WEEK 13 (LAST ONE!)

We have shown that a map of complexes induces a map on homology, now more on this:

<u>Def 2.3.5</u> Let $h:(F_\circ,\partial^F_\circ)\to(G_\circ,\partial^G_\circ)$ be a map of complexes. We call the map
$$H_n(h): H_n(F_\circ)\to H_n(G_\circ)$$
$$a+B_n(F)\mapsto h_n(a)+B_n(G)$$
the induced <u>map in homology</u>, and sometimes denote it by h_*.

One can show that H_n preserves compositions, and the map in homology induced by the identity is the identity. $\Rightarrow$
Taking n-th homology is a functor
$$H_n: Ch(R)\to R\text{-Mod}$$
which takes each map of cx's $h:F_\circ\to G_\circ$ to the R-module homom
$$H_n(h): H_n(F_\circ)\to H_n(G_\circ).$$

<u>Def</u> 2.3.6 A map of chain complexes h is a <u>quasi-isomorphism</u> (or <u>quism</u>) if it induces an isomorphism in homology, i.e., $H_n(h)$ is an isom. of R-modules $\forall n$.
If $\exists$ a quasi-isom. between two complexes C and D, then C and D are <u>quasi-isomorphic</u>, written $C\simeq D$.

Here note: If we say f is a quism between $F_\circ$ and $G_\circ$, then not only $H_n(F_\circ)\cong H_n(G_\circ)$, but we have the stronger statement that the isomorphisms $H_n(F_\circ)\cong H_n(G_\circ)$ are all induced by f.

Not all quisms are isos:
<u>Example</u> 2.3.7: Let $\pi:\mathbb{Z}\to\mathbb{Z}/2\mathbb{Z}$ be the projection. Then

the chain map h:

$$\begin{tikzcd}
\cdots \arrow[r] & \overset{n+1}{0} \arrow[r, "\partial_{n+1}"] \arrow[d, "0"] & \overset{n}{\mathbb{Z}} \arrow[r, "\partial_n = 2"] \arrow[d, "0"] & \overset{n-1}{\mathbb{Z}} \arrow[r, "\partial_{n-1}"] \arrow[d, "\pi"] & \overset{n-2}{0} \arrow[r] \arrow[d] & \cdots & F_\bullet \\
& 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots & G_\bullet
\end{tikzcd}$$

is a quasiisom,

$H_k(F_\bullet) = 0$ $\left(= \ker\partial_k / \operatorname{im}(\partial_{k+1})\right)$ $k \neq n-1$ $(\ker(\cdot 2) = 0)$

$H_{n-1}(F) = \mathbb{Z}/2\mathbb{Z}$

$H_{n-1}(h) = \mathbb{Z}/2\mathbb{Z} \xrightarrow[\cong]{\pi \bmod 2} \mathbb{Z}/2\mathbb{Z}$

$a + 2\mathbb{Z} \mapsto \pi(a) + 2\mathbb{Z}$

$H_k(G_\bullet) = 0$ $k \neq n-1$

$H_{n-1}(G_\bullet) = \mathbb{Z}/2\mathbb{Z}$

Def 2.3.8 Let $f, g : F_\bullet \to G_\bullet$ be maps of complexes. A (chain) homotopy between f and g is a sequence of maps $h_n : F_n \to G_{n+1}$

$$\begin{tikzcd}
\arrow[r, "\partial_{n+2}"] & F_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[dl, "h_{n+1}"'] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & F_n \arrow[r, "\partial_n"] \arrow[dl, "h_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & F_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[dl, "h_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & \arrow[dl, "h_{n-2}"'] \cdots \\
\cdots \arrow[r] & G_{n+1} \arrow[r, "\partial_{n+1}"'] & G_n \arrow[r, "\partial_n"'] & G_{n-1} \arrow[r, "\partial_{n-1}"'] & {}
\end{tikzcd}$$

s.t. $h_{n-1} \circ \partial_n + \partial_{n+1} \circ h_n = f_n - g_n \quad \forall n$.

If there exists a homotopy between f and g, we say that f and g are <u>homotopic</u> (or have the <u>same homotopy type</u>). We write $f \simeq g$ in this case. If $f \simeq 0$, we say that f is <u>nullhomotopic</u> (not to be confused with $C \simeq D$ for cx's !!)

