(X,p)$, that is, $\mathcal{O}_{X,p} := \mathcal{O}_{S,p}/I$ is a Gorenstein ring, and suppose that* $\mathrm{depth}(I,\mathcal{O}) = 2$. *Then $\int I = I^2$ holds.*

*Proof.* By definition of $\int I$, the inclusion $I^2 \subseteq \int I$ always holds. Since $I$ has height 2 and $\mathcal{O}/I$ is Gorenstein, by Serre's theorem, $I$ defines a complete intersection. Thus one can assume that $I$ is generated by a regular sequence $g_1, g_2 \in \mathcal{O}$. Let now $f$ be an element of $\int I$, then $f$ is clearly also contained in $I$. Hence $f = a_1 g_1 + a_2 g_2$ for some $a_i \in \mathcal{O}$. By definition of the primitive ideal, $J_f$ is also contained in $I$, which means that $\partial_{x_j} f \in I$ for all $j = 1, \dots, n$. Differentiating $f$ yields

$$\partial_{x_j} f = (\partial_{x_j} a_1) g_1 + (\partial_{x_j} a_2) g_2 + a_1(\partial_{x_j} g_1) + a_2(\partial_{x_j} g_2),$$

hence $a_1(\partial_{x_j} g_1) + a_2(\partial_{x_j} g_2) \in I$. Denote $\overline{a} = (a_1 \mod I, a_2 \mod I)$ and consider the exact sequence

$$\mathcal{O}^2_{X,p} \xrightarrow{dg} \mathcal{O}^n_{X,p} \longrightarrow \Omega_{X,p} \longrightarrow 0\ ,$$

where $dg : (a,b) \mapsto (a(\partial_{x_1} g_1) + b(\partial_{x_1} g_2), \dots, a(\partial_{x_n} g_1) + b(\partial_{x_n} g_2))^T$. One sees that $\overline{a}$ is contained in $\ker(dg)$. But by [60, 6.B] $dg$ is injective and hence $a_1, a_2$ have to be contained in $I$. This implies $f \in I^2$. □

*Remark* 2.26. The preceding proposition can be generalized in two ways: first if $I$ is radical of arbitrary height $\leq n$ and defines a complete intersection, then with the analogous proof one can show that $I^2 = \int I$. Second if $I$ is radical of depth $\leq 3$ and $\mathcal{O}/I$ is Gorenstein then, with an argument in [51] one can also show that $I^2 = \int I$.

**Lemma 2.27.** *Let $f \in \mathcal{O}_{S,p}$ be a non-unit and $(x_1, \dots, x_n)$ complex coordinates around $p$.*
*(i) $\partial_{x_s x_t} f \notin \mathfrak{m}$ if and only if in the equation of $f$ the monomial $x_s x_t$ has a non-zero coefficient. (ii) $f$ is of the form (possibly after a linear change of coordinates)*

$$f = \sum_{i=1}^n a_i x_i + \sum_{i=1}^n b_i x_i^2 + \tilde{f},$$

*where $a_i, b_i \in \mathbb{C}$ and $\tilde{f} \in \mathfrak{m}^3$, that is, $f$ contains no mixed quadratic terms.*

*Proof.* Straightforward computation. □

*Second Proof of Prop. 2.18.* The Jacobian ideal $J_h = (\partial_{x_1} h, \ldots, \partial_{x_n} h)$ in $\mathcal{O}_{S,p}$ is radical, of depth 2 on $\mathcal{O}_{S,p}$ and $\mathcal{O}_{\operatorname{Sing} D,p}$ is Gorenstein. We show that $(D,p)$ is analytically trivial near $p$ along the subspace $\{x_3 = \cdots = x_n = 0\}$ and thus reduce the problem to dimension 2. For this the triviality criterion A.44 is used: it must be shown that $\partial_{x_i} h \in \mathfrak{m} J_h$ for $i \geq 3$. By Lemma 2.27 we may suppose that $h$ has no mixed quadratic terms, that is, $\partial_{x_i x_j} h \in \mathfrak{m}$. From Lemma 2.22 it follows that $J_h$ can be generated by two derivatives, wlog. $J_h = (\partial_{x_1} h, \partial_{x_2} h)$. Since $h \in \int J_h$, Prop. 2.25 implies that $h$ is contained in $J_h^2$. Thus it is of the form

$$h = a(\partial_{x_1} h)^2 + b(\partial_{x_1} h)(\partial_{x_2} h) + c(\partial_{x_2} h)^2,$$

with $a, b, c \in \mathcal{O}_{S,p}$. But then for all $i \geq 3$, it follows that

$$\begin{aligned}\partial_{x_i} h =& \partial_{x_i} a(\partial_{x_1} h)^2 + 2a(\partial_{x_1} h)(\partial_{x_1 x_i} h) + (\partial_{x_i} b)(\partial_{x_1} h)(\partial_{x_2} h) + \\ & b(\partial_{x_1 x_i} h)(\partial_{x_2} h) + b(\partial_{x_1} h)(\partial_{x_2 x_i} h) + (\partial_{x_i} c)(\partial_{x_2} h)^2 + 2c(\partial_{x_2} h)(\partial_{x_2 x_i} h).\end{aligned}$$

Noting that $J_h^2 \subseteq \mathfrak{m} J_h$, it is easily seen that each summand of $\partial_{x_i} h$ is in $\mathfrak{m} J_h$ and thus $\partial_{x_i} h \subseteq \mathfrak{m} J_h$. This shows that $D$ is trivial along $\{x_3 = \cdots = x_n = 0\}$, which implies that locally at $p$ one can find coordinates $(y_1, \ldots, y_n)$ such that $h(y_1, \ldots, y_n) = h(y_1, y_2, 0, \ldots, 0)$. Thus the problem has been reduced to $\dim S = 2$. Then Prop. 2.15 shows that locally at $p$ the divisor $D$ is isomorphic to the union of two transversally intersecting hyperplanes. □

Unfortunately not all Cohen–Macaulay rings are Gorenstein. One could try to construct a hypersurface, whose singular locus $\mathcal{O}_{\operatorname{Sing} D}$ is reduced of dimension $(n-2)$ but locally not a complete intersection Cohen–Macaulay ring. In concrete examples one sees that this will not be the case:

*Example* 2.28. (The singular cubic space curve) One of the first non-trivial examples for this situation would be a surface in $\mathbb{C}^3$, whose singular locus is a singular Cohen–Macaulay curve, whose ideal can minimally generated by more than two elements. A classical example for such a curve is the *singular cubic space curve.* Recall, that the singular cubic is given by the ideal $I = (x^3 - yz, y^2 - xz, z^2 - x^2 y)$ in $\mathcal{O} = \mathbb{C}\{x, y, z\}$. Its coordinate ring $\mathcal{O}/I$ is Cohen–Macaulay but ideal-theoretically $I$ is not a complete intersection. We show that the ideal $I$ defining the twisted cubic space curve cannot be the Jacobian ideal of a surface $D$ in $(\mathbb{C}^3, 0)$.
Direct computation, using an analogous argument as in Example 2.19,

shows that $f_1 = x^3 - yz, f_2 = y^2 - xz, f_3 = z^2 - x^2y$ cannot be partial derivatives of an $h \in \mathcal{O}$: if there exists an $h$ such that $I = (\partial_x h, \partial_y h, \partial_z h)$ then

$$(\partial_x h, \partial_y h, \partial_z h)^T = A\underline{f}^T,$$

where $A$ is a $3 \times 3$ matrix with entries in $\mathcal{O}$ and $\underline{f}$ denotes the vector $(f_1, f_2, f_3)$. By Nakayama's lemma $A$ is even contained in $GL_3(\mathcal{O})$, which implies that $A(0)$ (the evaluation of the matrix $A$ at 0) is in $GL_3(\mathbb{C})$. Using the three necessary and sufficient conditions $\partial_{xy}h = \partial_{yx}h, \partial_{xz}h = \partial_{zx}h, \partial_{yz}h = \partial_{zy}h$ (see Lemma 3.5) one finds that the constant terms of the entries $a_{12}, a_{22}, a_{32} \in \mathcal{O}$ of $A$ are equal to zero. Hence $A(0)$ cannot be invertible and there does not exist an $h$ such that $I = J_h$.

### 2.3.2 Hyperplane arrangements

Hyperplane arrangements are finite unions of hyperplanes in a vector space. They can be described by combinatorial means by their so-called intersection lattice and are object of study in many fields of mathematics. For an introduction to hyperplane arrangements see e.g. [74, 91]. For hyperplane arrangements one can often find formulas to explicitly compute singularity invariants, like multiplier ideals, zeta-functions or $b$-functions, see e.g. [16] and references therein. Some of these invariants are even combinatorial, that is, they only depend on the lattice associated to the arrangement. An open question in this context is if the freeness of an hyperplane arrangement is a combinatorial property, see [85]. *Free* arrangements were first studied by Terao [94], where he also proved the Cohen–Macaulayness of the Jacobian ideal of a free hyperplane arrangement. Wakefield and Yoshinaga [99] have proved that a central hyperplane arrangement can be reconstructed from its Jacobian ideal.

Here we prove Theorem 2.1 for hyperplane arrangements and a slight generalization thereof. First some terminology: A *hyperplane arrangement* $D$ is a finite collection of affine hyperplanes in an $n$-dimensional vector space $V$ over a field $k$. When each hyperplane contains the origin, one speaks of a *central* arrangement. One fixes affine coordinates $(x_1, \ldots, x_n)$ for $V^*$, where $V^*$ denotes the dual vector space to $V$. Then one considers $S := \mathrm{Sym}(V^*) \cong k[x_1, \ldots, x_n]$. The hyperplane arrangement $D = \bigcup_{i=1}^m H_i$ is defined by a (reduced) equation $\{\prod_{i=1}^m l_i = 0\}$

where each $l_i$ is a polynomial of degree 1 in $k[x_1, \ldots, x_n]$ and corresponds to the hyperplane $H_i$.
Logarithmic differential forms, freeness, etc. are defined according to the general case, which we have already presented in Chapter 1.

**Proposition 2.29.** *Let $D$ be a central hyperplane arrangement in $\mathbb{C}^n$, defined by the reduced equation $h = h_1 \ldots h_m$ where each $h_i$ is a homogeneous polynomial of degree 1 in $k[x_1, \ldots, x_n]$. Let $J = (\partial_{x_1} h, \ldots, \partial_{x_n} h)$ be the Jacobian ideal of $h$. Then $\mathcal{O}_{\mathrm{Sing}\, D} = k[x_1, \ldots, x_n]/J$ is Cohen–Macaulay, $J$ has depth 2 on $k[x_1, \ldots, x_n]$ and is radical if and only if $D$ is a normal crossings arrangement.*

*Remark* 2.30. Note that Aleksandrov's theorem also holds in the affine case, see [4]. Also by Terao's result [94] the Jacobian ideal of a free hyperplane arrangement is Cohen–Macaulay.

*Proof.* Since the statement is local, we may assume that $D$ is defined at a point $p$ by some $h = h_1 \cdots h_m$, each $h_i$ linear and corresponding to $D_i$. The number $m$ depends on the chosen point $p$. By definition the components of $D$ are smooth and any two of them meet transversally. In order to apply Thm. 1.42 we only have to show that the dimension of an intersection $D_i \cap D_j \cap D_k$ is less than or equal to $n-3$ for $i \neq j \neq k$ and $i, j, k \in \{1, \ldots, m\}$. Let $\mathrm{Sing}\, D = \bigcup_{i=1}^l C_i$ be the decomposition of $\mathrm{Sing}\, D$ into irreducible components $C_i$, where each $C_i$ is defined by a prime ideal $\mathfrak{p}_i$ of depth 2 on $\mathcal{O}_{S,p}$. Suppose that $C_1$ were the intersection of $k \geq 3$ hyperplanes. Since $\dim(C_i \cap C_j) \leq n-3$ for all $i \neq j$ one can find a point $p$ in $C_1 \backslash \bigcup_{i=2}^l C_i$. Since $C_1$ is the intersection of linear subspaces of $\mathbb{C}^n$, it is again a linear subspace and hence smooth. Thus w.l.o.g. at $p$, we can choose $\mathfrak{p}_1 = (x_1, x_2)$ to be the defining ideal of $C_1$, where $(x_1, \ldots, x_n)$ are the affine coordinates of $\mathbb{C}^n$ at $p$. We can also assume that $h_1 = x_1$, $h_2 = x_2$ and $h_i = a_i x_1 + b_i x_2$ with $a_i, b_i \neq 0$. The defining ideal of $\mathrm{Sing}\, D$ at $p$ is

$$\begin{aligned} J =&(x_2 h_3 \cdots h_m + a_3 x_1 x_2 h_4 \cdots h_m + \ldots + a_m x_1 x_2 h_3 \cdots h_{m-1}, \\ &x_1 h_3 \cdots h_m + \ldots + b_m x_1 x_2 h_3 \cdots h_{m-1}). \end{aligned}$$

Clearly $J \subseteq (x_1, x_2)^{m-1} \subsetneq (x_1, x_2)$, which implies that $J$ is not radical at $p$. Contradiction.
Now all conditions of Thm. 1.42 (iv) are satisfied, that is, the components $D_i$ of $D$ are smooth (and thus normal), $D_i$ and $D_j$ intersect transversally and $\dim(D_i \cap D_j \cap D_k) \leq n-3$. Thus, by this theorem,

$\Omega^1_{\mathbb{C}^n,p}(\log D)$ is generated by closed forms for all $p$. Since by assumption $\mathcal{O}_{\operatorname{Sing} D,p}$ is Cohen–Macaulay and $\operatorname{depth}(J, \mathcal{O}_{S,p}) = 2$, Thm. 2.6 ensures that $D$ is a free divisor. Note that by Lemma 1.53 one can find a basis of $\Omega^1_{\mathbb{C}^n,0}$ consisting of closed forms. Hence by Thm. 1.52 these conditions imply that $D$ has normal crossings. □

*Remark* 2.31. Splayed divisors provide an alternative proof of this result, see the second corollary to Prop. 2.48.

As a generalization we can prove Theorem 2.1 for a divisor $D$ that is locally the union of normal divisors with essentially the same method as in the hyperplane arrangement case.

**Proposition 2.32.** *Let $D$ be a divisor in $S$, $\dim S = n$, that has locally at a point $p$ irreducible components $(D_1, p) \cup \ldots \cup (D_m, p)$ such that each $D_i$ is normal. If $D$ is a free divisor with radical Jacobian ideal then $D$ has normal crossings at $p$.*

*Proof.* Another application of Theorems 1.42 and 1.52 proves the assertion. For Thm. 1.42 (iv) it only remains to show that any two components $D_i$, $D_j$ intersect transversally outside an $(n-3)$-dimensional closed analytic subset and that $\dim(D_i \cap D_j \cap D_k) \leq n-3$ for different $i, j, k$. Denote by $D_i \cap D_j = C_{ij}$ the $(n-2)$-dimensional intersection of $D_i$ and $D_j$. Since the Jacobian ideal is radical and of depth 2, $C_{ij}$ is a union of irreducible $(n-2)$-dimensional irreducible components and we can find a smooth point $q$ near $p$ on $C_{ij}$. Moreover, the smooth points on $C_{ij}$ form an open dense subset of dimension $n-2$. If $D_i$ and $D_j$ meet tangentially at $q$ then we may assume that $D_i = \{x_1 = 0\}$ and $D_j = \{x_2^m - x_1 = 0\}$. Then the Jacobian ideal is $J_{h,q} = (x_2^m - 2x_1, x_1 x_2^{m-1})$, which is clearly not radical. Contradiction. If $\dim(D_i \cap D_j \cap D_k) = n-2$ then $D_i \cap D_j \cap D_k = C_{ijk}$ would be a union of irreducible components of $\operatorname{Sing} D$. Again, we can find a smooth point $q \in C_{ijk}$ near $p$ where wlog. $J_{h,q} = (x_1, x_2)$. But then we are in the 2-dimensional case and Lemma 2.17 or alternatively Prop. 2.15 shows a contradiction. Hence all conditions of Theorem 1.42 (iv) are satisfied and the rest of our argument is the same as in the hyperplane arrangement case: one can find a basis of closed forms of $\Omega^1_{S,p}(\log D)$ and by Thm. 1.52 $D$ has normal crossings at $p$. □

## 2.4 The general case of Thm. 2.1

We are now approaching the proof of the general case of the implication (2) $\Rightarrow$ (1) of Theorem 2.1. Therefore we reduce the problem in this section to the case of an irreducible divisor, which will be treated similarly like the results on the logarithmic residue from Chapter 1. The goal is to show that if a reducible divisor is free and has radical Jacobian ideal then already each of its irreducible components has both properties. This is essentially the content of Prop. 2.48. In order to achieve a proof of this statement, we introduce so-called splayed divisors, which are divisors whose defining equation $h$ can be factored into $h = h_1 h_2$ such that the $h_i$ have separated variables (probably after a coordinate change). Thus splayed divisors are a generalization of the union of transversally intersecting smooth divisors. First it is shown that a splayed divisor is free and has radical Jacobian ideal if and only if its splayed components have these two properties (Lemma 2.38). Then we show that a divisor that is a union of two components and that has radical Jacobian ideal is splayed (Lemma 2.40). Along the way we obtain a characterization of splayed divisors in terms of their Jacobian ideals (see Thm. 2.43), namely, $h_1 h_2$ defines locally a splayed divisor if and only if its Jacobian ideal satisfies the so-called Leibniz property

$$J_{h_1 h_2} = h_1 J_{h_2} + h_2 J_{h_1}.$$

For this section we use the following notation: if a divisor $D$ is the union of some $\bigcup_{i=1}^m D_i$, where the $D_i$ do not have to be irreducible but have no common components, then we denote their respective equations at a point $p$ by $h_1, \ldots, h_m$, where $h_i \in \mathcal{O}_{S,p}$. Then $D = \{h = h_1 \cdots h_m = 0\}$. The Jacobian ideal of $D_i$ is denoted by $J_{h_i} = (\partial_{x_1} h_i, \ldots, \partial_{x_n} h_i)$ and the Jacobian ideal of $D$ is denoted by $J_{h_1 \cdots h_m} = J_h = (\partial_{x_1} h, \ldots, \partial_{x_n} h)$.

### 2.4.1 Splayed divisors

Here splayed divisors are introduced and we show the two properties we are interested in: a splayed divisor is free if and only if its components are and a splayed divisor has radical Jacobian ideal if and only if all its splayed components have radical Jacobian ideal. However, splayed divisors are certainly interesting in their own right since they are a natural generalization of normal crossing divisors to divisors with singular components. In Chapter 3, we will consider singularity invariants of splayed divisors, in particular, we find that their Hilbert–Samuel polynomials satisfy an additivity condition.

**Definition 2.33.** Let $D$ be a divisor in a complex manifold $S$, $\dim S = n$. The divisor $D$ is called *splayed* at a point $p \in S$ (or $D$ is a *splayed divisor* at $p$) if one can find coordinates $(x_1, \dots, x_n)$ at $p$ such that $(D,p) = (D_1,p) \cup (D_2,p)$ is defined by

$$h(x) = h_1(x_1, \dots, x_k)h_2(x_{k+1}, \dots, x_n),$$

$1 \leq k \leq n-1$, where $h_i$ is the defining reduced equation of $D_i$. Note that the $h_i$ are not necessarily irreducible. The *splayed components* $D_1$ and $D_2$ are not unique. Splayed means that $D$ is the union of two products: since $h_1$ is independent of $x_{k+1}, \dots, x_n$, the divisor $D_1$ is locally at $p$ a product $(D_1', 0) \times (\mathbb{C}^{n-k}, 0)$, where $(D_1', 0) \subseteq (\mathbb{C}^k, 0)$ (and similar for $D_2$).

*Example* 2.34. (1) Let $(D,0)$ be the divisor in $(\mathbb{C}^2, 0)$ defined by $h_1h_2 = x(y-x^2)$. Since $D$ has normal crossings at the origin, $D$ is splayed.
(2) Let $D = \{(x^3-y^2)(z^2-w^2) = 0\} \subseteq \mathbb{C}^4$. Then $D$ is splayed with splayed components $h_1 = x^3 - y^2$ and $h_2 = z^2 - w^2$.
(3) The divisor $D = \{(x-y^2)zw = 0\}$ is splayed in $(\mathbb{C}^4, 0)$ but its splayed components are not unique, e.g. $h_1 = x - y^2$ and $h_2 = zw$ or $h_1 = (x-y^2)w$ and $h_2 = z$.
(4) The divisor $D = \{(x-y^2)yz = 0\}$ is also splayed in $(\mathbb{C}^3, 0)$ with components given by $h_1 = (x-y^2)y$ and $h_2 = z$.

Let $S, T$ be complex manifolds of dimensions $n, m$ and suppose that $(S \times T, 0) \cong (\mathbb{C}^{n+m}, 0)$, with complex coordinates $(x,y) = (x_1, \dots, x_n, y_1, \dots, y_m)$ at the origin. Let $(D_1^x, 0)$ be a divisor in $(S, 0)$, which is defined by a reduced $g'(x) \in \mathcal{O}_{S,0} \cong \mathbb{C}\{x_1, \dots, x_n\}$ and which has a logarithmic derivation module over $\mathbb{C}\{x\}$ denoted by $\mathrm{Der}_{S,0}(\log D_1^x)$. Then we may consider the cylinder over $D_1^x$ in the $T$-direction in $(S \times T, 0)$, namely the hypersurface $D_1$ defined by $g(x,y) = g(x,0) := g'(x) \in \mathbb{C}\{x,y\}$. It is easy to see that

$$\mathrm{Der}_{S\times T,0}(\log D_1) = (\mathrm{Der}_{S,0}(\log D_1^x) \otimes_{\mathbb{C}\{x\}} \mathbb{C}\{x,y\}) \oplus (\mathrm{Der}_{T,0} \otimes_{\mathbb{C}\{y\}} \mathbb{C}\{x,y\}).$$

Similarly define $D_2^y$ and $D_2$ with equations $h'(y) = h(x,y)$ and also $\mathrm{Der}_{S\times T,0}(\log D_2)$. Note that both $g, h$ are reduced and have no common factor. Thus we define the (splayed) divisor $D = D_1 \cup D_2$ in $S \times T$ that is given at 0 by the equation $gh = 0$. Since $g$ and $h$ have separated variables, there is a natural splitting of $\mathrm{Der}_{S\times T,0}(\log D)$: by definition for any element $\delta = \delta_g + \delta_h$ of $\mathrm{Der}_{S\times T,0}(\log D)$, where $\delta_g := \sum_{i=1}^n a_i \partial_{x_i}$ and $\delta_h := \sum_{j=1}^m b_j \partial_{y_j}$ for some $a_i, b_j \in \mathbb{C}\{x,y\}$, one has

$$\delta(gh) = h\delta_g(g) + g\delta_h(h) = agh, \tag{2.1}$$

for some $a \in \mathcal{O}_{S,p}$. Dividing (2.1) through $g$ or $h$ this implies that $\delta_g(g)$ is divisible by $g$, that is, $\delta_g(g) \in (g)$ in $\mathbb{C}\{x,y\}$ and also that $\delta_h(h) \in (h)$ in $\mathbb{C}\{x,y\}$. Therefore each element $\delta$ of $\mathrm{Der}_{S\times T,0}(\log D)$ can be written uniquely as $\delta = \delta_g + \delta_h$. Conversely, a computation shows that for any $\eta_1 \in \mathrm{Der}_{S,0}(\log D_1^x) \otimes_{\mathbb{C}\{x\}} \mathbb{C}\{x,y\}$ and $\eta_2 \in \mathrm{Der}_{T,0}(\log D_2^y) \otimes_{\mathbb{C}\{y\}} \mathbb{C}\{x,y\}$ the vector field $\eta_1 + \eta_2$ is contained in $\mathrm{Der}_{S\times T,0}(\log D)$. Hence it follows that

$$\mathrm{Der}_{S\times T,0}(\log D) = (\mathrm{Der}_{S,0}(\log D_1^x) \otimes_{\mathbb{C}\{x\}} \mathbb{C}\{x,y\}) \oplus (\mathrm{Der}_{T,0}(\log D_2^y) \otimes_{\mathbb{C}\{y\}} \mathbb{C}\{x,y\}). \tag{2.2}$$

*Remark* 2.35. The concept of splayed divisors was also studied by J. Damon under the name product union, see [24].

**Lemma 2.36.** *Let $D_1$, $D_2$ be divisors in some $S \times T \cong \mathbb{C}^n \times \mathbb{C}^m$ and $D = D_1 \cup D_2$ be splayed at a point $p = (x_1, \ldots, x_n, y_1, \ldots y_m)$ defined locally by $D_1 = \{g(x) = 0\}, D_2 = \{h(y) = 0\}$ resp. $D = \{g(x)h(y) = 0\}$ with $g, h \in \mathcal{O}_{S\times T,p} \cong \mathbb{C}\{x,y\}$ . If $J_g$ and $J_h$ are both radical ideals then*

$$J_{gh} = (g,h) \cap J_g \cap J_h.$$

*Proof.* First note that $(g,h)$ is a radical ideal (see Remark 2.37). As $D$ is splayed, it follows that $J_{gh} = gJ_h + hJ_g$. From $J_g$ and $J_h$ radical follows $g \in J_g$ and $h \in J_h$ (see Prop. 2.13). Thus it is clear that $J_{gh}$ is contained in $(g,h) \cap J_g \cap J_h$. Conversely, suppose that $\alpha$ is an element in $(g,h) \cap J_g \cap J_h$. Then $\alpha$ can be written as

$$\alpha = ag + bh = \sum_{i=1}^{n} a_i \partial_{x_i} g = \sum_{j=1}^{m} b_i \partial_{y_j} h$$

for some $a, b, a_i, b_j \in \mathbb{C}\{x,y\}$. The element $\alpha - ag = bh$ is contained in $J_g$ since $g \in J_g$. The ideal $J_g$ can be written as an intersection of prime ideals $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$ where all $\mathfrak{p}_\mathfrak{i}$ are in $\mathbb{C}\{x\}$. Because $g$ and $h$ have separated variables, we have $h \notin \mathfrak{p}_i$ for any $i = 1, \ldots, s$. However, $bh$ must be contained in each of the $\mathfrak{p}_i$. Thus it follows that $b$ has to be contained in each $\mathfrak{p}_i$, which means nothing else but $b \in J_g$. Interchanging the role of $g$ and $h$ yields $a \in J_h$. Hence $\alpha$ is contained in $hJ_g + gJ_h = J_{gh}$, which is what had to be shown. □

*Remark* 2.37. In most textbooks it is shown that the tensor product $A \otimes_k B$ of two reduced *finitely generated* $k$ algebras $A, B$ is again reduced. Here, one has to assume that $k$ is a perfect field. A sketch of the proof is as follows: if $A$ is reduced, then also $K \otimes_k A$ is reduced for

all extension fields $K \supseteq k$, see [10, ch. 5,§15]. So choose a $k$-basis $(v_i)$ of $A$ (as a vector space) and suppose that $\alpha = \sum_i v_i \otimes b_i$ is a non-zero nilpotent element in $A \otimes_k B$. We may suppose that there exists a maximal ideal $\mathfrak{m}$ in $B$, which does not contain $b_1 \neq 0$ (this holds because for a reduced finitely generated algebra the intersection of its prime ideals is just 0). Then $\overline{\alpha} \in A \otimes_k (B/\mathfrak{m})$ is nilpotent and not equal to 0. But $B/\mathfrak{m}$ is a field and so this is a contradiction to the fact that for all field extensions of $k$ the tensor product is reduced.
A general proof that the tensor product of reduced $k$ algebras is again reduced can be found in Bourbaki [10, Ch. 5, §15,Thm. 3]. For reduced analytic algebras there is also a proof with the help of Grauert's division theorem, which can be found in [27, Thm. 7.3.5.]. For local analytic algebras $A, B$ a theory about the analytic tensor product was developed in [44, III, §5].