<u>Exercise</u>: Homotopy is an equivalence relation.

Equivalence classes under this relation are called <u>homotopy classes</u>.

Homotopy is interesting for us, since homotopic maps induce the same maps on homology:

<u>Lemma 2.3.9</u> Let $f, g : (F_\bullet, \partial_\bullet^F) \to (G_\bullet, \partial_\bullet^G)$ be maps of cxs. If f is homotopic to g, then $H_n(f) = H_n(g)$ $\forall n$.

In particular, every nullhomotopic map induces the 0-map on homology.

Pf: Let $f, g : (F_\bullet, \partial^F_\bullet) \to (G_\bullet, \partial^G_\bullet)$ be homotopic maps of cx's, and h a homotopy between f and g. Consider the map of cx's $(f-g): F_\bullet \to G_\bullet$ (defined in obvious way)

Claim: $(f-g)$ sends cycles (i.e. elts of $Z_n(F_\bullet)$) to boundaries (i.e. elts of $B_n(G_\bullet)$).

$\therefore$ If $a \in Z_n(F_\bullet)$: $(f-g)_n(a) \overset{\text{Def. of chain homotopy}}{=} \partial^G_{n+1} \circ h_n(a) + h_{n+1} \circ \underbrace{\partial^F_n(a)}_{=0, \text{ since } a \in \ker(\partial_n)}$

$$= \partial^G_{n+1}(h_n(a)) \in \operatorname{Im}(\partial^G_{n+1}) = B_n(G_\bullet). \checkmark$$

$\Rightarrow$ The map on homology induced by $f-g$ must then be the 0-map so f and g induce the same map on homology. (Here we are using that $H_n(f+g) = H_n(f) + H_n(g)$ ← ok, since all R-mod. maps!) $\square$

Not done in lecture!

Note: the converse is false: the induced map in homology can be 0 even if the original map is not nullhomotopic

Example: Consider the map of cx's: (!)

$$\begin{tikzcd}
F & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, "\cdot 0"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "\cdot 2"] & 0 \arrow[r] \arrow[d, "\cdot 0"] & \cdots \\
G & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "\cdot 2"'] & \mathbb{Z}/4\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Here $f: F \to G$ is not nullhom, but 0 on homology (!)

Def 2.3.10 If $f : (F_\bullet, \partial^F_\bullet) \to (G_\bullet, \partial^G_\bullet)$ and $g : (G_\bullet, \partial^G_\bullet) \to (F_\bullet, \partial^F_\bullet)$ are maps of cx's, s.t. $fg \simeq 1_{G_\bullet}$ on $(G_\bullet, \partial^G_\bullet)$ and $gf \simeq 1_{F_\bullet}$ on $(F_\bullet, \partial^F_\bullet)$ then f and g are called homotopy equivalences and $(F_\bullet, \partial^F)$ and $(G_\bullet, \partial^G)$ are homotopy equivalent.

Cor 2.3.11 Homotopy equivalences are quasi-isos.

Pf If $f:(F_\circ,\partial^F)\to(G_\circ,\partial^G)$ and $g:(G_\circ,\partial^G)\to(F_\circ,\partial_\circ^F)$ are s.t. $fg\simeq 1_{G_\circ}$ and $gf\simeq 1_{F_\circ}$, then by Lemma 2.3.9. the map fg induces the identity map on homology. So $\forall n$:

$$H_n(f)\circ H_n(g)\underset{\substack{\uparrow\\ H_n \text{ is functor}}}{=}H_n(fg)=H_n(1_{G_n})=1_{H_n(G)}$$

$G_n \to G_n$; $Z_n/B_n \xrightarrow{id} Z_n/B_n$

Exchanging roles of f and g shows that $H_n(f)$ an $H_n(g)$ must be isomorphisms. $\square$

The converse is false.

Not done in lecture

Exercise Let π denote the projection $\pi:\mathbb{Z}\to\mathbb{Z}/2\mathbb{Z}$. The chain map

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\pi"] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is a quism (see example above), but not a homotopy equivalence.

Rmk 2.3.12 Note that the relation $F\simeq G$, means "$\exists$ quasi-iso from F to G", is not symmetric (see exercise above: no quism in opp. direction is given.