**Lemma 2.38.** *Let $D_1$, $D_2$ and $D$ be splayed divisors in $S$ defined as in Lemma 2.36.*
*(a) The Jacobian ideal of $D$, denoted by $J_{gh} = (\partial_{x_1} gh + g\partial_{x_1} h, \ldots, \partial_{y_m} gh + g\partial_{y_m} h)$ is radical if and only if both $J_h$ and $J_g$ are also radical.*
*(b) The splayed divisor $D = \{g(x)h(y) = 0\}$ is free if and only if $D_1 = \{g(x) = 0\}$ and $D_2 = \{h(y) = 0\}$ are both free.*

*Proof.* (a): Suppose that $J_g$ and $J_h$ are radical. By Lemma 2.36 the ideals $J_{gh}$ and $(g,h) \cap J_g \cap J_h$ are equal. We compute its radical

$$\sqrt{J_{gh}} = \sqrt{(g,h) \cap J_g \cap J_h} = \sqrt{(g,h)} \cap \sqrt{J_g} \cap \sqrt{J_h} = (g,h) \cap J_g \cap J_h = J_{gh},$$

where the second equality holds because the radical of an intersection of ideals is equal to the intersection of the radicals of these ideals (easy computation) and the third equality because of our assumptions. Conversely, suppose that $J_{gh} = \sqrt{J_{gh}}$. Then $gh$ is an element of $J_{gh}$ and the ideal $J_{gh}$ can be generated by

$$J_{gh} = (gh, (\partial_{x_1} g)h, \ldots, (\partial_{x_n} g)h, (\partial_{y_1} h)g, \ldots, (\partial_{y_m} h)g).$$

Localization of $\mathbb{C}\{x,y\}$ in $g$ yields $(J_{gh})_g = ((h) + J_h)_g$, which is radical, since $J_{gh}$ is radical. Note that for an ideal $I \subseteq \mathbb{C}\{x,y\}$, we denote by $I_g$ the localization of $I$ in $g$ (cf. Appendix A). The ideal $((h) + J_h)$ in $\mathbb{C}\{x,y\}$ can be written as a minimal irredundant primary decomposition $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$ of primary ideals with associated prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$. Since $h$ only depends on $y$, all the $\mathfrak{p}_i$ and $\mathfrak{q}_i$ are ideals of $\mathbb{C}\{y_1, \ldots, y_m\}$. Hence no power of $g$ is contained in any of the $\mathfrak{p}_i$, and it follows that

$$((h) + J_h)_g = (\mathfrak{q}_1)_g \cap \cdots \cap (\mathfrak{q}_m)_g = (\mathfrak{p}_1)_g \cap \cdots \cap (\mathfrak{p}_m)_g,$$

where no $(\mathfrak{q}_i)_g$ is the whole ring. Let now $\alpha \in \mathcal{O}_{S,p}$ be an element of the radical of $((h)+J_h)$. This means that there exists an integer $k$ such that $\alpha^k \in ((h) + J_h)$. Then $(\alpha/1)^k \in ((h) + J_h)_g$ and since this ideal is radical also $\alpha \in ((h) + J_h)_g$ holds. Thus $\alpha$ is contained in any $(\mathfrak{p}_i)_g$. Therefore (by definition of localization) the equality

$$\alpha = \frac{a_i}{g^t}$$

holds for some $a_i \in \mathfrak{p}_i$ and some $t \in \mathbb{N}$ and there exists a $u \in \mathbb{N}$ such that $g^u(\alpha g^t - a_i) = 0$. This implies $\alpha g^{u+t} \in \mathfrak{p}_i$. But $g$ and $h$ have separated variables, hence $\alpha$ is contained in $\mathfrak{p}_i$ for any $i$. This shows the radicality of $((h) + J_h)$. Similarly one proves $((g) + J_g) = \sqrt{((g) + J_g)}$.
(b): If both $D_1$ and $D_2$ are free then there exist bases of $\mathrm{Der}_{S\times T,p}(\log D_1)$ and $\mathrm{Der}_{S\times T,p}(\log D_2)$ of the form

$$\delta_1 = \sum_{i=1}^{n} a_{1i}\partial_{x_i}, \ldots, \delta_n = \sum_{i=1}^{n} a_{ni}\partial_{x_i}, \delta_{n+1} = \partial_{y_1}, \ldots, \delta_{n+m} = \partial_{y_m}$$

and

$$\varepsilon_1 = \partial_{x_1}, \ldots, \varepsilon_n = \partial_{x_n}, \varepsilon_{n+1} = \sum_{i=1}^{m} b_{n+1,i}\partial_{y_i}, \ldots, \varepsilon_{n+m} = \sum_{i=1}^{m} b_{n+m,i}\partial_{y_i}.$$

It is easy to see that any $\delta_i$ for $1 \leq i \leq n$ and any $\varepsilon_j$ for $n + 1 \leq j \leq n + m$ is also an element of $\mathrm{Der}_{S\times T,p}(\log D)$ (direct computation, using separated variables, see the discussion at the beginning of this section). By Saito's criterion (Thm. 1.19) it follows that $\delta_1, \ldots, \delta_n, \epsilon_{n+1}, \ldots, \epsilon_{n+m}$ form a basis of $\mathrm{Der}_{S\times T,p}(\log D)$. Conversely, suppose that $\mathrm{Der}_{S\times T,p}(\log D)$ is free. From (2.2) we know that

$$\mathrm{Der}_{S\times T,p}(\log D) \cong (\mathrm{Der}_{\mathbb{C}^n,0}(\log D_1^x) \otimes_{\mathbb{C}\{x\}} \mathbb{C}\{x,y\}) \oplus (\mathrm{Der}_{\mathbb{C}^m,0}(\log D_2^y) \otimes_{\mathbb{C}\{y\}} \mathbb{C}\{x,y\}).$$

Since $\mathrm{Der}_{S\times T,p}(\log D)$ is free, it follows that $\mathrm{Der}_{\mathbb{C}^n,0}(\log D_1^x) \otimes_{\mathbb{C}\{x\}} \mathbb{C}\{x,y\}$ and $\mathrm{Der}_{\mathbb{C}^m,0}(\log D_2^y) \otimes_{\mathbb{C}\{y\}} \mathbb{C}\{x,y\}$ are projective $\mathcal{O}_{S\times T,p}$-modules. Since the notion of projective and free module over regular local rings coincide (see Appendix A), these two modules are even free. □

### 2.4.2 Killing components of divisors

In this section we show that the properties of freeness and radical Jacobian ideal of a divisor are preserved under adding to or subtracting components from the divisor. Suppose that the divisor $D \subseteq S$, with $\dim S = n$, is given locally at a point $p = (x_1, \dots, x_n)$ by

$$D = \{g(x_1, \dots, x_n)h(x_1, \dots, x_n) = 0\},$$

with $g, h \in \mathcal{O}_{S,p}$ reduced but not necessarily irreducible and with no common factors. Then $(D, p)$ is a union $(D_1, p) \cup (D_2, p)$ of $D_1 = \{g = 0\}$ and $D_2 = \{h = 0\}$ near $p$. Here we ask for conditions and a characterization when $D$ is splayed.

**Definition 2.39.** Let $D_1 = \{g = 0\}$, $D_2 = \{h = 0\}$ and $D = \{gh = 0\}$ at $p$ be defined as above. We say that $J_{gh}$ satisfies the *Leibniz property* if

$$J_{gh} = gJ_h + hJ_g.$$

We show a characterization of splayedness by Jacobian Ideals, by the Leibniz property. This property makes it easy to check in concrete examples whether a divisor is splayed.

The goal of this section is to show that a reducible free divisor with radical Jacobian ideal is splayed. First an ideal-theoretic characterization of splayedness is proven (Lemma 2.40). Then it is shown that a divisor is splayed if and only if it has the Leibniz property (Theorem 2.43). Finally we show that if $J_{gh}$ is radical then it satisfies the Leibniz property and is thus splayed (Prop. 2.48).

**Lemma 2.40.** *Let* $\dim S = n$ *and at a point* $p = (x_1, \dots, x_n)$ *denote by* $\mathcal{O}_{S,p} = \mathbb{C}\{x_1, \dots, x_n\}$ *(in short:* $\mathcal{O} = \mathbb{C}\{x\}$*) the local ring at* $p$*. Let* $D_1 = \{g(x) = 0\}$*,* $D_2 = \{h(x) = 0\}$ *and* $D = \{gh(x) = 0\}$ *be divisors, where we assume that* $g, h \in \mathcal{O}_{S,p}$ *are reduced and have no common factors. Then* $D$ *is locally at* $p$ *splayed if and only if*

$$(g) \cap ((gh) + J_{gh}) = g((h) + J_h).$$

*Remark* 2.41. The idea to consider the equality of these two ideals comes from the case when one component is smooth, that is, if $g = x_1$. Then it is rather easy to see that a splayed divisor $D = \{x_1 h = 0\}$ satisfies $(x_1) \cap ((x_1 h) + J_{x_1 h}) = x_1((h) + J_h)$.

*Proof.* If $D$ is splayed, we can suppose wlog. that $D_1 = \{g(x,0)=0\}$ and $D_2 = \{h(0,x)=0\}$ where $(x) = (x_1,\dots,x_k,x_{k+1},\dots,x_n)$. In this case (separated variables) it is easy to see that $J_{gh} = gJ_h + hJ_g$. An element $\alpha \in g((h)+J_h)$ can be written as $agh + g\sum_{i=k+1}^n a_i\partial_{x_i}h$. Clearly $\alpha$ is contained in the ideal $(g)$ and $g\sum_{i=k+1}^n a_i\partial_{x_i}h \subseteq gJ_h$ and this ideal is contained in $J_{gh}$. Thus $\alpha$ is contained in $(g)\cap(gh+J_{gh})$. If $\alpha \in (g)\cap((gh)+J_{gh})$ we can write it as $h\sum_{i=1}^k a_i\partial_{x_i}g + g\sum_{i=k+1}^n a_i\partial_{x_i}h + agh$ and since $\alpha \in (g)$, it follows that $g$ divides $\sum_{i=1}^k a_ih\partial_{x_i}g$. Therefore $\alpha = gh\tilde{a} + g\sum_{i=k+1}^n a_i\partial_{x_i}h$ for some $\tilde{a}\in\mathcal{O}$. Hence $\alpha$ is also contained in $g((h)+J_h)$.
Conversely, suppose that

$$(g)\cap((gh)+J_{gh}) = g((h)+J_h). \tag{2.3}$$

The assertion is shown in two steps: first one can rectify $h$ and second is the rectification of $g$. We remark that if $\varphi:\mathcal{O}\to\mathcal{O}$ is a local isomorphism, then $\varphi((g)\cap((gh)+J_{gh})) = \varphi(g((h)+J_h))$ is isomorphic to $(g\circ\varphi)\cap((gh\circ\varphi)+J_{gh\circ\varphi}) = (g\circ\varphi)\cdot((h\circ\varphi)+J_{h\circ\varphi})$. This means that (2.3) is stable under a local algebra isomorphism of $\mathcal{O}$. Moreover, (2.3) is stable under multiplication with units.

*First Step*: We show that one can assume $h(x_1,\dots,x_n) = h(0,\dots,0,x_{k+1},\dots,x_n)$ and that

$$\partial_{x_i}h \notin (h,\partial_{x_{k+1}}h,\dots,\widehat{\partial_{x_i}h},\dots,\partial_{x_n}h) \tag{2.4}$$

for all $i\in\{k+1,\dots,n\}$. If not so, suppose that e.g. $\partial_{x_1}h$ is contained in $(\partial_{x_2}h,\dots,\partial_{x_n}h)$. Then by the triviality lemma A.44 there exists an algebra isomorphism $\varphi:\mathcal{O}\to\mathcal{O}$ such that $\varphi(x) = (x_1,\varphi_2(x),\dots,\varphi_n(x))$ and $h\circ\varphi(x) = v(x)h(0,x_2,\dots,x_n)$, with $v\in\mathcal{O}^*$. Then set $\widetilde{h} := h(0,x_2,\dots,x_n)$ and $\widetilde{g} := g\circ\varphi$. The divisor defined by $\widetilde{g}\cdot\widetilde{h}$ is clearly isomorphic to $D$, and $D_1$ is isomorphic to $\{\tilde{g}=0\}$ and $D_2$ is isomorphic to $\{\widetilde{h}=0\}$. By the above remarks, equation (2.3) also holds for $\widetilde{g}$ and $\widetilde{h}$ instead of $g$ and $h$. If another $\partial_{x_i}\widetilde{h}\neq 0$ were contained in the ideal $(\partial_{x_2}\widetilde{h},\dots,\widehat{\partial_{x_i}\widetilde{h}},\dots,\partial_{x_n}\widetilde{h})$ the triviality lemma could again be applied to $\widetilde{h}$.

*Second Step:* We may assume that $h(x) = h(0,\dots,0,x_{k+1},\dots,x_n)$ and that (2.4) holds. Suppose that $i$ is an element of $\{k+1,\dots,n\}$. Since any $g\partial_{x_i} h \in g((h) + J_h)$, we can also write it (by (2.3)) as

$$g(\partial_{x_i} h) = a_i' gh + \sum_{j=1}^{n} a_{ij} g(\partial_{x_j} h) + \sum_{j=1}^{n} a_{ij} h(\partial_{x_j} g).$$

Division through $g$ shows that $\sum_{j=1}^{n} a_{ij} h(\partial_{x_j} g) = \tilde{a}_i gh$ for some $\tilde{a}_i \in \mathcal{O}$. Hence reduction of the above equation by $g$ yields

$$\partial_{x_i} h = a_i h + \sum_{j=k+1}^{n} a_{ij}(\partial_{x_j} h),$$

where $a_i := a_i' + \tilde{a}_i$. But this equation implies

$$(1 - a_{ii})(\partial_{x_i} h) = a_i h + \sum_{j=k+1, j\neq i}^{n} a_{ij}(\partial_{x_j} h).$$

Then $(1 - a_{ii}) \in \mathfrak{m}$, that is, $a_{ii} \in \mathcal{O}^*$, and $a_{ij} \in \mathfrak{m}$ for all $i, j = k+1, \ldots, n$, otherwise (2.4) would be contradicted. Again from (2.3), namely,

$$0 = a_i' gh + \sum_{j=1, j\neq i}^{n} a_{ij} g(\partial_{x_j} h) + (a_{ii} - 1) g(\partial_{x_j} h) + h \sum_{j=1}^{n} a_{ij}(\partial_{x_j} g)$$

we get

$$-a_{ii} h(\partial_{x_i} g) = a_i' hg + g \sum_{j=1}^{n} \tilde{a}_{ij}(\partial_{x_j} h) + h \sum_{j=1, j\neq i}^{n} a_{ij}(\partial_{x_j} g),$$

for any $i = k+1, \ldots, n$. Reduction of these $(n-k)$ equations by $h$ yields

$$a_{ii}(\partial_{x_i} g) = \tilde{a}_i g - \sum_{j=1, j\neq i}^{n} a_{ij}(\partial_{x_j} g), \text{ for some } \tilde{a}_i \in \mathcal{O}.$$

Keeping in mind that the $a_{ij}$ for $i, j \geq k+1$ are in $\mathfrak{m}$ we manipulate these $(n-k)$ equations (substituting $\partial_{x_{k+1}}(g)$ the second equation, $\partial_{x_{k+1}}(g)$ and $\partial_{x_{k+2}}(g)$ in the third, and so on) such that we arrive at an equation

$$\partial_{x_n} g = b_n g + \sum_{j=1}^{k} b_{nj}(\partial_{x_j} g),$$

with some coefficients $b_n, b_{nj} \in \mathcal{O}$. Substituting back in all $(n-k)$ equations yields

$$\partial_{x_i} g \in (g, \partial_{x_1} g, \dots, \partial_{x_k} g) \text{ for all } i = k+1, \dots, n.$$

By the triviality lemma there exists an algebra isomorphism $\psi : \mathcal{O} \to \mathcal{O}$ with $\psi(x) = (\psi_1(x), \dots, \psi_k(x), x_{k+1}, \dots, x_n)$ such that $\psi(x_1, \dots, x_k, 0) = (x_1, \dots, x_k, 0)$. Moreover there exists $v(x_1, \dots, x_k, 0) \equiv 1$ such that $g \circ \psi$ is equal to $vg(x_1, \dots, x_k, 0)$. Set $\tilde{g} := v^{-1}(g \circ \psi)$ and $\tilde{h} := h \circ \psi = h = h(0, \dots, 0, x_{k+1}, \dots, x_n)$. By construction $\tilde{g}\tilde{h}$ defines a splayed divisor that is isomorphic to $D$ such that the assertion has been shown. □

*Example* 2.42. Let $D$ be the divisor in $\mathbb{C}^3$ given at a point $p$ by $x(x + y^2 - z^3)$. Then $D$ is the union of two smooth components $D_1 = \{h = x + y^2 - z^3 = 0\}$ and $H = \{x = 0\}$. The ideal $(x) \cap (x(x+y^2-z^3), J_{xh}) = (xy, x^2, xz^2)$ is strictly contained in $(x(x + y^2 - z^3), xJ_h) = (x)$. Thus $D$ is not a splayed divisor.

**Theorem 2.43.** *Let $(S, D)$ be a complex manifold $S$, $\dim S = n$, together with a divisor $D \subseteq S$ that is locally at a point $p = (x_1, \dots, x_n) \in S$ defined by $\{gh = 0\}$, where $g$ and $h$ are reduced elements of $\mathcal{O}_{S,p}$ that are not necessarily irreducible but have no common factor. Then $D$ is splayed at $p$ if and only if $J_{gh}$ satisfies the Leibniz property*

$$J_{gh} = gJ_h + hJ_g.$$

*Proof.* First suppose that $gJ_h + hJ_g = J_{gh}$. By Lemma 2.40 the equality $(g) \cap ((gh) + J_{gh}) = g((h) + J_h)$ has to be shown. So take an $\alpha \in g((h) + J_h)$, which is of the form

$$\alpha = agh + g \sum_{i=1}^{n} a_i(\partial_{x_i} h),$$

for $a, a_i \in \mathcal{O}_{S,p}$. One sees that $gh \in ((gh) + J_{gh})$ and $g \sum_{i=1}^{n} a_i(\partial_{x_i} h) \in gJ_h \subseteq J_{gh}$ and hence $\alpha \in ((gh) + J_{gh})$ and also in $(g)$. Now take a $\beta \in (g) \cap ((gh) + gJ_h + hJ_g)$, which can be written as

$$\beta = agh + g \sum_{i=1}^{n} a_i(\partial_{x_i} h) + h \sum_{i=1}^{n} b_i(\partial_{x_i} g).$$

Since $\beta$ is also contained in $(g)$ it follows that $\sum_{i=1}^{n} b_i(\partial_{x_i} g) = g\tilde{b}$ for some $\tilde{b} \in \mathcal{O}_{S,p}$. Hence $\beta \in (gJ_h + (gh)) = g(J_h + h)$.

Conversely, let $D$ be splayed. Then we may assume that $D$ is given locally by

$$g(x_1,\dots,x_k,0,\dots,0)h(0,\dots,0,x_{k+1},\dots,x_n).$$

A direct computation shows that

$$\partial_{x_i}(gh) = \partial_{x_i}(g)h \text{ for } i=1,\dots,k \text{ and}$$

$$\partial_{x_i}(gh) = g\partial_{x_i}(h) \text{ for } i=k+1,\dots,n.$$

Thus clearly $J_{gh}$ is equal to $gJ_h + hJ_g$. □

*Remark* 2.44. Note that with Theorem 2.43 we have obtained an algebraic description of splayed divisors by their Jacobian ideals, namely that their Jacobian ideals satisfy the Leibniz property $J_{gh} = gJ_h + hJ_g$. In Chapter 3, we will also derive an algebraic characterization of $\operatorname{Sing} D$ defined by $((gh)+J_{gh})$ of a splayed divisor, namely, $D=\{gh=0\}$ at a point $p$ is splayed if and only if $((gh)+J_{gh}) = (g,h)\cap((g)+J_g)\cap((h)+J_h)$.

**Lemma 2.45.** *Let $D \subseteq S$ be a divisor given at $p \in S$ by $\{gh=0\}$ with $gh \in \mathcal{O}_{S,p}$ reduced and suppose that $J_{gh}$ is radical. Then*

$$J_{gh} = gJ_h + hJ_g.$$

*Proof.* By definition $J_{gh} = (\partial_{x_1}gh + g\partial_{x_1}h, \dots, \partial_{x_n}gh + g\partial_{x_n}h)$. Since $J_{gh}$ is radical, it follows that $gh \in J_{gh}$ (Prop. 2.13). The ideal $J_{gh}$ can be written uniquely as an irredundant intersection of prime ideals $\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_k \cap \mathfrak{p}_{k+1} \cap \dots \cap \mathfrak{p}_m$, where we may assume that $g \not\subseteq \mathfrak{p}_i$ for all $i=1,\dots,k$ and that $g$ is contained in the remaining $\mathfrak{p}_i$. Localizing in $g$ yields $(J_{gh})_g = (h, \partial_{x_1}h, \dots, \partial_{x_n}h)_g = ((h)+J_h)_g \subseteq (\mathcal{O}_{S,p})_g$. By Prop. A.1 one has $(J_{gh})_g = \bigcap_{i=1}^m (\mathfrak{p}_i)_g = \bigcap_{i=1}^k (\mathfrak{p}_i)_g$, since the $\mathfrak{p}_i$ with $i=k+1,\dots,m$ contain the unit of the localization. Thus $\partial_{x_j}h$ is contained in $(\mathfrak{p}_i)_g$ for $i=1,\dots,k$ and for all $j=1,\dots,n$. Hence

$$\frac{\partial_{x_j}h}{1} = \frac{p_i}{g^{t_i}}$$

for all $i=1,\dots,k$, where $p_i \in \mathfrak{p}_i$ and $t_i \in \mathbb{N}$. This means that there exists an $g^{l_i}$, $l_i \in \mathbb{N}$, such that $g^{l_i}(\partial_{x_j}(h)g^{t_i} - p_i) = 0$. Hence it follows that $\partial_{x_j}(h)g^{t_i+l_i} \in \mathfrak{p}_i$. Since by assumption, $g$ is not contained in any of the $\mathfrak{p}_i$ for $i=1,\dots,k$, it follows that $\partial_{x_j}h \in \mathfrak{p}_i$. Thus $g\partial_{x_j}(h)$ is contained in all $\mathfrak{p}_i$ with $i = 1, \ldots, k$. Further, $g$ is contained in the remaining $\mathfrak{p}_i$, which implies that $g(\partial_{x_j} h)$ is contained in all associated primes of $J_{gh}$, and thus $g(\partial_{x_j} h) \in J_{gh}$ for all $j = 1, \ldots, n$. This yields

$$J_{gh} = (g\partial_{x_1}(h), \ldots, g\partial_{x_n}(h), h\partial_{x_1}(g), \ldots, h\partial_{x_n}(g)) = gJ_h + hJ_g.$$

□

*Example* 2.46. A splayed divisor need not have a radical Jacobian ideal, as the following example shows. Let $D$ be the divisor in $(\mathbb{C}^3, 0)$ with coordinates $(x, y, z)$ at 0, that is defined by $gh = x(y^2 + z^3)$. Then clearly $D$ is splayed. The Jacobian ideal is $J_{gh} = (y^2 + z^3, xy, xz^2) = (y, z^2) \cap (x, y^2 + z^3)$, which is not radical. Note that $D$ is a free divisor.

*Example* 2.47. Let $D \subseteq \mathbb{C}^3$ be given at the origin by $gh = x(x + y^2 + z^3) = 0$. Then $D$ is not splayed at the origin. Here the intersection of the two components is given by the ideal $(g, h) = (x, y^2 + z^3)$. Also consider the divisor $D' \subseteq \mathbb{C}^3$ that is given by $g'h' = x(y^2 + z^3)$. Clearly $D'$ is splayed at the origin and the intersection of the two components is given by the ideal $(g', h') = (x, y^2 + z^3)$. Here one sees that splayedness cannot be determined by just the knowledge of the ideal of the intersection of the two components, in contrast to the case of two smooth divisors intersecting transversally, see Chapter 3.

**Proposition 2.48.** *Let $D = D_1 \cup D_2$ be a divisor in an $n$ dimensional complex manifold $S$ and let $D$, $D_1$ and $D_2$ at a point $p \in S$ be defined by the equations $gh$, $g$ and $h$, respectively. Suppose that $J_{gh}$ is radical. Then $D$ is splayed and $J_h$ and $J_g$ are also radical. If moreover $D$ is free at $p$ then also $D_1$ and $D_2$ are free at $p$.*

*Proof.* By Theorem 2.43 and Lemma 2.45 it follows that $D$ is locally splayed. By Lemma 2.38 also $J_g$ and $J_h$ are radical and from the same lemma follows that $D_1$ and $D_2$ are free if $D$ is free. □

**Corollary.** *Let $D' = \{h = 0\}$ be a free divisor in $S$ with $J_h = \sqrt{J_h}$ and $D$ the union of $D'$ with a smooth component, wlog. $D = \{x_1 h = 0\}$. Then $D$ is free and $J_{x_1 h}$ is radical if and only if $D$ is splayed.*

*Proof.* Follows directly from Lemma 2.38 and Prop. 2.48. □

**Corollary.** *Let $(S, D)$ be a complex manifold,* $\dim S = n$, *together with a divisor $D \subseteq S$ and suppose that locally at a point $p \in S$ the divisor $(D, p)$ has the decomposition into irreducible components $\bigcup_{i=1}^m (D_i, p)$*

*such that each $(D_i, p)$ is smooth. Let the corresponding equation of $D$ at $p$ be $h = h_1, \dots, h_m$. If $D$ is free at $p$ and $J_h = \sqrt{J_h}$ then $D$ has normal crossings at $p$.*

*Proof.* We use induction on $n$. If $n = 2$, then apply Prop. 2.15. Now suppose the assertion is true for divisors in manifolds of dimension $n-1$. For a smooth component $D_1$ of $D$, one can find local coordinates $(x_1, \dots, x_n)$ such that $D_1 = \{h_1(x_1, \dots, x_n) = x_1 = 0\}$. Prop. 2.48 shows that the divisor $(D \setminus D_1) := h_2 \cdots h_m$ is also free and has a radical Jacobian ideal. Moreover, $D$ is locally splayed, that is, $D \setminus D_1$ is locally isomorphic to some divisor depending only on the last $n-1$ coordinates. Thus by induction hypothesis $D \setminus D_1$ is isomorphic to a normal crossings divisor $y_2 \cdots y_m = 0$, where the $y_i$ are the result of a coordinate transformation of $(x_1, \dots, x_n)$ such that $x_1 = y_1$. Thus $x_1, y_2, \dots, y_n$ are also local coordinates at $p$. This implies that $m \leq n-1$. Hence $D$ is isomorphic to the normal crossings divisor $x_1 y_2 \cdots y_m$. □

In this section the problem of proving Thm. 2.1 has been reduced to the "irreducible" case: by Prop. 2.48 a divisor $D$, which is a union of irreducible components, is free and has radical Jacobian ideal if and only if all its components have these properties. Thus $D$ can only have smooth components and/or irreducible components with $(n-2)$-dimensional singular locus. For the irreducible case we have to show that $D$ is free, has radical Jacobian ideal at a point and its normalization is Gorenstein if and only if it is smooth at this point.