Next we go onto consider sequences of α's, first some defs:

Def 2.3.13 Given complexes $B_\circ$ and $C_\circ$, $B_\circ$ is a subcomplex of $C_\circ$ if $B_n\subseteq C_n$ is a submodule $\forall n$, and the inclusion maps $i_n: B_n\hookrightarrow C_n$ define a map of complexes $i:B\to C$. Given a subcomplex B of C, the quotient of C by B is the complex C/B that has C_n/B_n in hom. degree n, with differential induced by the differential on C_n.

Exercise: The def of the differential of the quotient complex C/B makes sense.

Def 2.3.14 Given any map of complexes $f: B_\bullet \to C_\bullet$, the <u>kernel</u> of f is the subcomplex $\ker(f)$ of $B_\bullet$ that we get from the kernels $\ker(f_n)$:

$$\cdots \to \ker(f_n) \xrightarrow{\partial} \ker(f_{n-1}) \xrightarrow{\partial} \ker(f_{n-2}) \to \cdots \subseteq B_n$$

where the differentials are $\partial^B_\bullet|_{\ker(f)_\bullet}$, i.e. $\partial_n = \ker(f_n) \to B_{n-1}$, $a \mapsto \partial^B_n(a)$.

The <u>image</u> of f is the subcomplex $\operatorname{im}(f)_\bullet \subset C_\bullet$, where again differentials are given by the restriction of ∂^C to $\operatorname{im}(f)_\bullet$.

$$\cdots \to \operatorname{im}(f_n) \to \operatorname{im}(f_{n-1}) \to \operatorname{im}(f_{n-2}) \to \cdots$$

The <u>cokernel</u> of f is the quotient complex $C_\bullet / \operatorname{im} f_\bullet$.

Again check:

<u>Exercise</u>: Show that the kernel, image, and cokernel of a α are indeed complexes.

Def 2.3.15 A <u>complex</u> in $Ch(R)$ is a sequence of α's of R-modules C_n and chain maps $d_n: C_n \to C_{n-1}$ between them

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots$$

s.t. $d_n \circ d_{n+1} = 0$ $\forall n$. A complex of complexes is a diagram of the form

$$\begin{array}{ccccccc}
 & \downarrow & & \downarrow & & \downarrow & \\
\to & C_{n+1,i+1} & \xrightarrow{d_{n+1}} & C_{n,i+1} & \xrightarrow{d_n} & C_{n-1,i+1} & \to \\
 & \downarrow \partial_{i+1} & & \downarrow \partial_{i+1} & & \downarrow \partial_{i+1} & \\
\to & C_{n+1,i} & \xrightarrow{d_{n+1}} & C_{n,i} & \xrightarrow{d_n} & C_{n-1,i} & \to \\
 & \downarrow \partial_i & & \downarrow \partial_i & & \downarrow \partial_i & \\
\to & C_{n+1,i-1} & \xrightarrow{d_{n+1}} & C_{n,i-1} & \xrightarrow{d_n} & C_{n-1,i-1} & \to \\
 & \downarrow & & \downarrow & & \downarrow &
\end{array}$$

where $C_{i,j}$ is the module in hom. degree j in the complex C_i.
The n-th column $\leftrightarrow$ complex $(C_n)_\bullet$

Every row is also a cx. The vertical maps are diff. on each individual cx. and the horizontal maps are diff. on the complex of complexes.

Given a cx $C \in Ch(R)$, we can talk about cycles + boundaries and its homology. Such a cx. is exact if $im(d_{n+1}) = \ker(d_n)$ ∀n.

<u>Def 2.3.16</u> A <u>short exact sequence</u> of cx.'s is an exact cx. in $Ch(R)$ of the form:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

Equivalently a s.e.s. of cx's is a comm. diag:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, "\partial_{i+1}"'] & B_{i+1} \arrow[r, "g_{i+1}"] \arrow[d, "\partial_{i+1}"] & C_{i+1} \arrow[r] \arrow[d, "\partial_{i+1}"] & 0 \\
0 \arrow[r] & A_i \arrow[r, "f_i"] \arrow[d, "\partial_i"'] & B_i \arrow[r, "g_i"] \arrow[d, "\partial_i"] & C_i \arrow[r] \arrow[d, "\partial_i"] & 0 \\
0 \arrow[r] & A_{i-1} \arrow[r, "f_{i-1}"] & B_{i-1} \arrow[r, "g_{i-1}"] & C_{i-1} \arrow[r] & 0 \\
 & \vdots & & \vdots &
\end{tikzcd}$$

where rows are exact and cols are cx's.