### 2.4.3 Proof of Theorem 2.1

If $D$ has normal crossings at $p$, then $D$ is free at $p$, that is, it is either smooth or $\operatorname{depth}(J_h, \mathcal{O}_{S,p}) = 2$ and $\mathcal{O}_{\operatorname{Sing} D,p}$ is Cohen–Macaulay (Aleksandrov's theorem). The normalization of a normal crossing divisor $D = \bigcup_{i=1}^m D_i$ is smooth since it is the disjoint union of the smooth components $D_i$ (cf. Example A.27). So it remains to show that for a point $p \in \operatorname{Sing} D$ the ideal $J_h$ is radical at $p$. This is done by direct computation: since $D$ has normal crossings at $p \in \operatorname{Sing} D$, we can assume that $D = \bigcup_{i=1}^m (D_i, p)$ is given by the equation $h = x_1 \cdots x_m$, $1 < m \leq n$ where each $x_i$ corresponds to an irreducible component $D_i$

passing through $p$. Then

$$J_h = \sum_{i=1}^{m} (x_1 \cdots \hat{x}_i \cdots x_m).$$

Using facts about primary decomposition of monomial ideals, see e.g. [54], it follows that

$$\begin{aligned} J_h =& (x_2, x_1x_3 \cdots x_m) \cap (x_3 \cdots x_m, x_1x_2x_4 \cdots x_m, \dots, x_1 \dots x_{m-1}) \\ =& \bigcap_{\substack{i=1 \\ i\neq 2}}^{m} (x_i, x_2) \cap \bigcap_{\substack{j=1 \\ j\neq 3}}^{m} (x_j, x_3) \cap (x_4 \cdots x_m, x_1 \cdots \hat{x}_4 \cdots x_m, \dots, x_1 \cdots x_{m-1}) \\ =& \bigcap_{1 \leq i < j \leq m} (x_i, x_j). \end{aligned}$$

This irredundant primary decomposition shows that $J_h$ is the intersection of prime ideals of height 2. Thus $J_h$ is clearly radical.
Conversely, suppose that $J_h = \sqrt{J_h}$ and $\mathcal{O}_{\operatorname{Sing} D,p}$ is Cohen–Macaulay of dimension $(n-2)$ and moreover that the normalization $\mathcal{O}_{\widetilde{D}}$ is Gorenstein (here $\pi : \widetilde{D} \to D$ denotes the normalization morphism). Prop. 2.48 implies that each $D_i$ is free at $p$ and has a radical Jacobian ideal. So we may assume that $D$ is irreducible. By our hypothesis, Piene's theorem A.42 and Remark A.43 yield the equality of ideals

$$C_D I_\pi \mathcal{O}_{\widetilde{D},p} = J_h \mathcal{O}_{\widetilde{D},p}.$$

Since by Lemma 1.69 one has $J_h = C_D$ in $\mathcal{O}_{D,p}$, this implies $C_D = C_D I_\pi$ in $\pi_* \mathcal{O}_{\widetilde{D},p}$. By Nakayama's lemma, it follows that $I_\pi = \mathcal{O}_{\widetilde{D},p}$. Hence $\Omega^1_{\widetilde{D}/D} = 0$. If $\widetilde{D}$ is smooth at $\pi^{-1}(p)$ then a similar argument as in the proof of Thm. 1.63 yields that $D$ is already smooth at $p$: then $\mathcal{O}_{\widetilde{D}} \cong \mathbb{C}\{z_1, \dots, z_{n-1}\}$ for some independent variables $z_1, \dots, z_{n-1}$. Hence one has an inclusion of rings

$$\mathcal{O}_{D,p} = \mathbb{C}\{f_1, \dots, f_r\} \subseteq \mathbb{C}\{z_1, \dots, z_{n-1}\},$$

where $f_1, \dots, f_r \in \mathcal{O}_{\widetilde{D}_i}$ and $r \geq n-1$. By definition one can write

$$0 = \Omega^1_{\widetilde{D}/D} = \bigoplus_{i=1}^{n-1} \mathcal{O}_{\widetilde{D}} dz_i / \sum_{j=1}^{r} \mathcal{O}_{\widetilde{D}} df_j.$$

By Nakayama's lemma one finds $n-1$ generators of $\mathcal{O}_{D,p}$, w.l.o.g., $f_1, \dots f_{n-1}$ such that the Jacobian determinant $\frac{\partial(f_1,\dots,f_{n-1})}{\partial(z_1,\dots,z_{n-1})} \neq 0$. By

the implicit function theorem, $f_1, \ldots, f_{n-1}$ are independent variables and hence $\mathcal{O}_{D,p} \cong \mathcal{O}_{\widetilde{D}}$ is smooth.
If $\pi^{-1}(p) \in \operatorname{Sing} \widetilde{D}$, then because $n-2 \geq \dim(\operatorname{Sing} \widetilde{D})$ and $\pi$ is a finite map, one finds that $\dim(\operatorname{Sing} D) = \dim(\pi(\operatorname{Sing} \widetilde{D})) = \dim \widetilde{D} \leq n-2$. By Theorem A.24, $D$ is normal at $p$. By Aleksandrov's theorem, $D$ is then already smooth at $p$. For $(D,p) = \bigcup_{i=1}^{m}(D_i,p)$ this means that we are in the situation of the second corollary of Proposition 2.48 and the assertion follows. □

*Remark* 2.49. We can also give a different proof of (2) ⇒ (1) of Thm. 2.1 using the characterization of normal crossings by the logarithmic residue of Thm. 1.63: let $(D,p) = \bigcup_{i=1}^{m}(D_i,p)$ be the decomposition into irreducible components and suppose that $J_h = \sqrt{J_h}$. Then the singular locus of the singular locus $\operatorname{Sing}(\operatorname{Sing} D)$ is of dimension less than or equal to $(n-3)$. By Lemma 2.17, $D$ has normal crossings at smooth points of $\operatorname{Sing} D$. Hence $D$ has normal crossings in codimension 1. From Lemma 1.80 it follows that the logarithmic residue is holomorphic on the normalization, that is, $\rho(\Omega^1_S(\log D)) = \pi_* \mathcal{O}_{\widetilde{D}}$. Then Theorem 1.63 shows that $D$ is a normal crossing divisor.

*Remark* 2.50. We do not know whether the condition on the normalization of $D$ in Theorem 2.1 is necessary. If $(D,p)$ is free and has a radical Jacobian ideal, then by Lemma 1.65 the normalization $(\widetilde{D}, \pi^{-1}(p))$ is Cohen–Macaulay. One can use Piene's theorem only if $\widetilde{D}$ is Gorenstein because then the canonical sheaf $\omega_{\widetilde{D}}$ is invertible. More precisely, one can prove the following: $\omega_{\widetilde{D}} = C_D \mathcal{O}_{\widetilde{D}}$ if and only if $\widetilde{D}$ is Gorenstein (see Prop. 3.5 of [66]). Moreover, $\widetilde{D}$ is Gorenstein if and only if it is isomorphic to the blowup of $D$ in the conductor $C_D$ (by Thm. 2.7 of [101]).

**Question 2.51.** *Let $D \subseteq S$ be a divisor in a complex manifold $S$ that is locally at a point $p$ given by $h = 0$ and denote by $\pi : \widetilde{D} \to D$ its normalization. Suppose that $D$ is free at $p$ and that $J_h = \sqrt{J_h}$. Is then the normalization $\widetilde{D}$ of $D$ already Gorenstein at $\pi^{-1}(p)$?*

# Chapter 3

# Jacobian ideals of hypersurfaces

In this chapter we have two different aims: the first one is to classify divisors with radical Jacobian ideals. The second one is to study two possible generalizations of normal crossing divisors, namely splayed divisors and mikado divisors. We consider some of their properties and also try to characterize them in terms of their singular loci given by their Jacobian ideals.
First we ask for an analogue of Theorem 2.1 for radical Jacobian ideals of higher codimension. In low ambient dimension, that is, $\dim S \leq 3$ divisors with radical Jacobian ideal can be described with the help of Thm. 2.1 (see Prop. 3.2). However, it is not clear how to classify divisors with radical Jacobian ideal in higher dimensional ambient spaces, since then also embedded components of the Jacobian ideal have to be taken into account. Here we have results in special cases and conjectures for more general situations. The second topic of this chapter is splayed divisors (also see Chapter 2), which are a natural generalization of normal crossing divisors. The difference between the two classes of divisors is that irreducible components of splayed divisors may have singularities. Here we present a characterization of splayed divisors in terms of their Jacobian ideals (corresponding to the geometry) and compute their Hilbert–Samuel polynomials, which satisfy a certain additivity property. Finally we consider another generalization of normal crossing divisors, so-called mikado divisors. The irreducible components of a mikado divisor are smooth and all possible intersec-

tions between them are also smooth. The difference to normal crossing divisors is that more than $n$ components can meet at a point. We give a characterization of a mikado divisor $D \subseteq S$ in terms of its Jacobian ideal for $\dim S = 2$. Finally we ask for a generalization to higher dimensions.

## 3.1 Radical Jacobian ideals

In Chapter 2 it was shown that if a free divisor with a Gorenstein normalization in a complex manifold has a *radical* Jacobian ideal, then it is already a normal crossing divisor. Now we consider a more general problem. Suppose that $D$ is a divisor in a smooth complex $n$-dimensional manifold $S$ that is locally at a point $p$ given by a reduced equation $h \in \mathcal{O}_{S,p} = \mathbb{C}\{x_1, \ldots, x_n\}$. Denote by $J_h = (\partial_{x_1} h, \ldots, \partial_{x_n} h)$ its Jacobian ideal and suppose that $J_h$ is radical. Which ideals $I \subseteq \mathcal{O}_{S,p}$ can be such radical Jacobian ideals $J_h$? More precisely: given a radical ideal $I \subseteq \mathcal{O}_{S,p}$, when does there exist a divisor $(D,p) = \{h = 0\}$ such that $I = J_h$?

The case of $\dim S = 2$ was treated in Chapter 2: if $D$ is a reduced curve in $S$, then its singular locus consists of isolated points. Thus locally at such a singularity, the ideal $J_h$ is an $\mathfrak{m}$-primary ideal and if $J_h = \sqrt{J_h}$ is radical then it has to be the maximal ideal. With the theorem of Mather–Yau (or with the Corollary of Theorem 3.49) it follows that $(D,p)$ is a normal crossing singularity. For $\dim S = 3$ we need a little preparation.

**Lemma 3.1.** *Let $(R, \mathfrak{m})$ be an $n$-dimensional regular local ring, $I \subseteq R$ an ideal of height $(n-1)$ and suppose that $R/I$ is reduced. Then $R/I$ is a one-dimensional Cohen–Macaulay ring.*

*Proof.* Since $I$ has height $(n-1)$ in $R$ and $R$ is Cohen–Macaulay, it follows from the height-equality that $R/I$ is of dimension 1. Since $R/I$ is reduced, $I$ is radical and can be written as as a finite intersection of minimal prime ideals $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$, where height$(\mathfrak{p}_i) \geq n-1$. If height$(\mathfrak{p}_i) = n$ holds for some $i$, then $\mathfrak{p}_i = \mathfrak{m}$ (any prime ideal is contained in a maximal ideal). But $\mathfrak{m}$ cannot be a minimal element of the primary decomposition of $I$ since at least one $\mathfrak{p}_i$ is of height $(n-1)$ and hence strictly contained in $\mathfrak{m}$. Thus all $\mathfrak{p}_i$ have height $(n-1)$ and $I$ is equidimensional. Now it remains to show that the depth of

$R/I$ is 1. The maximal ideal of $R/I$ is $\overline{\mathfrak{m}}$, where $\overline{\mathfrak{m}}$ is the image of $\mathfrak{m}$ under the canonical projection. We show that $\mathfrak{m}$ is not contained in $\mathrm{Ass}(R/I) = \{\mathfrak{p} \in R \text{ prime: } \mathfrak{p} = \mathrm{ann}(\bar{a}), \text{ for an } \bar{a} \in R/I\}$. Suppose therefore that $\mathfrak{m}$ were contained in $\mathrm{Ass}(R/I)$. This means that there exists an $\bar{a} \neq 0$ such that $\overline{\mathfrak{m}} \cdot \bar{a} = \bar{0}$. If $\bar{a} \neq \bar{0}$ were also contained in $\overline{\mathfrak{m}}$, then $\bar{a}^2 = \bar{0}$ would hold, which is a contradiction to $R/I$ reduced. Hence $\bar{a} \in (R/I)^*$, that is, there exists some $\bar{b} \in R/I$ such that $\overline{a \cdot b} = \bar{1}$. Then $\overline{ab} \cdot \overline{\mathfrak{m}} = \overline{\mathfrak{m}}$ and by Nakayama's lemma $\overline{ab} = \bar{0}$. Contradiction. Thus there exists a $\bar{c} \in \overline{\mathfrak{m}}$ such that for all $\bar{a} \in R/I$ we have $\overline{ac} \neq \bar{0}$, that is, $R/I$ contains a nonzerodivisor. Hence $\bar{c}$ is regular on $R/I$, so $\mathrm{depth}(R/I) \geq 1$ and since its dimension is already one, it follows that $R/I$ is Cohen–Macaulay. □

**Proposition 3.2.** *Let $S$ be a 3-dimensional manifold and let $D \subseteq S$ be a divisor such that at a point $p$, $D$ is defined by $h \in \mathcal{O}_{S,p}$ and has radical Jacobian ideal $J_h \neq (1)$. Suppose moreover that the normalization $\widetilde{D}$ of $D$ is Gorenstein. Then one of the two cases occurs:*
*(i)* $\mathrm{depth}(J_h, \mathcal{O}_{S,p}) = 3$ *and $(D,p)$ is an $A_1$-singularity.*
*(ii)* $\mathrm{depth}(J_h, \mathcal{O}_{S,p}) = 2$ *and $D$ has normal crossings at $p$.*

*Proof.* (i) Since $J_h$ is of depth 3 on a three-dimensional local ring, it defines an isolated singularity. From the radicality of $J_h$ follows that $J_h$ has to be the maximal ideal $\mathfrak{m} \subseteq \mathcal{O}_{S,p}$. The rest is the content of Prop. 3.9.
(ii) If $\mathrm{depth}(J_h, \mathcal{O}_{S,p}) = 2$, then $J_h$ defines a reduced curve $C$ in $S$. Since $\mathcal{O}_{S,p}/J_h$ is a reduced one-dimensional local ring, it is Cohen–Macaulay by Lemma 3.1. Hence it follows by Theorem 2.1 that $D$ has normal crossings at $p$. □

**Question 3.3.** *Does there exist a surface $(D,p) \subseteq (\mathbb{C}^3, p)$ such that $(D,p)$ is free and $J_h = \sqrt{J_h} \neq (1)$ but $(\widetilde{D}, \tilde{p})$ is not Gorenstein?*

For $\dim S \geq 4$ the situation is more complicated. We split this part into two subsections.

### 3.1.1 Singular locus of codimension 1

This question was already considered, in case $J_h = \sqrt{J_h}$, $\mathrm{depth}(J_h) = 2$ and $\mathcal{O}_{S,p}/J_h$ is Cohen–Macaulay: then the equation $h = 0$ locally defines a normal crossing divisor $D$, that is, there exist complex coordinates $(x_1, \ldots, x_n)$ such that $h = x_1 \cdots x_m$, for some $m \leq n$. Then

the Jacobian ideal has the prime decomposition $J_h = \bigcap_{i<j\leq m}(x_i, x_j)$, which means that $\operatorname{Sing} D$ is locally at $p$ the union of $\binom{m}{2}$ smooth codimension 2 subvarieties of $S$. However, if we drop the Cohen–Macaulay condition, we know less. If $J_h$ is of depth 2 on $\mathcal{O}_{S,p}$ and $\mathcal{O}_{S,p}/J_h$ is not Cohen–Macaulay, then this means either that $J_h$ is not equidimensional or if $J_h$ is equidimensional, then $\operatorname{projdim}(\mathcal{O}/J_h) \neq 2$. Another way to phrase this is: $J_h$ is not perfect (see Appendix A).
For the equidimensional case we have a conjecture based on the following type of example:

*Example* 3.4. Consider a manifold $(S,p) \cong (\mathbb{C}^4, 0)$ with coordinates $p = (x,y,z,w)$. Then the ideal $I = (x,y) \cap (z,w) = (xz, xw, yz, yw) \subseteq \mathcal{O}_{S,p}$ is radical and defines an equidimensional 2-dimensional analytic space germ $(Z,p)$. In Remark A.5 it is shown that $\mathcal{O}_{S,p}/I$ is not Cohen–Macaulay, which implies that $I$ is not a complete intersection. By computation we show that there does not exist an $h \in \mathcal{O}_{S,p}$ such that $I = J_h$: first note that $I$ is the Jacobian ideal of a divisor defined by some $h \in \mathcal{O}_{S,p}$ if and only if there exists a matrix $A \in GL_4(\mathcal{O}_{S,p})$ such that

$$(\partial_x h, \partial_y h, \partial_z h, \partial_w h)^T = A\underline{f}^T, \tag{3.1}$$

where $\underline{f}$ is the vector $(f_1, \ldots, f_4) := (xz, yz, xw, yw)$. This follows from Nakayama's lemma. Since $A \in GL_4(\mathcal{O}_{S,p})$, the matrix $A(0)$ has to be in $GL_4(\mathbb{C})$. We will show that this cannot be the case. Therefore Lemma 3.5 is used: the partial derivatives of $h$ have to satisfy six equations, namely

$$\begin{aligned} &\partial_{xy} h = \partial_{yx} h, \partial_{xz} h = \partial_{zx} h, \partial_{xw} h = \partial_{wx} h, \\ &\partial_{yz} h = \partial_{zy} h, \partial_{yw} h = \partial_{wy} h, \partial_{zw} h = \partial_{wz} h. \end{aligned}$$

From (3.1) it follows that these equations are of the form (e.g., for $\partial_{xz} h = \partial_{zx} h$)

$$\partial_x a_{3-} \cdot \underline{f} + a_{31} z + a_{33} w = \partial_z a_{1-} \cdot \underline{f} + a_{11} x + a_{12} y, \tag{3.2}$$

where $\partial_x a_{i-} \cdot \underline{f}$ stands for $\sum_{j=1}^4 \partial_x a_{ij} \cdot f_j$, and $a_{ij} \in \mathcal{O}_{S,p}$ are the entries of $A$. Denote by $\alpha_{ij} \in \mathbb{C}$ the constant term of $a_{ij}$. The order of $\partial_z a_{1-} \cdot \underline{f}$ (and similarly of $\partial_x a_{3-} \cdot \underline{f}$) is greater or equal 2 (since $\operatorname{ord}(f_i) = 2$). Thus for the order 1 term of (3.2) the equation $\alpha_{31} z + \alpha_{33} w = \alpha_{11} x + \alpha_{12} y$ holds. This implies $\alpha_{11} = \alpha_{12} = \alpha_{31} = \alpha_{33} = 0$. Similarly follows from $\partial_{xw} h = \partial_{wx} h$ that $\alpha_{13} = \alpha_{14} = 0$. Thus in the matrix $A(0)$ the first row is zero, which means that $A \notin GL_4(\mathcal{O}_{S,p})$. Hence there does not exist an $h$ as asserted and the ideal $I$ cannot be the Jacobian ideal of a divisor $D$.

**Lemma 3.5.** *Let $f_1, \ldots, f_n$ be in $\mathcal{O}_{S,p}$. Then there exists a $g \in \mathcal{O}_{S,p}$ such that $f_i = \partial_{x_i} g$ if and only if for all $1 \leq i, j \leq n$ we have*

$$\partial_{x_i} f_j = \partial_{x_j} f_i.$$

*Proof.* Let $f_1, \ldots, f_n$ be such that $\partial_{x_i} f_j = \partial_{x_j} f_i$. Define the differential form $\omega = \sum_{i=1}^n f_i dx_i$. By Poincaré's lemma $\omega = dg$ for some $g \in \mathcal{O}_{S,p}$ if and only if $d\omega = 0$. A computation shows

$$d\omega = \sum_{i<j} (\partial_{x_i} f_j - \partial_{x_j} f_i) dx_i \wedge dx_j,$$

which is (using the relation between the $\partial_{x_j} f_i$) equal to zero. The other implication follows by reading the argument backwards. □

**Conjecture 3.6.** *Let $D \subseteq S$ be a divisor defined at a point $p$ by $h \in \mathcal{O}_{S,p}$. If $J_h$ is radical, of depth 2 on $\mathcal{O}_{S,p}$ and equidimensional, then $\mathcal{O}_{S,p}/J_h$ is already Cohen–Macaulay. In other words: we conjecture that if a divisor that has locally at a point $p \in S$ an equidimensional radical Jacobian ideal of depth 2 is already free at $p$.*

If $J_h$ is not equidimensional, the only thing we can say is that it is the intersection of some prime ideals whose minimal height is 2.

*Example* 3.7. (The Jacobian ideal can be of height 2 and radical but it may not be equidimensional) Consider $S = \mathbb{C}^5$ at the origin with coordinates $(x, y, z, s, t)$. Let the divisor $D$ be locally defined by $h = (x^2 + y^2 + z^2)(s^2 - t^2) \in \mathcal{O} = \mathbb{C}\{x, y, z, s, t\}$. Note that $D$ is splayed and the union of a normal crossing divisor and a cone. The Jacobian ideal $J_h = (xs^2 - xt^2, ys^2 - yt^2, zs^2 - zt^2, x^2 s + y^2 s + z^2 s, x^2 t + y^2 t + z^2 t)$ is radical, its height is 2 and it has the prime decomposition

$$(x, y, z) \cap (s - t, x^2 + y^2 + z^2) \cap (s, t) \cap (s + t, x^2 + y^2 + z^2).$$

The ideal $J_h$ is not unmixed and hence $\mathcal{O}_{\operatorname{Sing} D} = \mathcal{O}/J_h$ is not Cohen–Macaulay.

**Question 3.8.** *Suppose that $J_h$ of a divisor $D$ is radical and of height 2 but not equidimensional. Which $J_h$ are possible?*

### 3.1.2 Higher codimensional singular locus

We start to tackle this question with special cases that are easy generalizations of the free divisor case from Chapter 2:

**Proposition 3.9.** *Let $D$ be a divisor in an $n$-dimensional complex manifold $S$, locally at a point $p = (x_1, \ldots, x_n)$ defined by a reduced $h \in \mathcal{O}_{S,p}$. Let $J_h = \sqrt{J_h}$ be the Jacobian ideal and denote by $(\mathrm{Sing}\, D, p)$ the singular locus of $D$ at $p$ with associated ring $\mathcal{O}_{\mathrm{Sing}\, D,p} = \mathcal{O}/J_h$. Suppose that $\mathrm{codim}_p(\mathrm{Sing}\, D, S) = k$ and that $(\mathrm{Sing}\, D, p)$ is smooth. Then $D$ is locally isomorphic to $\{x_1^2 + \cdots + x_k^2 = 0\}$, that is, $(D, p)$ is isomorphic to a Cartesian product $(V \times \mathbb{C}^{n-k}, (0,0))$ where $(V, 0) = \{x_1^2 + \cdots + x_k^2 = 0\}$ is an $A_1$-singularity in $(\mathbb{C}^k, 0)$.*

*Proof.* This is a generalization of the smooth case, see Lemma 2.17. Since $\mathrm{Sing}\, D$ is smooth and of codimension $k$ at $p$, we may assume $J_h = (x_1, \ldots, x_k)$. Since $J_h$ is radical, $h$ is also contained in $J_h$ and can be written as $h = \sum_{i=1}^k f_i x_i$. For any $j \leq k$ the $\partial_{x_j} h = \sum_{i=1}^k \partial_{x_j} f_i x_i + f_j$ are in $J_h$. This implies that all $f_j$, $1 \leq j \leq k$ are also contained in $J_h$. Hence $h$ lies in $(x_1, \ldots, x_k)^2$ and it can be written as $h = \sum_{i=1}^k a_i x_i^2 + \sum_{1 \leq i < j \leq k} b_{ij} x_i x_j$. Computing $\partial_{x_j} h$ for $j > k$ yields $\partial_{x_j} h \in \mathfrak{m} J_h$. An application of Nakayama's lemma yields

$$(\partial_{x_1} h, \ldots, \partial_{x_k} h) = J_h = (x_1, \ldots, x_k).$$

Hence using the Triviality lemma A.44, one finds that $D$ is locally analytically trivial along $\{x_{k+1} = \ldots = x_n = 0\}$ and we may assume that $h(x_1, \ldots, x_n) = h(x_1, \ldots, x_k, 0)$. Thus $D$ may be considered in $(\mathbb{C}^k, 0)$ defined by $h^*(x_1, \ldots, x_k) := h(x_1, \ldots, x_k, 0)$. But in this situation $J_{h^*} = (x_1, \ldots, x_k)$ defines an isolated singularity and by the theorem of Mather–Yau (or direct computation) it follows that $\{h = 0\} \cong \{x_1^2 + \cdots + x_k^2 = 0\}$. □

**Proposition 3.10.** *Let $J_h = \sqrt{J_h}$ be the Jacobian ideal of the divisor $D \subseteq S$ and denote by $\mathrm{Sing}\, D$ its singular locus with associated ring $\mathcal{O}_{\mathrm{Sing}\, D,p} = \mathcal{O}/J_h$ at $p$. Suppose that $\mathrm{codim}_p(\mathrm{Sing}\, D, S) = k$ and that $(\mathrm{Sing}\, D, p)$ is a complete intersection. Then $D$ is isomorphic to $\{x_1^2 + \cdots + x_k^2 = 0\}$, that is, $D$ has locally along $\mathrm{Sing}\, D$ an $A_1$-singularity.*

*Proof.* The proof is again similar to the free divisor case, see Prop. 2.18. Since $(\mathrm{Sing}\, D, p)$ is a complete intersection, there exist $f_1, \ldots, f_k \in \mathcal{O}$

such that $J_h = (f_1, \ldots, f_k)$. Since the $f_i$ generate $J_h$, there is an $n \times k$ matrix $A$ with entries in $\mathcal{O}$ such that

$$A(f_1, \ldots, f_k)^T = (\partial_{x_1} h, \ldots, \partial_{x_n} h)^T.$$

Consider the $\mathcal{O}/\mathfrak{m} = \mathbb{C}$ module $J_h/\mathfrak{m}J_h$: the $f_i$ are a minimal set of generators of $J_h$, thus their residues modulo $\mathfrak{m}J_h$, denoted by $\bar{f}_1, \ldots, \bar{f}_k$, form a basis of the $\mathbb{C}$-vector space $J_h/\mathfrak{m}J_h$ and the matrix $\bar{A}$ has entries in $\mathbb{C}$. Hence the linear system of equations

$$\bar{A}(\bar{f}_1, \ldots, \bar{f}_k)^T = (\overline{\partial_{x_1} h}, \ldots, \overline{\partial_{x_n} h})^T$$

is solvable over $\mathbb{C}$, that is, there exist $k$ partial derivatives, wlog. $\partial_{x_1} h$, $\ldots, \partial_{x_k} h$ such that $J_h = (\partial_{x_1} h, \ldots, \partial_{x_k} h)$. But then the remaining $\partial_{x_i} h$, $i = k+1, \ldots, n$ are contained in $(\partial_{x_1} h, \ldots, \partial_{x_k} h)$ and once more an application of the Triviality lemma shows that $D$ is trivial along $\{x_{k+1} = \ldots = x_n = 0\}$. Hence $D$ may be considered in $(\mathbb{C}^k, 0)$ and defined by $h^*(x_1, \ldots, x_k) = h(x_1, \ldots, x_k, 0)$. Then since $J_{h^*}$ is a complete intersection of codimension $k$ in $(\mathbb{C}^k, 0)$, it defines an isolated singularity. Since $J_{h^*}$ is by assumption radical, the only possibility is $J_{h^*} = (x_1, \ldots, x_k)$. Like in the previous proposition we find that $D$ is locally isomorphic to $\{x_1^2 + \cdots x_k^2 = 0\}$. $\square$

**Corollary.** *Let $J_h = \sqrt{J_h}$ be the Jacobian ideal of the divisor $D$ and denote by $(\operatorname{Sing} D, p)$ its singular locus with associated ring $\mathcal{O}_{\operatorname{Sing} D, p} = \mathcal{O}/J_h$. Suppose that $\operatorname{codim}_p(\operatorname{Sing} D, S) = k$. Then $(\operatorname{Sing} D, p)$ is a complete intersection if and only if $\operatorname{Sing} D$ is locally at $p$ smooth and thus $D$ is isomorphic to $\{x_1^2 + \cdots + x_k^2 = 0\}$.*

*Remark* 3.11. Note that if $J_h$ is radical of height $n$ in $\mathcal{O}$, then $J_h$ is already equal to the maximal ideal of $\mathcal{O}$ and hence the corresponding germ $(D, p)$ is an $A_1$-singularity. Moreover, in Lemma 3.1 it is shown that if $J_h = \sqrt{J_h}$ has height $n-1$ in $\mathcal{O}$, then $\mathcal{O}/J_h$ is already Cohen–Macaulay. However, in this case we cannot ensure a priori that $J_h = \sqrt{J_h}$ is a complete intersection.

It is not clear how to treat non-complete intersection radical Jacobian ideals of height $k$, $3 \leq k < n$ in $\mathcal{O}$, we do not even know if there exist divisors $D = \{h = 0\}$ such that $J_h = \sqrt{J_h}$ is not a complete intersection.

*Example* 3.12. Consider $S = \mathbb{C}^6$ at the origin with coordinates $(x_1, \ldots, x_6)$. The ideal $I = (x_1, x_2, x_3) \cap (x_4, x_5, x_6) \subseteq \mathcal{O} = \mathbb{C}\{x_1, \ldots, x_6\}$ is of height

3, so $\dim(\mathcal{O}/I) = 3$. A computation with SINGULAR [98] shows that $\mathrm{projdim}_{\mathcal{O}}(\mathcal{O}/I) = 5$ and by Auslander–Buchsbaum $\mathrm{depth}(\mathcal{O}/I) = 1$. Hence $\mathcal{O}/I$ is not Cohen–Macaulay and $I$ is not a complete intersection ideal. We show that $I$ cannot be the Jacobian ideal of some $h \in \mathcal{O}$. If $I$ were $J_h$ for some (reduced) $h \in \mathcal{O}$, then

$$(\partial_{x_1} h, \ldots, \partial_{x_6} h)^T = A\underline{f}^T \tag{3.3}$$

would hold, where $\underline{f} = (x_1x_4, x_1x_5, x_1x_6, x_2x_4, x_2x_5, x_2x_6, x_3x_4, x_3x_5, x_3x_6)$ and $A$ is an $6 \times 9$ matrix with entries in $\mathcal{O}$. Considering $I$ as an $\mathcal{O}$-module, take the equation modulo $\mathfrak{m}I$ and obtain the equation $\overline{A}\overline{\underline{f}}^T = \overline{\partial_x h}^T$, where the entries of $\overline{A}$ are in $\mathbb{C}$. Using Nakayama's lemma, it follows that the rank of $\overline{A}$ is 6. Denote again by $\alpha_{ij}$ the constant parts of the entries of $A$. Apply Lemma 3.5: from $\partial_{x_4x_1} h = \partial_{x_1x_4} h$, $\partial_{x_1x_5} h = \partial_{x_5x_1} h$ and $\partial_{x_6x_1} h = \partial_{x_1x_6} h$ it follows that all $\alpha_{1i}$ must be zero and hence the rank of $\overline{A}$ is strictly smaller than 6. (Using the remaining relations, all $\alpha_{ij}$ are found to be zero).

The propositions and examples above motivate the following

**Conjecture 3.13.** *Let $D$ be a divisor in a complex manifold $S$, defined locally at a point $p$ by a reduced $h \in \mathcal{O}_{S,p}$. Suppose that the Jacobian ideal $J_h$ is radical, equidimensional and of depth $\geq 3$ on $\mathcal{O}_{S,p}$. Then the variety* $\mathrm{Sing}\, D$ *with coordinate ring $\mathcal{O}/J_h$ is a complete intersection, that is,* $\mathrm{Sing}\, D$ *is Cohen–Macaulay and must even be smooth by Prop. 3.10.*

## 3.2 Properties of splayed divisors

Splayed divisors were introduced in Chapter 2 to prove Thm. 2.1. But splayed divisors are interesting in their own right, in particular for computational reasons. We start here a study of properties of splayed divisors by considering their Hilbert–Samuel polynomials. We find that multiplicities behave the same for splayed as for non–splayed divisors but that the Hilbert–Samuel polynomials for splayed divisors are additive, which means the following: let $(D,p) = (D_1,p) \cup (D_2,p)$ be a splayed divisor at a point $p$ in a complex manifold. Then from the exact sequence

$$0 \to \mathcal{O}_{D,p} \to \mathcal{O}_{D_1,p} \oplus \mathcal{O}_{D_2,p} \to \mathcal{O}_{D_1 \cap D_2,p} \to 0$$

follows

$$\chi_{D,p} + \chi_{D_1 \cap D_2,p} = \chi_{D_1,p} + \chi_{D_2,p},$$

where $\chi_{D,p}$ denotes the Hilbert–Samuel polynomial of $D$ at $p$. Then a "geometric" characterization of splayed divisors in terms of their Jacobian ideals is given (see Prop. 3.37). It would be interesting to compute other singularity invariants for splayed divisors, therefore we list some questions at the end of this section.

We mostly use notation from [62] and [27]. Often the term "additive function" is used. With this we mean the following: let $R$ be a noetherian ring and let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of finitely generated $R$-modules. Then a function $\lambda : \mathbf{Rmod} \to \mathbb{Z}$ is called *additive* if $\lambda(M_1) - \lambda(M_2) + \lambda(M_3) = 0$.
Recall that a divisor $D$ in a complex manifold $S$ with $\dim S = n$ is called *splayed* at a point $p$ if there exist complex coordinates $(x_1, \ldots, x_n)$ at $p$ such that $(D,p) = (D_1,p) \cup (D_2,p)$ is defined by

$$h(x) = h_1(x_1, \ldots, x_k, 0)h_2(0, x_{k+1}, \ldots, x_n),$$

$1 \leq k \leq n-1$, where $h_i$ is the defining reduced equation of $D_i$. When working with splayed divisors, we will simplify the notation and write for the coordinates $(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = (x, y)$ and $h(x,y) = h_1(x)h_2(y)$. Moreover, we will also abbreviate $\mathcal{O} := \mathcal{O}_{S,p} = \mathbb{C}\{x, y\}$.

## 3.2.1 The polynomial case

First we consider graded modules over polynomial rings. We introduce the Hilbert function, the Hilbert–Poincaré series and the Hilbert polynomial. Then we compute these objects for divisors $D$ in some $\mathbb{P}^n_{\mathbb{C}}$ defined by polynomial equations. For the results of the computations it makes no difference whether $D$ is splayed or not.

### Hilbert function, Hilbert–Poincaré series and Hilbert polynomial

In this section the above notions are introduced and some useful results are given for the computation of Hilbert–Poincaré series and Hilbert polynomials. Let $k$ be a field and let $A = \bigoplus_{n \geq 0} A_n$ be a noetherian graded $k$-algebra and let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated $A$-module. The *Hilbert-function* $H_M : \mathbb{Z} \to \mathbb{Z}$ of $M$ is defined by

$$H_M(n) = \dim_k(M_n).$$

The *Hilbert–Poincaré series* $P_M$ of $M$ is the formal power series defined by

$$P_M(t) = \sum_{n\in\mathbb{Z}} H_M(n)t^n \in \mathbb{Z}[[t]][t^{-1}].$$

One can easily show the following properties of $H_M$ and $P_M$:
Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely generated graded $A$-modules (where $A$ is as above). Then

$$H_M(n) = H_{M'}(n) + H_{M''}(n)$$

for all $n$, that is, $H$ is additive. This implies that

$$P_M(n) = P_{M'}(n) + P_{M''}(n).$$

For an integer $d$ one has

$$H_{M(d)}(n) = H_M(n+d),$$

where $M(d)$ is the $d$-shifted graded module $M$, that is, $(M(d))_n = M_{d+n}$. From this follows

$$P_{M(d)}(t) = t^{-d}P_M(t).$$

Keeping the notation from above and assuming that $A_1 = (x_1, \ldots, x_n)A_0$ generates $A$ as an $A_0$-algebra, i.e., $A = A_0[x_1, \ldots, x_n]$, one finds that there exists a polynomial $Q_M(t) \in \mathbb{Z}[t]$ such that $P_M(t) = \frac{Q_M(t)}{(1-t)^n}$, see e.g. [27, Prop. 4.2.10].
One can show that $H_M(d)$ behaves like a polynomial for $d \gg 0$. Then the *Hilbert-Polynomial* $\varphi_M \in \mathbb{Q}[d]$ is defined as the polynomial such that $\varphi_M(d) = H_M(d)$ for all positive integers $d \gg 0$. One can show that $\deg \varphi_M \leq n-1$, see e.g., [62]. The Hilbert-Poincaré series determines the Hilbert-polynomial in the following way (see [45, Definition 5.1.4]): Write $P_M(t)$ as

$$P_M(t) = \frac{G(t)}{(1-t)^s}, \quad 0 \leq s \leq n, \quad G(t) = \sum_{i=0}^{d} g_i t^i \in \mathbb{Z}[t],$$

such that $g_d \neq 0$ and $G(1) \neq 0$. This means that the order of the pole of $P_M(t)$ at $t = 1$ is equal to $s$. Then the Hilbert-polynomial of $M$ is

$$\varphi_M(t) = \sum_{i=0}^{d} g_i \binom{s-1+t-i}{s-1} \in \mathbb{Q}[t].$$

For the computation of the Hilbert–Poincaré series the following lemma is useful.

**Lemma 3.14.** *(a) Let $I \subseteq k[x_1, \ldots, x_n]$ be a homogeneous ideal and let $f$ be a homogeneous element of $k[x_1, \ldots, x_n]$ of degree $\deg(f) = d$. Then one has*

$$P_{k[x]/I}(t) = P_{k[x]/(I,f)}(t) + t^d P_{k[x]/(I:(f))}(t).$$

*(b) Let $>$ be a monomial ordering on $k[x]$. Then*

$$P_{k[x]/I}(t) = P_{k[x]/L(I)}(t),$$

*where $L(I)$ denotes the leading ideal of $I$, that is, the ideal generated by the leading monomials of elements in $I$ (see definition 3.23).*

*Proof.* (a) is Lemma 5.2.2 of [45] and (b) is Theorem 5.2.6. of loc. cit. □

*Example* 3.15. (a) Let $k[x] = k[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables with the standard grading. Then the Hilbert function of $k[x]$ is given as $H_{k[x]}(d) = \binom{n+d-1}{n-1}$ because there are $\binom{n+d-1}{n-1}$ monomials of degree $d$ in $k[x]$. Hence $P_{k[x]}(t) = \frac{1}{(1-t)^n}$.
(b) Let $f$ be a homogeneous polynomial of degree $d$. Using (a) of the preceding lemma with $I = (0)$ one finds $P_{k[x]}(t) = P_{k[x]/(f)}(t) + t^d P_{k[x]}$ and hence

$$P_{k[x]/(f)} = \frac{1-t^d}{(1-t)^n} = \frac{\sum_{i=1}^{d-1} t^i}{(1-t)^{n-1}}.$$

Using the procedure to compute the Hilbert polynomial of $k[x]/(f)$ described above, yields $\varphi_{k[x]/(f)}(t) = \sum_{i=1}^{d-1} \binom{n-2+t-i}{n-2}$.

*Remark* 3.16. The Hilbert polynomial of a hypersurface $D = \{f = 0\}$ depends only on the degree of $f \in k[x]$.

The following lemma will be used in the sequel to compute Hilbert–Poincaré series and Hilbert–Samuel polynomials.

**Lemma 3.17.** *Let $R$ be any commutative ring and let $I, J$ be two ideals in $R$. Then the sequence*

$$0 \to R/(I \cap J) \to R/I \oplus R/J \to R/(I+J) \to 0$$

*with the first map: $a \mod (I \cap J) \mapsto (a \mod I, a \mod J)$, and the second map: $(a \mod I, b \mod J) \mapsto (a-b) \mod (I+J)$, is exact.*

*Proof.* Computation. □

**Application to divisors**

Now the Hilbert–Poincaré series and the Hilbert-polynomial are computed for divisors defined by homogeneous polynomials. Let therefore $k[x] = k[x_1, \ldots, x_n]$ be the polynomial ring over a field $k$ in $n$ variables considered as graded algebra with the standard grading. Let $D = \{g(x)h(x) = 0\}$ be a divisor with $g$ and $h$ homogeneous of degree $m$ and $k$. Using Lemma 3.14 with $I = (gh)$ and $f = g$ and the example (b) above, the Hilbert–Poincaré series has the following form

$$P_{k[x]/(gh)}(t) = P_{k[x]/(g)}(t) + t^m P_{k[x]/(h)}(t) = \frac{1 - t^{m+k}}{(1-t)^n}$$

and the Hilbert-polynomial $\varphi_{k[x]/(gh)}(t) = \sum_{i=0}^{m+k-1} \binom{n-2+t-i}{n-2}$. We are also able to compute the Hilbert–Poincaré series for the intersection of $\{g(x) = 0\}$ and $\{h(x) = 0\}$: from Lemma 3.14 (a) with $I = (g)$ and $f = h$ follows $P_{k[x]/(g)}(t) = P_{k[x]/(g,h)}(t) + t^k P_{k[x]/(h)}$, and hence

$$P_{k[x]/(g,h)}(t) = \frac{(1-t^k)(1-t^m)}{(1-t)^n}.$$

Combining the explicit formulas one sees that the Hilbert–Poincaré series is additive:

$$P_{k[x]/(gh)}(t) + P_{k[x]/(g,h)}(t) = P_{k[x]/(g)}(t) + P_{k[x]/(h)}(t).$$

The Hilbert-polynomial can easily be computed using the exact sequence from Lemma 3.17 and the obvious exact sequence

$$0 \to k[x]/(g) \to k[x]/(g) \oplus k[x]/(h) \to k[x]/(h) \to 0.$$

Using the additivity of the Hilbert-polynomials, it follows that

$$\varphi_{k[x]/(g,h)}(t) = \varphi_{k[x]/(g)}(t) + \varphi_{k[x]/(h)}(t) - \varphi_{k[x]/(gh)}(t).$$

If $g$ and $h$ from above are not homogeneous, Lemma 3.14 (b) can be used to compute the Hilbert–Poincaré series of $k[x,y]/(gh)$ explicitly.

Up to now the splayed property has played no role!

### 3.2.2 Dimension, multiplicity and beyond

Our main objective is to study the local case, namely the Hilbert–Samuel polynomials of finite modules over local rings. Therefore we introduce the local counterparts to the Hilbert-function and the Hilbert polynomial, namely the Hilbert–Samuel function and the Hilbert–Samuel polynomial. These do not only depend on the module one is considering but also on a chosen filtration. However, the degree and leading coefficient of the Hilbert–Samuel polynomial are independent of the filtration. So one gets characterizations of dimension and multiplicities of modules over a local ring. Here the definitions are given (mostly without proofs, which can be found for example in [27, 45, 62]).
First we compute multiplicities of divisors $D = D_1 \cup D_2$. Then we see that the Hilbert–Samuel polynomials of splayed divisors are additive.

#### Hilbert–Samuel polynomials and standard bases

Let $R$ be a noetherian local ring with maximal ideal $\mathfrak{m}$. Let $I \subseteq R$ be an ideal and let $M$ be a module over $R$. A set $\{M_n\}_{n\geq 0}$ of submodules of $M$ is called an *$I$-filtration* of $M$ if $M = M_0 \supset M_1 \supset M_2 \supset \ldots$ and $IM_n \subseteq M_{n+1}$ for all $n \geq 0$. An $I$-filtration is called *stable* if $IM_n = M_{n+1}$ for $n \geq 0$ sufficiently large.
Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of $R$ and let $\{M_i\}$ be a $\mathfrak{q}$-filtration. Then the *Hilbert–Samuel function of the filtration* $\{M_i\}$ is

$$HS_{\{M_i\}} : \mathbb{N} \to \mathbb{N}, d \mapsto \mathrm{length}_{R/\mathfrak{m}} M/M_d.$$

In order to see that this definition makes sense, namely that the length of $(M/\mathfrak{q}^n)$ over $R/\mathfrak{m}$ is finite, one considers the associated graded module $\mathrm{Gr}_{\mathfrak{q}}(M) = \bigoplus_{n=0}^{\infty} M_n/M_{n+1}$ and reduces everything to the homogeneous case, see e.g. [27, 4.2] or [62, §13]. Further one can show that there exists a polynomial $\chi_{\{M_i\}}$ with rational coefficients such that $HS_{\{M_i\}}(d) = \chi_{\{M_i\}}(d)$ for $d$ sufficiently large. We call $\chi_M^{\mathfrak{q}} := \chi_{\{\mathfrak{q}^n M\}_{n\geq 0}}$ the *Hilbert–Samuel polynomial of $M$ with respect to $\mathfrak{q}$*. The degree of $\chi_M^{\mathfrak{q}}(k) = \sum_{i=0}^{d} a_i k^i$ only depends on $M$ and not on $\mathfrak{q}$. Therefore we denote the degree $d$ of $\chi_M^{\mathfrak{q}}$ by $d(M)$. By a result of dimension theory, the Krull-dimension $\dim(M)$ is equal to the degree of the Hilbert–Samuel polynomial $d(M)$. The leading coefficient $a_d$ of the Hilbert-Samuel polynomial depends on $\mathfrak{q}$ and the positive integer $d!a_d$ is called the *multiplicity of* $M$ *with respect to* $\mathfrak{q}$ and is denoted by $\mathfrak{e}(M, \mathfrak{q})$. The multiplicity $\mathfrak{e}(M, \mathfrak{m})$ is simply called the *multiplicity* of $M$.

In general, there is no straightforward way to compute Hilbert–Samuel polynomials. However, when considering Hilbert–Samuel polynomials of modules over a local ring $\mathcal{O} = \mathbb{C}\{x_1, \ldots, x_n\}$ w.r.t. the maximal ideal $\mathfrak{m}$, one can use standard bases to simplify computations. In particular one obtains an analogue to the graded case, namely that the Hilbert–Samuel function of some $\mathcal{O}$-module $\mathcal{O}/I$ is equal to the Hilbert–Samuel function of $\mathcal{O}/L(I)$. Here we will only give the definitions and theorems that we need to state this result and to prove the additivity for Hilbert–Samuel polynomials of splayed divisors.

**Definition 3.18.** Let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial in $\mathbb{C}[x]$ and denote by $S = \{x^\alpha : \alpha \in \mathbb{N}^n\}$ the set of all monomials in $\mathbb{C}[x]$. A *monomial ordering* is a total ordering on $S$ (resp. $\mathbb{N}^n$) compatible with the semigroup structure, i.e., from $x^\alpha > x^\beta$ follows $x^{\alpha+\gamma} > x^{\beta+\gamma}$ for all $\gamma \in \mathbb{N}^n$. (We will always assume that the considered orderings are well-orderings, i.e., every non-empty set of monomials has a minimal element w.r.t. to the ordering.)

**Definition 3.19.** Let $>$ be a monomial ordering and let $f \in \mathbb{C}\{x\}$ be a non-zero power series. Then $f$ can be written in the form

$$f = \sum_{i \geq 1} a_i x^{\alpha_i},$$

such that $a_i \neq 0 \in \mathbb{C}$ and $x^{\alpha_i} < x^{\alpha_{i+1}}$ for all $i$. Then $L(f) = x^{\alpha_1}$ is called the *leading monomial* of $f$. For an ideal $I \in \mathbb{C}\{x\}$ we call $L(I) = (\{L(f) : f \in I, f \neq 0\})$ the *leading ideal.* Note that $L(I)$ is the ideal generated by all leading monomials of nonzero $f \in I$.

*Remark* 3.20. In general the leading ideal $L(I)$ is not equal to the ideal generated by the leading monomials of a set of generators of $I$. However, if the generators $f_1, \ldots, f_k$ of $I$ form a standard basis (see next definition), then $L(I) = (L(f_i))$.

In the following we use the *degree lexicographical ordering* with a weight vector $w = (w_1, \ldots, w_n) \in \mathbb{R}_+^n$. For $\alpha \in \mathbb{N}^n$ define $|\alpha|_w := \sum_{i=1}^n \alpha_i w_i$. Then $x^\alpha < x^\beta$ if and only if $|\alpha|_w < |\beta|_w$ or if $|\alpha|_w = |\beta|_w$, then $\alpha < \beta$

with respect to the lexicographical ordering. In order to define standard bases an analogous of division with remainder of polynomials for power series is needed. Therefore we recall

**Theorem 3.21** (Grauert's division theorem)**.** *Let* $f_1, \ldots, f_m \in \mathbb{C}\{x\}$*. Then for any* $g \in \mathbb{C}\{x\}$ *there exist elements* $q_1, \ldots, q_m \in \mathbb{C}\{x\}$ *and an element* $r \in \mathbb{C}\{x\}$ *such that*

$$g = \sum_{i=1}^{m} q_i f_i + r,$$

*satisfying the two conditions:*
*(1) No monomial of* $r$ *is divisible by* $L(f_i)$ *for* $i = 1, \ldots, m$*,*
*(2)* $L(q_i f_i) \geq L(g)$ *for* $i = 1, \ldots, m$*.*

*Proof.* See [27, Theorem 7.1.9]. □

*Remark* 3.22. Denote $S = \{f_1, \ldots, f_m\}$ the ordered set formed by the $f_i$ of the theorem. In the proof of Grauert's division theorem one constructs the $r$ of the theorem explicitly and this $r$ is uniquely determined with respect to $S$. Then one calls $NF(f|S) := r$ (or just $NF(f)$ if $S$ is fixed) the *normal form* of $f$.

**Definition 3.23.** Let $I$ be an ideal in $\mathbb{C}\{x\}$. A set $S = \{f_1, \ldots, f_m\}$, with all $f_i \in I$ is called a *standard basis* of $I$ if

$$L(I) = (L(f_1), \ldots, L(f_m)).$$

*Remark* 3.24. One can show that any ideal $I$ of $\mathbb{C}\{x\}$ has a standard basis and if $S = \{f_1, \ldots, f_m\}$ is such a standard basis, then $I = (f_1, \ldots, f_m)$. For two standard bases $S = \{f_1, \ldots, f_m\}$ and $T = \{g_1, \ldots g_k\}$ of $I$, the equality $NF(f|S) = NF(f|T)$ holds for any $f \in \mathbb{C}\{x\}$. This means that $NF(f)$ is independent of the chosen standard basis. Then one can construct (again in analogy to the polynomial case of Gröbner bases) standard bases via syzygy polynomials and *Buchberger's criterion*, but this is not needed here. For details see [27].

**Lemma 3.25.** *Let* $f, g \in \mathbb{C}\{x\}$ *and assume that* $L(f)$ *and* $L(g)$ *are coprime. Then* $L((f, g)) = (L(f), L(g))$*, that is,* $f, g$ *are a standard basis of the ideal* $(f, g)$*.*

*Proof.* (Notations from [27]) Choose a monomial ordering $<$. In order to show that $f$ and $g$ form a standard basis, we have to show that

$NF(\mathrm{spoly}(f,g)|\{f,g\}) = 0$ (for the definition of $\mathrm{spoly}(f,g)$, the $S$-polynomial of the leading coefficients of $f$ and $g$, see [27]). Suppose that $f = x^\alpha + P(x)$ and $g = x^\beta + Q(x)$ with $L(f) = x^\alpha$ and $L(g) = x^\beta$ are coprime. Then of course $x^\alpha < P$ and $x^\beta < Q$. The power series $\mathrm{Syz}(f,g) = x^\beta P - x^\alpha Q$ can be written in the form of Grauert's division theorem as $x^\beta f - x^\alpha g$. Hence $NF(\mathrm{Syz}(f,g)|\{f,g\}) = 0$. □

The following Proposition 3.26 requires the weight-vector $w \in \mathbb{R}^n_+$ the chosen monomial ordering $<$ to be equal to $(1,\ldots,1)$. So from now on we assume $w = (1,\ldots,1)$.

**Proposition 3.26.** *Let $I \subseteq \mathbb{C}\{x\}$ be an ideal. Then*

$$HS_{\mathbb{C}\{x\}/I,\mathfrak{m}}(k) = HS_{\mathbb{C}\{x\}/L(I),\mathfrak{m}}(k).$$

*In particular, $\mathbb{C}\{x\}/I$ and $\mathbb{C}\{x\}/L(I)$ have the same Hilbert–Samuel polynomial with respect to $\mathfrak{m}$.*

*Proof.* See [27]. □

In contrast to the graded case, the Hilbert–Samuel polynomial is not additive on exact sequences, one has a certain error polynomial, whose degree can be determined with a theorem of Flenner and Vogel, see Thm. 3.28 below.

**Lemma 3.27.** *Let $(R,\mathfrak{m})$ be a noetherian local ring, $0 \to M' \to M \to M'' \to 0$ an exact sequence of finitely generated $R$-modules and $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal. Then*

$$\chi^{\mathfrak{q}}_M = \chi^{\mathfrak{q}}_{M'} + \chi^{\mathfrak{q}}_{M''} - S,$$

*where $S$ is a polynomial, whose degree is strictly smaller than the degree of $\chi^{\mathfrak{q}}_{M'}$.*

**Theorem 3.28** (Flenner–Vogel)**.** *Notation as in the lemma. Denote further*

$$\mathrm{Gr}_{\mathfrak{q}}(M) = \bigoplus_{i=0}^{\infty} \mathfrak{q}^i M/\mathfrak{q}^{i+1} M$$

*the associated graded ring of a finite $R$-module $M$. Then, for the exact sequence of the lemma, the following holds:*
*(a)* $\mathrm{supp}\ker(\mathrm{Gr}_{\mathfrak{q}}(M') \to \mathrm{Gr}_{\mathfrak{q}}(M)) = \mathrm{supp}\ker(\mathrm{Gr}_{\mathfrak{q}}(M)/\mathrm{Gr}_{\mathfrak{q}}(M') \to$

$\mathrm{Gr}_{\mathfrak{q}}(M''))$.
*(b) Denote d the dimension of these supports. Then for all* $n \geq 0$

$$S(n) := \chi^{\mathfrak{q}}_{M'}(n) + \chi^{\mathfrak{q}}_{M''}(n) - \chi^{\mathfrak{q}}_{M}(n),$$

*where* $S(n)$ *is a polynomial of degree* $d-1$ *for* $n \gg 0$*. In particular, if* $d = 0$*, then set* $S = 0$*.*

*Proof.* See [34]. □

### Multiplicities – Additivity of Hilbert–Samuel polynomials

First a well-known general result about multiplicities of divisors is shown. Then we see that for splayed divisors the Hilbert–Samuel polynomial is additive (Prop. 3.33). Let now $D = D_1 \cup D_2 \subseteq \mathbb{C}^n$ be a not necessarily splayed divisor that is locally at a point $p = (x, y)$ defined by $h_1(x)h_2(x) \in \mathcal{O} = \mathbb{C}\{x\}$, with components $(D_1, p) = \{h_1(x) = 0\}$ and $D_2 = \{h_2(x) = 0\}$. We assume here that the $h_i$ are not necessarily irreducible but have no common irreducible factor. The multiplicities of $D_i$ at $p$ are denoted by $m_p(D_i) := \mathfrak{e}(\mathcal{O}_{S,p}/(h_i), \mathfrak{m})$. The Hilbert–Samuel polynomial of $D_i$ at $p$ is denoted by $\chi_{D_i,p} := \chi^{\mathfrak{m}}_{\mathcal{O}/(h_i)}$, and similarly the multiplicity and Hilbert–Samuel polynomial for $D$.

*Remark* 3.29. In order to compute the Hilbert–Samuel polynomial of $\mathcal{O}/I$ for any ideal $I \subseteq \mathcal{O}$ we can consider $\mathcal{O}/I$ either as ring or as an $\mathcal{O}$-module. This does not make a difference for the Hilbert–Samuel functions, since they only depend on the graded structure of $\mathcal{O}/I$ and this is the same for a $\mathfrak{q} \in \mathcal{O}$ or the corresponding $\overline{\mathfrak{q}} \in \mathcal{O}/I$.