<u>Short exact sequences</u>

We will study s.e.s. in more detail ⇝ Snake lemma, study long exact sequences (homology)

Everything here can be extended to s.e.s of complexes, just replace modules by complexes and R-homom. by maps of complexes.

<u>Ex</u>: (Trivial s.e.s) Let R be a ring, A and C be R-modules. $i: A \hookrightarrow C$, $a \mapsto a$ inclusion, $\pi: A \oplus C \to C$, $(a,c) \mapsto c$ are R-module maps.

Fit into s.e.s: $0 \to A \xrightarrow{i} A \oplus C \xrightarrow{\pi} C \to 0$.

Def 2.3.17 A s.e.s of R-modules *splits* or is a *split s.e.s* if

$$0 \to A \to B \to C \to 0$$

is isomorphic to $0 \to A \xrightarrow{i} A \oplus B \xrightarrow{\pi} C \to 0$

where i, π are as above.

Lemma 2.3.18 @Def (Splitting lemma) Consider the s.e.s.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "\varphi"] \arrow[r, shift left, "g"] & C \arrow[l, shift left, "r"] \arrow[r] & 0 \end{tikzcd}$$

of R-modules. TFAE:

(a) $\exists$ an R-module hom: $\varphi: B \to A$ s.t. $\varphi \circ f = Id_A$

(b) $\exists$ an R-module hom: $r: C \to B$ s.t. $g \circ r = Id_C$.

(c) The s.e.s splits.

The maps r, φ are called *splittings*.

Pf: $\underline{(c) \Rightarrow \overset{(b)}{(a)}}$: If

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"', "\cong"] & B \arrow[r, "g"] \arrow[d, "\beta"', "\cong"] & C \arrow[r] \arrow[d, "\gamma"', "\cong"] & 0 \\ 0 \arrow[r] & A \arrow[r, shift left, "i"] & A \oplus C \arrow[l, shift left, dashed, "p_A"] \arrow[r, shift left, "\pi"] & C \arrow[l, shift left, dashed, "i_C"] \arrow[r] & 0 \end{tikzcd}$$

splits, then:

Def: $p_A: A \oplus C \to A$, $(a,c) \mapsto a$ $\qquad i_c: C \to A \oplus C$, $c \mapsto (0,c)$

Now define maps

$$r := \beta^{-1} \circ i_c \circ \gamma$$

$$\varphi := \alpha^{-1} \circ p_A \circ \beta$$

Then $\varphi \circ f = \alpha^{-1} \circ p_A \circ \underbrace{\beta \circ f}_{\substack{i \circ \alpha \\ \text{commut.}}} = \alpha^{-1} \circ \underbrace{p_A \circ i}_{p_A \circ i = id_A} \circ \alpha = \alpha^{-1} \circ \alpha = Id_A$.

$$g \circ r = g \beta^{-1} i_c \gamma = (\gamma^{-1}\gamma) g \beta^{-1} i_c \gamma \underset{\text{comm}}{=} \gamma^{-1} \pi \beta \beta^{-1} i_c \gamma = \gamma^{-1} \underbrace{(\pi i_c)}_{Id_C} \gamma = Id_c.$$

$\Rightarrow$ (a) and (b).

$\underline{(b) \Rightarrow (c)}$: First show $B \cong A \oplus C$ $\qquad$ $\left[(b) \quad \begin{tikzcd} 0 \arrow[r] & A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "\varphi"] \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd} \quad \varphi f = Id_A. \right]$

We can write $\forall b \in B$: $b = (b - f\varphi(b)) + f\varphi(b)$, where $f\varphi(b) \in im(f) \cong A$.

$\Rightarrow \varphi(b - f\varphi(b)) = \varphi(b) - \underbrace{\varphi f}_{1_A}\varphi(b) = 0. \quad \Rightarrow b - f\varphi(b) \in \ker(\varphi)$

$\Rightarrow B = \operatorname{Im}(f) + \ker(\varphi).$

If $f(a) \in \ker(\varphi)$, then $a = f\varphi(a) = f(0) = 0. \Rightarrow \operatorname{im} f \cap \ker g = 0$ and

$$B = \operatorname{im} f \oplus \ker(\varphi).$$

If we restrict g to $\ker(\varphi)$, it becomes injective.