We have the following:

**Proposition 3.30.** *Let* $D$*,* $D_i$ *be defined as above. Then* $m_p(D) = m_p(D_1) + m_p(D_2)$*.*

*Proof.* Since $D_1$ and $D_2$ are assumed to have no common components, it is clear that $\dim_p(D_1 \cap D_2) \leq n-2$. Consider the exact sequence from Lemma 3.17:

$$0 \to \mathcal{O}/(h_1h_2) \to \mathcal{O}/(h_1) \oplus \mathcal{O}/(h_2) \to \mathcal{O}/(h_1, h_2) \to 0. \tag{3.4}$$

By Lemma 3.27 it follows that $\chi^{\mathfrak{m}}_{\mathcal{O}/(h_1)\oplus\mathcal{O}/(h_2)} = \chi_{D,p} + \chi^{\mathfrak{m}}_{\mathcal{O}/(h_1,h_2)} - R$, where $R$ is a polynomial in $\mathbb{Q}[t]$ of degree strictly smaller than that of $\chi_{D,p}$. The exact sequence

$$0 \to \mathcal{O}/(h_1) \to \mathcal{O}/(h_1) \oplus \mathcal{O}/(h_2) \to \mathcal{O}/(h_2) \to 0$$

yields that $\chi^{\mathfrak{m}}_{\mathcal{O}/(h_1)\oplus\mathcal{O}/(h_2)} = \chi_{D_1,p} + \chi_{D_2,p}$, see Lemma 3.32. Since the Krull-dimensions of $D$ and the $D_i$ are all $(n-1)$, the degrees of $\chi_{D_i,p}$ and $\chi_{D,p}$ are equal to $(n-1)$ and the degree of $\chi_{\mathcal{O}/(h_1,h_2)}$ is strictly less than $n-1$. Combining the equalities of the Hilbert–Samuel polynomials, one finds that

$$\chi_{D,p} = \chi_{D_1,p} + \chi_{D_2,p} - \chi^{\mathfrak{m}}_{\mathcal{O}/(h_1,h_2)} - T,$$

with $\deg(T) \leq n-2$. Thus it follows that the leading coefficient of $\chi_{D,p}$ is the sum of the leading coefficients of the $\chi_{D_i,p}$ and hence $m_p(D) = m_p(D_1) + m_p(D_2)$. □

*Remark* 3.31. This proposition can be easily generalized to $m \geq 2$ components $(D_1, p), \ldots, (D_m, p)$.

We now ask if in the case of splayed divisors the additivity formula from Lemma 3.27 holds without remainder: is it true that for the germ of a splayed divisor $(D, p) = (D_1, p) \cup (D_2, p)$, one has

$$\chi_{D,p} = \chi_{D_1,p} + \chi_{D_2,p} - \chi_{D_1\cap D_2,p}? \tag{3.5}$$

First consider the problem for arbitrary local rings $(R, \mathfrak{m})$. If

$$0 \to N \to M \to M/N \to 0$$

is an exact sequence of finitely generated $R$-modules and $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal, then

$$0 \to N/(\mathfrak{q}^n M \cap N) \to M/\mathfrak{q}^n M \to (M/N)/\mathfrak{q}^n(M/N) \to 0$$

is an exact sequence, which implies

$$\chi^{\mathfrak{q}}_M = \chi^{\mathfrak{q}}_{M/N} + \chi_{\{\mathfrak{q}^n M\cap N\}}.$$

But in general $\chi^{\mathfrak{q}}_N \neq \chi_{\{\mathfrak{q}^n M\cap N\}}$, where $\chi_{\{\mathfrak{q}^n M\cap N\}}$ denotes the Hilbert–Samuel polynomial w.r.t. the filtration $N/(\mathfrak{q}^n M \cap N)$.
However, for split exact sequences the Hilbert–Samuel polynomial is always additive:

**Lemma 3.32.** *Let $(R, \mathfrak{m})$ be a local ring and let $M, N$ be finitely generated $R$-modules. Consider the exact sequence*

$$0 \to M \to M \oplus N \to N \to 0.$$

*Then $\chi^{\mathfrak{q}}_{N\oplus M} = \chi^{\mathfrak{q}}_M + \chi^{\mathfrak{q}}_N$ for any $\mathfrak{m}$-primary ideal $\mathfrak{q}$.*

*Proof.* Above we have seen that there is an exact sequence

$$0 \to M/M \cap \mathfrak{q}^n(M \oplus N) \to (M \oplus N)/\mathfrak{q}^n(M \oplus N) \to N/\mathfrak{q}^n N \to 0.$$

So this lemma is shown if $M \cap \mathfrak{q}^n(M \oplus N) = \mathfrak{q}^n M$ for all $n$. We may consider $M \cap \mathfrak{q}^n(M \oplus N)$ as $(M \oplus 0) \cap \mathfrak{q}^n(M \oplus N)$ and $\mathfrak{q}^n M$ as $\mathfrak{q}^n(M \oplus 0)$. Then

$$(M \oplus 0) \cap \mathfrak{q}^n(M \oplus N) \subseteq (M \oplus 0) \cap (\mathfrak{q}^n M \oplus \mathfrak{q}^n N) = (\mathfrak{q}^n M \oplus 0) = \mathfrak{q}^n(M \oplus 0).$$

Conversely, take a $(q\alpha, 0) \in \mathfrak{q}^n(M \oplus 0)$, where $\alpha \in M$ and $q \in \mathfrak{q}^n$. Since $(q\alpha, 0) = q(\alpha, 0) \in \mathfrak{q}^n(M \oplus N)$ and $(q\alpha, 0) \in M \oplus 0$, it is shown that $M \cap \mathfrak{q}^n(M \oplus N) = \mathfrak{q}^n M$. Hence it follows that

$$\chi^{\mathfrak{q}}_{N \oplus M} = \chi^{\mathfrak{q}}_M + \chi^{\mathfrak{q}}_N.$$

□

**Proposition 3.33.** *Let $(D, p) = (D_1, p) \cup (D_2, p)$ be splayed at $p \in S$, where $(S, p) \cong (\mathbb{C}^n, 0)$. Then the Hilbert–Samuel polynomials of the components $D_1$ and $D_2$ are additive, that is,*

$$\chi_{D,p}(t) + \chi_{D_1 \cap D_2, p}(t) = \chi_{D_1,p}(t) + \chi_{D_2,p}(t).$$

*Proof.* Denote by $\mathcal{O} := \mathbb{C}\{x, y\} = \mathbb{C}\{x_1, \ldots, x_k,\ y_{k+1}, \ldots, y_n\}$ the coordinate ring of $(\mathbb{C}^n, 0)$. With Prop. 3.26 the question can be reduced to leading ideals because the exact sequence (3.4) remains exact if we just consider the leading ideals. This is not true in general because $L(h, g) \neq (L(g), L(h))$ is possible! The divisor $D$ is splayed, so we can assume that it is defined by $g(x)h(y)$. Choosing any valid monomial ordering one finds $L(g) = x^\alpha$, $L(h) = y^\beta$, $L((gh)) = x^\alpha y^\beta$, and by Lemma 3.25 follows $L((g, h)) = (x^\alpha, y^\beta)$. From the exact sequence

$$0 \to \mathcal{O}/(x^\alpha) \to \mathcal{O}/(x^\alpha) \oplus \mathcal{O}/(y^\beta) \to \mathcal{O}/(y^\beta) \to 0$$

and Lemma 3.32 it follows that

$$\chi^{\mathfrak{m}}_{\mathcal{O}/(x^\alpha)} + \chi^{\mathfrak{m}}_{\mathcal{O}/(y^\beta)} = \chi^{\mathfrak{m}}_{\mathcal{O}/(x^\alpha) \oplus \mathcal{O}/(y^\beta)}.$$

Thus it remains to prove that the Hilbert–Samuel polynomials w.r.t. $\mathfrak{m}$ of the exact sequence

$$0 \to \mathcal{O}/(x^\alpha \cdot y^\beta) \to \mathcal{O}/(x^\alpha) \oplus \mathcal{O}/(y^\beta) \to \mathcal{O}/(x^\alpha, y^\beta) \to 0$$

are additive. In order to apply the theorem of Flenner–Vogel we show that the map

$$\mathrm{Gr}_{\mathfrak{m}}(\mathcal{O}/(x^\alpha y^\beta)) \xrightarrow{\varphi} \mathrm{Gr}_{\mathfrak{m}}(\mathcal{O}/(x^\alpha) \oplus \mathcal{O}/(y^\beta))$$

is injective. The map $\varphi$ clearly preserves the degree, so it is enough to show the assertion for a homogeneous element of degree $d$, that is, to show that

$$\mathfrak{m}^d(\mathcal{O}/(x^\alpha y^\beta))/\mathfrak{m}^{d+1}(\mathcal{O}/(x^\alpha y^\beta)) \to \mathfrak{m}^d(\mathcal{O}/(x^\alpha)\oplus\mathcal{O}/(y^\beta))/\mathfrak{m}^{d+1}(\mathcal{O}/(x^\alpha)\oplus\mathcal{O}/(y^\beta))$$

is an injection. Therefore take some $\overline{a} \in \mathfrak{m}^d(\mathcal{O}/(x^\alpha y^\beta))/\mathfrak{m}^{d+1}(\mathcal{O}/(x^\alpha y^\beta))$. This means that $\overline{a}$ is the representative of the element $a = \overline{a} + f x^\alpha y^\beta$ of $\mathcal{O}$, where $f$ with the right degree is in $\mathcal{O}$. Consider $\overline{a}$ as an element in $\mathcal{O}$: from Grauert's division theorem follows that $\overline{a}$ can be written as $\alpha + \alpha_1 x^\alpha + \alpha_2 y^\beta$, where $\alpha$ is the unique remainder from the division through $x^\alpha$ and $y^\beta$ and $\alpha_1 \in \mathcal{O}$ is not divisible by $y^\beta$ (else $\alpha_1$ would be $\overline{0}$ in $\mathcal{O}/(x^\alpha y^\beta)$) and $\alpha_2$ is not divisible by $x^\alpha$ (for the analogue reason). Suppose that $\varphi(\overline{a}) = (0,0)$. Then write $\varphi(\overline{a}) = \varphi(\alpha + \alpha_1 x^\alpha + \alpha_2 y^\beta) = (\alpha + \alpha_2 y^\beta, \alpha + \alpha_1 x^\alpha)$ in $\mathcal{O}/(x^\alpha) \oplus \mathcal{O}/(y^\beta)$. In $\mathcal{O}$ this reads as $\alpha + \alpha_2 y^\beta = c x^\alpha$ and $\alpha + \alpha_1 x^\alpha = c' y^\beta$ for some $c, c' \in \mathcal{O}$ with the right order. Taking one of these two equations one sees that $\alpha \in (x^\alpha, y^\beta)$. But $\alpha$ is the unique remainder from the division through the standard basis $(x^\alpha, y^\beta)$, so $\alpha = 0$ in $\mathcal{O}$. Hence there are two relations between $x^\alpha, y^\beta$, namely

$$\alpha_2 y^\beta - c x^\alpha = 0 \quad \text{and} \quad \alpha_1 x^\alpha - c' y^\beta = 0.$$

Since $x^\alpha, y^\beta$ are clearly a regular sequence in $\mathcal{O}$, their syzygies are trivial and from the conditions on $\alpha_1, \alpha_2$ it follows that $\alpha_1 = \alpha_2 = 0$. But this means nothing else but $\overline{a} = \overline{0}$, that is, $\ker(\varphi) = (\overline{0})$ and $\varphi$ is injective. Hence by the theorem of Flenner–Vogel, the remainder polynomial $S$ is the zero polynomial and the assertion of the proposition follows. □

*Remark* 3.34. If we just consider the question for the leading ideals, then we are in the case of monomial ideals and one can argue that $\varphi$ is injective by looking at the Newton polyhedra of these monomial ideals.

In general the Hilbert–Samuel polynomial of a divisor $(D, p) = (D_1, p) \cup (D_2, p)$ is not additive, as is seen in the following example.

*Example* 3.35. By Lemma 4.2.20 of [27] one can explicitly compute the Hilbert–Samuel polynomial of $\mathcal{O}/(f)$, where $\mathcal{O} = \{x_1, \ldots, x_n\}$ and

$\operatorname{ord}(f) = m$, namely

$$\chi^{\mathfrak{m}}_{\mathcal{O}/(f)}(d) = \sum_{j=1}^{m} \binom{n+d-j-1}{n-1}. \tag{3.6}$$

Consider now $(\mathbb{C}^2, 0)$ with coordinate ring $\mathcal{O} = \mathbb{C}\{x, y\}$ and with $h_1 = x^2 - y$ and $h_2 = y$. Then the germ of the divisor $(D, 0) = (D_1, 0) \cup (D_2, 0)$ that is locally given by $\{y(x^2 - y) = 0\}$ with $(D_1, 0) = \{x^2 - y = 0\}$ and $(D_2, 0) = \{y = 0\}$ is not splayed. The intersection $(D_1 \cap D_2, 0)$ is locally given by the ideal $(x^2, y)$ and coordinate ring $\mathcal{O}/(x^2, y) = \mathbb{C}\{x\}/(x^2)$. By formula (3.6) we can compute the Hilbert–Samuel polynomials of $D, D_1, D_2$ and $D_1 \cap D_2$ and obtain $\chi_{D,p}(t) = 2t - 1$, which is clearly not equal to $\chi_{D_1,p}(t) + \chi_{D_2,p}(t) - \chi_{D_1 \cap D_2,p}(t) = t + t - 2$.

One might ask if the additivity of the Hilbert–Samuel polynomials characterizes splayed divisors. However, here is a counterexample to this assertion:

*Example* 3.36. Consider $D \subseteq \mathbb{C}^3$ locally defined by $gh = (x^2 - y^3)(y^2 - x^2 z)$. Then $(D, p)$ is the union of the cylinder over a cusp $(D_1, p)$ and of the Whitney Umbrella $(D_2, p)$. Clearly $(D, p)$ is not splayed (use for example the Leibniz–property). However, $L(g) = x^2$ and $L(h) = y^2$, so the leading monomials of $g$ and $h$ are coprime and one can repeat the argument in the proof of the preceding proposition to find that

$$\chi_{D,p} + \chi_{D_1 \cap D_2,p} = \chi_{D_1,p} + \chi_{D_2,p}.$$

### 3.2.3 A characterization of splayed divisors by their singular locus

In the spirit of our characterization of normal crossing divisors we want to characterize a splayed divisor $D \subseteq S$, locally at a point $p$ given by a $gh \in \mathcal{O}_{S,p}$, by $\mathcal{O}_{\operatorname{Sing} D,p} = \mathcal{O}_{S,p}/((gh) + J_{gh})$.

**Proposition 3.37.** *The divisor $D$, defined at $p$ as above, is splayed if and only if*

$$((gh) + J_{gh}) = (g, h) \cap ((g) + J_g) \cap ((h) + J_h). \tag{3.7}$$

*Remark* 3.38. Here it can be seen that for splayed divisors the Jacobian ideal is the intersection of the two ideals defining the singular loci of the splayed components $D_1$ and $D_2$ plus the intersection of $D_1$ and $D_2$. For two smooth divisors $D_1$ and $D_2$ this means nothing else but that $D = D_1 \cup D_2$ is a splayed divisor if and only if the scheme-theoretical intersection $D_1 \cap D_2$ is smooth, which is in turn equivalent to saying that $D_1$ and $D_2$ intersect transversally.

*Proof.* Recall here that a divisor $\{gh = 0\}$ is splayed if and only if

$$(g) \cap ((gh) + J_{gh}) = g((h) + J_h) \tag{3.8}$$

First suppose that (3.7) holds. We have to show that (3.8) holds. Let therefore $\alpha$ be an element of $g((h) + J_h)$, that is $\alpha = agh + g\sum_{i=1}^n a_i \partial_{x_i} h$. Clearly $\alpha \in (g)$. But it is immediately seen that $\alpha \in ((h) + J_h)$, and hence $\alpha \in (g, h) \cap ((g) + J_g) \cap ((h) + J_h))$. By (3.7) this means that $\alpha \in ((gh) + J_{gh})$. Conversely, let $\alpha \in (g) \cap ((gh) + J_{gh})$. In Chapter 2 it was seen that (without further conditions on $((gh)+J_{gh})$ such an $\alpha$ is also contained in $g((h) + J_h)$.
For the other implication we use Grauert's division theorem: Suppose that $D = \{gh = 0\}$ is splayed. Then wlog. $g(x, y) = g(x)$ and $h(x, y) = h(y)$ in $\mathbb{C}\{x, y\} = \mathbb{C}\{x_1, \dots, x_n, y_1, \dots, y_m\}$. Clearly $((gh) + J_{gh}) \subseteq (g, h) \cap ((g) + J_g) \cap ((h) + J_h)$. So let $\alpha$ be an element of the right-hand side, that is, $\alpha = ag + bh = cg + \sum_{i=1}^n a_i \partial_{x_i} g = dh + \sum_{j=1}^m b_j \partial_{y_j} h$ for some $a, b, c, d, a_i, b_i \in \mathcal{O}$. Then also $\alpha - ag = bh = (c-a)g + \sum_{i=1}^n a_i \partial_{x_i} g$ is contained in $((g) + J_g)$. By Grauert's division theorem here exist some $\tilde{a}, \tilde{a}_i, r, \tilde{r}_i$ such that for all $i = 1, \dots, n$ one has $c - a = \tilde{a}h + r$ and $a_i = \tilde{a}_i h + r_i$ and the leading monomial $L(h)$ does not divide any monomial of the unique remainders $r, r_i$. Then write

$$(b - \tilde{a}g - \sum_{i=1}^n \tilde{a}_i(\partial_{x_i} g))h = rg + \sum_{i=1}^n r_i(\partial_{x_i} g).$$

Since $h$ only depends on $y$ and $g$ only on $x$, it follows that $L(h)$ does also not divide any of the monomials of the right-hand side of the equation. But this is only possible if the right-hand side of the equation is 0 (otherwise at least one monomial of $h$, which is a multiple of $L(h)$ would occur on the right-hand side). Hence $(b - \tilde{a}g - \sum_{i=1}^n \tilde{a}_i(\partial_{x_i} g))h = 0$ and since $h$ is not a zero-divisor in $\mathcal{O}$, it follows that $b = \tilde{a}g + \sum_{i=1}^n \tilde{a}_i(\partial_{x_i} g)$ is contained in $((g) + J_g)$. Interchanging the roles of $g$ and $h$ yields that $a \in ((h) + J_h)$ and thus $\alpha = ag + bh \in (gh, gJ_h + hJ_g)$. The Leibniz property of $J_{gh}$ implies that $\alpha \in ((gh) + J_{gh})$. ☐

### 3.2.4 Questions about splayed divisors

1. Can we separate the splayed components of a divisor by some algebraic procedure? (The normalization does too much and the blowup of the intersection scheme may be singular. We search for something like a deformation, where $h_1(x)h_2(y) \longmapsto h_1(x)(z-1) + h_2(y)z$ but multiplicatively)

2. Does the additivity of Hilbert–Samuel polynomials (equation (3.5)) characterize a particular class of divisors? Is it enough for the additivity that the leading monomials of the defining equations of $D_1$ and $D_2$ are coprime?

3. Can we compute other singularity invariants of splayed divisors in terms of their splayed components, e.g., Milnor fibres, Zeta functions, jumping numbers?

## 3.3 Mikado divisors

In this section we consider another generalization of normal crossing divisors, so-called mikado divisors. The idea behind mikado divisors is to allow more smooth components meeting at a point than indicated by the dimension of the ambient space. However, the irreducible components of a mikado divisor still have to be smooth and the divisor has to satisfy the additional property that it is closed under taking scheme-theoretical intersections (see definition below). Mikado singularities appear in the study of arrangements of subvarieties, namely, a collection of smooth algebraic subvarieties in a smooth ambient space form an arrangement of subvarieties in the sense of [58] if and only if their maximal members form a mikado variety. Here one is interested e.g. in wonderful compactifications of these arrangements, see [58] for details. Mikado divisors are also present in resolution of singularities, see [49] and [33].
First we study mikado divisors in a complex manifold of dimension 2,that is, plane mikado curves, and give a characterization of their singular loci in terms of their Jacobian ideals. Therefore we need some theory about generalized Milnor numbers, see [93]. Then we give some examples and ask questions related to mikado divisors.

Let $X$ and $Y$ be two irreducible algebraic subvarieties of $\mathbb{C}^n$ defined by radical ideals $I_X$ and $I_Y$. Then their intersection $Z := X \cap Y$ is *scheme-*

*theoretically smooth* if $Z$ is set-theoretically smooth and its defining ideal $I_Z$ is equal to $I_X + I_Y$. In particular this means that $I_X + I_Y$ is radical. A collection of smooth algebraic subvarieties $X_1, \dots X_k$ in $S$ is called *mikado* if all possible intersections

$$\bigcap_{j \in J} X_j, \text{ with } J \subseteq \{1, \dots, k\}$$

are scheme-theoretically smooth. We can also make the analogue definition locally at a point $p$ for analytic space germs $(X_1, p), \dots (X_k, p)$. We then say that the germ of a variety $(X, p) = \bigcup_{i=1}^{k} (X_i, p)$, where the $X_i$ are the irreducible components at $p$, is a *mikado singularity* at $p$ (or is *mikado* at $p$) if and only if the $X_1, \dots, X_k$ are mikado at $p$.

*Example* 3.39. (1) The divisor $D = D_1 \cup D_2 \cup D_3 = \{(x - y^3)(y - x^2)(y - x) = 0\}$ in $\mathbb{C}^2$ is mikado at 0: with the normal crossings criterion of Theorem 2.1 one sees that the ideals of the pairwise intersections of the irreducible components of $D$ are reduced and hence $D_i \cup D_j$ has normal crossings everywhere. The ideal of the triple intersection at the origin is easily seen to be reduced.
(2) All hyperplane arrangements in $\mathbb{C}^n$ are mikado divisors: let $H = \bigcup_{i=1}^{m} H_i$ be a hyperplane arrangement in $\mathbb{C}^n$ with defining polynomial $Q(x) = \prod_{i=1}^{m} l_i(x)$, where each $l_i$ is a linear polynomial. Clearly, any intersection $\bigcap_{i \in I} H_i$ for some $I \in \{1, \dots, m\}$ is again a linear space with defining ideal $(l_i, i \in I) = \sum_{i \in I} I_{H_i}$ and hence smooth.
(3) The divisor Tülle $D = D_1 \cup D_2 \cup D_3$ in $\mathbb{C}^3$ defined by $h = h_1 h_2 h_3 = xz(x + z - y^2)$ is not mikado at the origin. All components are smooth, their pairwise intersections are transversal but the ideal of their triple intersection is $I_{D_1} + I_{D_2} + I_{D_3} = (x, y^2, z)$, which is not the reduced ideal defining the origin.

### 3.3.1 Plane mikado curves

In dimension two one can give a characterization of mikado divisors in terms of the Jacobian ideal (see Thm. 3.49). For the proof we need some properties of the generalized Milnor number, which was first investigated by Teissier. We closely follow the exposition in [93].

Note again that we always work in the analytic context. For plane curves $(X, p) = \bigcup_{i=1}^{k} (X_i, p)$ being mikado at $p$ is equivalent to the fact that the irreducible components $(X_i, p)$ are all smooth and that $X_i$ and $X_j$ intersect transversally at $p$ for $i \neq j$. This means that the tangent cone $TK(X)_p$ of $X$ at $p$ is reduced (see the next lemma). Recall here that for $X = \{h = 0\}$, with $h = \sum_{i \geq o \geq 1} h^i$, where $h^i$ is the homogeneous part of $h$ of degree $i$, the *tangent cone* $TK(X)_p$ of $(X, p)$ is the homogenous part of $h$ of smallest degree, that is, $h^o$.

**Lemma 3.40.** *Let $(X, p) = \bigcup_{i=1}^{k}(X_i, p)$ be a mikado curve-singularity in $(\mathbb{C}^2, 0)$ defined locally at $p$ by $h = h_1 \cdots h_k$. Then the tangent cone $TK(X)_p$ of $X$ at $p$ is reduced and consists of $o = k$ lines meeting at the origin.*

*Proof.* First note that $TK(X)_p = h^o$, where $h^o$ is a homogeneous polynomial of degree $o$ in two variables. Since the $h_i$ define nonsingular curves $X_i$, the order of each $h_i = 1$, so $o = k$. By dehomogenizing $h^o$ we obtain a polynomial of degree $k$ in one variable over $\mathbb{C}$ that has $k$ different zeros (since the $k$ tangent directions of $X$ at $p$ are distinct). Homogenizing again it follows that $TK(X)_p$ is the product of $k$ linear forms and is reduced because of the mikado condition. □

*Remark* 3.41. In 3.2 the Hilbert–Samuel polynomial $\chi_M^{\mathfrak{q}}$ of a finite module $M$ over a local ring $(R, \mathfrak{m})$ w.r.t. an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ and the multiplicity $\mathfrak{e}(M, \mathfrak{q})$ of $M$ w.r.t. $\mathfrak{q}$ was defined. Using Koszul homology, one gets the formula $\mathfrak{e}(M, \mathfrak{q}) = \sum_{i=0}^{n}(-1)^i h_i(M, \mathfrak{q})$ where $h_i(M, \mathfrak{q})$ denotes the dimension of the $i$-th Koszul-homology group. If $\mathfrak{q}$ is generated by an $M$-regular sequence, then $h_i = 0$ for $i \geq 1$ by [88, Prop. 3, IV]. By definition the 0-th Koszul-homology group is $M/\mathfrak{q}M$ and hence

$$h_0 = \mathfrak{e}(M, \mathfrak{q}) = \text{length}(M/\mathfrak{q}M).$$

**Theorem 3.42** (Rees)**.** *Let $(R, \mathfrak{m})$ be an equidimensional analytic algebra and $\mathfrak{q}_1$ and $\mathfrak{q}_2$ be two $\mathfrak{m}$-primary ideals with $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$. Then their multiplicities are equal if and only if their integral closures agree, that is,*

$$\mathfrak{e}(R, \mathfrak{q}_1) = \mathfrak{e}(R, \mathfrak{q}_2) \Longleftrightarrow \overline{\mathfrak{q}_1} = \overline{\mathfrak{q}_2}.$$

*Proof.* See for example [55] or [93]. □

Let $D$ be a divisor in $\mathbb{C}^n$, that is, $D$ is defined at a point $p$ by a reduced holomorphic equation $h = 0, h \in \mathcal{O} = \mathbb{C}\{x_1, \ldots, x_n\}$. Recall that the germ $(D, p)$ is an isolated singularity if $D - \{p\}$ is non-singular (in a sufficiently small neighbourhood of $p$). One can show that this is equivalent to the fact that the Jacobian ideal $J_h = (\partial_{x_1} h, \ldots, \partial_{x_n} h)$ is $\mathfrak{m}$-primary. Denote the multiplicity of $J_h$ in $\mathcal{O}$ by $\mu_p(D)$. Note that it does neither depend on the choice of coordinates of $p$ nor on the choice of the equation $h$. The Jacobian ideal $J_h$ is generated by a regular sequence because it is a complete intersection. Thus, by remark 3.41 also

$$\mu_p(D) = \dim \mathbb{C}\{x_1, \ldots, x_n\}/J_h.$$

So $\mu_p(D)$ is the *Milnor number* of the isolated singularity $(D, p)$, see [27, 64].