<u>Claim</u>: $g : \ker(\varphi) \to C$ is surjective and thus an iso.

$\therefore$: $\forall c \in C : \exists b \in B$ s.t. $g(b) = c$ (since g is surjective).

We can also write $b = f(a) + k$ for some $k \in \ker(\varphi)$.

Then $g(k) = \underbrace{gf}_{=0}(a) + g(k) = g(f(a) + k) = g(b) = c.$ $\square$

Finally, note that $\operatorname{im}(f) \cong A \Rightarrow B \cong A \oplus C$ via isom ψ given by

$$\begin{aligned} B &\to \operatorname{im} f \oplus \ker\varphi \to A \oplus C \\ b &\mapsto (f\varphi(b), b - f\varphi(b)) \mapsto (\varphi(b), g(b)) \end{aligned} \quad (!)$$

$\rightsquigarrow$
$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "\psi"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] & A \oplus C \arrow[r, "\pi"] & C \arrow[r] & 0
\end{tikzcd}$$

Since ψ is an iso, so is our map of complexes, and thus our original sequence is a split exact sequence.

<u>(b) $\Rightarrow$ (c)</u>: Similar. $\square$

<u>Example 2.3.19</u>

(a) $0 \to K[x] \xrightarrow{\cdot x = f} K[x] \xrightarrow{\pi} K[x]/(x) \to 0$ is not split exact. ($K[x]/(x) \cong K$)

Show that there is no homom: $\alpha : K[x] \to K[x]$ s.t. $\alpha f(P) = P \ \forall P \in K[x]$

($\alpha(f(P)) = \alpha(xP) = x\alpha(P)$. If $x\alpha(P) = P \ \forall P$, then also for $P = 1$

But: $x \cdot \alpha(1) = 1 \Rightarrow \alpha(1) = \frac{1}{x} \notin K[x]$ ⚡).

(b) $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is not split exact: since $\mathbb{Z} \ncong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

($\mathbb{Z}$ has no torsion elements!)

(c) caution: $0 \to A \xrightarrow{f} A \oplus C \xrightarrow{g} C \to 0$ is only split if f is canon. inclusion and g is projection.

Exercise 2.3.20 (The 5-Lemma) Consider the comm. diag. of R-modules with exact rows:

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "a"'] & B' \arrow[r] \arrow[d, "b"] & C' \arrow[r] \arrow[d, "c"] & D' \arrow[r] \arrow[d, "d"] & E' \arrow[d, "e"] \\ A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E \end{tikzcd}$$

Show that if a, b, d, and e are isos, then c is an iso.

Exercise 2.3.21 Let R be a ring. Show that if F is a free R-module, then every s.e.s. of R-modules splits

$$0 \to A \to B \to F \to 0.$$

Long Exact sequences

⇝ Building l.e.s. from s.e.s., use the snake lemma.

Thm 2.3.22 (Snake Lemma) Consider the comm. diagram of R-modules

$$\begin{tikzcd} & A' \arrow[r, "i'"] \arrow[d, "f"'] & B' \arrow[r, "p'"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] & C & \end{tikzcd}$$

Suppose that the rows of this diagram are exact. There exists a map $\partial : \ker h \to \operatorname{coker}(f)$ and $\exists$ an exact sequence

$$\ker f \to \ker g \to \ker h \xrightarrow{\partial} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h).$$

∂ is constructed as follows: given $c' \in \ker h$, pick $b' \in B'$ s.t. $p'(b') = c'$ and $a \in A$ s.t. $i(a) = g(b')$, then

$$\partial(c') = a + \operatorname{im}(f) \in \operatorname{coker}(f).$$

Moreover: • If i' is injective, then we can extend our ex. seq. to

$$0 \to \ker f \to \ker g \to \dots$$

• If p is surjective, then we can extend our ex. seq. to

$$\dots \to \operatorname{coker} g \to \operatorname{coker} h \to 0.$$

Pic:

$$\begin{tikzcd}
 & \ker f \arrow[r] \arrow[d] & \ker g \arrow[r] \arrow[d] & \ker h \arrow[d] \arrow[ddddll, bend left, "\partial"] & \\
 & A' \arrow[r, "i'"] \arrow[d, "f"] & B' \arrow[r, "p'"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[d] & B \arrow[r, "p"] \arrow[d] & C \arrow[d] & \\
 & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h &
\end{tikzcd}$$

$c' \in \ker h$

$\exists b' \in B' : p'(b') = c'$

by comm: $\underbrace{h(p'(b'))}_{h(c') = 0} = p(g(b'))$ $\Rightarrow g(b') \in \ker p$

$\Rightarrow \exists a \in A : i(a) = g(b')$

The map ∂ is called connecting homom.

Pf: See the movie 'It's my turn' 1981 → youtube.

[Idea: First show that we get ex. seq.

$\ker f \xrightarrow{i'} \ker g \xrightarrow{p'} \ker(h)$ by restriction to ker's.

Similarly $\operatorname{im} f \xrightarrow{i} \operatorname{im} g \xrightarrow{p} \operatorname{im}(h)$ exact.

$\rightsquigarrow$ $\operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h)$ exact.

- Construction of ∂: ∂ is well-defined
- Show that seq.: $\ker g \to \ker h \xrightarrow{\partial} \operatorname{coker} f \to \operatorname{coker} g$ is exact

(⇒ Lemma!) $\square$]

This is proof (idea) by diagram chase. Alternatively: proof using pullback / pushout squares.

Thm 2.28 (Long exact sequence in homology) Given a short exact sequence in $Ch(R)$ $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$

there are connecting homoms $\partial : H_n(C) \to H_{n-1}(A)$ s.t.

$$\cdots \to H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

is an exact sequence.

Pf: For each n we have s.e.s.

$$0 \to A_n \to B_n \to C_n \to 0$$

The cond. that f, g are maps of cx, implies by Lemma 2.3.4,

that f,g take cycles to cycles. This gives us an induced map (11)
$Z_n(A) \to Z_n(B)$, and since this map is the restriction of an inclusion, it must also be an inclusion. $\Rightarrow$ Get exact seq.

$$0 \to Z_n(A) \to Z_n(B) \to Z_n(C)$$

Again, by Lemma 2.3.4, the cond., that f,g are maps of cx's $\Rightarrow$ f and g take boundaries to boundaries $\Rightarrow$ ex. seq.

$$A_n/\operatorname{im} d^A_{n+1} \to B_n/\operatorname{im} d^B_{n+1} \to C_n/\operatorname{im} d^C_{n+1} \to 0.$$

Now use the general fact:
If $\mathcal{F}$ is any cx., the boundary maps on $\mathcal{F}$ induce maps $\mathcal{F}_n \to Z_{n-1}(\mathcal{F})$ that send $d^{\mathcal{F}}_{n+1}$ to 0, so we get all induced maps

$$\mathcal{F}_n/\operatorname{im} d^{\mathcal{F}}_{n+1} \to Z_{n-1}(\mathcal{F}).$$

Now in our situation to A, B, C, we get a comm. diag

$$\begin{tikzcd}
 & A_n/\operatorname{im} d^A_{n+1} \arrow[r] \arrow[d, "d^A_n"] & B_n/\operatorname{im} d^B_{n+1} \arrow[r] \arrow[d, "d^B_n"] & C_n/\operatorname{im} d^C_{n+1} \arrow[r] \arrow[d, "d^C_n"] & 0 \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r] & Z_{n-1}(B) \arrow[r] & Z_{n-1}(C) &
\end{tikzcd}$$

For any cx. $\mathcal{F}$,

$$\ker(\mathcal{F}_n/\operatorname{im} d^{\mathcal{F}}_{n+1} \xrightarrow{d^{\mathcal{F}}_n} Z_{n-1}(\mathcal{F})) = H_n(\mathcal{F})$$

and

$$\operatorname{coker}(\mathcal{F}_n/\operatorname{im} d^{\mathcal{F}}_{n+1} \xrightarrow{d^{\mathcal{F}}_n} Z_{n-1}(\mathcal{F})) = Z_{n-1}(\mathcal{F})/\operatorname{im} d^{\mathcal{F}}_n = H_{n-1}(\mathcal{F}).$$

The snake lemma gives us exact sequences

$$H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to.$$

Finally: Glue these together to get long exact seq. on homology.
$\square$