Teissier has shown how the notion of the Milnor number $\mu$ can be generalized by general hyperplane sections [93, chapitre 1]: Let $D$ be a hypersurface in $\mathbb{C}^n$ and $p \in D$. For any $1 \le i \le n$ there exists a neighbourhood $V$ of $x$ in $\mathbb{C}^n$ and a Zariski-open dense set $U_0^{(i)}$ of the Grassmannian $G^{n-1,i-1}$ of $i$-planes of $\mathbb{C}^n$ passing through $p$ such that for any $i$-plane $H \in U_0^{(i)}$ one has

$$V \cap \mathrm{Sing}(D \cap H) = V \cap H \cap \mathrm{Sing}(D).$$

If $i_0$ is the codimension of $(\mathrm{Sing}(D), p)$ in $\mathbb{C}^n$ then for any $0 \le i \le i_0$ there exists a Zariski-open dense $U_1^{(i)}$ of $G^{n-1,i}$ such that for $H \in U_1^{(i)}$, $D \cap H$ has an isolated singularity in $p$. Thus, for $i \le i_0$ the $i$-planes of $\mathbb{C}^n$ passing through $p$ and cutting $\mathrm{Sing}(D)$ in $p$ in a neighbourhood of $p$ form a Zariski-open, dense subset of $G^{n-1,i}$.
If $(D, p) \subseteq (\mathbb{C}^n, 0)$ is a reduced hypersurface, then for any $0 \le i \le n$ there exists an open Zariski-dense subset $U_2^{(i)}$ of $G^{n-1,i}$ such that the topological type of $(D \cap H, p)$ is independent of $H \in U_2^{(i)}$. Thus one can speak of the so-called topological type of a generic plane section of $D$ (or general $i$-plane), see [93].

**Definition 3.43.** Let $(D, p) \subseteq (\mathbb{C}^n, 0)$ be the germ of an analytic hypersurface. Let $i_0$ be the codimension of the singular locus $\mathrm{Sing}(D)$ in $\mathbb{C}^n$. By the preceding considerations one can speak of the *Milnor number of a generic i-plane section*, if $i \le i_0$. We denote this number by $\mu_p^{(i)}$. For $i_0 < i \le n$ we set $\mu_p^{(i)} = +\infty$ and we define the vector

$$\mu_p^* = (\mu_p^{(n)}, \ldots, \mu_p^{(0)}).$$

Note that $\mu_p^{(n)} < +\infty$ if and only if $(D, p)$ is an isolated singularity and if that is the case then $\mu_p^{(n)} = \mu_p(D)$, that is, $\mu_p^{(n)}$ is the usual

Milnor–number of $(D,p)$.
Furthermore, $\mu_p^{(1)} = m_p(D) - 1$, where $m_p(D)$ denotes the multiplicity of the hypersurface germ $(D,p)$. Moreover, $\mu_p^{(0)} = 1$ always holds.

**Proposition 3.44.** *Let $(D,p) \subseteq (\mathbb{C}^n, 0)$ be an isolated hypersurface singularity with local equation $h = 0$. The following are equivalent:*
*(i) $\mu_p^{(n)} = {\mu_p^{(1)}}^n$,*
*(ii) $\overline{J_h} = \mathfrak{m}^{\mu^{(1)}}$,*
*(iii) $(D,p)$ is isomorphic to the general fibre of a one-parameter $\mu^*$-constant deformation of a cone with isolated singularity. This cone is the tangent cone of $(D,p)$.*

*Proof.* We show here only (i) ⇔ (ii). For the other equivalences see [93, ch. II, Prop. 2.7]. The multiplicity of the ideal $\mathfrak{m}^k$ in $\mathcal{O} = \mathbb{C}\{x_1, \ldots, x_n\}$ can be calculated by using the formula $\mathfrak{e}(\mathcal{O}, \mathfrak{m}^k) = k^n$ $\mathfrak{e}(\mathcal{O}, \mathfrak{m})$, which holds for $\mathfrak{m}$-primary ideals, see [62]. Since $\mathfrak{e}(\mathcal{O}, \mathfrak{m}) = 1$, it follows that $\mathfrak{e}(\mathcal{O}, \mathfrak{m}^k) = k^n$. By definition of $\mu_p^{(1)}$ one has $J_h \subseteq \mathfrak{m}^{\mu_p^{(1)}}$ and by Remark 3.41 follows $\mathfrak{e}(\mathcal{O}, J_h) = \mu_p^{(n)}$. By the above $\mathfrak{e}(\mathcal{O}, \mathfrak{m}^{\mu_p^{(1)}}) = {\mu_p^{(1)}}^n$. Application of Rees' theorem (Thm. 3.42) shows the equivalence of $(i)$ and $(ii)$. □

**Definition 3.45.** Let $(C,p) \subseteq (\mathbb{C}^n, 0)$ be a reduced curve. Denote by $\pi : (\widetilde{C}, \tilde{p}) \to (C,p)$ the normalization of $(C,p)$. We denote by $\delta_p(C)$ the $\mathbb{C}$-vector space dimension of $\pi_* \mathcal{O}_{\widetilde{C},\tilde{p}} / \mathcal{O}_{C,p}$. The nonnegative integer $\delta_p(C)$ is called the *$\delta$-invariant* of $C$ at $p$. It is sometimes also called the order of singularity of $C$ at $p$. Let $(C',p)$ be another curve in $\mathbb{C}^n$. Let $C$ and $C'$ be given locally at $p$ by ideals $I_C$ and $I_{C'}$, where $I_C, I_{C'} \subseteq \mathcal{O}_{\mathbb{C}^n,0}$. Then we define

$$(C \cdot C')_p := \operatorname{length}(\mathcal{O}_{\mathbb{C}^n,0}/(I_C + I_{C'})),$$

the *intersection multiplicity* of $C$ and $C'$ at $p$.

**Lemma 3.46.** *Let $(C,p)$ be a reduced curve singularity in $\mathbb{C}^2$. Suppose that locally at $p$ the curve $C$ has $m$ irreducible analytic components. Then*

$$\mu_p(C) = 2\delta_p(C) - m + 1.$$

*Remark* 3.47. This lemma is due to Milnor [64]. It can be generalized to complete intersection curves. For more information and a proof see [14].

The next lemma of Hironaka, see [52], is useful to compute the $\delta$-invariant for reducible curves:

**Lemma 3.48** (Hironaka)**.** *Let $(C, p)$ be a reduced curve in $\mathbb{C}^2$ that has locally $m$ components: $(C, p) = (C_1, p) \cup \cdots \cup (C_m, p)$. Then we have*

$$\delta_p(C) = \sum_{i=1}^{m} \delta_p(C_i) + \sum_{i,j=1, i<j}^{m} (C_i \cdot C_j)_p.$$

Now we are ready to prove the characterization of mikado singularities by Jacobian ideals in two dimensions:

**Theorem 3.49.** *Let $(X_1, p), \ldots, (X_k, p)$ with $k \geq 2$ be smooth curves in $(\mathbb{C}^2, p)$. Suppose, that they are locally defined by (reduced) equations $h_1 = 0, \ldots, h_k = 0$ with $h_i \in \mathbb{C}\{x_1, x_2\}$. Let $(X, p) = \bigcup_{i=1}^{k} (X_i, p)$ with equation $h = h_1 \cdots h_k$. Denote $J_h$ the Jacobian ideal of $h$. Then $X_1, \ldots, X_k$ are mikado at $p$ if and only if $\overline{J_h} = \mathfrak{m}^{k-1}$.*

*Proof.* Suppose that $(X, p)$ is mikado. Then $\mu_p^{(1)} = k - 1$ by definition. According to Prop. 3.44 it must be shown that $\mu_p(X) = \mu_p^{(2)} = (k-1)^2$. By Lemma 3.46 we have $\mu_p(X) = 2\delta_p(X) - k + 1$. Note that $\delta_p(X_i) = 0$, since $X_i$ is smooth at $p$ and that $(X_i \cdot X_j)_p = 1$, since $X_i$ and $X_j$ intersect transversally at $p$ for $j \neq i$. Then Hironaka's lemma says that $\delta_p(X) = \binom{k}{2}$ and plugging this into Milnor's $\delta$ formula yields $\mu_p(X) = (k-1)^2$. This finishes one implication.
The other implication follows by argumenting backwards and noting that $\delta_p(C) = 0$ if and only if $C$ is smooth at $p$. □

With Thm. 3.49 we obtain a simpler proof than that of Chapter 2 of the fact that $J_h = \mathfrak{m}$ if and only if the plane curve $(X, p)$ has normal crossings at $p$ (and thus we avoid the theorem of Mather–Yau):

**Corollary.** *Let $(X, p)$ be a curve in $(\mathbb{C}^2, p)$ defined by $h \in \mathcal{O}_{\mathbb{C}^2, p}$ such that $J_h = \mathfrak{m}$. Then $(X, p)$ has normal crossings at $p$.*

*Proof.* If $J_h = \mathfrak{m}$, then from $\mu_p^{(2)} = \dim_{\mathbb{C}}(\mathcal{O}/J_h) = 1$ it follows that the ordinary Milnor number of $(X, p)$ is 1. By Prop. 3.44 we have $1 = \mu_p^{(2)} = (\mu_p^{(1)})^2$ if and only if $\overline{J_h} = \mathfrak{m}$. By definition of $\mu_p^{(1)} = m_p(X) - 1$ it follows that $X$ is of multiplicity 2 at $p$ and has $k \leq 2$ irreducible components. By the $\delta$ formula one can discard the possibility $k = 1$ and by Theorem 3.49 follows the assertion. □

### 3.3.2 Mikado divisors in higher dimensions

A satisfying characterization of mikado divisors in a complex manifold $S$ of complex dimension 2 was found. Naturally, one asks for a similar criterion for mikado divisors in higher dimensional manifolds. However, it is not quite clear how to generalize Theorem 3.49, since in higher dimension the singularities of a mikado divisor are no longer isolated. New phenomena occur and the integral closure of the Jacobian ideal will certainly not be an $\mathfrak{m}$-primary ideal. One idea of generalization would be to take generic hyperplane sections (in the spirit of [93]). But this seems not to be the right approach:

*Example* 3.50. Let $(D, 0) \subseteq (\mathbb{C}^3, 0)$ be the divisor *Tülle*, that is given by $D = D_1 \cup D_2 \cup D_3 = \{xz(x + z - y^2) = 0\}$ (also cf. Example 1.43, where we have shown that $D$ is free and mikado everywhere but at the origin). The common intersection of $D_1 = \{x = 0\}$, $D_2 = \{z = 0\}$ and $D_3 = \{x + z - y^2 = 0\}$ is not scheme-theoretically smooth, thus $(D, 0)$ is not mikado. The Jacobian ideal $J_h$ is integrally closed. However, taking the hyperplane section with $H = \{x = y + z\}$ yields $D \cap H = \{(y + z)z(y + 2z - y^2\} \subseteq (\mathbb{C}^2, 0)$, which is mikado.

*Example* 3.51. Let $D \subseteq \mathbb{C}^3$ be the divisor $D = D_1 \cup D_2 \cup D_3$ defined by $h = xy(x + y)$. Clearly $D$ is a mikado divisor. Taking the generic hyperplane section with $H = \{z = 0\}$, one sees that $D \cap H = \{xy(x + y) = 0\} \subseteq \mathbb{C}^2$ is a mikado divisor.

Hence we pose the following

**Question 3.52.** *Let $D \subseteq S$ be a divisor in a complex manifold $S$ of complex dimension $n \geq 3$ and suppose that $D$ is mikado at a point $p$. Is there a characterization of the mikado property in terms of the singular locus* $(\operatorname{Sing} D, p)$ *given by the Jacobian ideal $J_h$ of $D$? In the same vein, another interesting question is if mikado is stable under generic hyperplane sections.*

# Appendix A

# Algebraic and complex analytic basics

Here we collect the most important notions and theorems that are used in the text and we also fix our notation. So this appendix is thought to serve as a reference to previous chapters. All results presented here are covered in textbooks. Therefore we only prove statements which we think are interesting for this thesis and give references to the remaining proofs. Nonetheless, we try to exhibit the beautiful description of normal varieties in the analytic context and some important theorems connected with it.
This appendix is divided into two sections, commutative algebra and local analytic geometry. However, commutative algebra heavily plays into local analytic geometry, therefore it is at the beginning.

## A.1 Commutative algebra

In this section we recall some results from commutative algebra that are mostly used in Chapter 2. In particular, we define depth of a module and Cohen–Macaulay modules, pass by projective dimension and perfect modules and quote some important theorems connected with these notions. Finally the integral closure of ideals and rings is considered, which leads the way into the analytic geometry section.
The first result tells us how primary decomposition behaves under localization (this is used in the section on splayed divisors in Chapter 2).

Then we state Nakayama's lemma, since it is frequently used throughout the text.

**Proposition A.1.** *Let $R$ be a commutative ring and let $S$ be a multiplicative set in $R$. Then all ideals of $R_S$, the localization of $R$ in $S$, are of the form $IR_S$ where $I$ is an ideal in $R$. Every prime ideal of $R_S$ is of the form $\mathfrak{p}R_S$ with $\mathfrak{p}$ a prime ideal in $R$ and $\mathfrak{p} \cap S = \emptyset$. Conversely $\mathfrak{p}R_S$ is prime in $R_S$ for any such $\mathfrak{p}$. The same holds for primary ideals. If $I$ is an ideal of $R$ then the set of associated primes $\operatorname{Ass}_R(I)$ is equal to $\operatorname{Ass}_{R_S}(I)$. If $R$ is noetherian then we have $\operatorname{Ass}(IR_S) = \operatorname{Ass}(I) \cap \operatorname{Spec}(R_S)$. In particular, if $I = \bigcap_{i=1}^k \mathfrak{q}$ is the primary decomposition of $I$ in $R$, then $I_S = \bigcap_{i=1}^k (\mathfrak{q}_i)_S$ (we write $I_S$ for the ideal $IR_S$) is the primary decomposition of $I_S$ in $R_S$.*

*Proof.* See [62] Thm. 4.1 and Thm. 6.2. □

**Theorem A.2** (Nakayama's lemma)**.** *(i) Let $R$ be a commutative ring, $M$ a finite (=finitely generated) $R$-module and $I$ an ideal of $R$. If $M = IM$ then there exists an element $x \in R$ such that $xM = 0$ and $x \equiv 1 \mod I$. If moreover $I$ is contained in the radical of $R$ (the intersection of all maximal ideals of $R$), then $M = 0$.*
*(ii) Let $M$ be an $R$ module, $I$ an ideal contained in the radical of $R$ and $N \subseteq M$ a submodule such that $M/N$ is finite over $R$. Then from $M = N + IM$ follows $M = N$.*
*(iii) Let $(R, \mathfrak{m})$ be a regular local ring and $M$ a finite $R$-module. Denote $k = R/\mathfrak{m}$ and $\overline{M} = M/\mathfrak{m}M$. Then $\overline{M}$ is a finite-dimensional $k$-vector space of some dimension $n$. Then one has:*
*(a) If $\{\bar{u}_1, \ldots, \bar{u}_n\}$ is a basis for $\overline{M}$ over $k$, then choosing inverse images $u_i \in M$ for each $\bar{u}_i \in \overline{M}$ yields a minimal system of generators $\{u_1, \ldots, u_n\}$ of $M$,*
*(b) conversely, any minimal system of generators of $M$ is obtained in this way and thus has $n$ elements,*
*(c) if $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ are both minimal systems of generators of $M$, and $v_i = \sum_{j=1}^n a_{ij} u_j$ with matrix $A = (a_{ij})_{1 \leq i,j \leq n}$, then $\det(A)$ is a unit in $R$, that is, $A$ is an invertible matrix over $R$.*

*Proof.* See [62] Thm. 2.2 and Thm. 2.3. □

The next theorem is a useful characterization of free modules over 2-dimensional regular local rings, which, in Chapter 1, provides the proof that any divisor in a 2-dimensional complex manifold is free.

**Theorem A.3.** *Let $R$ be a regular local ring of Krull-dimension 2 and $M$ be a finite $R$-module. The following are equivalent:*
*(i) $M$ is free.*
*(ii) $M$ is reflexive, that is, the canonical map $M \to \operatorname{Hom}(\operatorname{Hom}(M,R),R)$ is an isomorphism.*

*Proof.* See Corollary 6 of Theorem 9 of Chapter IV of [88]. □

## A.1.1 Cohen–Macaulay rings and modules

Let $R$ be a noetherian ring and $M$ an $R$-module. A sequence of elements $x_1,\dots,x_n \in I$, where $I$ is an ideal in $R$, is called a *regular $M$-sequence in $I$* if $(x_1,\dots,x_n)M \neq M$ and if for $i=1,\dots,n$ the element $x_i$ is a nonzerodivisor in $M/(x_1,\dots,x_{i-1})M$. If $M \neq IM$, then the length of a maximal $M$-sequence in $I$ is well-defined, and is called the *depth of $I$ on $M$* (or the *$I$-depth of $M$*) and denoted by $\operatorname{depth}(I,M)$. (If $IM = M$ we define the $I$-depth of $M$ to be $\infty$). If $M = R$ we simply speak of the *depth* of $I$ and write $\operatorname{depth}(I)$ or $\operatorname{depth}(I,R)$. For a local ring $(R,\mathfrak{m})$ and an $R$-module $M$ we denote the *depth* of $M$ by $\operatorname{depth}(M) := \operatorname{depth}(\mathfrak{m},M)$.
The *height* of a prime ideal $\mathfrak{p} \in R$ is the maximal length $m$ of a chain of prime ideals $\mathfrak{p}_0 \subseteq \ldots \subseteq \mathfrak{p}_m = \mathfrak{p}$ with $\mathfrak{p}_i \in R$. The *(Krull-)dimension* of a ring $R$ is the maximal height of a prime ideal in $R$ and denoted by $\dim(R)$. A noetherian local ring $(R,\mathfrak{m})$ is called *Cohen–Macaulay* if $\operatorname{depth}(R) = \dim(R)$. A finite $R$-module $M$ is called a *Cohen–Macaulay module* if $M \neq 0$ and $\operatorname{depth}(M) = \operatorname{depth}(\mathfrak{m},M) = \dim(M)$ or if $M = 0$. A noetherian ring $R$ is a Cohen–Macaulay ring if $R_\mathfrak{m}$ is a Cohen–Macaulay local ring for every maximal ideal $\mathfrak{m}$ of $R$. One can show that a noetherian local ring $R$ is Cohen–Macaulay if and only if its completion $\widehat{R}$ is Cohen–Macaulay (see e.g. [62, Theorem 17.5]) and that a localization $S^{-1}R$ of a Cohen–Macaulay ring $R$ is again Cohen–Macaulay.
Cohen–Macaulay is an algebraic condition and cannot be interpreted geometrically in a satisfying way. However, one geometric property of Cohen–Macaulayness is *equidimensionality*, that is, the scheme corresponding to a Cohen–Macaulay ring is always equidimensional:

**Lemma A.4.** *Let $(R,\mathfrak{m})$ be a Noetherian local ring and let $M$ be a finitely generated $R$-module. If $M$ is a Cohen–Macaulay module then for any $\mathfrak{p} \in \operatorname{Ass}(M)$ one has*

$$\dim(R/\mathfrak{p}) = \dim(M) = \operatorname{depth}(M).$$

*Hence $M$ has no embedded associated primes. In particular, if $M = R/I$ for an ideal $I \subseteq R$ is Cohen–Macaulay, then $I$ is equidimensional, that is, in an irredundant primary decomposition $I = \bigcap \mathfrak{q}_i$, with $\mathfrak{p}_i$ the associated prime to $\mathfrak{q}_i$, the $\mathfrak{p}_i$'s are all of the same height.*

*Proof.* If $M$ is a Cohen–Macaulay module then by definition we have for any $\mathfrak{p} \in \mathrm{Ass}(M)$ that $\dim(R/\mathfrak{p}) = \dim M = \operatorname{depth} M$. Thus $M$ has no embedded primes. The assertion for $M = R/I$ is clear. □

*Remark* A.5. We remark here that it is not enough to check equidimensionality if one wants to prove that a local ring is Cohen–Macaulay. An example therefore:
Let $R = \mathbb{C}\{x_1, x_2, x_3, x_4\}$ and $I = (x_1, x_2) \cap (x_3, x_4)$ an ideal in $R$. Then the dimension of $R/I$ is 2 whereas the depth of $R/I$ is only one. The depth can be computed by the Auslander–Buchsbaum formula (see below):

$$\mathrm{projdim}_R(R/I) + \mathrm{depth}(\mathfrak{m}, R/I) = \mathrm{depth}(\mathfrak{m}, R).$$

Since $\mathrm{depth}(\mathfrak{m}, R) = 4$ and $\mathrm{projdim}_R(R/I) = 3$ (by a Singular [98] computation), one obtains $\mathrm{depth}(\mathfrak{m}, R/I) = 1$.
In Example 3.4 we show that there does not exist a divisor in $\mathbb{C}^4$ that has $I \in \mathbb{C}\{x_1, x_2, x_3, x_4\}$ as Jacobian ideal at the point $p = (x_1, \ldots, x_4)$.

## A.1.2 Projective modules and some homological algebra

Let $R$ be a ring. An $R$-module $P$ is called *projective* if $P$ is a direct summand of a free $R$-module. There are also a few equivalent characterizations, see e.g. [32, Prop. A3.1]. A *projective resolution* of an $R$-module $M$ is a complex

$$\mathcal{F} : \cdots \longrightarrow F_n \xrightarrow{\varphi_n} \cdots \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0$$

of projective $R$-modules $F_i$, such that $\mathcal{F}$ has no homology, i.e., is exact. If all $F_i$ are free $R$-modules, then $\mathcal{F}$ is called a *free resolution.* If for some $n < \infty$ one has $F_{n+1} = 0$ but $F_i \neq 0$ for any $0 \leq i \leq n$, then $\mathcal{F}$ is called a *finite* resolution of *length* $n$. One defines the *projective dimension* $\mathrm{projdim}_R M$ (also written $\mathrm{projdim} M$, if the ring is clear from the context) to be the minimum length of a projective resolution of $M$. One sets $\mathrm{projdim}_R M = \infty$ if $M$ has no finite projective resolution.
Since we mostly deal with local rings, the following lemma is very important:

**Lemma A.6.** *Let* $(R, \mathfrak{m})$ *be a local ring. Then a projective module over* $R$ *is free.*

*Proof.* See e.g. [62, Theorem 2.5.] or [56]. □

The following results are used to prove Aleksandrov's theorem in Chapter 2. We also introduce perfect modules, which are important in the theory of Cohen–Macaulay modules.

**Theorem A.7** (Auslander–Buchsbaum)**.** *Let* $(R, \mathfrak{m})$ *be a noetherian local ring and* $M \neq 0$ *a finite* $R$*-module. Suppose that* $\mathrm{projdim}_R M < \infty$*; then*

$$\mathrm{projdim}_R M + \mathrm{depth}(\mathfrak{m}, M) = \mathrm{depth}(\mathfrak{m}, R) = \mathrm{depth}(R).$$

*Proof.* See e.g. [32, Theorem 19.9] or [62, Theorem 19.1]. □

**Definition A.8.** Let $R$ be a noetherian ring and $M$ be a non-zero, finite $R$-module. Then $M$ is called *perfect* if

$$\mathrm{projdim} M = \mathrm{depth}(\mathrm{Ann} M, R).$$

An ideal $I \subseteq R$ is called *perfect* if $R/I$ is a perfect module.

**Lemma A.9.** *Let* $R$ *be a noetherian ring and let* $I \subseteq R$ *be an ideal such that* $\mathrm{proj\,dim}_R\, R/I = \mathrm{depth}(I, R)$*, that is,* $I$ *is perfect. If* $R$ *is Cohen–Macaulay then* $R/I$ *is also Cohen–Macaulay.*

*Proof.* First we can reduce the problem to $R$ local since the following holds: $R$ is Cohen–Macaulay if and only if $R_\mathfrak{p}$ is Cohen–Macaulay for all maximal ideals $\mathfrak{p}$ in $R$. For depth the inequality $\mathrm{depth}(I, M) \leq \mathrm{depth}(I_\mathfrak{p}, M_\mathfrak{p})$ holds for any ideal $I \subseteq \mathfrak{p}$, $\mathfrak{p}$ prime in the support of a finite $R$-module $M$ [32, Lemma 18.1]. Further, by [32, Cor. 18.5], the inequality $\mathrm{projdim}_R M \geq \mathrm{depth}(\mathrm{Ann} M, R)$ holds for finite $M$. The localization functor is exact, hence $\mathrm{projdim}_{R_\mathfrak{p}} M_\mathfrak{p} \leq \mathrm{projdim}_R M$. Plugging $M = R/I$ into the first inequality, everything localized in the second inequality and using that $I$ is perfect in $R$ follows that $I_\mathfrak{p}$ is perfect in $R_\mathfrak{p}$ for $\mathfrak{p}$ maximal in $R$.
Now let $R$ be local with maximal ideal $\mathfrak{m}$. We have to show that $\dim(R/I) = \mathrm{depth}(R/I)$. By the theorem of Auslander–Buchsbaum (Thm. A.7) the following equality holds: $\mathrm{projdim}_R R/I = \mathrm{depth}(\mathfrak{m}, R) - \mathrm{depth}(\mathfrak{m}, R/I)$. Since $I$ is perfect, this equality becomes

$$\mathrm{depth}(I, R) = \mathrm{depth}(\mathfrak{m}, R) - \mathrm{depth}(\mathfrak{m}, R/I).$$

From $R$ local and Cohen–Macaulay follows $\dim(R) = \dim(R/I) + \mathrm{ht}(I)$. A combination of these two equalities and using Cohen–Macaulayness of $R$, that is, the height of any ideal in $R$ is equal to its depth on $R$, yields $\dim(R/I) = \mathrm{depth}(\mathfrak{m}, R/I)$. Hence

$$\mathrm{depth}(R/I) \geq \mathrm{depth}(\mathfrak{m}, R/I) = \dim(R/I),$$

which completes the proof. □

*Remark* A.10. One can show that for a local Cohen–Macaulay ring and a finite $R$-module $M$ of finite projective dimension the following holds: $M$ is a Cohen–Macaulay module if and only if it is perfect, see [12, Theorem 2.1.5.].

The next theorem characterizes modules of the form $R/I$, where $R$ is a local ring and $I \subseteq R$ is an ideal such that projdim$R/I = 2$.

**Theorem A.11** (Hilbert–Burch)**.** *Let $R$ be a local ring.*
*(a) If a complex*

$$\mathcal{F} : 0 \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0$$

*is exact, $F_2$ is free and $F_1 \cong R^p$, then $F_2 \cong R^{p-1}$ and there exists a nonzerodivisor $x \in R$ such that $I = xI_{p-1}(\varphi_2)$. Here $I_{p-1}(\varphi_2)$ denotes the ideal generated by the $(p-1) \times (p-1)$-minors of the $p \times (p-1)$-matrix representing $\varphi_2$. Moreover, the ideal $I_{p-1}(\varphi_2)$ has depth exactly 2 in $R$.*
*(b) Conversely, given any $p \times (p-1)$ matrix $\varphi_2$ such that* depth $I_{p-1}(\varphi_2)$ *is greater than or equal to 2 and a nonzerodivisor $x$, the map $\varphi_1$, obtained as in (a) makes of $\mathcal{F}$ a free resolution of $R/I$, with $I = xI_{p-1}(\varphi_2)$.*

*Proof.* See e.g. [32, Theorem 20.15]. □

## A.1.3 Integral closure of rings and ideals

We need normalization of analytic spaces as well as the integral closure of ideals. Here we describe the commutative algebra behind the geometry. Recall that a reduced commutative ring $R$ is called normal if it is integrally closed in its total ring of fractions. In the following we denote the normalization of a ring $R$ by $\widetilde{R}$.

**Theorem A.12** (Splitting of normalization)**.** *Let $R$ be a reduced noetherian ring and $(0) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$ be an irredundant primary decomposition. Then there is a canonical isomorphism between $\widetilde{R}$, the normalization of $R$, and the direct sum of the normalizations of $R/\mathfrak{p}_i$, that its,*

$$\widetilde{R} \cong \bigoplus \widetilde{R/\mathfrak{p}_i}.$$

*Proof.* See Theorem 1.5.20 of [27]. □

*Remark* A.13. The geometric content of this theorem is that normalization separates the analytic branches of a variety (see next section).

**Definition A.14.** Let $R$ be a commutative ring and let $I \subseteq R$ be a proper ideal. One says that an element $f \in R$ is *integral over $I$* if there exists a relation

$$f^k + a_1 f^{k-1} + \cdots + a_n = 0,$$

where $a_i$ are elements of $I^i$. Then the *integral closure* of the ideal $I$ is defined to be set of all integral elements over $I$ and is denoted by $\overline{I}$.

One can easily show that $\overline{I}$ is also an ideal of $R$ and that one always has the chain of inclusions $I \subseteq \overline{I} \subseteq \sqrt{I}$ and $(\overline{I})^n \subseteq \overline{I^n}$ for all $n \geq 0$. Integral closure of ideals was first defined in [104, Appendix]. In [57] various characterizations and applications of integral closure, in particular in the analytic case are discussed.

# A.2 Complex analysis – local analytic geometry

In local analytic geometry one uses concepts from complex analysis as well as from commutative algebra. First we recall the definitions of the objects we work with in the text. These definitions and more background info and proofs can be found in many textbooks, e.g. [27,69,70, 97].

## A.2.1 Analytic spaces, sheaves and notation

The notions of *sheaf* and *locally ringed space* can be found in any textbook on algebraic or analytic geometry. Here we remark that instead of the usual definition via open sets one can also define a sheaf via

its stalks, as e.g. in [87]: Let $X$ be a topological space. A sheaf $\mathcal{F}$ of abelian groups (modules, rings, ...) consists of
(a) a function $x \mapsto \mathcal{F}_x$ that corresponds to each $x \in X$ an abelian group (module, ring, ...),
(b) a topology on $\mathcal{F} = \bigcup_{x \in X} \mathcal{F}_x$, the disjoint union of the stalks $\mathcal{F}_x$.
We use this characterization of sheaves when we define the sheaves of logarithmic differential forms and vector fields in Chapter 1.
Let $(X, \mathcal{O}_X)$ be a locally ringed Hausdorff space. Then $(X, \mathcal{O}_X)$ is called a *complex manifold* if for any $p \in X$ there exists a neighbourhood $U \subseteq X$ such that $(U, \mathcal{O}_{X|U})$ is isomorphic to $(V, \mathcal{O}_V)$, where $V$ is a domain in $\mathbb{C}^n$. The integer $n$ is called the *(complex) dimension* of $X$ and denoted by $\dim X$. One also has an equivalent differential-geometric definition: a complex manifold $M$ is a topological manifold equipped with a system of local charts $\varphi_i : U_i \to \mathbb{C}^n$, $\varphi_i$ a diffeomorphism such that the open sets $U_i$ cover $M$ and the change of charts morphisms

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

are holomorphic. The $\varphi_i = (\varphi_{i1}, \ldots, \varphi_{in})$ on $U_i$ are called *(local) complex coordinates on $M$* and are mostly denoted by $(x_1, \ldots, x_n)$. Further, a locally ringed space $(X, \mathcal{O}_X)$ is called an *analytic space* if any $p \in X$ has a neighbourhood $U$ such that $(U, \mathcal{O}_{X|U})$ is isomorphic to $(V, \mathcal{O}_V)$, where $V$ is an analytic subset of an open set $W \subseteq \mathbb{C}^n$ for some $n$, and the ring $\mathcal{O}_V = (\mathcal{O}_W / \mathscr{I}(V))|_V$.
By definition, an analytic space is always *reduced*. Sometimes we need a non-reduced structure on an analytic space (e.g. when dealing with Jacobian ideals in Chapters 2 and 3): a locally ringed Hausdorff space $(X, \mathcal{O}_X)$ is called a *complex space* if any $p \in X$ has a neighbourhood $U$ such that $(U, \mathcal{O}_{X|U})$ is isomorphic to $(V, \mathcal{O}_V)$, where $V$ is an analytic subset of an open $W$ in some $\mathbb{C}^n$ and $\mathcal{O}_V = (\mathcal{O}_W / J)_{|V}$, $J \subseteq \mathcal{O}_W$ an ideal sheaf such that for all $p \in V$, $\sqrt{J_p} = \mathscr{I}(V)_p$.

Recall that a sheaf $\mathscr{F}$ on a locally ringed space $(X, \mathcal{O}_X)$ is called *coherent* if it is finitely generated and of relation finite type, that is, for any point $x \in X$ there is an open neighborhood $U \subseteq X$ and a surjective morphism of sheaves $\mathcal{O}^q_{X|U} \to \mathscr{F}_{|U} \to 0$ and for any open set $U$ and any morphism of sheaves $\alpha : \mathcal{O}^q_{X|U} \to \mathscr{F}_{|U}$ the kernel $\ker(\alpha)$ is finitely generated. Sometimes we need

**Theorem A.15** (Meta-Theorem for coherent sheaves)**.** *Let $\mathscr{F}$ and $\mathscr{G}$ be coherent sheaves of $\mathcal{O}_X$-modules. Then every reasonable operation* *with $\mathscr{F}, \mathscr{G}$ (finitely generated subsheaves, taking kernel or cokernel, tensor product, ...) results again in a coherent sheaf.*

*Proof.* See [27, Theorem 6.2.3]. □

The following theorem due to G. Scheja is about the singularities of coherent sheaves is the main ingredient to answer Saito's question (Chapter 1) in general.
Let $(X, \mathcal{O}_X)$ be a complex space and $\mathscr{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules on $X$. Denote for a point $x \in X$

$$\operatorname{depth}_x \mathscr{F} = \begin{cases} \infty \text{ if } \mathscr{F}_x = 0, \\ \operatorname{depth}(\mathfrak{m}_x, \mathscr{F}_x) \text{ else (where } \mathfrak{m}_x \text{ is the maximal ideal of } \mathcal{O}_{X,x}). \end{cases}$$

Further define the *singular subvarieties* of $\mathscr{F}$ as

$$S_m(\mathscr{F}) = \{x \in X : \operatorname{depth}_x \mathscr{F} \leq m\}.$$

**Theorem A.16** (Scheja)**.** *Let $(X, \mathcal{O}_X)$ be a complex space and $\mathscr{F}$ a coherent sheaf of $\mathcal{O}_X$-modules on $X$. Then the sets $S_m(\mathscr{F})$ are subvarieties of $X$ and $\dim S_m(\mathscr{F}) \leq m$.*

*Proof.* See [84] and for this formulation (1.11) of [90]. □

A germ of an analytic space $(X, x)$ is called *normal* if the local ring $\mathcal{O}_{X,x}$ is a normal ring and the analytic space $X$ is called normal if for all $x \in X$ the germ $(X, x)$ is normal. Note that a normal germ of an analytic space is irreducible, see [27, Thm. 1.5.7].

We always work with a *divisor* $D$ in a complex manifold $S$ with $\dim(S) = n$. A divisor $D$ is by definition an analytic hypersurface in $S$, that is, locally at a point $p = (x_1, \ldots, x_n) \in S$, where $(x_1, \ldots, x_n)$ denote complex coordinates, $D$ is defined by an equation $h = 0$ (we also say: $D$ is given by $h$ or write $D = \{h = 0\}$ locally), where $h$ is a holomorphic function germ in $\mathcal{O}_{S,p} \cong \mathbb{C}\{x_1, \ldots, x_n\}$ (sometimes only denoted by $\mathcal{O}$). Note that a-priori we do not assume that $h$ is reduced. However, in our context we nearly always work with reduced defining equations, that is, $(D, p)$ is an analytic space. The next notion is crucial in the whole thesis, therefore it gets its own definition:

**Definition A.17.** Let $(S, D)$ be a complex manifold with $\dim(S) = n$ together with a divisor $D$. We say that $D$ has *normal crossings* at a point $p \in S$ if one can find local coordinates $(x_1, \ldots, x_n)$ such that $D$

is defined by the equation $x_1 \cdots x_r = 0$, where $0 \le r \le n$ depends on the considered point. In this case we also say that $(D, p)$ has normal crossings or is a *normal crossing singularity.* A divisor $D$ is called a *normal crossing divisor* if $D$ has normal crossings at any point $p \in S$.

*Example* A.18. Let $D \subseteq S = \mathbb{C}^2$ be globally defined by $h = x^3 - y^2$. At the origin $D$ does not have normal crossings since $(D, 0)$ is an $A_3$-singularity. At any point $p \in D$, $p$ not the origin, one finds that $(D, p)$ is smooth and hence one can find local coordinates $(x', y')$ at $p$, such that $D = \{x' = 0\}$. Thus $(D, p)$ is a normal crossing singularity. At a point $p \notin D$, $D$ is clearly defined by $h \equiv 0$, so in this case $D$ has also normal crossings at $p$.

Two divisors $D_1$ and $D_2$ in a complex manifold $S$ are said to intersect *transversally* at a point $p$ if they are both smooth at $p$ and their union $D_1 \cup D_2$ has normal crossings at $p$. Otherwise we say that $D_1$ and $D_2$ meet *tangentially* at $p$. The following fact about reduced divisors is used on various occasions throughout the text.

**Lemma A.19.** *Let $S$ be a complex manifold of complex dimension $n$ and $D$ a divisor in $S$. Suppose that at a point $p$ the divisor is defined by a reduced $h \in \mathcal{O}_{S,p}$. Then one can choose coordinates $(x_1, \dots, x_n)$ at $p$ such that* $\dim\{h = \partial_{x_n} h = 0\} \le n - 2$, *i.e., $h$ and $\partial_{x_n} h$ do not have a common factor.*

*Proof.* We may suppose that $h$ is a Weierstrass-polynomial in $\mathcal{O}_{S,p} \cong \mathbb{C}\{x_1, \dots x_n\}$, see [27]. Then

$$h = x_n^p + a_{n-1}(x_1, \dots, x_{n-1}) x_n^{p-1} + \cdots + a_0(x_1, \dots, x_{n-1}),$$

with $a_i \in \mathbb{C}\{x_1, \dots, x_{n-1}\}$. Let $h = h_1 h_2$, where the $h_i$ are not necessarily irreducible. Suppose that $h$ and $\partial_{x_n} h$ had some common components, namely the ones of $h_1$. Then one had $\partial_{x_n} h = h_1 g$ for some $g \neq 0 \in \mathcal{O}_{S,p}$. Note that $\partial_{x_n} h_1 \neq 0$ because otherwise it would follow (using that $h$ is a Weierstrass-polynomial) that $h_1$ is a unit in $\mathcal{O}_{S,p}$. Differentiating by $x_n$ yields

$$\partial_{x_n} h = (\partial_{x_n} h_1) h_2 + h_1 (\partial_{x_n} h_2) = h_1 g,$$

which implies

$$(g - \partial_{x_n} h_2) h_1 = (\partial_{x_n} h_1) \cdot h_2.$$

Since $\partial_{x_n} h_1$ is not equal to zero, $h_1$ divides $(\partial_{x_n} h_1) \cdot h_2$. But $h_1$ does not divide any factor of $h_2$, thus $h_1$ has to divide $\partial_{x_n} h_1$, which yields a contradiction. □

Next we recall some facts about vector fields and differential forms on manifolds: Let $S$ be a complex manifold and $p \in S$ a point. A *tangent vector* $v$ is an element of the tangent space $T_pS$. Equivalently, a tangent vector can be given as a *derivation* $\chi : \mathcal{O}_{S,p} \to \mathbb{C}$, that is, a $\mathbb{C}$-linear map such that for all $f, g \in \mathcal{O}_{S,p}$ we have $\chi(fg) = f\chi(g) + g\chi(f)$. The $T_pM$ make up the *tangent bundle* $\mathcal{T}_S$ of $S$, which is also denoted by $\mathrm{Der}_S$. A section of the tangent bundle is called *vector field*. Informally speaking, a vector field $\delta$ assigns continuously to each $p \in S$ a vector $\delta(p) \in T_pS$. The dual bundle $(\mathcal{T}_S)^* = \mathrm{Hom}(\mathcal{T}_S, \mathbb{C})$ is called the *cotangent bundle* and denoted by $\Omega^1_S$. A section of $\Omega^k_S := \bigwedge^k \Omega^1_S$ is called a *(holomorphic) differential $k$-form*. In local coordinates $(x_1, \ldots, x_n)$ in a neighbourhood $U$ of a point $p \in S$ a vector field $\delta$ can be expressed by $\delta = \sum_{i=1}^n a_i(x)\partial_{x_i}$ with $a_i \in \mathcal{O}_U$. The elements $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Omega^k_U$, $i_1 < \cdots < i_k$, provide a basis of $\Omega^k_U$. One can also consider *meromorphic* differential $k$-forms that are locally in $U$ of the form $\omega = \sum_I w_I dx_I$ with $w_I \in \mathcal{M}_U$, the meromorphic functions on $U$.

## A.2.2 Extension Theorems

In analytic geometry, one often wants to extend holomorphic functions that are only defined on a subset of an analytic space to the whole space. Extension theorems (Hartogs, Riemann) tell us when this is possible. In the following section we will also learn about weakly holomorphic functions, which are connected to Riemann's extension theorem.

**Theorem A.20** (Riemann's extension theorem)**.** *Let $U \subseteq \mathbb{C}^n$ be open and connected and $X \subseteq U$ an analytic set. Consider a holomorphic function*

$$f : U \backslash X \longrightarrow \mathbb{C}$$

*and suppose that $f$ is locally bounded, that is, for all $p \in X$ there exists a neighbourhood $V$ of $p$ such that $f|_{V \backslash (V \cap X)}$ is bounded. Then there exists a holomorphic extension of $f$ to $U$. This means that there exists a holomorphic function $F : U \to \mathbb{C}$ such that $F$ restricted to $U \backslash X$ is equal to $f$.*

**Theorem A.21** (Hartogs' theorem)**.** *Let $U \subseteq \mathbb{C}^n$ be an open set and consider an analytic set $A \subseteq U$ with $\dim_p(A) \leq n - 2$ for all $p \in A$, that is, the codimension of $A$ in $U$ is at least 2. Let*

$$f : U \backslash A \to \mathbb{C}$$

*be a holomorphic function. Then there exists a unique holomorphic extension of $f$ to $U$.*

*Proof.* [27, Theorem 3.1.15] and [27, Theorem 4.1.24]. □

This theorem can be stated in more generality for analytic spaces, namely, instead of $\mathbb{C}^n$ it is enough to have a normal ambient space:

**Theorem A.22.** *Let $X$ be a normal analytic space and $A$ be an analytic subset of $X$ with $\dim_p(A) \leq \dim_p(X) - 2$ for all $p \in X$. Then any holomorphic function on $X \backslash A$ can be extended to $X$.*

*Proof.* See [69, Proposition 4, Ch. VI]. □

### A.2.3 Normalization, universal denominators and weakly holomorphic functions

In this section we give a brief overview of some notions and results related to normalization of analytic spaces, which are used in chapter 1. Briefly, normalization of an analytic space separates its irreducible components and kills all singularities of codimension 1.

In order to define the normalization of an analytic space germ $(X, x)$, which is in general not again an analytic space germ, we need some more notions: a *multi-germ* $(X, x)$ of analytic spaces $(X_1, x_1), \ldots, (X_k, x_k)$, $k \leq \infty$ is the disjoint union $(X, x) = (X_1, x_1) \cup \cdots \cup (X_k, x_k)$. By definition, the ring $\mathcal{O}_{X,x}$ is $\bigoplus_{i=1}^k \mathcal{O}_{X_i,x_i}$. Note that $\mathcal{O}_{X,x}$ is a semi-local ring. Finally, let $(Y, y) = (Y_1, y_1) \cup \cdots \cup (Y_m, y_m)$ be another multi-germ. Suppose, we are given a system of maps of $\varphi_i : (X_i, x_i) \to (Y_{\alpha(i)}, y_{\alpha(i)})$ for $i = 1, \ldots, k$ and some $\alpha(i) \in \{1, \ldots, m\}$. Then a map $\varphi : (X, x) \to (Y, y)$ is given by this system and this map induces and is induced by a $\mathbb{C}$-algebra map $\varphi^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. One can define properties of this map (e.g. finite, proper) in an obvious way. A multi-germ $(X, x)$ is called *normal* if $\mathcal{O}_{X,x}$ is a normal ring. It is easy to see that the ring $\mathcal{O}_{X,x}$ is normal if and only if $\mathcal{O}_{X_i,x_i}$ is normal for $i = 1, \ldots, k$.

**Definition A.23.** Let $(X, x)$ be an analytic space germ. A *normalization* of $(X, x)$ is a multi germ $(\widetilde{X}, \tilde{x})$, which is normal, together with a proper map $\pi : (\widetilde{X}, \tilde{x}) \to (X, x)$ with finite fibers such that if $(\operatorname{Sing} X, x)$ denotes the singular set of $(X, x)$ and $(A, \tilde{x}) := (\pi^{-1}(\operatorname{Sing} X), \tilde{x})$, then

$(\widetilde{X}\backslash A, \tilde{x})$ is dense in $(\widetilde{X}, \tilde{x})$ and via $\pi|_{\widetilde{X}\backslash A}$ analytically isomorphic to $(X\backslash \operatorname{Sing} X, x)$.

One can prove that a normalization always exists (see Thm. 4.4.8. of [27]) and that it is uniquely determined. For the normalization of a germ $(X, x) = \bigcup_{i=1}^{k}(X_i, x)$, with $X_i$ irreducible, one obtains that $\widetilde{\mathcal{O}}_{X,x} = \bigoplus_{i=1}^{k} \widetilde{\mathcal{O}}_{X_i,x}$ (by the splitting of normalization theorem). In particular the normalization of an irreducible space germ $(X, x)$ is again an irreducible space germ $(\widetilde{X}, \tilde{x})$.

**Theorem A.24.** *Let $(X, x)$ be a normal analytic space germ. Denote* $\operatorname{Sing} X$ *the singular locus of $X$. Then one has*

$$\dim_x(\operatorname{Sing} X) \leq \dim_x(X) - 2.$$

*If $X$ is a hypersurface in a complex manifold, then the other implication holds.*

*Proof.* See Chapter VI, Theorem 2 of [69]. We prove here the second statement for hypersurfaces: Let $X$ be a hypersurface in a complex manifold $S$ of dimension $n$. Suppose that locally at a point $x$ the hypersurface is defined by a reduced $h \in \mathcal{O}_{S,x}$. Denote $(X, x)$ be the corresponding analytic germ at $x$ and suppose that $\dim_x(\operatorname{Sing} X) \leq n-2$. We show that $\mathcal{O}_{X,x}$ is a normal ring with Serre's characterization of normal rings (see [27, Thm. 4.4.11]); the ring $\mathcal{O}_{X,x}$ is normal if and only if the following two conditions hold:
(R1) For each prime ideal $\mathfrak{p} \in \mathcal{O}_{X,x}$ the ring $(\mathcal{O}_{X,x})_{\mathfrak{p}}$ is a regular local ring.
(S2) If $f \in \mathcal{O}_{X,x}$ is a nonzerodivisor, then the ideal $(f) \subseteq \mathcal{O}_{X,x}$ has no embedded primes.
The condition (R1) follows from the assumption on the singular locus of $X$ (using the Jacobian criterion for regularity [32, Cor. 16.20]). For (S2) we remark that $\mathcal{O}_{X,x}$ is a Cohen–Macaulay ring, since $X$ is locally a complete intersection (it is a hypersurface). Hence for a nonzerodivisor $f$, the ring $\mathcal{O}_{X,x}/(f)\mathcal{O}_{X,x}$ is also Cohen–Macaulay (see e.g. [32]). But this implies that the ideal $(f)$ is unmixed, that is, it has no embedded primes, and hence (S2) holds for $\mathcal{O}_{X,x}$. □

*Remark* A.25. In general the following holds: If $(X, x)$ is locally a Cohen–Macaulay singularity, that is, $\mathcal{O}_{X,x}$ is Cohen–Macaulay, and if $\dim_x(\operatorname{Sing} X) \leq \dim_x(X) - 2$, then $(X, x)$ is a normal analytic space germ. The proof is the same as in the hypersurface case since for (S2) we just need that $\mathcal{O}_{X,x}$ is a Cohen–Macaulay ring.

We gave the definition of a normal space from the algebraic point of view. But one also has an interpretation of a normal analytic space in complex analysis. Namely, an analytic space is normal if and only if the Riemann extension theorem holds for it. Therefore we need the notion of weakly holomorphic functions. Moreover, we will come across so-called universal denominators, that is, holomorphic functions $f$ such that the multiplication of a weakly holomorphic function on an analytic space germ with $f$ yields a holomorphic function. But first a few definitions.

**Definition A.26.** Let $X$ be an analytic space and denote by $\operatorname{Sing} X$ its singular locus. A function $f : X \setminus \operatorname{Sing} X \to \mathbb{C}$ is said to be *weakly holomorphic* on $X$ if the following two conditions hold:
(1) $f$ is holomorphic on $X \setminus \operatorname{Sing} X$.
(2) $f$ is locally bounded along $\operatorname{Sing} X$.
Let $(X, x)$ be an analytic space germ. For any $x \in X$ we may define the germ of a weakly holomorphic function at $x$. Obviously the germs of weakly analytic functions at $x$ form a ring, the *ring of weakly holomorphic functions*, which we denote by $\mathcal{O}'_{X,x}$.

*Example* A.27. Let $(X, x)$ be a normal crossing divisor in $\mathbb{C}^n$ defined by

$$\mathcal{O}_{X,x} = \mathbb{C}\{t_1, \dots, t_n\}/(t_1 \cdots t_d)$$

with $1 \leq d \leq n$. The coordinate ring of the normalization is then $\widetilde{\mathcal{O}}_{X,x} = \oplus_{i=1}^d \mathbb{C}\{t_1, \dots, t_n\}/(t_i)$. Geometrically, the normalization $(\widetilde{X}, \tilde{x})$ of $(X, x)$ consists of $d$ copies of smooth hyperplanes $(X_i, x_i)$ with coordinates $(t_1, \dots, \widehat{t_i}, \dots, t_n)$. The normalization map $\pi$ is given by the $d$ maps $\pi_i : (X_i, x_i) \to (X, x)$, sending $(t_1, \dots, \widehat{t_i}, \dots, t_n)$ to $(t_1, \dots, 0, \dots, t_n)$.

*Example* A.28. Let $(X, x)$ be the cusp with $\mathcal{O}_{X,x} = \mathbb{C}\{x, y\}/(y^2 - x^3)$. The element $t := \frac{y}{x}$ is integral over $\mathcal{O}_{X,x}$ since $t^2 = \frac{y^2}{x^2} = \frac{x^3}{x^2} = x$, and $t$ satisfies the integral equation $T^2 - x = 0$. The normalization ring is $\mathbb{C}\{x, y, t\}/(y^2 - x^3, t^2 - x) = \mathbb{C}\{t\}$. It is easy to see that the normalization map is given by $t \mapsto (t^2, t^3)$.

*Example* A.29. Let $(X, x)$ be the $E_8$-singularity in $(\mathbb{C}^3, 0)$ with coordinate ring $\mathcal{O}_{X,x} = \mathbb{C}\{x, y, z\}/(x^2 + y^3 + z^5)$. Since $\dim_x(\operatorname{Sing} X) = 2 - 2 = 0$, it follows from Thm. A.24 that $(X, x)$ is already normal. Hence the normalization map is the identity.

One sees that $\mathcal{O}_{X,x}$ embeds into $\mathcal{O}'_{X,x}$. On the other hand, one may even find a holomorphic $f \in \mathcal{O}_{X,x}$ such that $f\mathcal{O}'_{X,x} \subseteq \mathcal{O}_{X,x}$. Such an $f$ is called a "universal denominator". We list some facts about universal denominators before stating that $\mathcal{O}'_{X,x} = \widetilde{\mathcal{O}}_{X,x}$.

**Definition A.30.** Let $X$ be an analytic set in an open set $U$ in some $\mathbb{C}^n$. A holomorphic function $f$ on $U$ is called a *universal denominator* for $X$ at a point $x \in X$ if we can find a neighbourhood $V$ of $X$ in $U$ such that: if $g$ is a holomorphic function on the analytic set $X' = (X \setminus \operatorname{Sing} X) \cap V$ and if $g$ is bounded on $X'$, then there exists a neighbourhood $W$ of $x$ such that $fg$ is the restriction of a holomorphic function on $W$ to $X' \cap W$.

The previous definition can be used for analytic spaces in an obvious way.

**Theorem A.31.** *Let $X$ be an analytic set in $\mathbb{C}^n$ and $x \in X$. Then there exists a neighbourhood $V$ of $x$ and finitely many holomorphic functions $f_1, \ldots, f_m$ in $V$ such that:*
*(1) The set $\operatorname{Sing}(X \cap V) = \{p \in V : f_1(p) = \cdots = f_m(p) = 0\}$;*
*(2) Each $f_i$ is a universal denominator at every point of $V$.*

*Proof.* See [69, III, Thm. 6]. □

**Theorem A.32** (Tsikh)**.** *Let $U \subseteq \mathbb{C}^n$ be a domain and $X = \{z \in U : f_1(x) = \cdots = f_p(x) = 0\}$ for some $f_i \in \mathcal{O}_U$ be a complete intersection analytic subset of $U$, that is, $\dim X = n - p$ and $X$ is pure-dimensional. Define for any $I = (i_1, \ldots, i_p) \in \{1, \ldots, p\}$ the function $g_I = \frac{\partial(f_1,\ldots,f_p)}{\partial(x_{i_1},\ldots,x_{i_p})}$. If $g_I$ does not vanish on any irreducible component of $X$, then $g_I$ is a universal denominator forX.*

*Proof.* See [96, Thm. 1] □

**Theorem A.33.** *Let $(X, x)$ be an analytic space germ. A function germ $f$ is in the integral closure of $\mathcal{O}_{X,x}$ if and only if $f$ is a weakly holomorphic function germ, that is, one has a canonical isomorphism*

$$\mathcal{O}'_{X,x} \cong \widetilde{\mathcal{O}}_{X,x}.$$

*Moreover, $(X, x)$ is normal if and only if every germ of a weakly holomorphic function on $(X, x)$ can be extended to a holomorphic function on $(X, x)$.*

*Remark* A.34. This theorem means that an analytic space germ $(X, x)$ is normal if and only if the Riemann extension theorem holds for $(X, x)$.

We have another isomorphism, namely the direct image sheaf of the normalization of the analytic space $X$, denoted by $\pi_*\mathcal{O}_{\widetilde{X}}$, is isomorphic to the normalization sheaf $\widetilde{\mathcal{O}}_X$. To state this result properly we need some more facts about coherent sheaves on analytic spaces, in particular Oka's finite mapping theorem.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be analytic spaces and let $f : X \to Y$ be a holomorphic map. An *analytic sheaf* $\mathscr{F}$ on $X$ is a sheaf of $\mathcal{O}_X$-modules. We define a presheaf $f_*\mathscr{F}$ on $Y$ by the following rule: for an open set $U \subseteq Y$ set

$$f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U)).$$

The restriction maps are the obvious ones, and one can easily see that $f_*\mathscr{F}$ is a sheaf on $Y$. The sheaf $f_*\mathscr{F}$ is called the *direct image (sheaf)* of $\mathscr{F}$, with stalk $f_*\mathcal{F}_y$ at a point $y \in Y$.

*Example* A.35. Let $X$ be any analytic space with an analytic sheaf $\mathscr{F}$ and suppose that $Y = \{p\}$ is a point and $f : X \to Y$ is the map sending the whole of $X$ to $p$. Then $f_*\mathscr{F}$ is a sheaf on a point and can hence be identified with a ring. But we also have $f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U)) = \mathscr{F}(X)$. Thus the direct image sheaf is in this case just the ring of global sections of $\mathscr{F}$ over $X$.

*Remark* A.36. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be analytic spaces, such that for all $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring and let $f : X \to Y$ be a holomorphic map. Then by definition of a morphism of locally ringed spaces there is also a morphism $f^*$ of sheaves of $\mathcal{O}_Y$-modules:

$$f^* : \mathcal{O}_Y \to f_*\mathcal{O}_X,$$

such that the induced map on the stalks $f^*_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ sends the maximal ideal of $\mathcal{O}_{Y,f(x)}$ into the maximal ideal of $\mathcal{O}_{X,x}$.

**Theorem A.37** (Finite mapping theorem)**.** *Let $f : X \to Y$ be a finite mapping of analytic spaces, and let $\mathscr{F}$ be a coherent $\mathcal{O}_X$-sheaf. Then $f_*\mathscr{F}$ is a coherent $\mathcal{O}_Y$-sheaf.*

*Proof.* See [27, Thm. 6.3.5]. □

The finite mapping theorem is used to prove the following theorem, in whose proof one shows the equality of the normalization sheaf $\widetilde{\mathcal{O}}_X$ and the direct image sheaf of the normalization $\pi_*\mathcal{O}_X$.

**Theorem A.38** (Oka). *Let $(X, \mathcal{O}_X)$ be an analytic space and denote by $\pi : \widetilde{X} \to X$ its normalization. The normalization sheaf $\widetilde{\mathcal{O}}_X$ is the sheaf whose stalk at a point $x \in X$ is $\widetilde{\mathcal{O}}_{X,x}$. Then the normalization sheaf is $\mathcal{O}_X$-coherent.*

*Proof.* See [27, Thm. 6.3.7.]. □

From Theorems A.33 and A.38 we can conclude that the three rings $\mathcal{O}'_{X,x}$, $\pi_*\mathcal{O}_{\widetilde{X},x}$ and $\widetilde{\mathcal{O}}_{X,x}$ are isomorphic for any $x$ in an analytic space $X$.

*Remark* A.39. We remark here a fact about universal denominators (in $\mathcal{O}'_{X,x}$ and hence also in $\pi_*\mathcal{O}_{\widetilde{X},x}$). Obviously, the ring $\mathcal{O}_{X,x}$ is contained in $\mathcal{O}'_{X,x}$. If $g$ is a universal denominator at $x$ which does not vanish on any irreducible component of the analytic space germ $(X, x)$, then we obtain an exact sequence

$$0 \longrightarrow \mathcal{O}'_{X,x} \xrightarrow{\cdot g} \mathcal{O}_{X,x}.$$

This means that the $\mathcal{O}_{X,x}$-homomorphism $\mathcal{O}'_{X,x} \to \mathcal{O}_{X,x}$ given by the multiplication with $g$ is injective and maps $\mathcal{O}'_{X,x}$ onto some subring of $\mathcal{O}_{X,x}$.

**Definition A.40.** Let $(X, x)$ be the germ of an equidimensional analytic space with normalization $\pi : \widetilde{X} \to X$. Then the *conductor ideal* $C_{X,x}$ at $x$ is the largest ideal that is an ideal in $\mathcal{O}_{X,x}$ as well as in $\pi_*\mathcal{O}_{\widetilde{X},x}$ (we write $C_X$ if there is no danger of confusion regarding the point $x$). Alternatively, the conductor $C_{X,x}$ can be defined as the ideal quotient $(\mathcal{O}_{X,x} : \pi_*\mathcal{O}_{\widetilde{X},x}) = \{f \in \mathcal{O}_{X,x} : f\pi_*\mathcal{O}_{\widetilde{X},x} \subseteq \mathcal{O}_{X,x}\}$ or as $\mathrm{Hom}_{\mathcal{O}_{X,x}}(\pi_*\mathcal{O}_{\widetilde{X},x}, \mathcal{O}_{X,x})$.

*Remark* A.41. The conductor $C_X$ is a coherent sheaf of ideals over $\mathcal{O}_X$.

**Theorem A.42** (Piene's Theorem). *Let $X$ be a locally complete intersection variety of dimension $s$ over an algebraically closed field $k$. Let $f : Z \to X$ be a desingularization of $X$ and denote by $I_f = F_Z^0(\Omega^1_{Z/X})$ the ramification ideal of $f$ in $\mathcal{O}_Z$ and by $J_X$ the ideal $F_X^s(\Omega^1_{X/k})$. Suppose that $f$ is finite. Then there is an equality of ideals*

$$J_X\mathcal{O}_Z = I_f C_X \mathcal{O}_Z.$$

*Proof.* See Theorem 1 and Corollary 1 of [76]. □

*Remark* A.43. (1) The above theorem also holds in the analytic case since all constructions in the proof of Theorem 1 of [76] also work, cf. [4, 43, 66]. It also holds if we take as $Z$ the normalization $\widetilde{X}$ of $X$ and $\widetilde{X}$ is Gorenstein (because in the proof of Piene's Theorem one only needs that the dualizing sheaf $\omega_Z$ is invertible, cf. [43]).
(2) The ideal $J_X$ is sometimes also called "Jacobian ideal of $X$". We need the above theorem in the case where $X$ is a divisor $D$ in a complex manifold $S$ defined locally at a point $p$ by $\{h = 0\}$. Then $J_D$ is simply the ideal $J_h$ in $\mathcal{O}_{D,p}$ (resp. the ideal $((h) + J_h) \subseteq \mathcal{O}_{S,p}$) defining the singular locus $(\mathrm{Sing}(D), p)$. Clearly, $D$ is locally at $p$ a complete intersection.

## A.2.4 Cartesian products

Sometimes it is useful to know that an analytic space $(X, x)$ is a Cartesian product, which means that $(X, x)$ is locally isomorphic to some $(X' \times T, (x', t))$ where $(X', x')$ is of lower dimension than $(X, x)$ and $(T, t)$ is a smooth factor. Then one can read off properties of $X$ from $X'$ or apply induction on the dimension on $X$. Since we deal exclusively with hypersurfaces, the following is described only for them. A stronger form of the Cartesian product structure is analytic triviality, where one prescribes the structure of $(X', x')$. More precisely: let $T \cong \mathbb{C}^m$ and let $\{(X_t, 0)_{t \in T}\} = \{V(g_t), 0)\}_{t \in T}$ be a family of analytic hypersurface germs with $(X_t, 0) \subseteq (\mathbb{C}^n, 0)$ where $g_t := g(x_1, \ldots, x_n, t_1, \ldots, t_m) \in \mathbb{C}\{x_1, \ldots, x_n, t_1, \ldots, t_m\}$. The family $\{(X_t, 0)\}_{t \in T}$ is called *locally analytically trivial at* $t = 0 \in T$ if for all $t \in T$ there exists a biholomorphic map $\varphi_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ sending $(X_t, 0)$ to $(X_0, 0)$ that can additionally chosen to be analytic in $t$. This means nothing else but that $X = V(g(x, t)) \subseteq \mathbb{C}^{n+m}$ is locally isomorphic to $(X_0, 0) \times (T, 0) = (X_0 \times T, (0, 0))$.
The next lemma gives an ideal-theoretic characterization of Cartesian product structure resp. local analytic triviality. It is used frequently and can be found in various different formulations in the literature (e.g. in [22, 27, 39, 81]). We will give a partial proof and will refer to this lemma as the *triviality lemma.*

**Lemma A.44** (Triviality lemma)**.** *Let $(S, p)$ locally be isomorphic to $(\mathbb{C}^{n+m}, 0)$ and denote $\mathcal{O}_{S,p} = \mathbb{C}\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ (in short: $\mathcal{O} = \mathbb{C}\{x, y\}$) and let $h(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be an element of $\mathcal{O}$. Then the following are equivalent:*
*(a) The ideal $(\partial_{y_1} h, \ldots, \partial_{y_m} h)$ is contained in the ideal $(h, \partial_{x_1} h, \ldots, \partial_{x_n} h)$.*

*(b) There exists a local biholomorphic map* $\varphi : (\mathbb{C}^{n+m}, 0) \to (\mathbb{C}^{n+m}, 0)$ *and a holomorphic* $v(x, y) \in \mathcal{O}^*$ *such that*

$$\varphi(x, y) = (\varphi_1(x, y), \ldots, \varphi_n(x, y), y_1, \ldots, y_m),$$

$\varphi(x, 0) = (x, 0)$, $v(x, 0) \equiv 1$ *and* $h \circ \varphi(x, y) = v(x, y)h(x, 0)$.
*This means that* $D = \{h(x, y) = 0\}$ *is locally at* $p$ *isomorphic to some* $(D' \times \mathbb{C}^m, (0, 0))$ *where* $D'$ *is locally contained in* $\mathbb{C}^n$.
*Analytic triviality is characterized as follows. Under the same hypotheses as above the following are equivalent:*
*(a') The ideal* $(\partial_{y_1} h, \ldots, \partial_{y_m} h)$ *is contained in the ideal*

$$(x_1, \ldots, x_n, y_1, \ldots, y_m)(\partial_{x_1} h, \ldots, \partial_{x_n} h).$$

*(b') There exists a local biholomorphic map* $\varphi : (\mathbb{C}^{n+m}, 0) \to (\mathbb{C}^{n+m}, 0)$ *such that* $\varphi(x, y) = (\varphi_1(x, y), \ldots, \varphi_n(x, y), y_1, \ldots, y_m)$, $\varphi(x, 0) = (x, 0)$, $\varphi_i - x_i \in (x_1, \ldots, x_n)$ *and* $h \circ \varphi(x, y) = h(x, 0)$.
*This means that* $(D, p) \cong (D_0 \times \mathbb{C}^m, (p', 0))$ *where* $D_0 = \{h(x, 0) = 0\}$ *is the "fiber" at the origin.*

*Proof.* We prove $(a) \Leftrightarrow (b)$. First suppose (b): From $h \circ \varphi(x, y) = v(x, y)h(x, 0)$ follows $\partial_{y_i}(\frac{h\circ\varphi}{v}) = 0$ for all $i = 1, \ldots, m$. By chain and product rule one gets

$$\partial_{y_i}(h \circ \varphi)v = \partial_{y_i} v \cdot h \circ \varphi$$
$$\sum_{j=1}^{n} \partial_{x_j} h \circ \varphi \cdot \partial_{y_i} \varphi_j + \partial_{y_i} h \circ \varphi = \frac{\partial_{y_i} v}{v} h \circ \varphi.$$

Since $\varphi$ is biholomorphic one may substitute $(x, y)$ with $\varphi^{-1}(x, y)$. This yields
$\partial_{y_i} h \in (h, \partial_{x_1} h, \ldots, \partial_{x_n} h)$, what has to be shown.
Conversely, the statement is proven by induction on the number of $y_i$. We show the assertion for $m = 1$, i.e., $y = y_1$. Then (a) yields an equation

$$\partial_y h + \sum_{i=1}^{n} a_i \partial_{x_i} h = ah. \tag{A.1}$$

We define the vector field $\delta = \partial_y + \sum_{i=1}^{n} a_i \partial_{x_i}$, which satisfies $\delta(h) = ah$, $\delta(x_i) = a_i$ and $\delta(y) = 1$. Then consider its integral $\Psi$ which gives a biholomorphic map with a parameter $t$, namely $\Psi : (\mathbb{C}^{n+1+1}, 0) \to (\mathbb{C}^{n+1}, 0)$, which sends $(x, y, t)$ to $(\Psi_1(x, y, t), \ldots, \Psi_n(x, y, t), y+t)$ with

$\Psi_i(x,y,t) := \sum_{k=0}^{\infty} \frac{t^k}{k!}\delta^k(x_i)$ for $i = 1,\ldots,n$ and $\Psi_{n+1} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\delta^k(y)$ $= y + t$. On the algebra level the dual morphism for an $f \in \mathcal{O}$ is $\Psi^*(f,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\delta^k(f)t^k$ (a priori $\Psi^*$ is only formal but using Artin's approximation theorem, one finds that it is actually analytic). From equation (A.1) one gets

$$h(\Psi(x,y,t)) = e^{ta(x,y)}h(x,y). \tag{A.2}$$

Now define $\varphi_i(x,y) = \Psi_i(x,0,y)$ for $i = 1,\ldots,n$ and $\varphi_{n+1}(x,y) = \Psi_{n+1}(x,0,y) = y$. One immediately sees $\varphi_i(x,0) = x_i$ for $i = 1,\ldots,n$. Clearly $\varphi$ is a biholomorphic map and $\varphi^*(h) = h(\Psi(x,0,y))$. From equation (A.2) it follows that $h(\varphi_1,\ldots,\varphi_n,\varphi_{n+1}) = e^{ya(x,0)}h(x,0)$. Then putting $v(x,y) := e^{ya(x,0)}$ satisfies $v(x,0) \equiv 1$. Hence we have shown all conditions of (b). The induction from $k$ to $k+1$ for $k < m$ is done in an obvious way, see for example [81]. $\square$

# Appendix B

# Figures

In this appendix[3] illustrations of some examples of divisors in 2- and 3-dimensional manifolds appearing in the main text are shown. The divisors are visualized in $\mathbb{R}^2$ and $\mathbb{R}^3$. This realization may cause some geometric features of the divisors (originally defined over the complex numbers) to change. The pictures were produced by the author with the ray-tracing program POV-ray.

The main object of the thesis are divisors with normal crossings. In figure B.1 the typical normal crossing divisor in $\mathbb{R}^2$ and $\mathbb{R}^3$ is shown.

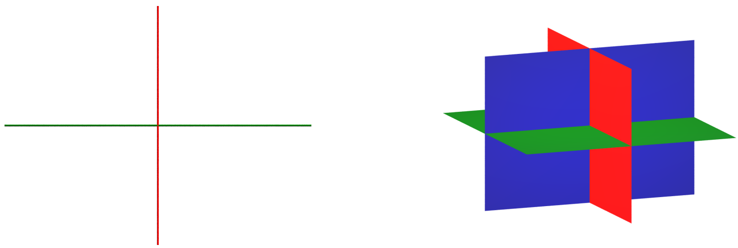

Figure B.1: Normal crossing divisor in $\mathbb{R}^2$ defined by $h = xy$ (left) and in $\mathbb{R}^3$ defined by $h = xyz$ (right)

In fig. B.2 the curves node and cusp are pictured. The node has normal crossings at the origin, i.e., it has an $A_1$-singularity. The cusp, on

[3] In order not to distract the reader's attention from the mathematics in the main text, we have chosen to defer the pictures to this appendix.

the other hand, has an $A_2$-singularity. Note that the cusp is even analytically irreducible.

Figure B.2: The node with equation $x^2 = y^2 + x^3$ and the cusp $x^3 = y^2$.

In chapter 1 free divisors are introduced. It is not easy to grasp the concept of freeness geometrically. In fig. B.3 the discriminant of a versal deformation of an $A_3$-singularity is shown (see Example 1.17). The singular locus of this free surface is one-dimensional and consists of a parabola and a cusp. Note that because of the visualization in $\mathbb{R}^3$ one cannot "see" the singular parabola. Another important example of a free divisor is the 4-lines divisor (fig. B.3) of Example 1.16: it consists of four smooth components and locally at the origin its singular locus is the $z$-axis. In fig. B.4 the free surface of Example 2.9 (2) is pictured. Here the singular locus consists of three smooth curves, along which the divisor does not have normal crossings.

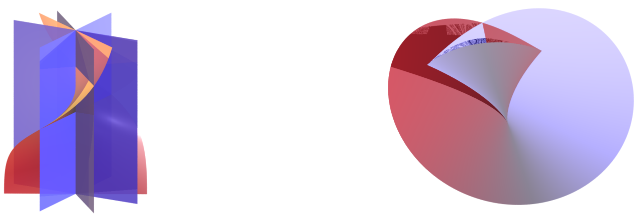

Figure B.3: The 4-lines: $xy(x+y)(x+yz)$ (left) and discriminant of versal deformation of an $A_3$-singularity (right).

The two divisors of fig. B.5 are everywhere free but at the origin. The Whitney Umbrella was considered in Example 1.13. It has the $z$-axis as singular locus but at the origin the Jacobian ideal has an embedded

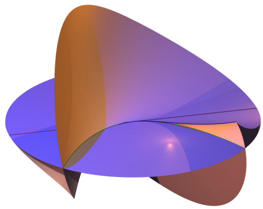

Figure B.4: Sekiguchi's $F_{B,1}$-example, with $h = z(x^2y^2 - 4y^3 - 4x^3z + 18xyz - 27z^2)$.

primary component. Outside the origin along the $z$-axis the Whitney Umbrella is analytically isomorphic to the union of two transversally intersecting hyperplanes, that is, it has normal crossings. The surface Tülle of Example 1.43 is the union of three smooth surfaces. In Example 3.50 it was shown that Tülle is not mikado at the origin.

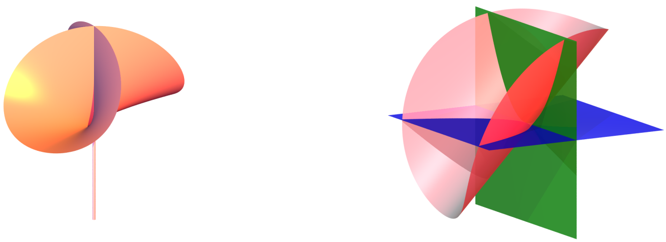

Figure B.5: The Whitney Umbrella: $x^2 - y^2z$ (left) and Tülle: $xz(x + z - y^2)$ (right).

In fig. B.6 the two surfaces from Example 2.47 are displayed: both have the cusp as singular locus. One surface is the union of the cylinder over the cusp with a transversal plane and is splayed and even free at the origin, whereas the other is neither free nor splayed at the origin.

Figure B.6: The cusp as singular locus: $h = x(y^2 - z^3$ (left) and $h = x(x + y^2 - z^3)$ (right).

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# Abstract English

The main objective of this thesis is to give an effective algebraic characterization of normal crossing divisors (= hypersurfaces) in complex manifolds. In order to obtain such a characterization we study logarithmic vector fields along a divisor, i.e., vector fields defined on the ambient space, which are tangent to the divisor at its smooth points, as well as logarithmic differential forms. Using the corresponding theory, which was developed by K. Saito, a characterization of a normal crossing divisor in terms of logarithmic differential forms (vector fields) is shown. Also a characterization of a normal crossing divisor in terms of the logarithmic residue is given (which is essentially due to Granger–Schulze). With this a question posed by K. Saito in 1980 can be answered.
In the second chapter we study singularities of normal crossing divisors, in particular we consider Jacobian ideals, which define the singular locus of a divisor. The main theorem is that a divisor has normal crossings at point if and only if it is free at the point, its Jacobian ideal is radical and its normalization is Gorenstein. Free divisors are defined via logarithmic vector fields and form a class of divisors containing normal crossing divisors. Since there exists an algebraic characterization of free divisors by their Jacobian ideals, our result yields a purely algebraic characterization of the normal crossings property. During the proof of the main theorem splayed divisors are introduced, which are a slight generalization of normal crossing divisors.
In the last part we consider further-reaching questions: first we ask, which radical ideals can be Jacobian ideals of divisors. Then splayed divisors are studied in more detail, in particular, we show that their Hilbert–Samuel polynomials satisfy a certain additivity property. Finally, we consider another generalization of normal crossing divisors, so-called mikado divisors. Here the plane curve case is studied and we characterize mikado curves by their Jacobian ideal.

# Zusammenfassung Deutsch

Das Hauptziel dieser Dissertation ist eine effektive algebraische Charakterisierung von Divisoren (= Hyperflächen) mit normalen Kreuzungen in komplexen Mannigfaltigkeiten anzugeben. Um eine derartige Charakterisierung zu finden, studieren wir sowohl logarithmische Vektorfelder entlang eines Divisors, d.h., Vektorfelder des umgebenden Raumes, die in allen glatten Punkten des Divisors tangential an ihn sind, als auch logarithmische Differentialformen. Mit Hilfe der zugehörigen Theorie, entwickelt von K. Saito, wird eine Charakterisierung von Divisoren mit normalen Kreuzungen durch logarithmische Differentialformen (Vektorfelder) gezeigt. Des weiteren wird eine Charakterisierung durch das logarithmische Residuum vorgestellt (diese beruht auf Ergebnissen von Granger und Schulze). Damit kann eine Frage von K. Saito beantwortet werden.

Im zweiten Kapitel werden Singularitäten eines Divisors mit normalen Kreuzungen untersucht, insbesondere betrachten wir das Jacobi Ideal, das den singulären Ort des Divisors definiert. Unser Hauptsatz besagt, dass ein Divisor genau dann normale Kreuzungen in einem Punkt besitzt, wenn er frei in diesem Punkt, sein Jacobi Ideal radikal und seine Normalisierung Gorenstein ist. Freie Divisoren werden durch logarithmische Differentialformen definiert und bilden eine Klasse von Divisoren, die insbesondere Divisoren mit normalen Kreuzungen enthält. Da eine algebraische Charakterisierung von freien Divisoren durch deren Jacobi Ideale existiert (nach A. G. Aleksandrov), ergibt sich aus unserem Resultat eine rein algebraische Charakterisierung der normalen Kreuzungsbedingung. Im Laufe des Beweises des Hauptsatzes werden gespreizte Divisoren eingeführt, die eine leichte Verallgemeinerung von Divisoren mit normalen Kreuzungen darstellen.

Im letzten Teil der Arbeit werden weiterreichende Probleme betrachtet: Zuerst fragen wir, welche radikalen Ideale Jacobi Ideale von Divisoren

sein können. Dann werden gespreizte Divisoren genauer untersucht, insbesondere zeigen wir, dass ihre Hilbert–Samuel Polynome eine gewisse Additivitätsbedingung erfüllen. Schließlich wird eine weitere Verallgemeinerung von Divisoren mit normalen Kreuzungen betrachtet, sogenannte Mikado Divisoren. Hier charakterisieren wir ebene Mikado Kurven durch ihr Jacobi Ideal.

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## Personal Information

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| Date of Birth | September 27, 1984 |
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## Education

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| 2007–present | **Ph.D. student in Mathematics**, *Universität Wien*, Fakultät für Mathematik, Vienna (Austria). |
| 2007–2008 | **Ph.D. student in Mathematics**, *Universität Innsbruck*, Institut für Mathematik, Innsbruck (Austria). |
| 2003–2007 | **MS in Technical Mathematics**, *Universität Innsbruck*, Institut für Mathematik, Innsbruck (Austria). passed with distinction |
| 1995–2003 | **High School**, education at the Wirschaftskundl. Realgymnasium der Ursulinen, Innsbruck (Austria). school leaving examination passed with distinction |

## Employment history

| | |
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| 01/2011–present<br>09/2009–12/2009 | **Ph.D. research assistant**, *Universität Wien*. in the frame of FWF-project P21461. |
| 12/2007–08/2009 | **Ph.D. research assistant**, *Universität Wien*. in the frame of FWF-project P18992. |

10/2006–02/2007 **Tutor**, *Universität Innsbruck, Institut für Mathematik.*

10/2005–09/2006 **Demonstrator**, *Universität Innsbruck, Institut für Mathematik.*

09/2005 **Vacation Job**, *Wohnheim Pradl*, Innsbruck.
work at the "Innsbrucker Menü-Serivce" and in the cafe

## Grants, prizes

2011 "For Women in Science" Award of L'Oréal Austria, the Austrian commission for UNESCO and the Austrian Academy of Sciences

2010 Recipient of a doc-fForte Fellowship of the Austrian Academy of Sciences

2009 Research Grant (Forschungsstipendium) F443 of the Universität Wien

2007 Grant for travelling expenses (Förderungsstipendium) of the Leopold-Franzens-Universität Innsbruck

03/2007–11/2007 FWF-grant (FWF-Forschungsbeihilfe für Diplomanden), Project P18992

2005,06,07 Scholarship of excellence (Leistungsstipendium) Leopold-Franzens-Universität Innsbruck

2003 Participation in the Mediterranean Mathematics Competition

2002,03 Participation in the final round of the Austrian Mathematical Olympiad (Advanced)

2001 1st prize in the final round of the Austrian Mathematical Olympiad (Beginners)

## Scientific Activities

2006 Participant in the 2nd YMIS *Combinatorial Convexity and Algebraic Geometry. Applications*, Sedano (Spain)
Participant in workshop *Algebraic Geometry and Singularities*, Obergurgl (Austria); Talk: *Toric Varieties*

2007 Participant in the 3rd YMIS *Algebra and Topology of Singularities*, Sedano (Spain)
Participant in the workshop *Algebraic Geometry and Singularities*, Nove Hrady (Czech Republic); Talk: *Newtonpolyeder und Explosionen*
Participant in the conference MEGA 2007 (Effective Methods in Algebraic Geometry), Strobl (Austria)
Organizer of *the 1st* $\alpha-\omega$*-conference in Algebraic Geometry*, Obergurgl (Austria)

2008 Participant in the 4th YMIS *Arc Spaces, Integration and Combinatorial Algebra*, Sedano (Spain); Talk: *Blowups in monomial ideals*
Participant in the conference *Mathematics in the world* (Meeting of the executive officers of the International Mathematical Union (IMU)), Budapest (Hungary)
Assistance at the organisation of *The Vienna Geometry Day*, ESI, Wien (Austria)
Participant in the workshop *Wien–Linz Workshop on Algebraic Geometry*, Spitz an der Donau (Austria); Talk: *Geometry and resolution of surfaces*
Participant in the conference *Seminar on Singularities: algebraic methods*, Garachico (Spain)
Participant in the conference *Deformations of Singularities*, Budapest (Hungary)
Organisation of *The Vienna Algebra Day*, ESI, Wien (Austria), (together with H. Hauser)
Participant in the conference *Workshop: On the Resolution of Singularities*, RIMS, Kyoto (Japan)
Research stay at Tokyo Institute of Technology, Tokyo (Japan), invited by Prof. S. Ishii; Talk: *Problems and phenomena in resolution of singularities*

2009 Participant in the 5th YMIS *Grothendieck Duality, Valuations, Resolution of Singularities and Tropical Geometry*, Sedano (Spain)
Assistance at the exhibition *IMAGINARY*, Vienna (Austria) and Berkeley (USA)
Participant in the workshop *Combinatorial, Enumerative and Toric Geometry*, MSRI, Berkeley (USA)

2010 Participant in the workshop *Wien–Linz Workshop on Algebraic Geometry*, Rastenfeld (Austria); Talk: *Normal crossings and logarithmic differential forms*

Participant in the workshop *Resolution of singularities*, Cuenca (Spain); Talk: *Blowups in tame monomial ideals*

Organizer of *The Vienna PDE–Day*, ESI, Wien (Austria), (together with H. Hauser)

Participant in the conference *Geometry at Large I + II*, Wien (Austria)

Research stay at Universidad de Sevilla, invited by Prof. L. Narváez-Macarro; Talk in the Seminar de Álgebra: *Normal crossings, free divisors and Jacobian ideals*

Research stay at Universidad Autónoma de Madrid, invited by Prof. O. Villamayor; Talk: *A conjecture about singularities of normal crossing divisors*

Participant in the workshop *Wien–Linz Workshop on Algebraic Geometry*, Krems (Austria); Talk: *About the logarithmic residue and a conjecture by K. Saito*

2011 Participant in the conference *Multiplier Ideals in Commutative Algebra and Singularity Theory*, CIRM, Luminy (France)

Organizer of the *Vienna ETH–Day*, ESI, Wien (Austria), (together with C. Bruschek and H. Hauser)

Participant in the workshop *Resolution of singularities II*, Cuenca (Spain); Talk: *An algebraic characterization of normal crossings*

Participant in the workshop *Free divisors*, Warwick (UK); Talk: *Normal crossing divisors and Jacobian ideals*

Participant in the summer school *D-modules and applications in singularity theory*, Sevilla (Spain)

Research stay at Universidad Autónoma de Madrid, invited by Prof. O. Villamayor

Organizer of the research programme *Algebraic vs. Analytic Geometry*, ESI, Wien (Austria), November–December 2011 (together with C. Bruschek, N. Lubbes and G. Rond)

## Publications


[1] E. Faber. Torische Varietäten, Explosionen und das Newtonpolyeder. Master's thesis (advisor: Ao.-Univ. Prof. Herwig Hauser), Universität Innsbruck, 2007.

[2] E. Faber and H. Hauser. Today's menu: geometry and resolution of singular algebraic surfaces. *Bull. Amer. Math. Soc.*, 47(3):373–417, 2010.

[3] E. Faber and D. Westra. Blowups in tame monomial ideals. *J. of Pure and Applied Algebra*, 215:1805–1821, 2011.